Monday Morning, June 24, Second Hour

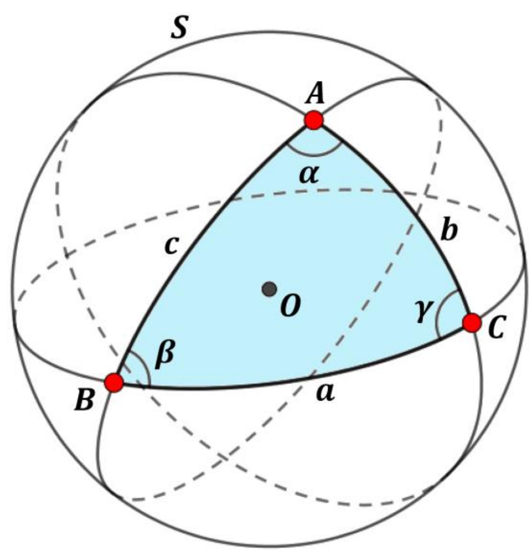
Distance Between Two Cities

An important application of the spherical law of cosines is to find the great circle distance from point B to point C from the latitude and longitude coordinates of these two points.

The key insight is to introduce another point A, namely the North Pole (if your problem is in the Northern Hemisphere).

As you will learn, the spherical triangle connecting points B and C with the North Pole fits naturally with the spherical law of cosines and spherical law of sines.

But before demonstrating how that happens, we will start with a quick review of how **latitude** and **longitude** are defined.



Review of Latitude and Longitude

In the exercises that follow you will need to remember the following definitions from geography (assuming we take Earth as a perfect sphere):

Meridian – any arc of a great circle on Earth connecting the North and South Poles.

Prime Meridian – the meridian which goes through the Royal Observatory in Greenwich (a section of London), England

Equator – the great circle of all points on Earth which are equally distant from the North and South Poles

Latitude – the standardized distance (in angular degrees North or South) a point on Earth is from the Equator

Longitude – the standardized distance (in angular degrees East or West) a point on Earth is from the Prime Meridian

Circle (or lines) of Latitude – a small circle on Earth connecting all points at a given latitude

Line of Longitude – another term for a meridian

Parallels – another term for a circle of latitude

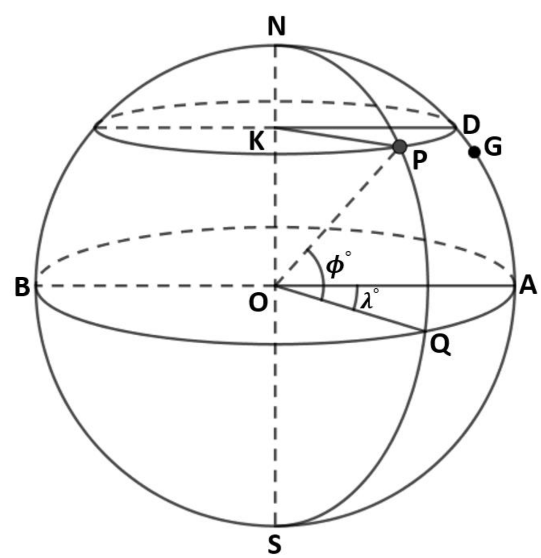
(Note: Lines of latitude and lines of longitude are perpendicular to each other.)

(Latitude, Longitude) Coordinates – every point of Earth is uniquely identified by its (Latitude, Longitude) or (ϕ, λ) coordinates in angular degrees.

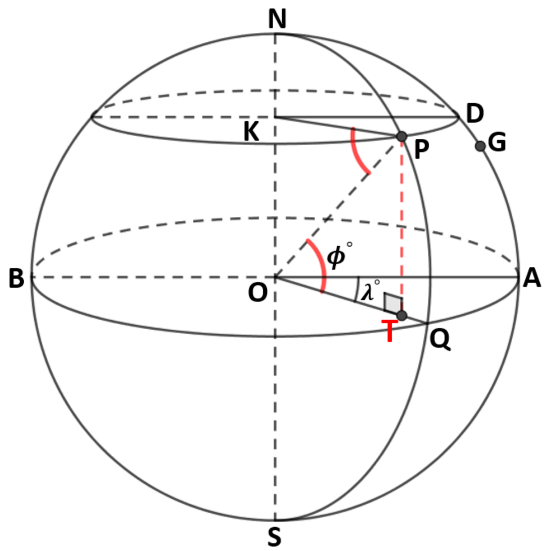
- Points north of the Equator have a latitude value ϕ in the range $0°$N to $90°$N while points south of the Equator have a latitude value in the range $0°$S to $90°$S.
- Points on the Equator have latitude value $\phi = 0°$.
- Points east of the Prime Meridian have a longitude value λ in the range $0°$E to $180°$E while points west of the Prime Meridian have a longitude value λ in the range $0°$W to $180°$W.
- Points on the Prime Meridian have longitude value $\lambda = 0°$.

Null Island – the fictious point where $(\phi, \lambda) = (0°, 0°)$.

Let $\angle AOQ = \phi°$ and $\angle QOP = \lambda°$. Then, with the notation as in the figure below, P is said to have **latitude $\boldsymbol{\phi}$° North and longitude $\boldsymbol{\lambda}$° West**.



