Number Theory

Study Notes

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For MN Students Preparing For
High School Mathematics
Contests

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Chapter 1 Positional Number Systems

When we write 235 we are using a **positional number system** because the position of each digit matters. In particular, we recognize that

$$235 = 200 + 30 + 5 = 2(10^2) + 3(10^1) + 5(10^0)$$

in the decimal (base 10) system. If there was any chance of ambiguity, we could be more explicit and write this as 235_{10} where the subscript "10" refers to the chosen base number. In the same manner we use 235_7 to represent the number 235 in base 7. What does mean?

In the **base *b* positional number system** the number $(c_n c_{n-1} \cdots c_1 c_0)_b$ represents $c_n b^n + c_{n-1} b^{n-1} + \cdots + c_1 b + c_0$ where by definition $c_n \neq 0$ and $c_j \in \{0,1, \dots, n-1\}$ for each j from 1 to n.

(In some texts, especially in computer science, the base number b is referred to as the ***radix*** of the number system and if you dig into pre-1970 math books you will see that base number problems were called ***scale of notation*** problems. So don't be throw off by these different names.)

If we simplify $c_n b^n + c_{n-1} b^{n-1} + \cdots + c_1 b + c_0$ to single number $\boldsymbol{a}$ using base 10 arithmetic, then we would can say that $a = c_n b^n + c_{n-1} b^{n-1} + \cdots + c_1 b + c_0$ is the **base 10 equivalent** of the base b number $(c_n c_{n-1} \cdots c_1 c_0)_b$.

For example, in base 7 the number 235_7 represents $2(7^2) + 3(7^1) + 5(7^0) = 98 + 21 + 5 = 124_{10}$ or just 124 in base 10. In general, if no base subscript is attached to a number, it is assumed that you are using base 10 notation.

In base 7 we only use the digits $\{0,1,2,3,4,5,6\}$. The next number after 6 in the base seven sequence would be $10_7 = 1(7^1) + 0(7^0) = 7 + 0 = 7_{10}$.

Notice that by these definitions, in base 7 (just for example), $n_7 = n_{10}$ for $n = 0,1,2,3,4,5,6$.

$*$ Notation: The b in n_b is understood to be a base 10 number

By definition, when we write 235_7, or more generally n_b, the base number 7 or b is to be interpreted as a base 10 number.

So, for example, when we write the hexadecimal (base 16) number n_{16}, the base number $b = 16$ is interpreted as $16_{10} = 1 \cdot 10^1 + 6 \cdot 10^0$ and ***<u>not</u>*** as $16_{16} = 1 \cdot 16^1 + 6 \cdot 16^0 = 22$.

Base Representation Theorem

Let b be an integer greater than 1. Then every $a > 0$ can be uniquely represented in the form

$$a = c_n b^n + c_{n-1} b^{n-1} + \cdots + c_1 b + c_0$$

with $c_n \neq 0$, $n \geq 0$, and $0 \leq c_i < b$ for $i = 0,1,2, \dots, n$.

1.1 Converting from Base b to Base 10

Example 1.1

Find the base 10 representation of 3201_6.

Solution

$$\begin{aligned} 3201_6 &= 3(6^3) + 2(6^2) + 0(6^1) + 1(6^0) \\ &= 3(216) + 2(36) + 0(6) + 1(1) \\ &= 648 + 72 + 0 + 1 \\ &= 721 \end{aligned}$$

That is, $3201_6 = 721_{10} = 721$.

■

Example 1.2

Find the base 10 representation of 54401_7.

Solution

$$\begin{aligned} 54401_7 &= 5(7^4) + 4(7^3) + 4(7^2) + 0(7^1) + 1(7^0) \\ &= 5(2401) + 4(343) + 4(49) + 0(7) + 1(1) = 13574 \end{aligned}$$

That is,

$$54401_7 = 13574_{10} = 13574.$$

■

1.2 Converting from Base 10 to Base b: Top-Down Method

Example 1.3

Find the base 5 representation of $1073_{10} = 1073$.

<u>Solution</u>

<u>Step 1.</u> Find the largest integer n for which $5^n \leq 1073$. We note that $5^4 = 625$ but $5^5 = 3125$. So $n = 4$.

Thus, by base representation theorem, there exists constants $c_i \in \{0,1,2,3,4\}$ for $i = 0,1,2,3,4$ and $c_4 \neq 0$ such that

$$1073 = c_4 5^4 + c_3 5^3 + c_2 5^2 + c_1 5^1 + c_0.$$

<u>Step 2.</u> c_4 equals the largest whole number of times $5^4 = 625$ goes into 1073.

$$\frac{1073}{5^4} = \frac{1073}{625} = \mathbf{1} + \frac{448}{625} \Longrightarrow c_4 = \mathbf{1}.$$

<u>Step 3.</u> Repeat Step 2 with $1073 - c_4 5^4 = 1073 - 625 = 448$.

$$\frac{448}{5^3} = \frac{448}{125} = \mathbf{3} + \frac{73}{125} \Longrightarrow c_3 = \mathbf{3}.$$

<u>Steps 4,5, …</u> Continue

$$1073 - c_4 5^4 - c_3 5^3 = 448 - 375 = 73.$$

$$\frac{73}{5^2} = \frac{73}{25} = \mathbf{2} + \frac{23}{25} \Longrightarrow c_2 = \mathbf{2}.$$

$$1073 - c_4 5^4 - c_3 5^3 - c_2 5^2 = 73 - 50 = 23.$$

$$\frac{23}{5^1} = \frac{23}{5} = \mathbf{4} + \frac{3}{5} \Longrightarrow c_1 = \mathbf{4}.$$

$$1073 - c_4 5^4 - c_3 5^3 - c_2 5^2 - c_1 5^1 = 23 - 20 = 3.$$

$$\frac{3}{5^0} = \mathbf{3} + \frac{0}{5} \Longrightarrow c_0 = \mathbf{3}.$$

Therefore,

$$1073 = (c_4c_3c_2c_1c_0)_5 = (13243)_5.$$

■

Check (time permitting this is always a good idea)

$$c_4 5^4 + c_3 5^3 + c_2 5^2 + c_1 5^1 + c_0 \stackrel{?}{=} 1073$$

$$1 \cdot 5^4 + 3 \cdot 5^3 + 2 \cdot 5^2 + 4 \cdot 5^1 + 3 \stackrel{?}{=} 1073$$

$$1073 \stackrel{\checkmark}{=} 1073.$$

■

Why it Works

Let d_4 equal the largest whole number of times $5^4 = 625$ goes into 1073.

Here we will only give a justification of the above Step 2 that $c_4 = d_4$. The justification of the remaining steps follows the same line of reasoning.

$$\boxed{c_4 \ngtr d_4}$$

Clearly $c_4 \ngtr d_4$ because in this case

$$c_4 5^4 + c_3 5^3 + c_2 5^2 + c_1 5^1 + c_0 \geq c_4 5^4 > 1073.$$

by definition.

$$\boxed{c_4 \nless d_4}$$

We know from the standard result on *geometry series* that

$$b^{n-1} + b^{n-2} + \cdots + b^1 + b^0 = \frac{b^n - 1}{b - 1}$$

for all $n \in \{1,2,3,\ldots\}$ and $b \in \{2,3,4,\ldots\}$. Hence

$$(b-1)\cdot b^{n-1}+(b-1)\cdot b^{n-2}+\cdots+(b-1)\cdot b^1+(b-1)=b^n-1.$$

Taking $b=5$ and $n=4$ in this identity we see that

$$4\cdot 5^3+4\cdot 5^2+4\cdot 5^1+4=5^4-1.$$

Notice that it follows from the restriction that $c_i\in\{0,1,2,3,4\}$ for $i=0,1,2,3,4$ and $c_4\neq 0$ that

$$c_3 5^3+c_2 5^2+c_1 5^1+c_0\leq 4\cdot 5^3+4\cdot 5^2+4\cdot 5^1+4=5^4-1<5^4.$$

Now suppose we take c_4 to be less than d_4 so that $d_4\geq c_4+1$. In this case

$$\begin{aligned}
&c_4 5^4+c_3 5^3+c_2 5^2+c_1 5^1+c_0\\
&\qquad=c_4 5^4+(c_3 5^3+c_2 5^2+c_1 5^1+c_0)\\
&\qquad<c_4 5^4+5^4\\
&\qquad=(c_4+1)5^4\\
&\qquad\leq d_4 5^4\\
&\qquad\leq 1073.
\end{aligned}$$

That is, if $c_4\nless d_4$ then $c_4 5^4+c_3 5^3+c_2 5^2+c_1 5^1+c_0<1073$.

But assuming c_4 exists (by the base representation theorem), the only remaining possibility is that $c_4=d_4$, the largest whole number of times $5^4=625$ goes into 1073.

■

1.3 Converting from Base 10 to Base b: Bottom-Up Method

In this section we will consider an alternative approach to converting from base 10 to base b that is *generally easier to implement* than the Top-Down Method considered in the previous section.

Example 1.4

Find the base 5 representation of 1073.

Answer

$$1073=1(5^4)+3(5^3)+2(5^2)+4(5^1)+3(5^0)=13243_5.$$

Solution

Step 1. Divide 1073 by 5 with remainder. That is, express 1073 in the form $1073 = 5 \cdot d_1 + r_1$ where $r_1 \in \{0,1,2,3,4\}$.

$$1073 = 214(5) + 3$$

Step 2. Divide d_1 by 5 with remainder. That is, express $d_1 = 214$ in the form $214 = 5 \cdot d_2 + r_2$ where $r_2 \in \{0,1,2,3,4\}$.

$$214 = 42(5) + 4$$

Step 3. Divide d_2 by 5 with remainder. That is, express $d_2 = 42$ in the form $42 = 5 \cdot d_3 + r_3$ where $r_3 \in \{0,1,2,3,4\}$.

$$42 = 8(5) + 2.$$

Steps 4,5, … Continue like this until you reach $d_k < 5$ and the line $d_k = 0(5) + r_{k+1}$.

$$8 = 1(5) + 3$$
$$1 = 0(5) + 1.$$

The remainders (shown in red), reading from the bottom up, reveal the digits of the base five representation of 1073.

■

Why it Works

Step 1.

If

$$1073 = c_4 5^4 + c_3 5^3 + c_2 5^2 + c_1 5^1 + c_0$$

then

$$1073 = 5(c_4 5^3 + c_3 5^2 + c_2 5^1 + c_1) + c_0.$$

So, when we divide 1073 by 5 the remainder equals c_0. That is,

$$1073 = 214(5) + \mathbf{3} \Longrightarrow c_0 = \mathbf{3}.$$

Step 2.

From Step 1 we have

$$d_1 = c_4 5^3 + c_3 5^2 + c_2 5^1 + c_1 = 5(c_4 5^2 + c_3 5 + c_2) + c_1.$$

So, when we divide d_1 by 5 the remainder equals c_1. That is,

$$d_1 = 214 = 42(5) + \mathbf{4} \Longrightarrow c_1 = \mathbf{4}.$$

The justification of the remaining steps follows the same line of reasoning.

■

Example 1.5

Find the base 2 representation of 1073 using the bottom-up method.

Solution

$$\begin{aligned}
1073 &= 536(2) + 1 \\
536 &= 268(2) + 0 \\
268 &= 134(2) + 0 \\
134 &= 67(2) + 0 \\
67 &= 33(2) + 1 \\
33 &= 16(2) + 1 \\
16 &= 8(2) + 0 \\
8 &= 4(2) + 0 \\
4 &= 2(2) + 0 \\
2 &= 1(2) + 0 \\
1 &= 0(2) + 1
\end{aligned} \quad \Bigg\uparrow$$

The remainders (shown in red), reading from the bottom up, reveal the digits of the base two representation of 1073. That is,

$$1073 = 10000110001_2.$$

■

1.4 Converting from Base a to Base b, $(a \neq 10, b \neq 10)$

Example 1.6

Find the base-nine number that is equivalent to 245_6.

Solution

First convert from base 6 to base 10.

$$245_6 = 2(6^2) + 4(6^1) + 5(6^0) = 101_{10} = 101.$$

Then convert from base 10 to base 9.

$$\begin{aligned} 101 &= 11(9) + \mathbf{2} \\ 11 &= 1(9) + \mathbf{2} \\ 1 &= 0(9) + \mathbf{1} \end{aligned} \quad \uparrow$$

Therefore

$$245_6 = 122_9.$$

■

1.5 Converting from Base a to Base a^k and Vice Versa

Section 1.5 covers a special case of the general base conversion problem introduced in Section 1.4.

The methods in Section 1.5 are a time saving shortcut you can use when the two bases of the conversion problem are powers of each other.

1.5.1 Base a to Base a^k

In this section we will develop a shortcut method for converting a number from base a to base a^k. The first step of the procedure is to partition the base a number into groups of size k starting from the right (the units digit). Add leading 0's (if needed) to make the last group a full group (of size k). The final step is to convert each group of size k from base a to base a^k.

We illustrate this procedure and explain why it works in the example that follows.

Example 1.7 (Source: MSHSML 1A984)

Using binary notation (base 2), let $N = 11110101$. Write N in octal notation (base 8).

Solution

Procedure: We want to go from base 2 to base $2^{\mathbf{3}}$. Group the base 2 number into sets of size **3**. Putting in leading 0's as needed to make the last group a full group of three digits.

$$11|110|101 = 011|110|101$$

added in a leading 0

Now convert each base 2 group of three digits into a single base 8 digit.

$$\begin{aligned} 011_2 &= 0(2^2) + 1(2^1) + 1(2^0) = 3 \\ 110_2 &= 1(2^2) + 1(2^1) + 0(2^0) = 6 \\ 101_2 &= 1(2^2) + 0(2^1) + 1(2^0) = 5 \end{aligned}$$

These are the digits of our base 8 conversion.

$$11110101_2 = 365_8.$$

■

Why it Works

Consider the grouping of k digits (going from right to left, starting with the unit's digit) of the base a number n_a

$$n_a = |\star_{rk-1} \cdots \star_{(r-1)k}| \cdots |\star_{mk-1} \cdots \star_{(m-1)k}| \cdots |\star_{2k-1} \cdots \star_k| \star_{k-1} \cdots \star_0|.$$

<u>Note</u>: Here we are using the notation $\star_j$ to represent the $(j+1)^{st}$ digit of the base a number n_a.

Expanding the group of k digits $(\star_{mk-1} \cdots \star_{(m-1)k})$ gives us

$$\begin{aligned} &(\star_{mk-1})a^{mk-1} + (\star_{mk-2})a^{mk-2} + \cdots + (\star_{(m-1)k})a^{(m-1)k} \\ &\quad = a^{(m-1)k}\left((\star_{mk-1})a^{k-1} + (\star_{mk-2})a^{k-2} + \cdots + (\star_{(m-1)k})a^0\right) \\ &\quad = (a^k)^{m-1}\left((\star_{mk-1})a^{k-1} + (\star_{mk-2})a^{k-2} + \cdots + (\star_{(m-1)k})a^0\right) \end{aligned}$$

$= (a^k)^{m-1} c_{m-1}$

where $c_{m-1} = (\star_{mk-1})a^{k-1} + (\star_{mk-2})a^{k-2} + \cdots + \left(\star_{(m-1)k}\right)a^0$.

The important fact to notice the lower and upper bounds on c_{m-1}. In particular, we have

$$c_{m-1} \geq 0 \cdot a^{k-1} + 0 \cdot a^{k-2} + \cdots + 0 \cdot a^0 = 0$$

and

$$\begin{aligned} c_{m-1} &\leq (a-1) \cdot a^{k-1} + (a-1) \cdot a^{k-2} + \cdots + (a-1) \cdot a^1 + (a-1) \cdot a^0 \\ &= (a^k - a^{k-1}) + (a^{k-1} - a^{k-2}) + \cdots + (a^2 - a^1) + (a^1 - a^0) \\ &= a^k - 1. \end{aligned}$$

That is, $c_{m-1} \in \{0,1,\dots,a^k-1\}$ for all $m-1$. It follows that

$$\begin{aligned} n_a &= \ | \star_{rk-1} \cdots \star_{(r-1)k} | \ \cdots \ | \star_{mk-1} \cdots \star_{(m-1)k} | \ \cdots \ | \star_{2k-1} \cdots \star_k | \star_{k-1} \cdots \star_0 | \\ &= c_{r-1}(a^k)^{r-1} + \cdots + c_{m-1}(a^k)^{m-1} + \cdots c_1(a^k)^1 + c_0(a^k)^0 \end{aligned}$$

where each $c_j \in \{0,1,\dots,a^k-1\}$. Thus we have verified that this procedure does convert the base a number into its equivalent base a^k form as claimed.

To make this more tangible we will illustrate these steps for the previous problem of converting 11110101_2 into base $2^3 = 8$.

$$11110101_2 = 1(2^7) + 1(2^6) + 1(2^5) + 1(2^4) + 0(2^3) + 1(2^2) + 0(2^1) + 1(2^0)$$

$$= \left(0(2^8) + 1(2^7) + 1(2^6)\right) + \left(1(2^5) + 1(2^4) + 0(2^3)\right) + \left(1(2^2) + 0(2^1) + 1(2^0)\right)$$

$$\begin{aligned} &= 2^6\left(0(2^2) + 1(2^1) + 1(2^0)\right) + 2^3\left(1(2^2) + 1(2^1) + 0(2^0)\right) \\ &\quad + 2^0\left(1(2^2) + 0(2^1) + 1(2^0)\right) \end{aligned}$$

$$\begin{aligned} &= (2^3)^2\left(0(2^2) + 1(2^1) + 1(2^0)\right) + (2^3)^1\left(1(2^2) + 1(2^1) + 0(2^0)\right) \\ &\quad + (2^3)^0\left(1(2^2) + 0(2^1) + 1(2^0)\right) \end{aligned}$$

$$\begin{aligned} &= \left(0(2^2) + 1(2^1) + 1(2^0)\right) \cdot 8^2 + \left(1(2^2) + 1(2^1) + 0(2^0)\right) \cdot 8^1 \\ &\quad + \left(1(2^2) + 0(2^1) + 1(2^0)\right) \cdot 8^0 \end{aligned}$$

$$= 3(8^2) + 6(8^1) + 5(8^0) = 365_8.$$

■

Example 1.8 (Source: 2008 Mu Alpha Theta Convention, Open Division Number Theory Test, Problem 5)

Find the sum of digits of the base 16 expression of $(1101010010001001)_2$. Express the sum in base 16.

Solution

Procedure: We want to go from base 2 to base $16 = 2^{\mathbf{4}}$. Group the base 2 number into sets of size $k = \mathbf{4}$ starting from the right.

$$1101 \mid 0100 \mid 1000 \mid 1001$$

In this problem the final group is a full group (contains $k = 4$ digits) and hence it is not necessary to put in leading 0's to make it a full group.

Convert each base 2 group of four digits into a single base 16 digit.

$$1001_2 = 1(2^3) + 0(2^2) + 0(2^1) + 1(2^0) = 9_{10} = 9_{16}$$
$$1000_2 = 1(2^3) + 0(2^2) + 0(2^1) + 0(2^0) = 8_{10} = 8_{16}$$
$$0100_2 = 0(2^3) + 1(2^2) + 0(2^1) + 0(2^0) = 4_{10} = 4_{16}$$
$$1101_2 = 1(2^3) + 1(2^2) + 0(2^1) + 1(2^0) = 13_{10} = D_{16}.$$

Notation: In these notes we adopt the convention of using letters (in alphabetic order) for the extra symbols needed beyond 9 for bases larger than ten. For example, in hex (base 16) the needed sixteen single digit numbers are $0,1,2,\ldots,9,\boldsymbol{A},\boldsymbol{B},\boldsymbol{C},\boldsymbol{D},\boldsymbol{E},\boldsymbol{F}$. With this convention we have:

$A_{16} = 10_{10}$ $B_{16} = 11_{10}$ $C_{16} = 12_{10}$ $D_{16} = 13_{10}$ $E_{16} = 14_{10}$ $F_{16} = 15_{10}$	$10_{16} = 1(16^1) + 0(16^0) = 16_{10}$ $11_{16} = 1(16^1) + 1(16^0) = 17_{10}$ $\vdots$ $1A_{16} = 1(16^1) + A(16^0)$ $= 16_{10} + 10_{10} = 26_{10}$ etc.

Thus,

$$(1101010010001001)_2 = D489_{16}$$

and the sum of the digits of $D489_{16}$ (expressed in base 16) equals

$$\begin{aligned} D_{16} + 4_{16} + 8_{16} + 9_{16} &= 13_{10} + 4_{10} + 8_{10} + 9_{10} \\ &= 34_{10} \\ &= 2(16^1) + 2(16^0) \\ &= 22_{16}. \end{aligned}$$

■

Example 1.9

Find the base 9 equivalent to 12012111_3.

Solution

We are converting from base 3 to base $3^{\mathbf{2}}$ so form groups of size **2** from right to left adding leading 0's if necessary to form a complete group of size 2.

$$12|01|21|11$$

Convert each group (base 3).

$$\begin{aligned} 12_3 &= 1(3^1) + 2(3^0) = 5 \\ 01_3 &= 0(3^1) + 1(3^0) = 1 \\ 21_3 &= 2(3^1) + 1(3^0) = 7 \\ 11_3 &= 1(3^1) + 1(3^0) = 4 \end{aligned}$$

These are the digits of our base 9 equivalent.

$$12012111_3 = 5174_9.$$

■

1.5.2 Base a^k to Base a

We can reverse the procedure in the previous section when we need to convert a number from base a^k to base a. The reverse of the final step in the previous section is to convert each of the base a^k digits into a set of k base a digits. (We include leading 0's (if necessary) to make each set have k digits.)

We illustrate this method in the next example.

Example 1.10

Find the base 3 equivalent to 5174_9.

Solution

To reverse the procedure in the previous example we want to convert each of the base $9 = 3^2$ digits $5,1,7$ and 4 into their base 3^1 equivalent. Because we are going from base $3^{\mathbf{2}}$ to base 3 so our group size is $k = 2$.

IMPORTANT! We must include leading 0's (if necessary) to make the equivalent number in base three a $k = 2$ digit number.

$$
\begin{aligned}
5_9 &= 1(3^1) + 2(3^0) && = \mathbf{12}_3 \\
1_9 &= 1(3^0) \quad = 1_3 && = \mathbf{01}_3 \quad \text{(put in the leading 0 as needed)} \\
7_9 &= 2(3^1) + 1(3^0) && = \mathbf{21}_3 \\
4_9 &= 1(3^1) + 1(3^0) && = \mathbf{11}_3
\end{aligned}
$$

These four two-digit groups when placed side by side form our base 3 equivalent.

$$5174_9 = 1\,2\,|\,0\,1\,|\,2\,1\,|\,1\,1_3 = 12012111_3$$

■

Analysis of a Mistake:

If we had forgotten to express 1_3 as 01_3 in this last example, we would have gotten

$$5174_9 \neq 1\,2\,|\,1\,|\,2\,1\,|\,1\,1_3 = 1212111_3.$$

which is the ***incorrect answer***.

■

Example 1.11 (Source: Alpha Mu Theta Florida State Convention 2005, Number Theory Test, Problem #4)

In a base-32 number system, the digits 0-9 represent themselves and the other digits are given by $\text{A} = 10, \text{B} = 11, \ldots, \text{V} = 31$. If you convert $(\text{MAO})_{32}$ into binary, what are the last 5 digits?

Solution

To convert from base $32 = 2^5$ to base 2^1 we must convert each base 32 digit in $(\text{MAO})_{32}$ to its base 2 equivalent. We must include leading 0's (if necessary) to each of these base 2 equivalents in order to make it into a $k = 5$ digit base 2 number.

This tells us that the last 5 digits of the base 2 equivalent of $(\text{MAO})_{32}$ will be the base 2 equivalent of O_{32} padded with leading 0's (if necessary) to make it a 5-digit base 2 number.

The following chart shows that the base 10 equivalent of O_{32} equals 24_{10}.

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
10	11	12	13	14	15	16	17	18	19	20	21	22	23	24

P	Q	R	S	T	U	V
25	26	27	28	29	30	31

Converting 24_{10} to base 2 we have

$$\begin{aligned} 24_{10} = 16 + 8 &= 1 \cdot 2^4 + 1 \cdot 2^3 + 0 \cdot 2^2 + 0 \cdot 2^1 + 0 \cdot 2^0 \\ &= (11000)_2. \end{aligned}$$

The base 2 number 11000 already has 5 digits so we do not need to add any leadings 0's.
It follows that when you the convert $(\text{MAO})_{32}$ into binary the last 5 digits are 11000.

■

Example 1.12 (Source: MSHSML 1T985)

A number written in base 3 has six "digits": $1 * 210 *$, where the asterisk's represent smudges that can't be read. What is the middle digit of this number when it is written in base 9 notation? Note: You should not assume the two smudges represented by asterisk's were the same number.

<u>**Solution**</u>

Let $N = (1 * 210 *)_3$. Following the procedure developed in Section 1.5.1 we break N into groups of size $k = 2$ digits.

$$1 * | 21 | 0 *$$

Now convert each base 3 group of two digits into a single base 9 digit.

$$
\begin{aligned}
(0\,*)_3 &= 0(3^1) + (*)(3^0) = (*)_{10} \\
(21)_3 &= 2(3^1) + 1(3^0) \quad = (7)_{10} \\
(1\,*)_3 &= 1(3^1) + (*)(3^0) = (3+*)_{10}
\end{aligned}
$$

Note that $* \in \{0,1,2\}$ because in both occurrences of $*$ in N it represents a single digit in a base three number.

It follows that $(3+*) \in \{3,4,5\}$. Hence,

$$
\begin{aligned}
(3+*)_{10} &= (3+*)_9 && \text{is a single digit number in base 9} \\
7_{10} &= 7_9 && \text{is a single digit number in base 9} \\
(*)_{10} &= (*)_9 && \text{is a single digit number in base 9}
\end{aligned}
$$

So $N = (1*210*)_3$ is a 3 digit number in base 9 and the middle digit is a 7.

■

1.6 Converting in a Variable Base

In this section we will consider some examples where the question is to carry out a base conversion in terms an unspecified base. This is a vague description but the examples should make the issue clear.

Example 1.13 (Source: MSHSML TA991)

In base b, c^2 is written 10. How do you write b^2 in case c?

Solution

$$c^2 = 10_b = 1(b^1) + 0(b^0) = b.$$

Therefore,

$$(c^2)^2 = (b)^2.$$

That is,

$$b^2 = c^4 = 1c^4 + 0c^3 + 0c^2 + 0c^1 + 0c^0.$$

Hence

$$b^2 = (10000)_c.$$

■

Example 1.14 (Source: Mu Alpha Theta 1991 National Convention, Number Theory Topic Test, Problem 16)

When the number n is written in base b its representation is the two-digit number $(AB)_b$ where $A = b - 2$ and $B = 2$. What is the base $b - 1$ representation of n?

Solution

The first point to notice is that the base $b - 1$ representation of n will be a three-digit number because the largest possible two-digit number in base $b - 1$ equals $(CD)_{b-1}$ where $C = b - 2$ and $D = b - 2$ which is less than $(AB)_b$. To see this, note that

$$\begin{aligned}(CD)_{b-1} &= (b-2)\cdot(b-1) + (b-2)\\ &= (b-2)\cdot b + (b-2)(-1) + (b-2)\\ &= (b-2)\cdot b\\ &< (b-2)\cdot b + 2\\ &= (AB)_b.\end{aligned}$$

So, the problem reduces to finding nonnegative integers c, d and e, all less than or equal $b - 2$ such that $(cde)_{b-1} = (AB)_b$. That is,

$$(cde)_{b-1} = c(b-1)^2 + d(b-1) + e = (AB)_b = (b-2)\cdot b + 2.$$

Expanding both sides, we have the identity

$$cb^2 + (-2c + d)b + (c - d + e) = b^2 - 2b + 2.$$

in the variable b.

Matching coefficients of like powers of b on both sides we have

$$c = 1, (-2c + d) = -2, \text{and } (c - d + e) = 2.$$

Solving this system we find

$$\begin{gathered}\boldsymbol{c = 1}\\ -2c + d = -2 \Longrightarrow -2(1) + d = -2 \Longrightarrow \boldsymbol{d = 0}\\ c - d + e = 2 \Longrightarrow 1 + 0 + e = 1 \Longrightarrow \boldsymbol{e = 1}.\end{gathered}$$

This shows

$$n = (AB)_b = (cde)_{b-1} = (101)_{b-1}.$$

■

Example 1.15

If $(2c+7)_{10}$ is written 13 in base b, how would you write $(2b-7)_{10}$ in base c? You can assume $b \geq 4$ and $c \geq 5$ are both integers.

Solution

$$\begin{gathered}(2c+7)_{10} = 13_b = (1b+3)_{10} \\ 2c+7 = b+3 \\ b = 2c+4 \\ 2b = 4c+8 \\ 2b-7 = 4c+1 = (41)_c\end{gathered}$$

■

1.7 Finding the base b such that …

In this section we will consider problems where the goal is to *solve* for the unknown number base(s) given information about a conversion problem(s).

Example 1.16 (Source: MSHSML TA092)

In what base b does the integer 63_b equal 117_{10}?

Solution

$$63_b = 6b^1 + 3b^0 = 6b + 3 = 117 \Longrightarrow 6b = 114 \Longrightarrow b = 19.$$

■

Example 1.17 (Source: Mu Alpha Theta 2011 National Convention, Open Division, Number Theory Topic Test, Problem 20)

Find the value of n such that $251_n + 121_{n-1} = 415_n$.

Solution

$$\left(2n^2 + 5n^1 + 1n^0\right) + \left(1(n+1)^2 + 2(n+1)^1 + 1(n+1)^0\right) = 4n^2 + 1n^1 + 5n^0$$

$$\Leftrightarrow 3n^2 + 9n + 5 = 4n^2 + n + 5$$

$$\Leftrightarrow n^2 - 8n = 0$$

$$\Leftrightarrow n = 0 \text{ or } n = 8.$$

But we do not define a base number $n = 0$ so $n = 8$.

$\blacksquare$

Example 1.18 (Source: MSHSML 1T122)

In some number base b, the number 121 is equal to the decimal (base-10) number 324. Calculate b.

Solution

$$\begin{aligned} 121_b &= 1b^2 + 2b^1 + 1 = 324 \\ &\Rightarrow (b+1)^2 = 324 \\ &\Rightarrow b + 1 = \sqrt{324} = 18 \\ &\Rightarrow b = 17. \end{aligned}$$

$\blacksquare$

Example 1.19 (Source: MSHSML TA013)

A certain integer is represented base 5 by 40142_5 and base b by 1583_b. Find b.

Solution

$$40142_5 = 4(5^4) + 0(5^3) + 1(5^2) + 4(5^1) + 2(5^0) = 2547$$

$$1583_b = 1(b^3) + 5(b^2) + 8(b^1) + 3(b^0)$$

$\Rightarrow b^3 + 5b^2 + 8b + 3 = 2547$
$\Rightarrow b^3 + 5b^2 + 8b - 2544 = 0$
$\Rightarrow b$ must divide $2544 = 2^4 \cdot 3 \cdot 54$ (rational root theorem).

Synthetic division shows $b = 8$ is not big enough

$$\begin{array}{r|rrrr} 8 & 1 & 5 & 8 & -2544 \\ \hline & & 8 & 104 & 896 \\ \hline & 1 & 13 & 112 & -1648 \end{array}$$

but that the next largest potential root is $b = 12$ is in fact a root.

$$\begin{array}{r|rrrr} 12 & 1 & 5 & 8 & -2544 \\ \hline & & 12 & 204 & 2544 \\ \hline & 1 & 17 & 112 & 0 \end{array}$$

We can also see from this division that

$$b^3 + 5b^2 + 8b - 2544 = (b - 12)(b^2 + 17b + 112)$$

and because $b^2 + 17b + 112$ is an irreducible quadratic (the discriminant is negative) there are no other real roots. So $b = 12$ is the only possible answer.

■

Example 1.20 (MSHSML 1T136)

Let N be a number in base b such that $N_b = 14_b \cdot 17_b$. What is the <u>greatest</u> base b for which N_b would be written with "2" as its left-most digit?

<u>Solution</u>

First note that $b \geq 8$ or 17_b could not be a base b number. Now

$$N_b = 14_b \cdot 17_b \Longrightarrow N = (b + 4)(b + 7) = b^2 + 11b + 28.$$

To get a sense for what is going on think base 10 for a moment. In this case ($b = 10$) we get

$$b^2 + 11b + 28 = 100 + 110 + 28 = 228_{10}$$

and we observe the left most digit is a "2" as required. In particular, it was necessary that $110 + 28 \geq 100$ so that our hundreds digit could increase from 1 to 2. In the general case this means that we must choose b such that

$$11b + 28 \geq b^2.$$

Solving this inequality, we will find that $b \leq 13$. Checking $b = 13$ we see that

$$N = 13^2 + 11(13) + 28 = 340 = 2(13^2) + 0(13^1) + 2(13^0) = 202_{13}.$$

Therefore $b = 13$.

■

Example 1.21 (Source: MSHSML TT095)

What is the smallest base b, with $b > 5$, for which $35_b \cdot 53_b$ is a 3-digit number in base b?

<u>Solution</u>

The base 10 expansion of this product becomes

$$35_b \cdot 53_b = (3b+5)(5b+3) = 15b^2 + 34b + 15.$$

Writing

$$\underset{b^2}{\underline{15}} \;\; \underset{b^1}{\underline{34}} \;\; \underset{b^0}{\underline{15}}$$

in standard base b notation will require some "carrying" unless $b \geq 35$.

For example, if $b = 22$, the 15 in the b^0 column is fine but the 34 in the b^1 column has to be split up and the extra must be "carried" into the b^2 column.

In particular, $34_{22} = 1 \cdot 22^1 + 12 \cdot 22^0$, so in standard base $b = 22$ notation we have to replace the 34 with 12 and carry the excess 1 group of 22^1 into the b^2 column.

$$\underset{22^2}{\underline{\overset{+1}{15}}} \;\; \underset{22^1}{\underline{12}} \;\; \underset{22^0}{\underline{15}} \quad = \quad \underset{22^2}{\underline{16}} \;\; \underset{22^1}{\underline{12}} \;\; \underset{22^0}{\underline{15}} \quad = \left(\underline{16}\;\underline{12}\;\underline{15}\right)_{22}$$

This shows that

$$\left(\underline{3}\;\underline{5}\right)_{22} \cdot \left(\underline{5}\;\underline{3}\right)_{22} = \left(\underline{16}\;\underline{12}\;\underline{15}\right)_{22}$$

is a three-digit number in base $b = 22$.