Now imagine dropping a perpendicular from P to the radius $\overline{OQ}$ intersecting at T. Then the quadrilateral KPTO is a rectangle and it follows that $m\angle KPO = m\angle QOP = \phi°$. Also notice that $OP = OQ = ON$ as all are radii of the sphere.



Therefore $\cos(\phi°) = KP/OP$ or $KP = OP\cos(\phi°)$. Hence, if the radius of the Earth equals r miles, then the circle of latitude containing P has radius

$$\mathbf{KP = r\cos(\phi°)\ miles.}$$

Now notice that the points D and A are both on the prime meridian containing G (Greenwich). And the points P and Q are on the same meridian. Therefore, points P and Q are said to have the same longitude. That is $m\angle DKP = m\angle POQ = \lambda°$. Hence,

$$\begin{aligned}\text{arc } \widehat{PD} &= \left(\frac{\lambda°}{360} \times 2\pi\right) \times KP \\ &= \left(\frac{\lambda°}{360} \times 2\pi\right) \times r\cos(\phi°) \text{ miles.}\end{aligned}$$

A ***nautical mile*** is the length of an arc of the meridian which subtends an angle of $1'$ at the center of the Earth.

Taking the radius of the Earth as $r = 3958.8$ statute miles, we see that

$$\begin{aligned}1 \text{ nautical mile} &= \frac{1}{60 \times 360} \times 2\pi \times 3958.8 \\ &= 1.15156824 \text{ statute miles} = 6080.280307 \text{ feet.}\end{aligned}$$

But the distance between two places on the same parallel of latitude, measured along the parallel, whose longitudes differ by, say, $1'$ *depends on their latitude*, for the distance is an arc of a circle of radius $r\cos(\phi°)$; the greater ϕ is, the smaller this distance is.

Example

A ship after sailing 200 (nautical) miles due West finds that her longitude has altered by $5°$. What is her latitude?

<u>Solution</u>

Without loss of generality, we can assume the ship started on the prime meridian. There are 1.15156824 statute miles (our regular mile) in one nautical mile. Therefore,

$$200 \text{ nautical miles} = 230.313648 \text{ statute miles.}$$

Taking the radius of the Earth as 3958.8 statute miles, we see that

$$\text{arc } \widehat{PD} = \left(\frac{\lambda°}{360} \times 2\pi\right) \times r\cos(\phi°) \text{ miles.}$$

Therefore,

$$230.313648 = \left(\frac{5}{360} \times 2\pi\right) \times 3958.8\cos(\phi°)$$

$$\phi° = \arccos\left(\frac{230.313648 \cdot 360}{3958.8 \cdot 5 \cdot 2\pi}\right) = \arccos(0.6666666664)$$

$$= 48.189685125°$$

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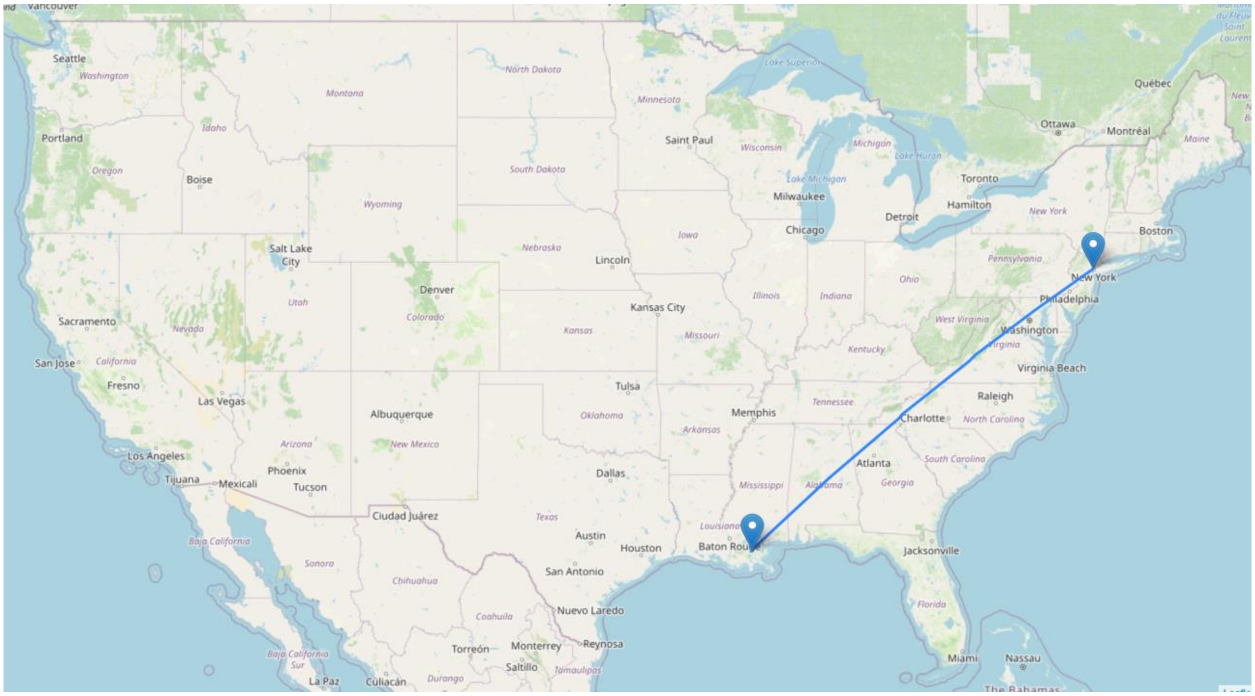
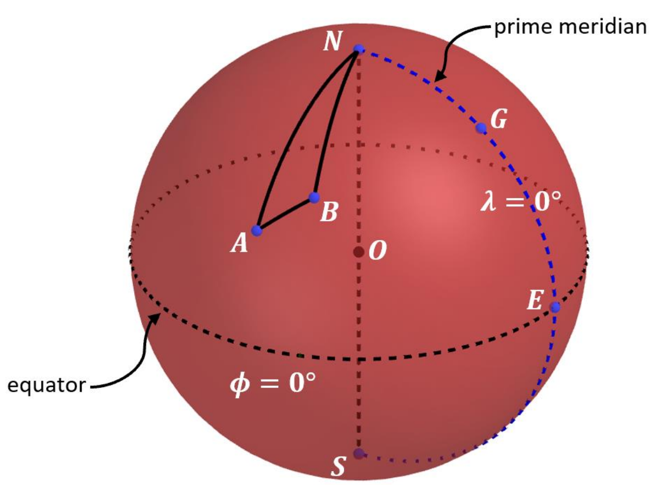
Example. The Great Circle (Geodesic) Distance From New Orleans to New York City

You've booked a flight from New Orleans, (latitude, longitude) $= (\phi, \lambda) = (30°\,\mathrm{N}, 90°\,\mathrm{W})$, to New York City, (latitude, longitude) $= (\phi, \lambda) = (41°\,\mathrm{N}, 74°\,\mathrm{W})$. Find the great circle distance of your flight.



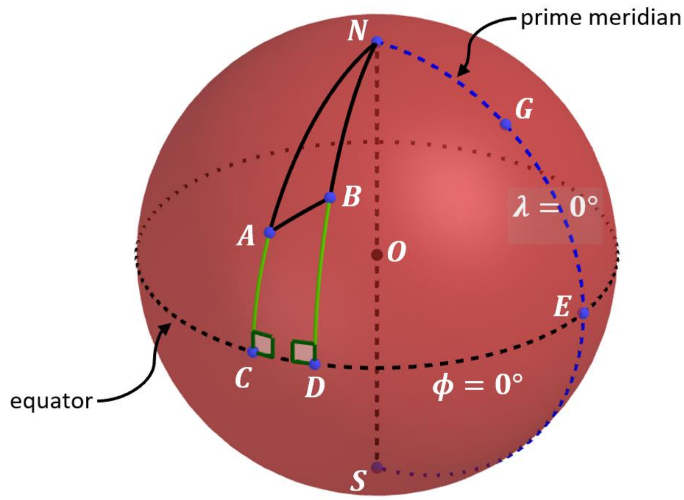
Solution

Consider the spherical triangle $\Delta_S(ABN)$ with A at New Orleans, B at New York and N representing the North Pole and S the South Pole. Let G be the Royal Observatory in Greenwich, England, let O be the center of Earth and let E be Null Island where $(\phi, \lambda) = (0°, 0°)$.



The problem is asking for $\widehat{AB}$.

Extend the great circle arcs $\widehat{NA}$ and $\widehat{NB}$ down to the equator to the points C and D. Then $\widehat{NC} = 90°$ and $\widehat{ND} = 90°$ because both are quadrants. We have $\angle NCE = 90°$ and $\angle NDE = 90°$ because lines of longitude are perpendicular to lines of latitude.



By definition of latitude, we have $\widehat{CA} = 30°$, the latitude of point A. In the same way $\widehat{DB} = 41°$, the latitude of point B.

It follows that $\widehat{NA} = 90° - 30° = 60°$ and $\widehat{NB} = 90° - 41° = 49°$.

By definition of longitude, we have $\angle GNB = 74°$, the longitude of point B. In the same way $\angle GNA = 90°$, the longitude of point A.

Therefore, $\angle BNA = 90° - 74° = 16°$.

Applying the spherical law of cosines for sides to spherical triangle $\Delta_S(ABC)$, we have

$$\cos\left(\widehat{AB}\right) = \cos\left(\widehat{NB}\right)\cos\left(\widehat{NA}\right) + \sin\left(\widehat{NB}\right)\sin\left(\widehat{NA}\right)\cos(\angle BNA)$$

$$= \cos(49°)\cos(60°) + \sin(49°)\sin(60°)\cos(16°) = 0.95630.$$

Thus,

$$\widehat{AB} = \cos^{-1}(0.9560) = 17.05962° = 0.29774653813 \text{ radians}.$$

Notice that I have expressed the distance $\widehat{AB}$ in terms of radians. To make that legitimate I have to work in "***standardized units of length***" where

$$\textbf{1 standarized unit} = r = 3{,}958.8 \text{ miles} = \text{radius of the Earth.}$$

Therefore,

$$\begin{aligned}\widehat{AB} &= 0.29774653813 \text{ standarized units of length}\\ &= 0.29774653813 \text{ standarized units} \cdot \frac{3{,}958.8 \text{ miles}}{1 \text{ standarized unit}}\\ &= 1178.72 \text{ miles.}\end{aligned}$$

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Angular Distance or Standardized Units of Distance

It simplifies notation when you adopt "standardized units" where

$$\textbf{1 standardized unit} = \boldsymbol{r}$$

where r is the radius of the sphere in your problem.