Now consider the case $b = 13$. In this case, "carrying" is required in all three columns.

$$\underset{13^2}{\underline{15}} \;\; \underset{13^1}{\underline{34}} \;\; \underset{13^0}{\underline{15}} \quad = \quad \underset{13^2}{\underline{15}} \;\; \underset{13^1}{\underline{\overset{+1}{34}}} \;\; \underset{13^0}{\underline{2}} \quad = \quad \underset{13^2}{\underline{\overset{+2}{15}}} \;\; \underset{13^1}{\underline{9}} \;\; \underset{13^0}{\underline{2}} \quad = \quad \underset{13^3}{\underline{1}} \;\; \underset{13^2}{\underline{4}} \;\; \underset{13^1}{\underline{9}} \;\; \underset{13^0}{\underline{2}}$$

15 in the 13^0 column gives us 2 plus a carry of 1 group of 13^1 into the 13^1 column.
$34 + 1$ in the 13^1 column gives us 9 plus a carry of 2 groups of 13^2 into the 13^2 column.
$15 + 2$ in the 13^2 column gives us 4 plus a carry of 1 group of 13^3 into the 13^3 column.

That is,

$$\left(\underline{3}\;\underline{5}\right)_{13} \cdot \left(\underline{5}\;\underline{3}\right)_{13} = \left(\underline{1}\;\underline{4}\;\underline{9}\;\underline{2}\right)_{13}$$

which is a four-digit number.

It was this final carry of 1 group of 13^3 into the 13^3 column that made this product a four-digit number in base $b = 13$.

To avoid making the product a four-digit number we need to *make b large enough to avoid carrying anything into the b^3 column.*

$\dfrac{15}{b^2}\ \dfrac{34}{b^1}\ \dfrac{15}{b^0}$	
$b = 16$	$15 = 0(16^1) + 15(16^0)$. No carry from the b_0 column into the b^1 column required. $34 = 2(16^1) + 2(16^0)$. Carry of 2 from the b^1 column into the b^2 column required. $15 + 2 = 1(16^1) + 1(16^0)$. Carry of 1 from the b^2 column into the b^3 column required.
$b = 17$	$15 = 0(17^1) + 15(17^0)$. No carry from the b_0 column into the b^1 column required. $34 = 2(17^1) + 0(17^0)$. Carry of 2 from the b^1 column into the b^2 column required. $15 + 2 = 1(17^1) + 0(17^0)$. Carry of 1 from the b^2 column into the b^3 column required.
$b = 18$	$15 = 0(18^1) + 15(18^0)$. No carry from the b_0 column into the b^1 column required. $34 = 1(18^1) + 16(18^0)$. Carry of 1 from the b^1 column into the b^2 column required. $16 + 1 = 0(18^1) + 17(18^0)$. No carry from the b^2 column into the b^3 column required.

So $b = 18$ is the smallest base that will not require any carry from the b^2 column into the b^3 column. That is $b = 18$ is the smallest base value where the product will be a three-digit number in that base.

■

Example 1.22 (Source: *Number Theory*, Freud and Gyarmati, Section 1.2, Exercise 18)

A positive integer n has four digits when expressed in base b but only two digits when expressed in base $b + 1$, $b \geq 2$. Determine n and b.

Solution

The smallest four digit number in base b is $1000_b = b^3$ and the largest four digit number in base b is

$$
\begin{aligned}
\big((b-1)(b-1)(b-1)(b-1)\big)_b &= (b-1)b^3 + (b-1)b^2 + (b-1)b^1 + (b-1)b^0 \\
&= b^4 - 1.
\end{aligned}
$$

Therefore,

$$b^3 \leq n \leq b^4 - 1.$$

Similarly, the smallest two digit number in base $b+1$ is $10_{b+1} = b+1$ and the largest two digit number in base $b+1$ is

$$(bb)_{b+1} = b(b+1)^1 + b(b+1)^0 = b^2 + 2b.$$

Therefore,

$$b + 1 \leq n \leq b^2 + 2b.$$

Hence, we are looking for integers n and b such that

$$\max(b^3, b+1) \leq n \leq \min(b^4 - 1, b^2 + 2b).$$

However, for all $b \geq 2$ we can see that

$$\max(b^3, b+1) = b^3$$

and

$$\min(b^4 - 1, b^2 + 2b) = b^2 + 2b.$$

So we want to find some positive integer n such that

$$b^3 \leq n \leq b^2 + 2b \quad \text{for all } b \geq 2.$$

However,

$$b^3 = b^2 + 2b = 8 \quad \text{for } b = 2$$

and

$$b^3 > b^2 + 2b \quad \text{for } b \geq 3.$$

Thus, $b = 2$ and $n = 8$ is the only possible solution.

■

Example 1.23 (Source: *School Science and Mathematics*, February 1972, Problem 3399, Charles Trigg.)

The three-digit positive number aba in base x equals the three-digit positive number bab in base y, $x \neq y$. If $x + y = 10$, find the two numbers and their bases of numeration.

Solution

Without loss of generality, we can assume that $x < y$. Also because aba is a three-digit number we can assume $a \neq 0$. Similarly $b \neq 0$ because bab is a three-digit number.

$$ax^2 + bx + a = by^2 + ay + b$$
$$a(x^2 - y + 1) = b(y^2 - x + 1)$$
$$a(x^2 - (10 - x) + 1) = b(y^2 - (10 - y) + 1)$$
$$a(x^2 + x - 9) = b(y^2 + y - 9).$$

Because $x < y$, we know that $x \in \{2,3,4\}$. But $(x, y) = (2,8)$ tells us that

$$a(2^2 + 2 - 9) = b(8^2 + 8 - 9)$$

or

$$-a = 21b.$$

But this contradicts the assumption that a and b are both positive. So $x \neq 2$.

Suppose $x = 3$. Then $y = 7$. $a(9 + 3 - 9) = b(49 + 7 - 9)$. $3a = 47b$. But in base 3, $a, b \in \{0,1,2\}$ and for no choice of a and b will $3a = 47b$. So $x \neq 3$.

Suppose $x = 4$. Then $y = 6$. $a(16 + 4 - 9) = b(36 + 6 - 9)$. $11a = 33b$. $a, b \in \{0,1,2,3\}$. In this case the only solution is $a = 3, b = 1$.

Therefore, the numbers are 313_4 and 131_6. As a check we note that

$$313_4 = 3(4^2) + 1(4) + 3 = 55$$

And

$$131_6 = 1(6^2) + 3(6) + 1 = 55.$$

■

1.8 Disguised polynomial factorization problems

Example 24.

The integer $N = 10100$ is expressed using base $b > 1$. Express N as a product of two integers, expressed as polynomials in b, that are both greater than 1. (Source: 2005-06, Meet 1, Event A)

Solution

$$\begin{aligned} 10100_b &= 1(b^4) + 0(b^3) + 1(b^2) + 0(b^1) + 0(b^0) \\ &= b^4 + b^2 \\ &= b^2(b^2 + 1) \end{aligned}$$

Note: $b > 1$ implies $b^2 > 1$ and $b^1 + 1 > 1$ as required.

■

Example 25.

Expressed using base $b > 3$, the integer $M = 231$. Write M as a product of two integers, also expressed using base b. (Note carefully, you are not being asked to express them as polynomials in b, but as integers, just as M is expressed.) (Source: 2005-06, State Tournament, Event D)

Solution

$$\begin{aligned} M = 231_b &= 2(b^2) + 3(b^1) + 1(b^0) \\ &= 2b^2 + 3b + 1 \\ &= (2b + 1)(b + 1) \\ &= (21_b)(11_b). \end{aligned}$$

■

1.9 Addition and Subtraction in Base b

Example 26.

Find $315_6 + 153_6$.

Solution

It can help to write out a base 6 addition table when you first learning to add in a different base.

Base 6 Addition Table

+	0_6	1_6	2_6	3_6	4_6	5_6
0_6	0	1	2	3	4	5
1_6	1	2	3	4	5	10
2_6	2	3	4	5	10	11
3_6	3	4	5	10	11	12
4_6	4	5	10	11	12	13
5_6	5	10	11	12	13	14

Using this table, we can see that

$$\begin{array}{cccc} & \color{red}{1} & \color{red}{1} & \\ & 3 & 1 & 5 \\ + & 1 & 5 & 3 \\ \hline & 5 & 1 & 2 \end{array}$$

That is, $315_6 + 153_6 = 512_6$.

■

Example 27.

Given the following summation in base 6

$$\begin{array}{ccccl} & b & 3 & 4 & _6 \\ + & a & a & c & _6 \\ \hline a & 0 & b & a & _6 \end{array}$$

find the sum of $a + b + c$ in base 6. (Source: Greater New Haven Mathematics League, 2009)

Solution

$$\begin{array}{cccc} & b & 3 & 4 \\ + & a & a & c \\ \hline a & 0 & b & a \end{array}$$

Forces that $a = 1$ because we can carry at most 1 in adding two numbers in the right most column of the addition.

$$\begin{array}{cccc} & b & 3 & 4 \\ + & 1 & 1 & c \\ \hline 1 & 0 & b & 1 \end{array}$$

After substituting for a we can see that $c = 3$ from the left most column of addition.

$$\begin{array}{cccc} & & {\color{red}1} & \\ & b & 3 & 4 \\ + & 1 & 1 & 3 \\ \hline 1 & 0 & b & 1 \end{array}$$

Now we can see that $b = 5$ from the second column from the left of addition, noticing that we also had to carry 1 from the previous column.

$$\begin{array}{cccc} & 5 & 3 & 4 \\ + & 1 & 1 & 3 \\ \hline 1 & 0 & 5 & 1 \end{array}$$

Therefore,

$$(a+b+c)_6 = (1+5+3)_6 = 13_6.$$

■

Example 28.

Add the following binary numbers and express the sum as a number in base three. (Source: 2000-01, Meet 4, Event C)

$$\begin{array}{ccccccccc} & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ + & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ \hline \end{array}$$

__Solution__

$$\begin{array}{ccccccccc} & {\color{red}1} & {\color{red}1} & {\color{red}1} & {\color{red}1} & {\color{red}1} & {\color{red}1} & {\color{red}1} & \\ & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ + & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ \hline 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \end{array}$$

$$101110100_2 = 2^8 + 2^6 + 2^5 + 2^2 = 372$$

But,

$$\begin{aligned} 372 &= 124(3) + 0 \\ 124 &= 41(3) + 1 \\ 41 &= 13(3) + 2 \\ 13 &= 4(3) + 1 \\ 4 &= 1(3) + 1 \\ 1 &= 0(3) + 1 \end{aligned}$$

Therefore,

$$\text{sum} = 111210_3.$$

■

Mu Alpha Theta Theta Washington State Convention 2007, Number Theory Test, Problem #8
What is $1321_4 + 11001001_2$ expressed as a base 8 number?
<u>Solution</u>

$$502_8$$

$$00,01,10,11$$
$$1_4 = 01_2, 2_4 = 10_2, 3_4 = 11_2, 1_4 = 01_2$$
$$1321_4 = 01\ 11\ 10\ \ 01_2$$
$$1111001_2 + 11001001_2 = 100110110_2$$

			1	1	1	1	0	0	1
+		1	1	0	0	1	0	0	1
	1	0	1	0	0	0	0	1	0

$$010_2 = 2_8$$
$$000_2 = 0_8$$
$$101_2 = 5_8$$
$$502_8$$

■

Example 1.29

Assuming GAB and BAG represent positive three-digit numbers in base r, find all possible solutions to the cryptarithm

$$\begin{array}{rccc} & G & A & B \\ + & G & A & B \\ \hline \end{array}$$

$$\begin{array}{ccc} \hline B & A & G \end{array}$$

for all possible positive integer bases $r \leq 10$.

Solution

Because BAG and GAB are positive, $BAG > GAB$ and it follows that $B > G$. Hence $B + B \neq G$ unless there is a carry from the r^0 column into the r^1 column. That is,

$$B + B = G + r.$$

By the same reasoning it follows that

$$A + A + 1 = A + r$$

and

$$G + G + 1 = B.$$

Solving for B, A and r in terms of G we have $B = 2G + 1$, $A = 3G + 1$, $r = 3G + 2$.

	$B = 2G + 1$ $A = 3G + 1$ $r = 3G + 2$	$\begin{array}{cccc} & G & A & B \\ + & G & A & B \\ \hline & B & A & G \end{array}$
$G = 0$	$B = 1$ $A = 1$ $r = 2$ (base number)	$\begin{array}{cccc} & 0 & 1 & 1 \\ + & 0 & 1 & 1 \\ \hline & 1 & 1 & 0 \end{array}$
$G = 1$	$B = 3$ $A = 4$ $r = 5$ (base number)	$\begin{array}{cccc} & 1 & 4 & 3 \\ + & 1 & 4 & 3 \\ \hline & 3 & 4 & 1 \end{array}$
$G = 2$	$B = 5$ $A = 7$ $r = 8$ (base number)	$\begin{array}{cccc} & 2 & 7 & 5 \\ + & 2 & 7 & 5 \\ \hline & 5 & 7 & 2 \end{array}$

For $G \geq 3$, we find that $r > 10$ and hence $G = 0, G = 1$ and $G = 2$ are the only possible cases.

■

1.10 Multiplication and Division in Base b

(1A964)
Let the distinct digits of the base twelve number system be $0,1,2,3,4,5,6,7,8,9,t,e$. Find $(te)_{12} \times (et)_{12}$, giving the answer in base twelve notation.
Solution

```
    te
 x  et
 ----
  ????
```

$$e_{12} \cdot t_{12} = (10)_{10} \cdot (11)_{10} = (110)_{10} = 9(12^1) + 2(12^0) \Longrightarrow \text{2 and carry 9}$$

$$t_{12} \cdot t_{12} + 9_{12} = (10)_{10} \cdot (10)_{10} + 9_{10} = (109)_{10} = 9(12^1) + 1(12^0) = (91)_{12}$$

So, our top row of the multiplication is $(912)_{12}$.

$$e_{12} \cdot e_{12} = (11)_{10} \cdot (11)_{10} = (121)_{10} = 10(12^1) + 1(12^0) \Longrightarrow \text{1 and carry 10}$$

$$e_{12} \cdot t_{12} + (10)_{12} = (10)_{10} \cdot (11)_{10} + (10)_{10} = (120)_{10} = 10(12^1) + 0(12^0) = (t0)_{12}$$

So, our bottom row of the multiplication is $(\mathrm{t}010)_{12}$.

Now we need to add (in base 12) $(912)_{12} + (\mathrm{t}010)_{12}$.

```
    912
 + t010
 ------
   ????
```

2+0=2 **2** with no carry
1+1=2 **2** with no carry
9+0=9 **9** with no carry
t=t **t**

That is,

```
   912
+ t010
------
  t922
```

Therefore, $(te)_{12} \times (et)_{12} = (t922)_{12}$.

Check (base ten)

$$\begin{array}{r} te \\ \underline{et} \\ 912 \\ \underline{t01} \\ t922 \end{array} \qquad \begin{array}{r} 131 \\ \underline{142} \\ 262 \\ 524 \\ \underline{131} \\ 18602 \end{array}$$

■

(1T962)

Let the digits of the base sixteen number system (the so-called hexadecimal system) be 0,1,2,3,4,5,6,7,8,9,a,b,c,d,e,f. Write (ab) × (ba) using base sixteen notation.

Solution

$$\begin{array}{r} ab \\ \underline{ba} \\ 6ae \\ \underline{759} \\ 7c3e \end{array}$$

Check Base Ten

$$\begin{array}{r} 171 \\ \underline{186} \\ 31806 \end{array}$$

$$\begin{array}{lcr} 7 \cdot 16^3 & = & 28672 \\ c \cdot 16^2 & = & 3072 \\ 3 \cdot 16 & = & 48 \\ e \cdot 1 & = & \underline{14} \\ & & 31806 \end{array}$$

■

(TA964)

Let the digits of the base sixteen number system (the so-called hexadecimal system) be 0,1,2,3,4,5,6,7,8,9,a,b,c,d,e,f. Write (db) $\times$ (bd) using base sixteen notation.

Solution

$$\begin{array}{r} \textbf{ab} \\ \textbf{x ba} \\ \hline \textbf{????} \end{array}$$

b $\cdot$ a $= 11 \cdot 10 = 110 = 6(16) + 14 \Longrightarrow$ e and carry 6 (e $= 14$)

a $\cdot$ a $+ 6 = 10 \cdot 10 + 6 = 106 = 6(16) + 10 =$ 6a (a $= 10$)

So, our top row of the multiplication is 6ae.

b $\cdot$ b $= 11 \cdot 11 = 121 = 7(16) + 9 \Longrightarrow$ 9 and carry 7

a $\cdot$ b $= 10 \cdot 11 + 7 = 117 = 7(16) + 5 \Longrightarrow 75$

So, our bottom row of the multiplication is 7590.

Now we need to add (in base 16) 6ae + 7590.

$$\begin{array}{r} \text{6ae} \\ +\ 7590 \\ \hline \text{????} \end{array}$$

e+0=e	**e** with no carry
a+9=10+9=19=1(16)+3	**3** carry 1
6+5+1=12=c	**c** with no carry
0+7=7	**7**

That is,

$$\begin{array}{r} \text{6ae} \\ +\ 7590 \\ \hline \textbf{7c3e} \end{array}$$

Therefore, (ab)·(ba)= $7c3e_{16}$

```
   db          219
   bd   Ck   × 189
  ---        -----
  b1f
 969
 ----
 a1af        41,319
```

■

Mathematics Teacher, Calendar Problem 28, November 1991

28 The following multiplication problem is correctly done if the problem is written in base b. What base is b?

$$15_b \times 15_b = 321_b$$

Solution

$$15_b \times 15_b = 321_b$$
$$\therefore (b+5)(b+5) = 3b^2 + 2b + 1$$
$$b^2 + 10b + 25 = 3b^2 + 2b + 1$$
$$0 = b^2 - 4b - 12$$
$$0 = (b-6)(b+2)$$
$$\therefore b = 6$$

■

Five Hundred Mathematical Challenges, Barbeau, Klamkin, Moser, Problem #299

Let the base two number a be $a = \left(\underbrace{11\cdots1}_{k\ 1's}\right)_2$. Find the last $(k+2)$ digits of a^2 expressed in base two notation.