Thinking in terms of "standardized units" is (basically) equivalent to ignoring r in your problem until the very end.

In standardized units, ***angles measure length***. In standardized units it makes sense to say "the distance" between points A and B, $\widehat{AB}$, equals $30°$. If we are on earth, saying $\widehat{AB} = 30°$ is a shortcut for saying

$$\widehat{AB} = 30° = 30° \cdot \frac{2\pi r}{360°} \text{ miles}$$

where $r = 3{,}958.8$ miles is the radius of the earth.

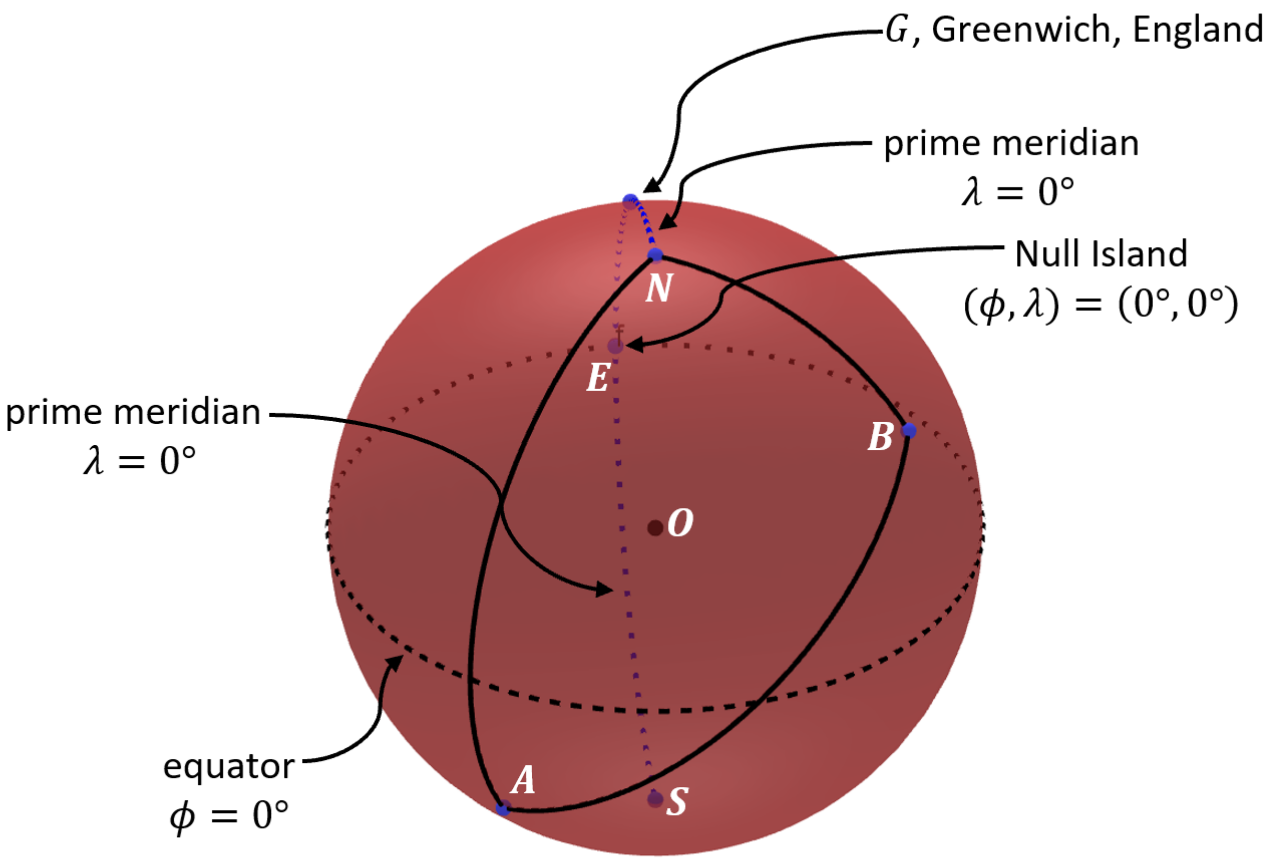
If we are working in radians, saying $\widehat{AB} = \pi/6$ radians is a short cut for saying

$$\widehat{AB} = \frac{\pi}{6} \text{radians} = \frac{\pi}{6} r \text{ miles.}$$

<u>Exercise</u>. The Great Circle (Geodesic) Distance from Los Angeles to Adelaide

You have a booked a flight from Los Angeles, (latitude, longitude) $= (\phi, \lambda) = (34°03'\text{ N}, 118°15'\text{ W})$, to Adelaide, Australia (latitude, longitude) $= (\phi, \lambda) = (34°55'42''\text{ S}, 138°36'02''\text{ E})$.

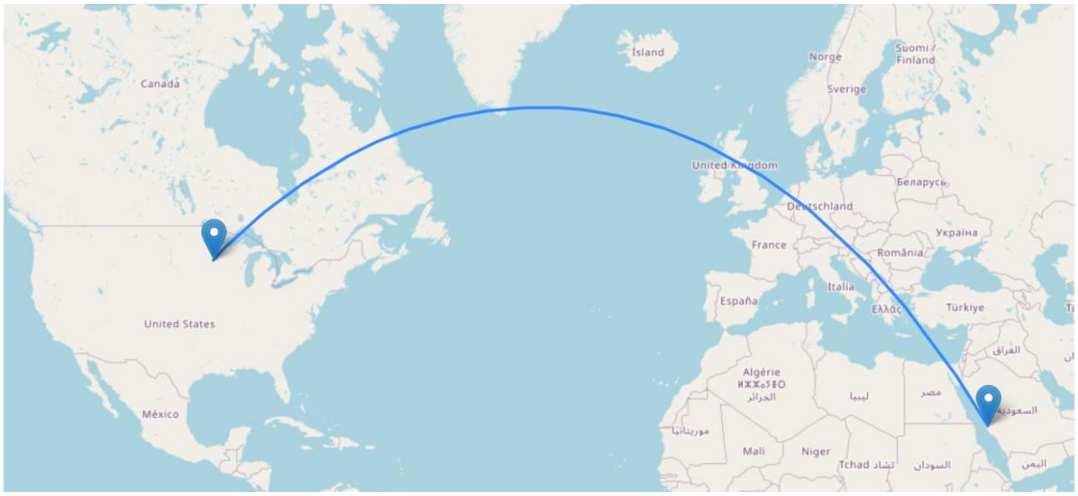
Consider the spherical triangle $\Delta_S(ABN)$ with A at Adelaide, B at Los Angeles with N representing the North Pole and S the South Pole. Let G be the Royal Observatory in Greenwich, England, let O be the center of Earth and let E be Null Island where $(\phi, \lambda) = (0°, 0°)$.

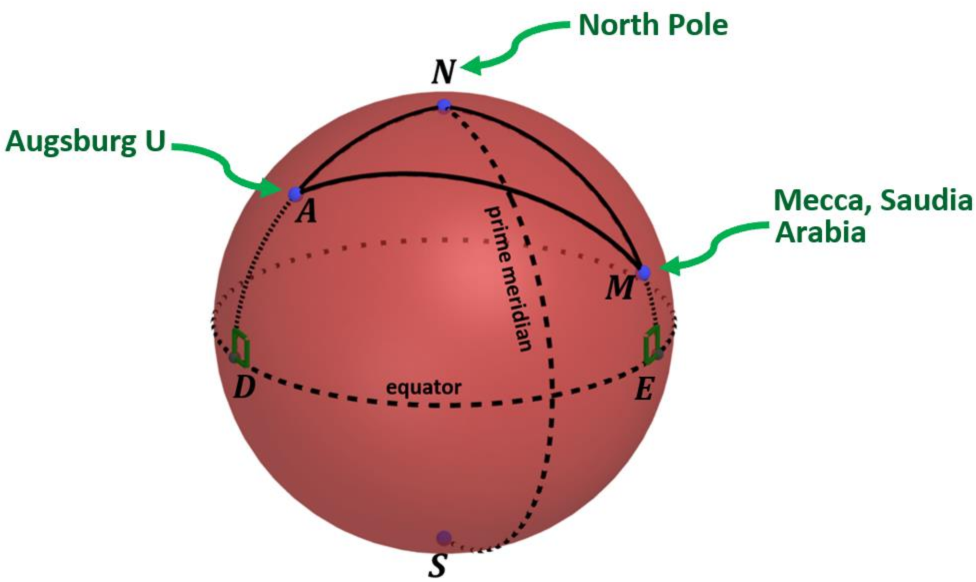


(a) Find $\widehat{BN}$.	(d) Find $\cos\left(\widehat{AB}\right)$.
(b) Find $\widehat{AN}$.	(e) Find $\widehat{AB}$ in miles.
(c) Find $\angle ANB$.	

The Quibla

According to the Islamic faith, a follower must offer their prayers facing Mecca by means of the most direct route (that is, the great circle route). This bearing is called the ***quibla***. For followers of Islam in Minneapolis, what is their quibla? In navigational settings, bearing is defined as the degrees off North, going clockwise.





A: Augsburg University ($\lambda = \text{latitude} = 44.9659°$ N, $\varphi = \text{longitude} = 93.2407°$ W)

M: Mecca, Saudia Arabia ($\lambda = 21.4241°$ N, $\varphi = 39.8173°$ E)

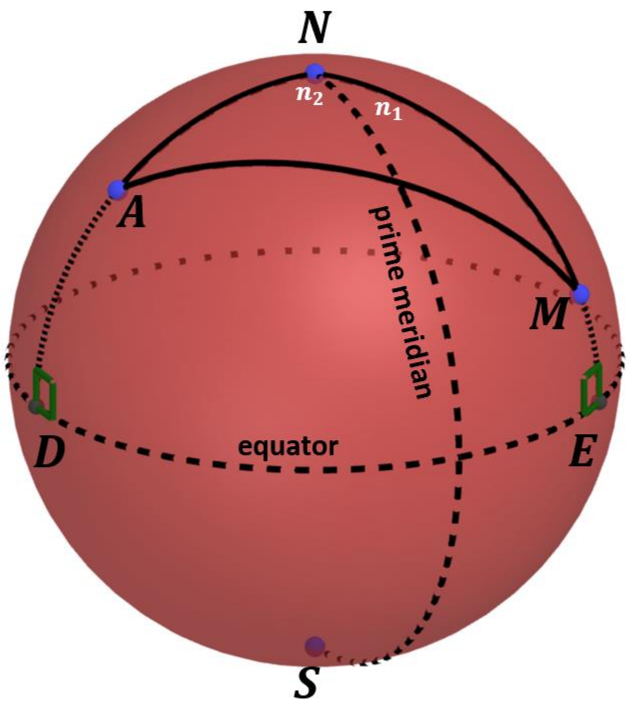
N: North Pole, S: South Pole.