Solution

We can get a sense for the problem by grinding out the base two arithmetic in the case $k = 4$.

$$
\begin{array}{cccccccc}
 & & & & 1 & 1 & 1 & 1 \\
 & & & \times & 1 & 1 & 1 & 1 \\
\hline
 & & & & 1 & 1 & 1 & 1 \\
 & & & 1 & 1 & 1 & 1 & \\
 & & 1 & 1 & 1 & 1 & & \\
 & 1 & 1 & 1 & 1 & & & \\
\hline
1 & 1 & 1 & 0 & 0 & 0 & 0 & 1
\end{array}
$$

In this case the last $k+2=4+2=6$ digits are 100001. A different approach will make it easier to sort out the problem for general k.

$$
\begin{aligned}
(11\cdots 1)_2 &= 1\cdot 2^{k-1}+1\cdot 2^{k-2}+\cdots 1\cdot 2^1+1\cdot 2^0 \\
&= \frac{2^k-1}{2-1}=2^k-1.
\end{aligned}
$$

Hence

$$
\begin{aligned}
a^2 &= (2^k-1)^2 = 2^{2k}-2^{k+1}+1 \\
&= 2^{k+1}(2^{k-1}-1)+1 \\
&= 2^{k+1}(2^{k-2}+2^{k-3}+\cdots+1)+1 \\
&= 1\cdot 2^{2k-1}+1\cdot 2^{2k-2}+\cdots+1\cdot 2^{k+1}+0\cdot 2^k+\cdots+0\cdot 2^1+1\cdot 2^0 \\
&= \Bigg(\underbrace{11\cdots 1}_{(k-1)\,1\text{'s}}\,\overbrace{00\cdots 0}^{k\ 0\text{'s}}\,1\Bigg)_2.
\end{aligned}
$$

Therefore, the last $(k+2)$ digits of a^2 expressed in base two are $1\,\overbrace{00\cdots 0}^{k\ 0\text{'s}}\,1$.

■

1.11 Carrying in Base b

1.12 Largest and Smallest Numbers in Base b

(4T165) Let b be a positive integer greater than 1. Determine all values of b such that there are exactly 306 two-digit numbers with a base of b.

<u>Solution</u>

The smallest two digit number would be 10_b *while the largest would be* $(b-1)(b-1)_b$. *Thus,*
$(b-1)\cdot b+(b-1)-(1\cdot b+0)+1=306 \Rightarrow b^2-b=306 \Rightarrow b^2-b-306=0 \Rightarrow (b-18)(b+17)=0$. *So b is* $\boxed{18}$.

■

1.13 Balanced Ternary System

45. Prove that any positive integer a can be uniquely expressed in the form

$$a = 3^m + b_{m-1}3^{m-1} + b_{m-2}3^{m-2} + \cdots + b_0$$

where each $b_j = 0, 1$, or -1.

■

1.14 Recreational Mathematics Involving Alternative Base Number Systems

Nim

In his 2008 recreational book entitled Group Theory in the Bedroom, Brian Hayes also mentions the design of the ubiquitous "menus" in automated telephone helplines: Directing a phone customer to one of *many* final destinations is best accomplished by offering successive options in groups of 3 (assuming we just want to minimize the total number of choices presented to stranded callers).
http://www.numericana.com/answer/numeration.htm#bestradix
Third Base. (*American Scientist*, Vol. 89, No. 6, November–December 2001, pages 488–492. Online version.)
People count by 10s and machines count by 2s—that pretty much sums up the way we do arithmetic on this planet. But there are countless other ways to count. Here I want to offer three cheers for base 3, the ternary system. The numerals in this sequence—beginning 0, 1, 2, 10, 11, 12, 20, 21, 22, 100, 101, 102—are not as widely known or widely used as their decimal and binary cousins, but they have charms all their own. They are the Goldilocks choice among numbering systems: when base 2 is too small and base 10 is too big, base 3 is just right.
On the Teeth of Wheels. (*American Scientist*, Vol. 88, No. 4, July–August 2000, pages 296–300. Online version.)
For many years, the basic raw material of the computer industry was not silicon but brass. Calculators built before 1700... were all based on the meshing of metal gears.

http://www.numericana.com/answer/magic.htm#ternary

http://www.solbakkn.com/math/triadic-nums.htm

A puzzle solved using a base 3 number system

I applied the above form of base 3 to a math puzzle discussed by Ken Pledge in the May, 1993 issue of Dramatic University (what he refers to as a 'reversed notation' on pages 23-25). The puzzle gives you a balance scale and some coins, from which you are to distinguish which coin is counterfeit and whether it is heavy or light. And you're to use as few weighings with the scale as possible. If the scale balances, you know all the coins on the scale are of the proper weight. If it does not, then either a coin on the heavy side is too heavy, or a coin on the light side is too light. Either could cause the imbalance. As I re-phrased the problem: If you have 13 coins, of which no more than one is counterfeit (too heavy or too light), and a 14th coin, known to be authentic, how many weighings will it take to determine if a coin is bad, and if so which, and whether it is heavy or light. There are 27 possible outcomes (all coins OK, coin one to thirteen light, coin one to thirteen heavy).

Three uses of the scale produce 3*3, or 27, results - making it theoretically possible to solve the above problem in three weighings.

A hint to solving the puzzle: arrange the digits expressing one to thirteen in three columns. Each column (9's, 3's, 1's) determines what gets weighed at one time. Each row determines whether a given coin will be weighed, and if so, on which side of the scale.

Answer to the Coin-Weighing Problem

Home • Triadic Numbers • N-Grams • Dimensionalities

1	0	0	+
2	0	+	-
3	0	+	0
4	0	+	+
5	+	-	-
6	+	-	0
7	+	-	+
8	+	0	-
9	+	0	0
10	+	0	+
11	+	+	-
12	+	+	0
13	+	+	+

To the left is a box with each of the numbers one to thirteen (one row for each coin) expressed in the zero-centered base three. The columns are the nine's, three's, and one's places. Each column determines how to position the coins for one of the weighings on the scale-zero for coins not being weighed, and + and - indicating on which side of the scale to put a coin. The fourteenth coin is used as necessary to equalize the number of coins on each side of the scale.

The trick here is to use one rule for odd-numbered coins and a different rule for even-numbered coins. For odd-numbered coins let + indicate the right-hand side of the scale and - the left-hand side. Reverse this rule for even-numbered coins. For the first weighing, then, coins 5, 7, 9, 11 and 13 would go on the right side of the scale, and coins 6, 8, 10, 12 and 14 (so that there are 5 coins on each side) would go on the left. For the second weighing, coins 3, 6, 11, 13 and 14 would go on the right, and coins 2, 4, 5, 7, and 12 would go on the left.

To convert the results of the three weighings into an answer to the puzzle, one takes the first weighing to have a value of +9 (if the right side is heavier) or -9 (if the left side is heavier). The second weighing would have a value +3 or -3, and the third weighting a value of +1 or -1. Add the results. If zero, no coins are counterfeit. For non-zero results, the absolute value of the answer indicates which coin is off. A negative result indicates a light odd-numbered coin or a heavy even-numbered coin. A positive result indicates the opposite.

Problem 1.1 Total Geek Magic
Write down a polynomial $p(x)$ with integer coefficients greater than or equal to zero, but don't show it to me. Do remember that the leading coefficient of a polynomial $p(x)$ cannot be zero. Then pick any integer you like which is strictly greater than each of coefficients of $p(x)$. Suppose you call this number a. As an example, you might choose to let $p(x) = 4x^4 + 2x + 5$ and then you might pick $a = 8$ (which is greater than all three coefficients $4, 2$ and 5 of $p(x)$ as required).

Now if you tell me just a and $p(a)$ I will reveal your polynomial $p(x)$ in its entirety. However, please do be kind to me and don't make the degree of $p(x)$ too high and don't pick your number a to be too large because you will likely get bored waiting for me to figure everything out!

Your question is to figure out how I manage to find your polynomial $p(x)$ from knowing just one point.

<u>Solution</u>
It is best understood by stepping through an example. Suppose you had picked $p(x) = 4x^4 + 2x + 5$ and then chose $a = 8$.
In this case you will reveal to me that your $a = 8$ and that $p(8) = 16{,}405$. At this point I know a little more than just these two numbers. I also know that $p(x)$ is a polynomial with nonnegative coefficients and all of these coefficients are strictly less than 8.
Suppose we let $c_0, c_1, c_2, \ldots$ be the unknown coefficients of your polynomial $p(x)$. So
$$p(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + \cdots .$$
Therefore, I know that
$$p(8) = c_0 + c_1 \cdot 8^1 + c_2 \cdot 8^2 + c_3 \cdot 8^3 + c_4 \cdot 8^4 + c_5 \cdot 8^5 + \cdots = 16{,}405$$
with each $c_j \leq 7$.
Do you see now that this is exactly the problem of converting 16405 into its base 8 equivalent! There is a standard method for doing this.
First find the highest power of 8 that is less than or equal to $16{,}405$. With a calculator I can easily find that $8^4 = 4096$ and $8^5 = 32{,}768$ so I know that $c_5, c_6, c_7, \ldots$ etc. all have to be 0 and that $c_4 \geq 1$.
How do I know that $c_4 \neq 0$. Notice that the maximum $c_0 + c_1 \cdot 8^1 + c_2 \cdot 8^2 + c_3 \cdot 8^3$ can be is

$$7 + 7 \cdot 8^1 + 7 \cdot 8^2 + 7 \cdot 8^3 = 4095 < 16405.$$
So, I cannot achieve 16405 if $c_4 = 0$.
By the way, is it just a coincident that the value of $7 + 7 \cdot 8^1 + 7 \cdot 8^2 s + 7 \cdot 8^3 = 4095$ and that $8^4 - 1 = 4095$?
No. This comes from the identity
$$(b-1) + (b-1)b^1 + (b-1)b^2 + \cdots + (b-1)b^k = b^{k+1} - 1.$$
Where does this come from? Write it out the left-hand side in full.
$$\begin{aligned}&(b-1) + (b^2 - b) + (b^3 - b^2) + (b^4 - b^3) + \cdots + (b^k - b^{k-1}) + (b^{k+1} - b^k)\\&\quad = -1 + (b - b) + (b^2 - b^2) + (b^3 - b^3) + \cdots + (b^k - b^k) + b^{k+1}\\&\quad = b^{k+1} - 1.\end{aligned}$$
Second, find out the largest whole number of times $8^4 = 4096$ goes into 16405. This will give us the value of c_4.
$$\frac{16405}{4096} = 4 + \frac{21}{4096} \Longrightarrow c_4 = 4.$$
Third, go back to step one using the remainder 21 as your new number.
$8^1 = 8 \leq 21$ but $8^2 = 64 > 21$. As a result, $c_2 = c_3 = 0$.
How many whole numbers of times does 8^1 go into 21? Twice, with a remainder of 5. So $c_1 = 2$.
Now start over with the remainder 5.
How many whole numbers of times does $8^0 = 1$ go into 5? Five times. So $c_0 = 5$.
Now I can state that your polynomial must have been
$$c_4 x^4 + c_3 x^3 + c_2 x^2 + c_1 x + c_0 = 4x^4 + 0x^3 + 0x^2 + 2x + 5 = 4x^4 + 2x + 5.$$
Everyone cheers!

■

1.15 Negative Integer Bases

Pg. 32, Ex. 8
Let b be greater than 1. Show that every integer a (positive, negative, or zero) can be represented uniquely in base $-b$, that is, in the form
$$a = c_n(-b)^n + c_{n-1}(-b)^{n-1} + \cdots + c_1(-b) + c_0$$
with $c_n \neq 0$ if $a \neq 0$ and $0 \leq c_i < b$ for $0 \leq i \leq n$. If $a < 0$, show that n is odd. If $a > 0$, show that n is even.

Ex. 9
Show that the method for changing from base 10 notation to base b is also valid for changing to base $-b$. For example, to write 392_{10} in base -10, we have that
$$\begin{aligned}392 &= (-10)(-39) + 2\\-39 &= (-10)(4) + 1\\4 &= (-10)(0) + 4\end{aligned}$$

So that $392_{10} = 412_{-10}$. In short form as above, this could have been written

$$\begin{array}{rlll} -10 \,|\, & 392 & & \\ -10 \,|\, & -39 & = q_1 & r_1 = 2 \\ -10 \,|\, & 4 & = q_2 & r_2 = 1 \\ & 0 & = q_3 & r_3 = 4 \end{array}$$

Ex. 10
Write (a) 82_{10} and (b) -761_{10} in base -10 notation.

Ex. 12
Show that, in addition or multiplying numbers in base -10, one must *subtract* the "carries." For example, 87_{-10} and 206_{-10} are added in the following way:

$$\begin{array}{ccc} & {\scriptstyle -1} & \\ & 8 & 7 \\ 2 & 0 & 6 \\ \hline 2 & 7 & 3 \end{array}$$

Write these numbers in base 10 notation and check that the addition is correct.

■

1.16 a_b is the Square of an Integer

AMC 1962 Problem #22
The number 121_b written in the integral base b, is the square of an integer for

(A) $b = 10$, only	(B) $b = 10$ and $b = 5$, only	(C) $2 \leq b \leq 10$
(D) $b > 2$	(E) no value of b	

Solution

■

AMC 1973 Problem #6
If 554 is the base b representation of the square of the number who's base b representation is 24, then b, when written in base 10, equals

(A) 6	(B) 8	(C) 12	(D) 14	(E) 16

Solution

■

AMC 1982 Problem #26

If the base 8 representation of a perfect square is $ab3c$, where $a \neq 0$, then c is

(A) 0	(B) 1	(C) 3	(D) 4	
(E) not uniquely determined				

Solution

(B) If $n^2 = (ab3c)_8$, let $n = (de)_8$. Then $n^2 = (8d + e)^2 = 64d^2 + 8(2de) + e^2$. Thus, the 3 in $ab3c$ is the first digit (in base 8) of the sum of the eights digit of e^2 (in base 8) and the units digit of $(2de)$ (in base 8). The latter is even, so the former is odd. The entire table of base 8 representations of squares of base 8 digits appears below.

e	1	2	3	4	5	6	7
e^2	1	4	11	20	31	44	61

The eights digit of e^2 is odd only if e is 3 or 5; in either case c, which is the units digit of e^2, is 1. (In fact, there are three choices for n: $(33)_8$, $(73)_8$ and $(45)_8$. The squares are $(1331)_8$, $(6631)_8$ and $(2531)_8$, respectively.)

(B) If $n^2 = (ab3c)_8$, let $n = (de)_8$. Then $n^2 = (8d + e)^2 = 64d^2 + 8(2de) + e^2$. Thus, the 3 in $ab3c$ is the first digit (in base 8) of the sum of the eights digit of e^2 (in base 8) and the units digit of $(2de)$ (in base 8). The latter is even, so the former is odd. The entire table of base 8 representations of squares of base 8 digits appears below.

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The eights digit of e^2 is odd only if e is 3 or 5; in either case c, which is the units digit of e^2, is 1. (In fact, there are three choices for n: $(33)_8$, $(73)_8$ and $(45)_8$. The squares are $(1331)_8$, $(6631)_8$ and $(2531)_8$, respectively.)

OR

We are given

$$n^2 = (ab3c)_8 = 8^3a + 8^2b + 8 \cdot 3 + c.$$

If n is even, n^2 is divisible by 4, and its remainder upon division by 8 is 0 or 4. If n is odd, say $n = 2k + 1$, then $n^2 = 4(k^2 + k) + 1$, and since $k^2 + k = k(k + 1)$ is always even, n^2 has remainder 1 upon division by 8. Thus in all cases, the only possible values of c are 0, 1 or 4. If $c = 0$, then $n^2 = 8(8K + 3)$, an impossibility since 8 is not a square. If $c = 4$, then $n^2 = 4(8L + 7)$ another impossibility since no odd squares have the form $8L + 7$. Thus $c = 1$.

OR

A perfect square will be $(8k+r)^2 = 64k^2 + 16kr + r^2 \equiv r^2 \pmod{16}$ where $r = 0, 1, ..., 7$.

Notice that $r^2 \equiv 1, 4, 9, 0 \pmod{16}$.

Now $ab3c$ in base 8 is $a8^3 + b8^2 + 3(8) + c \equiv 8 + c \pmod{16}$. It being a perfect square means $8 + c \equiv 1, 4, 9, 0 \pmod{16}$. That means that c can only be 1 so the answer is 1 = $\boxed{\textbf{(B)}.}$.

■

Challenging Problems in Algebra, Posamentier and Salkind
Chapter 4, Bases: Binary and Beyond

4-4 In what base b is 441_b the square of an integer?

4-4 *In what base* b *is* 441_b *the square of an integer?*

$441_b = 4b^2 + 4b + 1 = (2b+1)^2$. Therefore, 441_b is the square of an integer in all bases $b > 4$, as 4 must be a member of the set of digits to be used in base b.

ILLUSTRATION 1: $441_5 = (21_5)^2$

ILLUSTRATION 2: $441_{10} = (21_{10})^2$

ILLUSTRATION 3: $441_{12} = (21_{12})^2$

Challenge 1 If N is the base 4 equivalent of 441 written in base 10, find the square root of N in base 4.

Challenge 1 *If* N *is the base* 4 *equivalent of* 441 *written in base* 10, *find the square root of* N *in base* 4.

$$441_{10} = 12321_4 = 1 \cdot 4^4 + 2 \cdot 4^3 + 3 \cdot 4^2 + 2 \cdot 4 + 1$$
$$= (1 \cdot 4^2 + 1 \cdot 4 + 1)^2 = N$$
$$\therefore \sqrt{N} = 111_4$$

Challenge 2 Find the smallest base b for which 294_b is the square of an integer.