D: point where the great circle through A and N intersects the equator

E: point where the great circle through M and N intersects the equator

<u>Solution</u>

What do we know?



We know …

$n_1 = 39.8173$, the longitude of Mecca.

$n_2 = 93.2407$, the longitude of Augsburg University.

$\angle N = n_1 + n_2 = 133.058°$.

$\widehat{AD} = 44.9659°$ (the latitude of Augsburg) $\Longrightarrow \widehat{AN} = 90° - 44.9659° = 45.0341°$

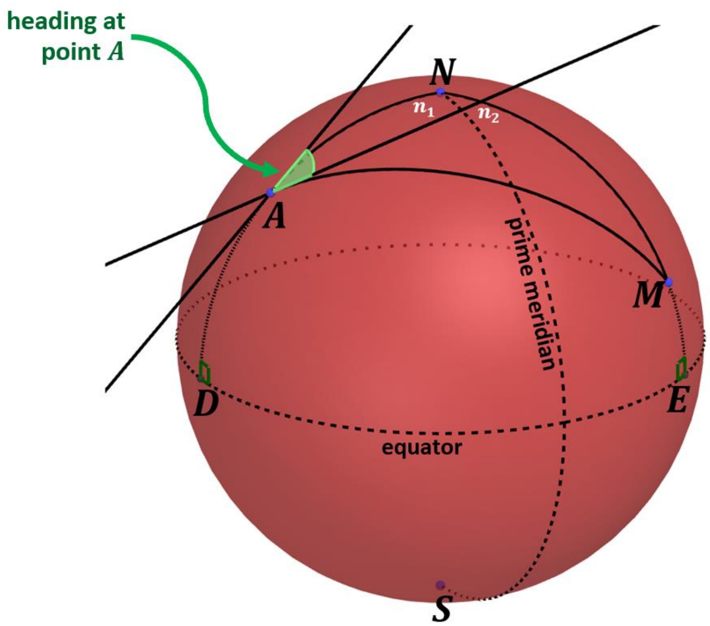
$\widehat{ME} = 21.4241°$ (the latitude of Mecca) $\Longrightarrow \widehat{MN} = 90° - 21.4241° = 68.5759°$

A: Augsburg University ($\lambda =$ latitude $= 44.9659°$ N, $\varphi =$ longitude $= 93.2407°$ W)

M: Mecca, Saudia Arabia ($\lambda = 21.4241°$ N, $\varphi = 39.8173°$ E)

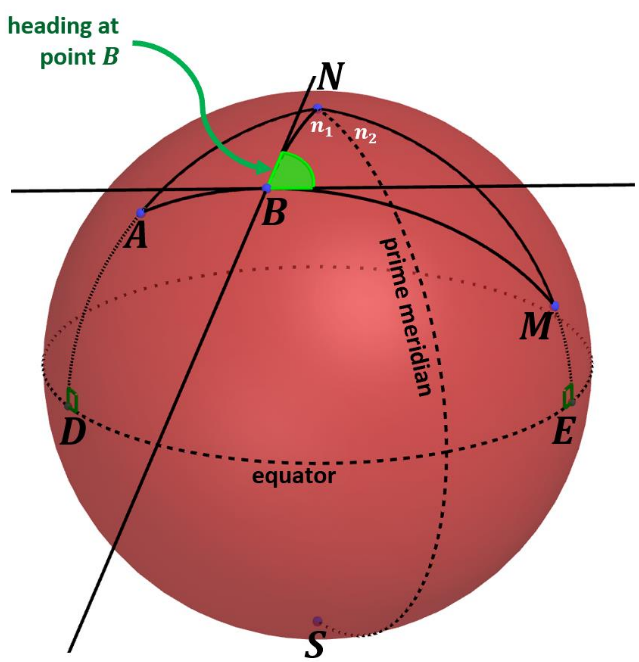
What do we want to find?

The heading between point A and point M. Heading is a technical term used in navigation and is defined as the spherical angle at A between the great arc connecting A to N (the North Pole) and the great arc connecting A to M. And by the definition of a spherical angle, this means that the initial heading equals the angle at A between the tangent lines to the great arc $\widehat{AN}$ and the great arc $\widehat{MN}$.



It should be made clear that heading along a great circle connecting A to M is not fixed. ***It changes continuously***.

That is the ***heading at a waypoint B*** between A and M will not be the same as the ***initial heading at A***.



So, to make this question precise, we should say that what we are looking to find is the ***initial heading*** at point A.

Do we have enough information to find $\angle NAM$?

A spherical triangle contains six pieces, ***three angles and three sides***. We can find the missing pieces ***if we know at least three of the six pieces***. (However, there are a few situations where we will get two possible solutions.)

We have three pieces of information in this problem, sides $\widehat{AN}$, $\widehat{MN}$ and $\angle ANM$ so we have enough information.

Finish this problem as your first homework assignment.

Step 1. Use the Spherical Law of Cosines to find $\widehat{AM}$, the *angular distance* between A and M.

Step 2. Use the Spherical Law of Cosines a second time to find $\angle NAM$.

AAA Is Solvable in Spherical Trigonometry

One striking difference between spherical and planar trigonometry is that the AAA (Angle, Angle, Angle) case is solvable in spherical trigonometry..

How? Think again about polar triangles where angles become sides and sides become angles. So, AAA is solvable by swapping angles for sides and then solving the SSS problem.

We will use this to our advantage in solving our next problem about spherical chalkboards.

Example 2. A Weighty Problem

You might find a spherical chalkboard such as the one shown below in American high school mathematics classrooms of yesteryear when spherical trigonometry was still a course option. They were an effective visual aid for students.

One practical difficulty with these chalkboard globes is that they could be *very* heavy, especially the oldest ones which were a solid piece of wood. How heavy were they? That's what this question is about. (Note: The chalkboard shown above is not actually solid but it is still somewhat heavy.)

Let r be the radius of this sphere in inches. Then the volume of this sphere is

$$V = \frac{4}{3}\pi r^3 \text{ in}^3.$$

Let ρ be the density in pounds per cubic inch of the wood used in making the globe. Assuming the globe is a solid piece of wood, then the weight of this globe is

$$W = \rho V = \frac{4}{3}\rho\pi r^3 \text{ lbs.}$$

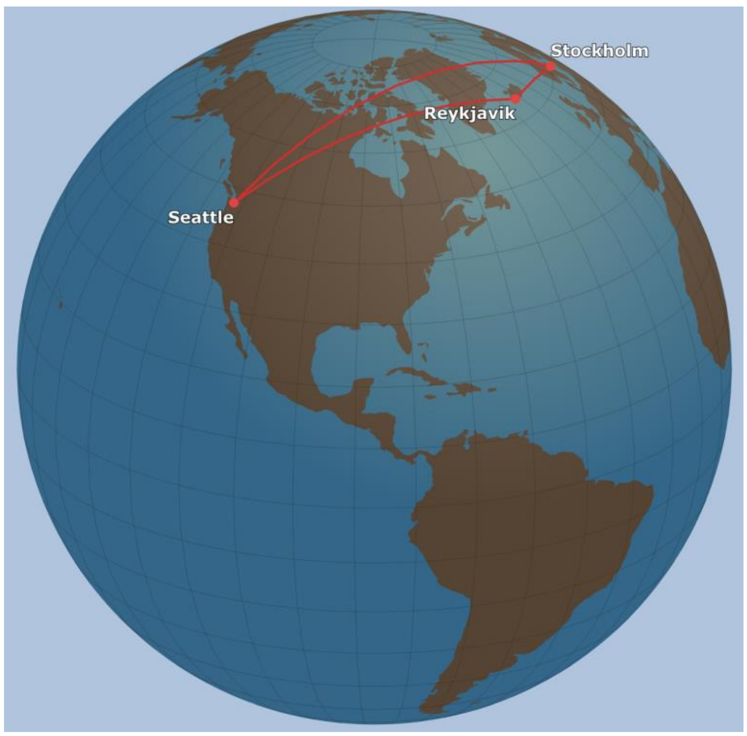
Suppose this spherical chalkboard is a solid piece of oak for which $\rho = 0.033$ pounds per cubic inch. Then use the information from the spherical triangle drawn on the blackboard in the figure below to find r and W. (Assume the sides are measured in inches and ignore the extra weight of the metal stand.)

Hint: Look at the Spherical Law of Cosines for Angles. What is the only unknown? It's a one equation one unknown situation.

Example

Imagine that you are a new college graduate living in Seattle ($\lambda = 47.6061°\text{ N}, \varphi = 122.3328°\text{ W}$) and you get a job interview in Stockholm ($\lambda = 59.3293°\text{ N}, \varphi = 18.0686°\text{ E}$). The flight over there will be direct. But on the flight home you have decided to make a stopover in Reykjavik, Iceland ($\lambda = 64.1470°\text{ N}, \varphi = 21.9408°\text{ W}$) to do some sightseeing. Perhaps you might decide to buy a lopi (lopapeysa) while there.

How many additional miles will the side trip to Reykjavik add to the direct route from Seattle to Stockholm? (Assume all routes follow great circles tracts and that the Earth is a sphere with a radius of 3949.9 statute miles where a ***statute*** mile is 5280 feet. Be cautioned that maps and the military, especially the navy, often express distance in terms of ***nautical*** miles, which equals one minute of latitude. For conversion between the two, take 1 nautical mile to equal 1.1508 statute miles. In these notes we will assume a "mile" is a statute mile unless otherwise stated. In a related note, a ***knot*** is a unit of speed equal to 1 *nautical* mile per hour.)