Challenge 2 *Find the smallest base* b *for which* 294_b *is the square of an integer.*

$294_b = 2b^2 + 9b + 4 = (2b+1)(b+4)$

Since 294_b is even and $2b+1$ is odd, then $b+4$ is even so that b is even, and $b \geq 10$. (Why?) It follows that each factor is the square of an integer.

If $b = 10$, then $2b + 1 = 21$, not the square of an integer.

If $b = 12$, then $2b + 1 = 25 = 5^2$ and $b + 4 = 16 = 4^2$.

VERIFICATION: $294_{12} = 2 \cdot 12^2 + 9 \cdot 12 + 4 = (1 \cdot 12 + 8)^2 = (18_{12})^2$

COMMENT: The next larger base is 60. Verify.

Problem Solving Through Recreational Mathematics
Bonnie Averbach and Orin Chein

More About Numbers: Bases and Cryptarithmetic

★ **5.28.** $\boxed{\text{H}}$ In what base is 11111 a perfect square? ([46], Vol. 15, Problem 149)

46. *National Mathematics Magazine*, Vol. 9–20 (1934–1945).

National mathematics magazine

5.28. **Hint**: If $(11111)_b = b^4 + b^3 + b^2 + b + 1 = x^2$, is x larger or smaller than $b^2 + \frac{b}{2}$?

What about $b^2 + \frac{b+1}{2}$? What must x be equal to?

National Mathematics Magazine, Vol. 15, No. 3 (Dec., 1940), pp. 145-153

No. 355. Proposed by *V. Thébault*, Le Mans, France.

Show that the three-digit number 111 is not a perfect square in any system of numeration. Is the same true of the five-digit number 11111 ?

Solution by the *Proposer*.

If the base of the system of numeration is represented by B, $B>1$, the number in question may be written $111=B^2+B+1$. Now

$$B^2<B^2+B+1<(B+1)^2.$$

Hence, whatever the value of B, B^2+B+1 is never a perfect square.

The same is not true of the number

$$11111=B^4+B^3+B^2+B+1.$$

Indeed, if $B=3$, $11111=(102)^2$. That this is the only value of B appears from the following relations:

$$(B^2+\tfrac{1}{2}B)^2<B^4+B^3+B^2+B+1<(B^2+\tfrac{1}{2}B+1)^2,$$

$$(B^2+\tfrac{1}{2}B+\tfrac{1}{2})^2=B^4+B^3+B^2+B+1 \text{ implies } B=3.$$

★ 5.29. [H] [A] In what base is 297 a factor of 792? ([41], Vol. 38, Problem 124)

41. *Mathematics Magazine,* Mathematical Association of America, Vol. 21– (1947–).

5.29. **Hint: What could the value of** 792 ÷ 297 **possibly be?**

5.29. 19.

Problem Department
National Mathematics Magazine, Vol. 15, No. 7 (Apr., 1941), pp. 378-383
National Mathematics Magazine, Vol. 16, No. 2 (Nov., 1941), pp. 102-109

No. 410. Proposed by *V. Thébault*, San Sebastián, Spain.

In what system of numeration, with base less than 100, is the three-digit number 333 a perfect square?

No. 410. Proposed by *V. Thébault*, Tennie(France-Sarthé).

In what system of numeration, with base less than 100, is the three-digit number 333 a perfect square?

Solution by *D. C. Binneweg*, student, Colgate University.

Let b be the base and x^2 the required square, obtaining

$$x^2=3b^2+3b+3.$$

Solving this equation for b (using the denary scale) we have

$$b=(-3+\sqrt{12x^2-27})/6.$$

A necessary condition is that $12x^2-27$ be a perfect square, say y^2. The least positive solution of $y^2=12x^2-27$ is (3,9), whence all solutions are given by

$$y+x\sqrt{12}=(9+3\sqrt{12})(7+2\sqrt{12})^k, \quad k=0,\ 1,\ 2,\ \cdots.$$

$k=0$ yields $b=1$, which is of no use. $k=1$ yields the solution $b=22$. Since $k=2$ gives $b=313$, 22 is the only solution less than 100.

Also solved by *A. B. Farnell*, *C. W. Trigg*, and the *Proposer*.

1.17 Extra Problems in Base Number Systems

1. (TT895)
The numeral $(ABC)_{nine}$ (that is, a numeral represented base nine) represents the same number as does $(CBA)_{seven}$. What is the base ten representation of this number?

2. School Science and Mathematics, Problem 1725, December 1941.
A number of three digits in base 7 when expressed in base 9 has the order of its digits reversed. What is the number in base 10?

3. *School Science and Mathematics*, Problem 3824, Volume 80, Issue 6, October 1980, page 525. Proposed by Herta Freitag.
Let N be a two-digit number in some base b, where b is an integer ≥ 2. Find all bases such that reversing the digits of N yields twice N.

4. Saint Mary's College Mathematics Contest Problems
48. Take any number in base 5. Rearrange the digits and find the difference between the original number and the rearranged number. What is the largest integer that ALWAYS divides the difference?

5. (1A173) What is the base b for which 68_b is 25% larger than 53_b? (Note that the percent is given in base 10.)

6. (1T163) Find the <u>smallest</u> integer x such that $\sin(124_x) = \sin(221_x)$, where both angles are measured in degrees in base x.

7. (4T165) Let b be a positive integer greater than 1. Determine all values of b such that there are exactly 306 two-digit numbers with a base of b.

8. (TI1313) Find the smallest positive integer n such that $n^2 = \left(\underline{4}\,\underline{7}\,\underline{6}\,\underline{X}\,\underline{Y}\right)_b$ for some number base b and base-b digits X and Y. (Express your value of n in base 10.)

10. (5T012) A popular method for multiplying numbers during the Renaissance was that of "geloxia," or grating. The grid in Figure 2a illustrates the multiplication of 2375 by 127. Figure out how it works, and then fill in the grid in Figure 2b (an essential part of the problem), showing how to multiply the *base eight* numbers 375 by 25, putting in the margin at the left your answer *in base eight*. [College Math Journal., 2000]

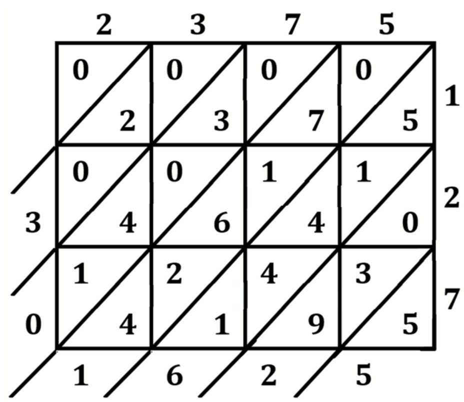

Figure 2a

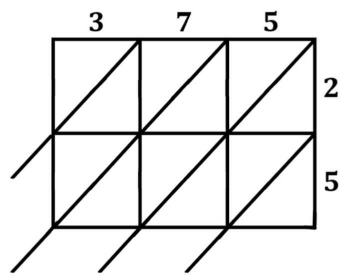

Figure 2b

11. (TA991)
In base b, c^2 is written 10. How do you write b^2 in base c?

12. (1A974)

4. According to the will, the entire estate of \$1,013,652, was to be divided among a list of potential heirs (in a given order of priority) according to the following stipulations:

1. **Each heir is to receive an amount (in dollars) equal to a power of 5 (that is, either \$1, \$5, \$25, \$125, ...).**
2. **No more than four people are to receive the same amount of money.**

How many heirs actually receive money from the estate? [*Mathematics*, Averbach, Chein]

13. National Mu Alpha Theta Convention 1991, Number Theory Topic Test, Problem 16
When the number n is written in base b its representation is the two-digit number AB_b where $A = b - 2$ and $B = 2$. What is the base $b - 1$ representation of n?

14. Jim Totten

(173) *Problem.* The expression of a positive integer n in base b is

$$n = (1254)_b.$$

It is known that the expression of the integer $2n$ in the same base is

$$2n = (2541)_b.$$

Determine the values of b and n in base 10.

15. AMC 1965 Problem #15

The symbol 25_b represents a two-digit number in the base b. If the number 52_b is double the number 25_b, then b is:

(A) 7	(B) 8	(C) 9	(D) 11	(E) 12

16. AMC 1978 Problem #14

If an integer n, greater than 8, is a solution of the equation $x^2 - ax + b = 0$ and the representation of a in the base n numeration system is 18, then the base n representation of b is

(A) 18	(B) 28	(C) 80	(D) 81	(E) 280

17. Mu Alpha Theta National Convention, 2001, Number Theory Test, Mu Division, Problem # 30

A number N expressed in base $(A+1)$ is $AAAA$. If $N = Q(Q-2)$, what is Q expressed in base $(A+1)$?

18. Mu Alpha Theta National Convention, 2001, Number Theory Test, Mu Division, Problem # 24

How many whole numbers are there less than $10{,}000$ which have units and tens digits of 1 when expressed in bases $4, 5$, and 6?

19. Mu Alpha Theta National Convention 2003, Number Theory Test, Alpha Division, Problem #23

How many base 10 counting numbers will have a three-digit representation in bases 4, 5 and 7?

20. Koshy

46. Suppose a space team investigating Venus sends back the picture of an addition problem scratched on a wall, as shown in Figure 2.4. The Venusian numeration system is a place value system, just like ours.

The base of the system is the same as the number of fingers on a Venusian hand. Determine the base of the Venusian numeration system. (This puzzle is due to H. L. Nelson.[†])

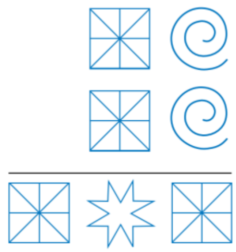

Figure 2.4

[†] M. Gardner, "Mathematical Games," *Scientific American*, 219 (Sept. 1968), 218–230.

21. Problem 9 (AHSME Dropped Problem)

In the small hamlet of Abaze, two base systems are in common use. Also, everyone speaks the truth. One resident said: "26 people use my base, base 10, and only 22 people speak base 14." Another said, "Of the 25 residents, 13 are bilingual and 1 cannot use either base." How many residents are there? (Use base 10 please!).

The following "caution" was *not* included in the statement of the original problem.

Caution: The wording in this problem can be misleading. To get to the root of the confusion let's think through the following imagined scenario.

Suppose the single digit numbers for Resident 1 of the hamlet of Abaze are $\{0,1,2,3,4,5,6,7,8,9\}$ and suppose that Resident 1 has no perception whatsoever of any base system other than their own.

Suppose the single digit numbers for Resident 2 of the hamlet of Abaze are $\{0,1,2,3,4,5,6,7\}$ and suppose that Resident 2 has no perception whatsoever of any base system other than their own.

Now imagine you asked both Resident 1 and Resident 2 what base they use. What would be their answers?

Assuming Resident 1 answers "base 10", what has Resident 1 actually told us from their own truthful perspective? They have told us that in their system the next number after 9 is 10.

Now assume that Resident 2 thinks in the same way. Then they would also answer "base 10" because in their system the next number after 7 is 10.

The confusion in the wording of this problem is that readers are naturally drawn to interpret a statement by what that statement means within their own personal system.

A very simple example would be if someone who "speaks in" base 5 tells me (a decimal speaker) there are 14 students in their math class. To correctly understand their statement, I have to interpret it from the perspective of their base system and not mine.

22. Problem 2

When the first Martian to visit Earth attended a high school algebra class, he watched the teacher show that the only solution of the equation $5x^2 - 50x + 125 = 0$ is $x = 5$.
"How strange," thought the Martian. "On Mars, $x = 5$ is a solution of this equation, but there is another solution."
If Martians have more fingers than Earthers, how many fingers do Martians have*?
* Historically, at least part of the reason that we have adopted the base ten for our number system is that humans have ten fingers; the implication for this problem then is that the number of fingers that Martians have is the base of their number system.

Mu Alpha Theta National Convention 2007, Mu Division, Number Theory Test, Problem #16

16) Each positive integer is coded using the following process: encode 1 as a, 2 as b, 3 as c, and continuing so that 26 is coded as z. After 26, encode 27 as aa, 28 as ab, and so forth. What is the code for 2007?
A) *bay* B) *bug* C) *bye* D) *bzi* E) NOTA

<u>Solution</u>

16) We solve this similar to a base-26 problem. $2007 \pmod{26} = 5$, $(2007-5)/26 = 77$
$77 \pmod{26} = 25$, $(77-25)/26 = 2$. $2 \Rightarrow b$, $25 \Rightarrow y$, $5 \Rightarrow e$ **C**

■

AMC 10A 2003 Problem 20

A base-10 three-digit number n is selected at random. What is the probability that the base-9 representation and the base-11 representation of n are both three-digit numbers?

<u>Solution</u>

Because the three-digit number n was selected at random from all numbers with 3 digits in base 10, the required probability equals

$$\frac{N(3\text{ digits in base 9 and base 10 and base 11})}{N(3\text{ digits in base 10})}.$$

The number n will have 3 digits in base 9 provided $100_9 \leq n \leq 888_9$, will have 3 digits in base 10 provided $100 \leq n \leq 999$ and will have 3 digits in base 11 provided $100_{11} \leq n \leq aaa_{11}$.

We note that

$$\begin{aligned}100_9 &= 1 \cdot 9^2 + 0 \cdot 9^1 + 0 \cdot 9^0 = 81\\ 888_9 &= 8 \cdot 9^2 + 8 \cdot 9^1 + 8 \cdot 9^0 = 728\\ 100_{11} &= 1 \cdot 11^2 + 0 \cdot 11^1 + 0 \cdot 11^0 = 121\\ aaa_{11} &= 10 \cdot 11^2 + 10 \cdot 11^1 + 10 \cdot 11^0 = 1330.\end{aligned}$$

Therefore,

$$\begin{aligned}&N(3\text{ digits in base 9 and base 10 and base 11})\\ &\quad = N\big((81 \leq n \leq 728)\text{ and }(100 \leq n \leq 999)\text{ and }(121 \leq n \leq 1330)\big)\\ &\quad = N(\max\{81,100,121\} \leq n \leq \min\{728,999,1330\})\\ &\quad = N(121 \leq n \leq 728)\\ &\quad = 728 - 121 + 1 = 608\end{aligned}$$

and

$$\begin{aligned}&N(3\text{ digits in base 10})\\ &\quad = N(100 \leq n \leq 999)\\ &\quad = 999 - 100 + 1 = 900.\end{aligned}$$

Hence, the required probability equals

$$\frac{N(3\text{ digits in base 9 and base 10 and base 11})}{N(3\text{ digits in base 10})} = \frac{608}{900} = \frac{152}{225}.$$

■

Example (Source:

A certain integer is represented base 5 by 40142_5 and base b by 1583_b. Find b. (Source: 2001-02, State Tournament, Event A)

<u>Solution</u>

$$40142_5 = 4(5^4) + 0(5^3) + 1(5^2) + 4(5^1) + 2(5^0) = 2547$$

$$1583_b = 1(b^3) + 5(b^2) + 8(b^1) + 3(b^0).$$

$\Longrightarrow b^3 + 5b^2 + 8b + 3 = 2547$

$\Longrightarrow b^3 + 5b^2 + 8b - 2544 = 0$

$\Longrightarrow b$ must divide $2544 = 2^4 \cdot 3 \cdot 54$ (rational root theorem).

Synthetic division shows $b = 8$ is not big enough

$$\begin{array}{r|rrrr} 8 & 1 & 5 & 8 & -2544 \\ \hline & & 8 & 104 & 896 \\ \hline & 1 & 13 & 112 & -1648 \end{array}$$

but that the next largest potential root is $b = 12$ is in fact a root.

$$\begin{array}{r|rrrr} 12 & 1 & 5 & 8 & -2544 \\ \hline & & 12 & 204 & 2544 \\ \hline & 1 & 17 & 112 & 0 \end{array}$$

We can also see from this division that

$$b^3 + 5b^2 + 8b - 2544 = (b - 12)(b^2 + 17b + 112)$$

and because $b^2 + 17b + 112$ is an irreducible quadratic (the discriminant is negative) there are no other real roots. So $b = 12$ is the only possible answer.

Chapter 2. Factoring, Prime Numbers and Prime Factorization

2.1 Factoring

Special Case Factoring Formulas

Difference of two powers	$a^n - b^n = (a-b)\left(a^{n-1} + a^{n-2}b + a^{(n-3)}b^2 + \cdots + ab^{n-2} + b^{n-1}\right)$
Sum of two odd powers	$a^n + b^n = (a+b)\left(a^{n-1} - a^{n-2}b + a^{(n-3)b^2} - \ldots + a^2b^{n-3} - ab^{n-2} + b^{n-1}\right)$ $n = 3,5,7,9,\ldots$

$$\left(x-(c+d)\right)\left(x-(-c+d)\right)\left(x-(c-d)\right)\left(x-(-c-d)\right)$$
$$= x^4 - 2(c^2+d^2)x^2 + (c^2-d^2)^2$$

Exercise 2.1
Fully factor $x^{12} - y^{12}$.