Solution

Let A: Seattle, B: Stockholm, C: Reykjavik, N: North Pole, r: radius of Earth

By the Law of Cosines for Sides, Seattle to Stockholm

$$\begin{aligned}\cos\left(\widehat{AB}\right) &= \cos\left(\widehat{NA}\right)\cos\left(\widehat{NB}\right) + \sin\left(\widehat{NA}\right)\sin\left(\widehat{NB}\right)\cos(\angle ANB) \\ &= \cos(90-47.6061)\cos(90-59.3293) \\ &\qquad + \sin(90-47.6061)\sin(90-59.3293)\cos(18.0686+122.3328) \\ &= 0.3702138705019533\end{aligned}$$

Angular Distance Between Seattle and Stockholm

$$\widehat{AB}_{angular\ radians} = \cos^{-1}(0.3702138705019533) = 1.191557087585942 \text{ radians}$$

$$\widehat{AB}_{angular\ degrees} = \cos^{-1}(0.3702138705019533) \cdot \frac{180}{\pi} = 68.27119216757463°.$$

Physical Distance Between Seattle and Stockholm

$$\begin{aligned}\widehat{AB}_{physical} &= \widehat{AB}_{angular\ radians} \cdot r \\ &= 1.191557087585942 \cdot 3949.9 \text{ miles} \\ &= 4706.53134025571 \text{ miles}\end{aligned}$$

or equivalently

$$\begin{aligned}\widehat{AB}_{physical} &= \widehat{AB}_{angular\ degrees} \cdot \frac{\pi}{180} \cdot r \\ &= 68.27119216757463° \cdot \frac{\pi}{180} \cdot 3949.9 \text{ miles} \\ &= 4706.53134025571 \text{ miles.}\end{aligned}$$

By the Law of Cosines for Sides, Stockholm to Reykjavik

$$\begin{aligned}\cos\left(\widehat{BC}\right) &= \cos\left(\widehat{NB}\right)\cos\left(\widehat{NC}\right) + \sin\left(\widehat{NB}\right)\sin\left(\widehat{NC}\right)\cos(\angle BNC) \\ &= \cos(90 - 59.3293)\cos(90 - 64.1470) \\ &\qquad + \sin(90 - 59.3293)\sin(90 - 64.1470)\cos(18.0686 + 21.9408) \\ &= 0.9444030180545421\end{aligned}$$

Angular Distance Between Stockholm and Reykjavik

$$\widehat{BC}_{angular\ radians} = \cos^{-1}(0.9444030180545421) = 0.3350221777055614 \text{ radians}$$

$$\widehat{BC}_{angular\ degrees} = \cos^{-1}(0.9444030180545421) \cdot \frac{180}{\pi} = 19.19535682581053°.$$

Physical Distance Between Stockholm and Reykjavik

$$\begin{aligned}\widehat{BC}_{physical} &= \widehat{BC}_{angular\ radians} \cdot r \\ &= 0.3350221777055614 \cdot 3949.9 \text{ miles} \\ &= 1323.304099719197 \text{ miles}\end{aligned}$$

or equivalently

$$\begin{aligned}\widehat{BC}_{physical} &= \widehat{BC}_{angular\ degrees} \cdot \frac{\pi}{180} \cdot r \\ &= 19.19535682581053° \cdot \frac{\pi}{180} \cdot 3949.9 \text{ miles} \\ &= 1323.304099719197 \text{ miles.}\end{aligned}$$

By the Law of Cosines for Sides, Reykjavik to Seattle

$$\begin{aligned}\cos\left(\widehat{CA}\right) &= \cos\left(\widehat{NC}\right)\cos\left(\widehat{NA}\right) + \sin\left(\widehat{NC}\right)\sin\left(\widehat{NA}\right)\cos(\angle CNA)\\ &= \cos(90 - 64.1470)\cos(90 - 47.6061)\\ &\qquad + \sin(90 - 64.1470)\sin(90 - 47.6061)\cos(122.3328 - 21.9408)\\ &= 0.6115791503329473\end{aligned}$$

Angular Distance Between Reykjavik and Seattle

$$\widehat{CA}_{angular\ radians} = \cos^{-1}(0.6115791503329473) = 0.9127413369078403 \text{ radians}$$

$$\widehat{CA}_{angular\ degrees} = \cos^{-1}(0.6115791503329473) \cdot \frac{180}{\pi} = 52.29622639194761°.$$

Physical Distance Between Reykjavik and Seattle

$$\widehat{CA}_{physical} = \widehat{CA}_{angular\ radians} \cdot r$$

$$= 0.9127413369078403 \cdot 3949.9 \text{ miles}$$

$$= 3605.237006652279 \text{ miles}$$

or equivalently

$$\widehat{CA}_{physical} = \widehat{CA}_{angular\ degrees} \cdot \frac{\pi}{180} \cdot r$$

$$= 52.29622639194761° \cdot \frac{\pi}{180} \cdot 3949.9 \text{ miles}$$

$$= 3605.237006652279 \text{ miles.}$$

Summary

Seattle to Stockholm (direct): $4706.53134025571 \text{ miles}$

Stockholm to Reykjavik to Seattle:

$$1323.304099719197 + 3605.237006652279 = 4928.541106371476 \text{ miles}$$

Difference:

$$4928.541106371476 - 4706.53134025571 = 222.0097661157661 \text{ miles}$$