<u>Solution</u>

$x^{12} - y^{12}$

$= (x^6)^2 - (y^6)^2$ — 2 is the smallest prime factor of 12

$= \left(x^6 - y^6\right)\left(x^6 + y^6\right)$ — Difference of two squares

$x^6 - y^6$

$= (x^3)^2 - (y^3)^2$ — 2 is the smallest prime factor of 6

$= \left(x^3 - y^3\right)\left(x^3 + y^3\right)$ — Difference of two squares

$x^6 + y^6$

$= (x^2)^3 + (y^2)^3$ — 3 is the smallest <u>odd</u> prime factor of 6

$= (x^2 + y^2)(x^4 - x^2y^2 + y^2)$ — Sum of two cubes

$x^3 - y^3$

$$= (x - y)(x^2 + xy + y^2) \qquad \text{Difference of two cubes}$$

$$x^3 + y^3$$
$$= (x + y)(x^2 - xy + y^2) \qquad \text{Sum of two cubes}$$

$$x^4 - x^2y^2 + y^2$$
$$= (x^4 - x^2y^2 + (\mathbf{3x^2y^2}) + y^2) - (\mathbf{3x^2y^2}) \qquad \text{Adding and subtracting like factor}$$
$$= (x^4 + 2x^2y^2 + y^2) - 3x^2y^2$$
$$= (x^2 + y^2)^2 - \left(\sqrt{3}xy\right)^2$$
$$= \left(x^2 + y^2 - \sqrt{3}xy\right)\left(x^2 + y^2 + \sqrt{3}xy\right) \qquad \text{Difference of two squares}$$

Piecing these results together, we have

$$x^{12} - y^{12} = (x - y)(x^2 + xy + y^2)(x + y)(x^2 - xy + y^2)(x^2 + y^2) \cdot \left(x^2 + y^2 - \sqrt{3}xy\right)\left(x^2 + y^2 + \sqrt{3}xy\right).$$

■

Sophie Germain Identity

$$a^4 + 4b^4 = (a^2 + 2b^2 + 2ab)(a^2 + 2b^2 - 2ab).$$

Fermat Factorization

Completing the Rectangle (Simon's Favorite Factoring Trick)

2.2 Prime Numbers

Pg. 17, Definition 1.2 (Prime and Composite Numbers)
If p is an integer greater than 1 whose only positive divisors are 1 and p itself, then p is called a ***prime***. If p exceeds 1 and is not a prime, then it is called ***composite***.

Pg. 17, Theorem 1.2
Every integer $n \geq 2$ is either a prime or can be represented as a product of primes.

Note: The positive integer 1 is neither prime nor composite by definition.

Lemma 1.5: Every integer greater than 1 has a prime divisor.

Theorem 1.6: (Euclid) There are infinitely many prime numbers.

Proposition 1.7: Let n be a composite number. Then n has a prime divisor p with $p \leq \sqrt{n}$.

- 2 is the only even prime number
- 2 and 3 are the only consecutive integers which are both prime
- *twin primes* : a pair of prime numbers whose difference equals two

  e.g. 3 and 5 are twin primes and 11 and 13 are twin primes because they are all prime numbers and $5 - 3 = 2$ and $13 - 11 = 2$.

Properties

Every prime number greater than 3 is of the form $6k + 1$ or $6k + 5$, where k is some integer.

(Equivalently, of the form $6k + 1$ or $6k - 1$. In this way it is clear that every prime (greater than 3) is an immediate neighbor of 6.)

$p^2 - 1$ is divisible by 24 for all prime $p > 3$.

Pg. 55, Fundamental Theorem of Arithmetic
Every integer $n \geq 2$ is either prime or a product of primes and the product is unique apart from the order in which the factors appear.

2.3 Prime Factorization

Pg. 55, Definition (Prime Factorization)

The Fundamental Theorem of Arithmetic tells us that there is a unique (up to the order in which the factors appear) way to express an integer $n \geq 2$ as a product of primes. This expression is called the **prime factorization** of that integer. For example,

$$2{,}094{,}840 = 5 \cdot 23 \cdot 2 \cdot 2 \cdot 3 \cdot 23 \cdot 11 \cdot 3 \cdot 2.$$

Of course, we could have represented 2,094,840 in the form

$$2{,}094{,}840 = 23 \cdot 11 \cdot 3 \cdot 2 \cdot 5 \cdot 2 \cdot 2 \cdot 3 \cdot 23.$$

The ***canonical*** (or standardized) **representation** is that representation where the primes are listed in increasing order and with exponents to account for a prime occurring multiple times. That is, in the form

$$2{,}094{,}840 = 2^3 \cdot 3^2 \cdot 5 \cdot 11 \cdot 23^2.$$

In a problem where we are comparing the prime factorization of two integers you will often see "**place holders primes**" included for those primes that are factors of one number but not the other.

For example, if we want to compare the integers 462 and 495 it can be visually helpful to include the place holder prime factor 5^0 in 462 and the place holder prime factors 2^0 and 7^0 in 495. That is, to express 462 and 495 in the forms

$$462 = 2^1 \cdot 3^1 \cdot 5^0 \cdot 7^1 \cdot 11^1$$
$$495 = 2^0 \cdot 3^2 \cdot 5^1 \cdot 7^0 \cdot 11^1.$$

Pg. 55, Theorem 2.23

Let $a = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r} = \prod_{i=1}^{r} p_i^{a_i}$ with $a_i > 0$ for each i be the canonical representation for a and let $b > 0$. Then $b|a$ if and only if $b = p_1^{b_1} p_2^{b_2} \cdots p_r^{b_r} = \prod_{i=1}^{r} p_i^{b_i}$ with $0 \leq b_i \leq a_i$ for each i.

2.4 Exercises in Prime Factorization

Calendar Problem 30, *Mathematics Teacher*, March 1987

a, b, and c are three positive integers such that $a^2, 2b^3$, and $3c^5$ are equal. Find the smallest values for a, b, c.

Solution

Let the prime factorizations of a, b and c be given by

$$\begin{aligned} a &= 2^{p_1} \cdot 3^{p_2} \cdot 5^{p_3} \cdot 7^{p_4} \cdots \\ b &= 2^{q_1} \cdot 3^{q_2} \cdot 5^{q_3} \cdot 7^{q_4} \cdots \\ c &= 2^{r_1} \cdot 3^{r_2} \cdot 5^{r_3} \cdot 7^{r_4} \cdots . \end{aligned}$$

Then the prime factorizations of $a^2, 2b^3$ and $3c^5$ would be

$$\begin{aligned} a^2 &= 2^{2p_1} \cdot 3^{2p_2} \cdot 5^{2p_3} \cdot 7^{2p_4} \cdots \\ 2b^3 &= 2^{3q_1+1} \cdot 3^{3q_2} \cdot 5^{3q_3} \cdot 7^{3q_4} \cdots \\ 5c^5 &= 2^{5r_1} \cdot 3^{5r_2+1} \cdot 5^{5r_3} \cdot 7^{5r_4} \cdots . \end{aligned}$$

We are given that $a^2 = 2b^3 = 3c^5$ and because prime factorizations are *unique* this requires that

$$\begin{array}{ccccc} 2p_1 & = & 3q_1 + 1 & = & 5r_1 \\ 2p_2 & = & 3q_2 & = & 5r_2 + 1 \\ 2p_3 & = & 3q_3 & = & 5r_3 \\ 2p_4 & = & 3q_4 & = & 5r_4. \\ & & \vdots & & \end{array}$$

Notice that we can take $p_j = q_j = r_j = 0$ for all $j \geq 3$ but this is not the case for $j = 1$ or 2. Notice that if $p_1 = 0$ then $2p_1 \neq 3q_1 + 1$. Similarly, if $q_1 = 0$ then $3q_1 + 1 \neq 2p_1$ and if $r_1 = 0$ then $5r_1 \neq 3q_1 + 1$.

So $p_1 \geq 1, q_1 \geq 1$ and $r_1 \geq 1$.

The same reasoning will also show that $p_2 \geq 1, q_2 \geq 1$ and $r_2 \geq 1$.

Because we are looking for the smallest positive a, b and c we will take $p_j = q_j = r_j = 0$ for all $j \geq 3$.

So, the entire problem is reduced to finding positive integers p_1, q_1, r_1, p_2, q_2, and r_2 such that

$$
\begin{aligned}
2p_1 &= 3q_1 + 1 &&= 5r_1 \\
2p_2 &= 3q_2 &&= 5r_2 + 1\,.
\end{aligned}
$$

We can immediately see that $p_1|5$ and $r_1|2$ so the minimum value of p_1 is 5 and the minimum value of r_1 is 2. Furthermore we can see that $q_1 = 1$ will not work because in this case we would need $3(1) + 1 = 5r_1$ which is impossible. Similarly, $q_1 = 2$ will not work because in this case $3(2) + 1 = 5r_1$ which is again impossible. So, the minimum value of q_1 is 3.

Notice that setting $p_1 = 5, q_1 = 3$ and $r_1 = 2$, their respective minimum values, will satisfy that the requirement that $2p_1 = 3q_1 + 1 = 5r_1$. Therefore, $p_1 = 5, q_1 = 3$ and $r_1 = 2$.
The same type of reasoning will also show that $p_2 = 3, q_2 = 2$ and $r_2 = 1$.

Therefore,

$$
\begin{aligned}
a &= 2^{p_1} \cdot 3^{p_2} = 2^5 3^3 \\
b &= 2^{q_1} \cdot 3^{q_2} = 2^3 3^2 \\
c &= 2^{r_1} \cdot 3^{r_2} = 2^2 3^1.
\end{aligned}
$$

■

Exercise

Find the prime factorization of 999936.

Solution

The leading four 9's is a clue to the direction to take*!* In particular, consider that

$$
\begin{aligned}
999936 = 100000 - 64 = 10^6 - 2^6 \\
&= 2^6(5^6 - 1) \\
&= 2^6(5^3 - 1)(5^3 + 1) \\
&= 2^6(5 - 1)(5^2 + 5(1) + 1^2)(5 + 1)(5^2 - 5(1) + 1^2) \\
&= 2^6(4)(31)(6)(21) \\
&= 2^9 \cdot 3^2 \cdot 7^1 \cdot 31.
\end{aligned}
$$

■

***Mathematics Teacher*, Calendar Problem 24, April 2004**

Positive integers B and C satisfy $B(B - C) = 23$. What is the value of C?

Solution

Because B is positive and the product $B(B-C)$ is positive, it follows that $B-C$ is positive. Therefore, we can conclude that $B > C > 0$. Because 23 is prime we know that $23 = B(B-C)$ implies

$$B = 1 \text{ and } B - C = 23$$

or

$$B = 23 \text{ and } B - C = 1.$$

But the former case makes $C = -22$ which contradicts $C > 0$. So, it follows from the latter case that $C = B - 1 = 23 - 1 = 22$.

■

National Mu Alpha Theta Convention 1991, Number Theory Topic Test, Problem #21

The number $2^{22} + 1$ has exactly one prime factor greater than 1000. Find it.

Solution

Let n be a positive odd integer and define a through the relationship $n + 1 = 2a$. In this case,

$$\begin{aligned} 2^{2n} + 1 &= (2^n + 1)^2 - 2^{n+1} \\ &= (2^n + 1)^2 - (2^a)^2 \\ &= (2^n + 1 + 2^a)(2^n + 1 - 2^a). \end{aligned}$$

Applying this factorization in the case $n = 11$, $a = (n+1)/2 = 12/2 = 6$, we have

$$\begin{aligned} 2^{22} + 1 &= (2^{11} + 1 + 2^6)(2^{11} + 1 - 2^6) \\ &= 2113 \cdot 1985 \\ &= 2113 \cdot 397 \cdot 5. \end{aligned}$$

This shows that if a prime factor greater than 1000 really does exist it will have to be 2113 because 2113 is clearly not divisible by 2 and $2113/3$ is less than 1000.

> In a test situation you would be "test smart" to just assume that the problem would not be worded as it is unless there was actually an answer and put 2113, and move on to the next problem.

If you want to verify that 2113 really is a number prime outside a test situation you will have to do test divisions of 2113 by all primes less than or equal to $\sqrt{2113}$ which is time consuming and tedious.

Mu Alpha Theta National Convention, 2001, Number Theory Test, Mu Division, Problem # 29

The product of two prime numbers between 60 and 80 is one less than a perfect square. What is the greater of the two primes?

Solution

29. The product can be written as $x^2 - 1$ and factored into (x + 1)(x – 1). If the only prime factors are between 60 and 80, they must have a difference of [(x + 1) – (x – 1)] = 2. The only twin primes in that range are 71 and 73. 73 is the larger.

■

Test #3 AMATYC Student Mathematics League **March/April 1997**

20. A picket fence has 100 vertical boards, numbered 1 through 100. Any board whose number is of the form $2k$-1 is painted red (1,3,5,7,...,99). Next any **unpainted** board whose number is of the form $3k$-1 is painted blue (2,8,14,...,98). If this process is continued, by increasing the coefficient of k by one and choosing a different color at each step, how many colors will be on the fence when every board has been painted?
A. 26 B. 29 C. 32 D. 35 E. none of these

Solution

20 A At step one ($2k$-1), we paint boards 1, 3, 5, 7, ... , 99.
At step two ($3k$-1), we paint boards 2, 8, 14, ... , 98.
At step three ($4k$-1), there are no boards to paint.
At step four ($5k$-1), we paint boards 6, 16, 26, ... , 96.
At step five ($6k$-1), there are no boards to paint.
There are only boards to paint when the coefficient of k is a prime.
The last board painted is board #100 at step one-hundred ($101k$-1).
The problem reduces to counting the number of primes less than or equal to 101.

■

Mu Alpha Theta National Convention 2005, Number Theory Test, Alpha Division, Problem #2

Find the prime factorization of 12221.

Solution

Taking advantage of the symmetry present is a good first step. We note that

$$12221 = 12100 + 121 = 121(100 + 1) = 11^2 \cdot 101.$$

101 is prime so we have the complete prime factorization.

■

Mu Alpha Theta 2019 National Convention, Number Theory Test, Open Division, Problem #27

27. How many distinct prime factors does $4^5 + 5^4$ have?

(A) 1 (B) 2 (C) 3 (D) 4 (E) NOTA

Solution

Using Sophie-Germaine factorization,

$$\begin{aligned}4^5+5^4 &= 5^4+4\cdot 4^4\\ &= (5^2+2\cdot 5\cdot 4+2\cdot 4^2)(5^2-2\cdot 5\cdot 4+2\cdot 4^2)\\ &= (25+40+32)(25-40+32)\\ &= (97)(17)\end{aligned}$$

Thus there are 2 prime factors of 5^4+4^5 $\boxed{B}$

■

https://artofproblemsolving.com/wiki/index.php/Sophie_Germain_Identity

ARML 2016 Individual #10

Find the largest prime factor of $13^4+16^5-172^2$, given that it is the product of three distinct primes.

Solution

Let $N=13^4+16^5-172^2$. Notice that $16^5=(2^4)^5=2^{20}$ and

$$\begin{aligned}13^4-172^2 &= 169^2-172^2\\ &= (169-172)(169+172)\\ &= -3\cdot 341 = 1-2^{10}.\end{aligned}$$

Thus

$$N=2^{20}-2^{10}+1.$$

Multiply by $1025=2^{10}+1$ to obtain

$$\begin{aligned}1025\cdot N &= (2^{20}-2^{10}+1)(2^{10}+1)=2^{30}+1\\ &= 4(2^7)^4+1.\end{aligned}$$

Now recall the **Sophie Germain Identity** which states that:

$$a^4+4b^4=(a^2+2b^2+2ab)(a^2+2b^2-2ab).$$

From this identity we have

$$\begin{aligned}1025\cdot N &= 4(2^7)^4+1\\ &= \left(1^2+2(2^7)^2+2(1)(2^7)\right)\left(1^2+2(2^7)^2-2(1)(2^7)\right)\\ &= (1+32768+256)(1+32768-256)\\ &= 33025\cdot 32513\\ &= (25\cdot 1321)(41\cdot 793)\end{aligned}$$

$$(25\cdot 41)\cdot N=(25\cdot 1321)(41\cdot 793)$$

$$\begin{aligned}N &= 1321\cdot 793\\ &= 1321\cdot 61\cdot 13.\end{aligned}$$

So 1321 is the largest prime factor of $N = 13^4 + 16^5 - 172^2$.

■

Mock AIME 5 2005-2006 Problems/Pro
Find the largest prime divisor of $5^4 + 4 \cdot 6^4$.
Solution

The **Sophie Germain Identity** states that:

$$a^4 + 4b^4 = (a^2 + 2b^2 + 2ab)(a^2 + 2b^2 - 2ab).$$

Therefore

$$\begin{aligned} 5^4 + 4 \cdot 6^4 &= (5^2 + 2 \cdot 6^2 + 2 \cdot 5 \cdot 6)(5^2 + 2 \cdot 6^2 - 2 \cdot 5 \cdot 6) \\ &= (25 + 72 + 60)(25 + 72 - 60) \\ &= 157 \cdot 37. \end{aligned}$$

157 and 37 are both prime numbers. Hence 157 is the largest prime divisor of $5^4 + 4 \cdot 6^4$.

■

Find the smallest prime divisor of $15^4 + 17^4$.
Solution
This is something of a "trick" question but I've variations used in many contests. The "trick" is to see if you might overthink this and waste time with a complicated approach.
15^4 and 17^4 are both odd numbers. Hence there sum is an even number. Thus $15^4 + 17^4$ is divisible by the smallest prime number 2.

■

Mu Alpha Theta National Convention: Denver, 2001
Number Theory Topic Test Solutions – Alpha Division

17. My father is less than forty years older than I am, but more than twenty years older than I am. The product of our ages is 1,288. How old is my father?

(A) 46 (B) 48 (C) 56 (D) 62 (E) NOTA

Solution

17. **Factoring 1,288 shows a prime factorization including 2 to the third power, 7, and 23. Since one of the ages is a multiple of 23, only a couple of attempts at dividing 1,288 need be made to see that (23, 56) and (28, 46) are the only age possibilities with differences less than 40. Only the first one has a difference greater than 20. My father is 56 -- and will be until a few days after this competition, though I will have turned 24 a few weeks before. ;-D**

■

Mu Alpha Theta, Number Theory Topic Test, 1991 National Convention, Problem #25

The number $2^{22}+1$ has exactly one prime factor greater than 1000. What is it?

Solution

$$\begin{aligned} 2^{22}+1 &= (2^{11}+1)^2 - 2^{12} \\ &= (2^{11}+1+2^6)(2^{11}+1-6) \\ &= 2113 \cdot 1985 \\ &= 2113 \cdot 5 \cdot 397. \end{aligned}$$

We could check to see if 2113 is prime (and it will be) but we can work backwards to see that it must be based on the wording of the question. Suppose 2113 isn't prime. That is, suppose $2113 = a \cdot b$ with $a \leq b$. Clearly $a \neq 2$ because 2113 is odd. But if $a > 2$, then $b < 1000$ and hence $2^{22}+1$ could not have a prime factor greater than 1000. But this contradicts the wording of the problem. So 2113 must be prime.

■

https://testbook.com/question-answer/if-n-314-313-12-then-what-is-the-larg--5c1b3b7d99958a038ab4be11

If N = $3^{14} + 3^{13} - 12$, then what is the largest prime factor of N?

Solution

$N = 3^{14} + 3^{13} - 12$

$\Rightarrow 3^{13} \times (3 + 1) - 12$

$\Rightarrow 3^{13} \times 4 - 12$

$\Rightarrow 3^{12} \times 12 - 12$

$\Rightarrow 12 \times (3^{12} - 1)$

$\Rightarrow 12 \times ((3^4)^3 - 1^3)$

$\Rightarrow 12 \times (3^4 - 1)\,(3^8 + 1 + 3^4)$

$\Rightarrow 12 \times 80 \times 6643$

$\Rightarrow 12 \times 80 \times 91 \times 73$

∴ Here we can see that largest prime factor is 73.

■

Chapter 3. Divisibility

3.1 Divisors, Factors and Multiples

Pg. 17, Definition 1.1 (Divides)
If a and b are integers with $a \neq 0$ and there exists an integer c such that $b = ac$, then we say that a **divides** b and write $a|b$.
We also call a a **divisor** or **factor** of b and b a **multiple** of a.
If a does **not divide** b, we write $a \nmid b$.

Pg. 17, Definition (Proper Divisor)
If $1 \leq a < b$ and $a|b$, then a is called a **proper divisor** of b. For example, 1,2 and 3 are proper divisors of 6, but 6 itself is not.

3.2 Divisibility and Primes

Pg. 45, Corollary 2.9 (Euclid)
If p is a prime and $p|bc$, then $p|b$, or $p|c$.

Lemma 1.19 (Euclid's Lemma). *If $a, b \in \mathbf{Z}$, and p is a prime such that $p \mid ab$, then $p \mid a$ or $p \mid b$. More generally, if $a_1, \ldots, a_n \in \mathbf{Z}$ and p is a prime such that $p \mid a_1 \cdots a_n$, then there exists an i with $1 \leq i \leq n$ such that $p \mid a_i$.*

Prove each, where a, b, c, d, m, and n are positive integers.

51. Let p be a prime such that $p|a^n$. Then $p^n|a^n$.

3.3 Divisibility Tests

An integer is divisible by 10 if and only if it ends in 0.
An integer is divisible by 5 if and only if it ends in 0 or 5.
An integer is divisible by 2^i if and only if the number formed by the last i digits is divisible by 2^i.
An integer is divisible by 3 if and only if the sum of its digits is divisible by 3.
An integer is divisible by 9 if and only if the sum of its digits is divisible by 9.
An integer is divisible by 11 if and only if the alternating sum of its digits is divisible by 11.
e.g. 918,082 is divisible by 11 because $9 - 1 + 8 - 0 + 8 - 2 = 22$ and 22 is divisible by 11

The integer $abcabc$ is divisible by 7, 11 and 13.

Proof

$$abcabc = abc(1000) + abc = abc(1001) = abc(7 \cdot 11 \cdot 13).$$

■

1A914
Write a six-digit number whose first three digits are 637... and which is divisible by 21, 22, 23, and 24.

Mu Alpha Theta National Convention 2005, Number Theory Test, Alpha Division, Problem #2

2. How many distinct positive prime numbers are divisors of 12221?

A) 2 B) 3 C) 4 D) 5 E) NOTA

Solution

2. A

We could use the Sieve of Eratosthenes to hunt for primes and we find that

$12221 = 11^2 \cdot 101^1$.

We might have also noticed the symmetry of the number. This might help us notice a number of shortcuts such as noticing 12221 = 12100 + 121 = 121(100 + 1) or that the divisibility rule for 11 is easy to apply.

■

3.4 Divisibility Tests in Base b

Even and Odd Numbers in Base b

We define a base 10 integer n as *even* if $n = 2m$ for some base 10 integer m. We say a base 10 integer n is *odd* if n is *not even*.

It is easy to establish from this definition that a base 10 integer n is even if the last digit of n (the units digit) belongs to the set $\{0,2,4,5,8\}$.

Why? If we expand n as $n = a_k 10^k + a_{k-1}10^{k-1} + \cdots + a_1 10^1 + a_0 10^0$ we can immediately see that $(a_k 10^k + a_{k-1}10^{k-1} + \cdots + a_1 10^1)$ is even because $10^j = 2(5 \cdot 10^{j-1})$ for each $j =$

$1,2,\ldots,k$. So, the parity (evenness or oddness) of n all comes down to whether a_0 is even or odd.

Evenness or Oddness of an Integer in Base b

Definition: We define an integer n in base b to be even if its base 10 equivalent is even.

By this definition, $n = 13_5$ is even because $13_5 = 1 \cdot 5^1 + 3 \cdot 5^0 = 5 + 3 = 8$ and 8 is even in base 10. But $n = 23_5$ is odd because $23_5 = 2 \cdot 5^1 + 3 \cdot 5^0 = 10 + 3 = 13$ and 13 is odd in base 10.

Thus, the obvious question, is there a "simple" way to tell if n base b is even or odd as we can for base 10 integers? The following theorem tells us there is.

Theorem

If b is even, then
- n_b is even if the last digit of n_b is even, and
- n_b is odd if the last digit of n_b is odd.

If b is odd, then
- n_b is even if the sum of all the digits in n_b is even, and
- n_b is odd if the sum of all the digits in n_b is odd*.

*An equivalently condition is that when b is odd, n_b is odd if and only if the number of odd digits in n_b is odd because a sum is odd if and only if there are an odd number of odd terms in that sum.

Also, remember ($*$ page 8) that, by definition, the base number b is to be understood as a base 10 integer in $\{2,3,4,5\ldots\}$. So, in this theorem when we ask "is b even" we are asking if the base 10 integer b is even.

Proof

The case of b even is tantamount to the above base 10 proof. Replacing 10 with the even number b we see that $(a_k b^k + a_{k-1} b^{k-1} + \cdots + a_1 b^1)$ is even because $b^j = 2\left(\frac{b}{2} \cdot b^{j-1}\right)$ for each $j = 1,2,\ldots,k$. So, as in the base 10 proof, the parity (evenness or oddness) of n all comes down to whether a_0 is even or odd.

The case of b odd is also easy to establish. Clearly, $n = a_k b^k + a_{k-1} b^{k-1} + \cdots + a_1 b^1 + a_0 b^0$ is odd if and only if an odd number of these $k + 1$ terms are odd.

But in the case of b odd, b^j is odd for each $j = 0,1,2, \dots, k$. Thus, the term

$a_j b^j$ is odd if and only if a_j is odd.

Hence,

$$n = a_k b^k + a_{k-1} b^{k-1} + \cdots + a_1 b^1 + a_0 b^0$$

is odd if and only there are an odd number of odd coefficients $a_0, a_1, \dots, a_k$. And as remarked above, there are an odd number of odd coefficients if and only if the sum of these coefficients is an odd number.

■

Example (Source: Mu Alpha Theta Log 1 Contest 2006-2007, Alpha Level, Number Theory Test, Problem #15)

Express $N = \dfrac{(23074356)_8}{2}$ in hexadecimal.

<u>Solution</u>

The first thing to notice is that example involves a ***division*** as well as a change of base. But we do not take up the general question of division in a non-base 10 system until Section 1.10.

Furthermore, at this point we don't know whether N is an integer or not. How would we convert a non-integer from base 8 to hexadecimal (base 16)? We will not take up that general question until Section 12.1.

So why are we taking up this example *now*? If the problem had been to convert $\dfrac{(23074356)_8}{\mathbf{3}}$ to hexadecimal we would need to wait until we have covered Section 12.1.

But the problem $\dfrac{(23074356)_8}{2}$ is a special case situation we can actually solve *now* with just a pointer to a result in Section 1.8 (Even and Odd Numbers in Base ***b***). Namely, an integer in an ***even* number base** (such as base 8 in this example) is **divisible by 2** if the last digit of that integer is an even number.

In this example, this means the integer $(23074356)_8$ is divisible by 2 because the last digit is 6 which is an even number.

The fastest approach to solving this problem

15.	$263c77_{16}$	This problem is best solved by realizing that a digit in base 8 is completely represented by 3 digits in base 2. For example: 6 base eight expands to 110. 5 base eight expands to 101. And so on. Once we have expanded our number into base two, we drop a digit for the division by 2. Finally to convert our number into hexadecimal, we use our original trick and take the digits in groups of 4: 0111 -> 7 0111 -> 7 1100 -> c

■

(see file "Divisibility and Bases Lesson")

Base systems: divisibility

SSM 3590. **by Herta T. Freitag**

In the base-ten system of numeration, divisibility of a number $N = a_n a_{n-1} \ldots a_2 a_1 a_0$ by 2 is tested by seeing if a_0 is divisible by 2; for divisibility by 4, one checks $a_1 a_0$; and if $a_2 a_1 a_0$ is divisible by 8, so is N.

Generalize these criteria for base system b and divisibility by a number m' where m and t are positive integers and m is greater than 1.

SSM, School Science and Mathematics

3590 75(1975)477 76(1976)84s, 442a

3590. *Proposed by Herta T. Freitag, Roanoke, Va.*

In the base-ten system of numeration, divisibility of a number $N = a_n a_{n-1} \ldots a_2 a_1 a_0$ by 2 is tested by seeing if a_0 is divisible by 2; for divisibility by 4, one checks $a_1 a_0$; and if $a_2 a_1 a_0$ is divisible by 8, so is N.

Generalize these criteria for base-system b and divisibility by a number m^t where m and t are positive integers and m is greater than 1.

3590. *Proposed by Herta T. Freitag, Roanoke, Va.*

In the base-ten system of numeration, divisibility of a number $N = a_n a_{n-1} \ldots a_2 a_1 a_0$ by 2 is tested by seeing if a_0 is divisible by 2; for divisibility by 4, one checks $a_1 a_0$; and if $a_2 a_1 a_0$ is divisible by 8, so is N.

Generalize these criteria for base-system b and divisibility by a number m^t where m and t are positive integers and m is greater than 1.

Solution by Bob Prielipp, Oshkosh, Wis.

Let N be an arbitrary number in a base-b system of numeration. Then $N = a_k a_{k-1} \ldots a_2 a_1 a_0 = a_k b^k + a_{k-1} b^{k-1} + \ldots + a_2 b^2 + a_1 b + a_0$. Let m be a positive integer greater than 1 which divides b. Then $m|N$ if and only if $m|a_0$, $m^2|N$ if and only if $m^2|(a_1 b + a_0)$ if and only if $m^2|(a_1 a_0)_{\text{base } b}$, $m^3|N$ if and only if $m^3|(a_2 b^2 + a_1 b + a_0)$ if and only if $m^3|(a_2 a_1 a_0)_{\text{base } b}$, etc. [Notice that since $m|b$, $m^j|b^j$ for each positive integer j.]

Solutions were also submitted by Anton Glaser, Shouthampton, Pa.; Fred A. Miller, Elkins, W. Va.; Gregory Wulczyn, Lewisburg, Pa.; and the proposer.

14. One digit of a number written in base 5 has been erased. The remaining digits are 4 , 2 , 0 , 1 , 3 , 3 , 2. If we know that the original number is even (divisible by 2), what are the possible values for the missing digit? Remember, the number is written in base 5.

http://mathcircle.berkeley.edu/sites/default/files/archivedocs/2011_2012/lectures/1112lecturespdf/BMC_Beg_Jan10_2012_Alien_Arithmetic.pdf

https://www.johndcook.com/blog/2011/08/17/odd-numbers-in-odd-bases/

In an odd base, a number is odd iff it has an odd number of odd digits.

(In case you haven't seen "iff" before, it's an abbreviation for "if and only if.")

So, for example, in base 7, the number 642341 is even because it contains two odd digits. And the number 744017 in base 9 is odd because it has three odd digits.

Why does this rule work? Suppose, for example, you have a 4-digit number number *pqrs* in base *b* where *b* is odd. Then *pqrs* represents

$p\, b^3 + q\, b^2 + r\, b + s$

All the powers of *b* are odd, so a number like *p* times a power of *b* is odd iff *p* is odd. So every odd digit in the number contributes an odd number to the sum that expands what the number means. Even digits contribute even terms. A sum is odd iff it has an odd number of odd terms, so a number in an odd base is odd iff it has an odd number of odd digits.

Dmitri Fomin, Sergey Genkin, and Ilia Itenberg, Mathematical circles (Russian experience), 1996

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Problem 3. State and prove a condition (involving the representation of a number) which allows us to determine whether the number is odd or even

a) in the base 3 system;

b) in the base n system.

Dmitri Fomin, Sergey Genkin, and Ilia Itenberg, Mathematical circles (Russian experience), 1996

In a base n system the representation of a number ends with zero if and only if this number is divisible by n.

Dmitri Fomin, Sergey Genkin, and Ilia Itenberg, Mathematical circles (Russian experience), 1996

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Problem 7. State and prove the test for divisibility by

a) a divisor of the number $n-1$ in the base n system (similar to the divisibility test for 3 in the decimal system);

b) the number $n+1$ in base n system (similar to the divisibility test for 11);

c) a divisor of the number $n+1$ in the base n system (there is no analog in base 10).

Dmitri Fomin, Sergey Genkin, and Ilia Itenberg, Mathematical circles (Russian experience), 1996

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Every natural number can be represented as the difference of two numbers whose base 3 representations contain only 0's and 1's.

6. a) In the base n number system the representation of a number ends with k zeros if and only if this number is divisible by n^k.

b) Let m be some divisor of n. The last digit of the base n representation of a number is divisible by m if and only if the number itself is divisible by m.

7. a) Let m be a divisor of $n-1$. Then the sum of the digits in the base n representation of a number is divisible by m if and only if the number itself is divisible by m.

b) The "alternating" sum (with alternating signs) of the digits in the base n representation of a number is divisible by $n+1$ if and only if the number itself is divisible by $n+1$.

c) Let m be some divisor of $n+1$. The alternating sum (with alternating signs) of the digits in the base n representation of a number is divisible by m if and only if the number itself is divisible by m.

Hall and Knight

39. In a scale whose radix is odd, shew that the sum of the digits of any number will be odd if the number be odd, and even if the number be even.

https://www.johndcook.com/blog/2011/05/12/casting-out-zs/

Casting out z's

casting out $(b-1)$'s in base b

Why can you cast out (b-1)'s in base b? First, a number written is base b is a polynomial in b. If the representation of a number x is $a_n a_{n-1} \dots a_1 a_0$ then

$$x = a_n b^n + a_{n-1} b^{n-1} + \dots + a_1 b + a_0.$$

Since

$$b^m - 1 = (b - 1)(b^{m-1} + b^{m-2} + \dots + 1)$$

it follows that b^m leaves a remainder of 1 when divided by $b - 1$. So $a_m b^m$ leaves the same remainder as a_m when divided by $b - 1$. If follows that

$$a_n b^n + a_{n-1} b^{n-1} + \dots + a_1 b + a_0$$

has the same remainder when divided by $b - 1$ as

$$a_n + a_{n-1} + \dots + a_1 + a_0$$

does when it is divided by $b - 1$.

Hall and Knight

82. *In any scale of notation of which the radix is* r, *the sum of the digits of any whole number divided by* r − 1 *will leave the same remainder as the whole number divided by* r − 1.

Bonus: Divisibility Rules

In the Solution, we used two divisibility rules:

- The sum of the digits of a number leaves the same remainder upon division by 9 as the number itself.
- The alternating sum of the digits of a number leaves the same remainder upon division by 11 as the number itself.

To prove the facts, suppose that a number n has the base-10 representation

$$n = d_k \dots d_2 d_1 d_0;$$

that is,

$$n = d_k 10^k + \cdots + d_2 10^2 + d_1 10 + d_0.$$

Reducing each term with respect to the moduli 9 and 11, we obtain

$$n \equiv d_k + \cdots + d_2 + d_1 + d_0 \pmod 9$$

and

$$n \equiv (-1)^k d_k + \cdots + d_2 - d_1 + d_0 \pmod{11}.$$

Hence, the remainder upon dividing n by 9 (respectively, 11) is the same as the remainder upon dividing the sum of the digits (respectively, the alternating sum of digits) of n by 9 (respectively, 11). This proves the divisibility rules for 9 and 11.

Note. For any base b, these same divisibility rules work for $b - 1$ (if $b > 2$) and $b + 1$.

Note. For any base b, these same divisibility rules work for $b - 1$ (if $b > 2$) and $b + 1$.

6.1 Divisibility rule for $n - 1$ in base n

The divisibility rule for 9 in base 10 can be extended to all bases. Here is the proof:

- Consider a number which can be written in base n as $x = A_j A_{j-1} A_{j-2} ... A_2 A_1 A_{0n}$.
- This number's value in base 10 is $A_j * n^j + A_{j-1} * n^{j-1} + A_{j-2} * n^{j-2} + ... + A_2 n^2 + A_1 n + A_0$.
- For any natural number k, $n - 1 | n^k - 1$.
- $A_j * n^j + A_{j-1} * n^{j-1} + A_{j-2} * n^{j-2} + ... + A_2 n^2 + A_1 n + A_0 = (A_j * (n^j - 1) + A_{j-1} * (n^{j-1} - 1) + A_{j-2} * (n^{j-2} - 1) + ... + A_2(n^2 - 1) + A_1(n - 1)) + (A_j + A_{j-1} + A_{j-2} + ... + A_2 + A_1 + A_0)$
- $A_j * n^j + A_{j-1} * n^{j-1} + A_{j-2} * n^{j-2} + ... + A_2 n^2 + A_1 n + A_0 = (n-1)Y + (A_j + A_{j-1} + A_{j-2} + ... + A_2 + A_1 + A_0)$
- Therefore, x is divisible by $n - 1$ if the sum of its digits in base n is divisible by $n - 1$.

This rule will work for all divisors of $n - 1$, because the term divisible by $n - 1$ will also have no effect for the divisors.

Mu Alpha Theta National Convention 2001, Number Theory Test, Alpha Division, Problem # 27

If $n = 6789_b$ and if n is a multiple of $b - 1$, what are the possible values for b less than 20?

Solution

$6 + 7 + 8 + 9 = 30 = 2 \cdot 3 \cdot 5$ is divisible by $b - 1$. Therefore $b - 1 \in \{1,2,3,5,6,10,15,30\}$ and $b \in \{2,4,6,7,11,16,31\}$. Because 6789_b exists we know $b \geq 10$. And we are given that $b < 20$. Therefore, $b \in \{11,16\}$.

■

Mu Alpha Theta National Convention, 2001, Number Theory Test, Mu Division, Problem # 28

What is the second smallest positive integer that is a multiple of 4 and has no digit greater than 1 when expressed in base 5?

Solution

28. An integer expressed in a base, B, is a multiple of (B – 1) if and only if the sum of the integer's digits is a multiple of (B – 1). This can be easily proven by induction (or other means) and is left as an exercise for the students. Now, we need only look at numbers with digit sums of 4 (8, 12, etc. would be far too large) in base 5. The second smallest of these is $10{,}111_5$ = 656.

■

6.2 Divisibility rule for $n + 1$ in base n

The divisibility rule for 11 in base 10 can be extended to all bases. Here is the proof:

- Consider a number which can be written in base n as $x = A_jA_{j-1}A_{j-2}...A_2A_1A_{0n}$. I'll assume j is even, but A_j can be zero if A_{j-1} is nonzero.
- This number's value in base 10 is $A_j * n^j + A_{j-1} * n^{j-1} + A_{j-2} * n^{j-2} + ... + A_2n^2 + A_1n + A_0$.
- For any natural number k, $n + 1|n^k + 1$ if k is odd and $n + 1|n^k - 1$ if k is even.
- $A_j * n^j + A_{j-1} * n^{j-1} + A_{j-2} * n^{j-2} + ... + A_2n^2 + A_1n + A_0 = (A_j * (n^j - 1) + A_{j-1} * (n^{j-1} + 1) + A_{j-2} * (n^{j-2} - 1) + ... + A_2(n^2 - 1) + A_1(n + 1)) + (A_j - A_{j-1} + A_{j-2} + ... + A_2 - A_1 + A_0)$
- $A_j*n^j+A_{j-1}*n^{j-1}+A_{j-2}*n^{j-2}+...+A_2n^2+A_1n+A_0 = (n+1)Y+(A_j-A_{j-1}+A_{j-2}+...+A_2-A_1+A_0)$
- Therefore, x is divisible by $n + 1$ if the sum of its digits in base n is divisible by $n + 1$.

This rule will work for all divisors of $n + 1$, because the term divisible by $n + 1$ will also have no effect for the divisors.

6.4 Grouping Digits in Base 10 to find more divisibility rules

These rules will also work on base 100 and base 1000. This means there is a divisibility rule for 101 in base 100, and a divisibility rule for 999 (and therefore 37, 27) and 1001 (and therefore 13) in base 1000.

A number can be easily converted from base 10 to base 100 or base 1000 by grouping digits into twos or threes, respectively, starting from the right.

Examples:
$190898 \rightarrow 19 - 08 - 98_{100}$
$190898 \rightarrow 190 - 898_{1000}$

This opens up some new divisibility rules:

- 101: The difference between alternating sums of pairs of digits (starting from the right) is a multiple of 101.
- 13: The difference between alternating sums of triplets of digits (starting from the right) is a multiple of 13.
- 27: The sum of triplets of digits (starting from the right) is a multiple of 27.
- 37: The sum of triplets of digits (starting from the right) is a multiple of 37.

These rules get to the point where dividing might be easier. However, they can be useful in certain circumstances.

(1A993)
Find binary digits x and y, so that the *binary* number $1x0y10$ will be divisible by 3.
Solution

$$\begin{aligned} 1x0y10_2 &= 1 \cdot 2^5 + x \cdot 2^4 + 0 \cdot 2^3 + y \cdot 2^2 + 1 \cdot 2^1 + 0 \cdot 2^0 \\ &= 32 + 16x + 8y + 2 \\ &= 34 + 15x + 8y \end{aligned}$$

where $x \in \{0,1\}$ and $y \in \{0,1\}$. Consider all four cases.

x	y	$34 + 15x + 8y$	Divisible by 3 ?
0	0	34	No
0	1	38	No
1	0	50	No
1	1	54	Yes

So $x = 1$ and $y = 1$.

How to Tell if a Binary Number is divisible by 3

https://web.archive.org/web/20171029092543/http://www.answermysearches.com/how-to-tell-if-a-binary-number-is-divisible-by-three/70/

https://stackoverflow.com/questions/39385971/how-to-know-if-a-binary-number-divides-by-3
Count the number of non-zero odd positions bits and non-zero even position bits **from the right**. If their difference is divisible by 3, then the number is divisible by 3.
For example:
$15_{10} = 1111_2$ has 2 odd and 2 even non-zero bits. The difference is 0. Thus 15 is divisible by 3.
$185_{10} = 10111001_2$ which has 2 odd non-zero bits and 3 even non-zero bits. The difference is 1. Thus 185 is not divisible by 3.

For $1x0y10_2$ the sum of bits in odd positions $= 0 + y + x$ and the sum of bits in the even positions $= 1 + 0 + 1 = 2$.
So by the above rule we need $x + y - 2 \in \{0,3,6, \dots\}$ for $1x0y10_2$ to be divisible by 3.
Obviously only 0 is possible and this can only happen if $x = y = 1$.

This is the same trick as for testing if a decimal number is divisible by 11. The keyword here is the *alternating digit sum*. If and only if the alternating digit sum in decimal radix is divisible by 11, so is the original number. It can be used when the number you want to test divisibility for is *one more* than the radix of the number system. TO test for divisibility of numbers one below the radix (e.g. 9 for the decimal system) use the ordinary digit sum. So one can also easily test divisibility by 17 in hexadecimal

https://math.stackexchange.com/questions/979274/determine-whether-or-not-a-binary-number-is-divisible-by-3
https://math.stackexchange.com/questions/2228122/general-rule-to-determine-if-a-binary-number-is-divisible-by-a-generic-number?rq=1

If the alternating sum of the binary expansion of n is divisible by 3 then n is. For example, let's check $228_{10} = 11100100_2$.

$$\begin{array}{ccccccccc}
+ & - & + & - & + & - & + & - & \\
1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & \\
\hline
 & & & & & & & & \\
1 & -1 & +1 & & & -1 & & & = 0
\end{array}$$

So 11100100_2 is divisible by 3.

■

Mu Alpha Theta National Convention 2007, Alpha Division, Number Theory Test, Problem #23

23. Which of the following base-7 numbers is divisible by 6?

(A) 33634_7 (B) 5553555_7 (C) 1111111_7 (D) 2534352_7 (E) NOTA

Solution

23. **(D)**. Any base-7 number can be written as $a_k\, 7^k + a_{k-1}\, 7^{k-1} + \ldots + a_0$. Taking this in modulo 6, we get $a_k + a_{k-1} + \ldots + a_0$. Thus, a base-7 number is divisible by 6 if the sum of its digits is divisible by 6. Answer Choice D qualifies for this since $2 + 5 + 3 + 4 + 3 + 5 + 2 = 24$, which is a multiple of 6.

■

NATIONAL CONVENTION SAN DIEGO 2013 Number Theory Open, Round 1

22. The four digit number $X45Y_{12}$ is divisible by 143_{10}. Evaluate $X + Y$ in base 10.

(A) 7 (B) 13 (C) 14 (D) 20 (E) NOTA

22. B

22) By the base 12 divisibility rule for 13 (same as 11 base 10), $X + 1 \equiv Y \pmod{13}$. By the base 12 divisibility rule for 11 (same as 9 base 10), $X + Y \equiv 2 \pmod{11}$. (6, 7) is the only solution that satisfies.

3.5 Division Algorithm

Pg. 24, Theorem 1.9 (Division Algorithm)

For any $b > 0$ and a, there exist unique integers q and r with $0 \le r < b$ such that $a = bq + r$.

Theorem 1.4 (Division Algorithm). *Given $a, b \in \mathbf{Z}$ with $b > 0$ there exist unique $q, r \in \mathbf{Z}$ such that $a = qb + r$ and $0 \le r < b$. Moreover, q and r are given by the formulas $q = [a/b]$ and $r = a - [a/b]b$.*

***Mathematics Teacher*, Calendar Problem 28, January 2004 (originally AMC 1976 Problem #15)**

If r is the remainder when each of $1059, 1417$, amd 2312 is divided by d (where d is greater than 1), compute $d - r$.

Solution

The question states that there exists an integer divisor $d > 1$, an integer remainder $r < d$, and integer quotients q_1, q_2 and q_3 such that

$$1059 = d \cdot q_1 + r$$
$$1417 = d \cdot q_2 + r$$
$$2312 = d \cdot q_3 + r$$

and asks you to find d and r from this information.

We know that the difference of any two of these numbers must be divisible by d. In particular,

$$1417 - 1059 = 358 = (dq_2 + r) - (dq_1 + r) = d(q_2 - q_1)$$

is divisible by d. But $358 = 2 \cdot 179$ and 179 is prime. Therefore $d \in \{1,2,179,358\}$.

We are given that $d > 1$ so we can remove that possibility. There are two more differences we can compute, namely

$$2312 - 1059 = 1253 = (dq_3 + r) - (dq_1 + r) = d(q_3 - q_1)$$

and

$$2312 - 1417 = 895 = (dq_3 + r) - (dq_2 + r) = d(q_3 - q_2).$$

We know that d must divide both of these differences. We can use this information to rule out the candidates 2 and 358 because both are even and the differences 1253 and 895 are both odd (and hence not divisible by any even number).

Therefore, by the process of elimination, $d = 179$. As a check we note that

$$1059 = 179 \cdot 5 + 164$$
$$1417 = 179 \cdot 7 + 164$$
$$2312 = 179 \cdot 12 + 164.$$

We can see that the remainder $r = 164$ is common to all three cases as required. Therefore,

$$d - r = 179 - 164 = 15.$$

■

AMC 1967 Problem #22

For natural numbers, when P is divided by D, the quotient is Q and the remainder is R. When Q is divided by D', the quotient is Q' and the remainder is R'. Then, when P is divided by DD', the remainder is:

(A) $R + R'D$	(B) $R' + RD$	(C) RR'	(D) R	(E) R'

Solution

■

AMC

64

3. When a positive integer x is divided by a positive integer y, the quotient is u and the remainder is v, u and v integers. What is the remainder when $x + 2uy$ is divided by y?

(A) 0 (B) $2u$ (C) $3u$ (D) v (E) $2v$

Solution

$$\frac{x}{y} = u + \frac{v}{y}$$

$$\frac{x + 2uy}{y} = 2u + \frac{x}{y} = 2u + \left(u + \frac{v}{y}\right) = 3u + \frac{v}{y}$$

Remainder is v.

■

3.6 Reducible Fractions and Reducible Rational Functions

Alpha Mu Theta Florida State Convention 2007, Number Theory Test, Problem #25

25. Find the largest integer n such that $n + 10$ divides $n^3 + 2007$ evenly.

A. 997 B. 1997 C. 2006 D. No maximum exists E. NOTA

Solution

■

Open Number Theory MAΘ National Convention 2016

14. The fraction $\frac{103_b}{136_b}$ can be reduced to $\frac{14_b}{18_b}$ (which is simplest since it is irreducible in base 10). What is the simplest form of the fraction $\frac{149_b}{338_b}$, written as a quotient of integers, each in base b?

A. $\frac{16}{37}$ B. $\frac{19}{39}$ C. $\frac{21}{52}$ D. $\frac{149}{338}$ E. NOTA

Solution

14. A Based on the information given,

$$\frac{b^2+3}{b^2+3b+6}=\frac{b+4}{b+8}$$

Cross multiply to get $b^3+8b^2+3b+24=b^3+7b^2+18b+24$, or $b^2-15b=0$. Therefore, $149_{15}=225+60+9=294$ and $338_{15}=675+45+8=728$. $\frac{294}{728}=\frac{21}{52}=\left(\frac{16}{37}\right)_{15}$.

■

Mu Alpha Theta Log1 Contest, 2011-2012, Round 2, Mu Division, Number Theory Test, Problem #15

What is the sum of the integer values of n for which $\frac{n^2+3n+3}{n-1}$ is an integer?

Solution

First note that

$$(n-1)(n+4)=n^2+3n-4.$$

Therefore,

$$\frac{n^2+3n+3}{n-1}=\frac{(n-1)(n+4)+7}{n-1}=(n+4)+\frac{7}{n-1}$$

which is an integer if and only if $(n-1)|7$. But $(n-1)|7$ if and only if $(n-1)\in\{\pm1,\pm7\}$.

$$\begin{aligned} n-1&=\ \ 1 \iff n=2\\ n-1&=-1 \iff n=0\\ n-1&=\ \ 7 \iff n=8\\ n-1&=-7 \iff n=-6. \end{aligned}$$

Therefore, the sum of the possible values of n will be $2+0+8-6=4$.

■

Open Number Theory 2018 MAΘ National Convention

20. How many positive integers N are there such that N^3+100 is divisible by $N+4$?

(A) 9 (B) 7 (C) 5 (D) 3 (E) NOTA

Solution

20. Solution: Note that $\frac{N^3+100}{N+4} = \frac{N^3+64+36}{N+4} = N^2 + 4N + 16 + \frac{36}{N+4}$. Therefore, $N + 4$ divides $N^3 + 100$ if and only if $N + 4$ divides 36. Factors of 36 are 1,2,3,4,6,9,12,18,36. Since N is a positive integer, there are five values of $N = 2,5,8,14,32$.

Answer: (C)

■

Mu Alpha Theta 2019 National Convention, Number Theory Test, Open Division, Problem #22

22. What is the sum of all positive integers n for which $(n-4)|(n^3 + 7n^2 - 13n + 19)$?

(A) 8 (B) 187 (C) 45 (D) 73 (E) NOTA

Solution

Since $n^3 + 7n^2 - 13n + 19 = (n-4)n^2 + (n-4)11n + (n-4)31 + 143$ (synthetic division), then $n - 4|n^3 + 7n^2 - 13n + 19 \to n - 4|143$. Then $n - 4 = -1, 1, 11, 13, 143 \to n = 3, 5, 15, 17, 147$. The sum of all possible values of n is $3 + 5 + 15 + 17 + 147 = 187$ $\boxed{B}$

■

2014 Lehigh University High School Math Contest, Problem #11

What is the number of integers n for which $\frac{7n+15}{n-3}$ is an integer?

Solution

A key step in problems of this type is to separate out the largest integer "hiding" in the expression. One way to do this is to carry out the polynomial division. In this problem it is a little faster to do a change of variable along with some algebra. Let $m = n - 3$. Then $n = m + 3$ and on simplification we see that

$$\frac{7n+15}{n-3} = \frac{7(m+3)+15}{m} = \frac{7m+36}{m} = 7 + \frac{36}{m} = 7 + \frac{36}{n-3}.$$

Now we can focus on finding the value of n where the simpler fraction $\frac{36}{n-3}$ is an integer. This reduced fraction will be an integer for all n such that $(n-3)|36$.

We know that $36 = 2^2 3^2$ has 9 positive integer divisors and hence 18 integer divisors. So there are 18 distinct values of $n - 3$ such that $(n-3)|36$ and hence there are $\mathbf{18}$ distinct values of $n = (n-3) + 3$ such that $(n-3)|36$.

■

AMC 1985 Problem #26

Find the least positive integer n for which $\frac{n-13}{5n+6}$ is a non-zero reducible fraction.

(A) 45	(B) 68	(C) 155	(D) 226	(E) none of these

Solution

https://math.stackexchange.com/questions/1041864/when-is-fracn-135n6-reducible

■

let n be the least positive integer for which n-10/9n+11 is a non zero reducible fraction. Find the value of n/3

https://www.gauthmath.com/solution/i248147501

Find the least positive integer n for which $\dfrac{3n+1}{n(2n-1)}$ is a non-zero reducible fraction. ( )

A. 4
B. 6
C. 5
D. 2
E. 3

Singapore Mathematical Olympiad (SMO) 2009

29 Find the least positive integer n for which $\dfrac{n-10}{9n+11}$ is a non-zero reducible fraction.

Solution

29 Answer: (111)

Consider $\dfrac{9n+11}{n-10} = 9 + \dfrac{101}{n-10}$. If $\dfrac{n-10}{9n+11}$ is a non-zero reducible fraction, then $\dfrac{101}{n-10}$ is also a non-zero reducible fraction $\Rightarrow$ Least positive integer $n = 111$.

■

35 m and n are two positive integers of reverse order (for example 123 and 321) such that $mn = 1446921630$. Find the value of $m + n$.

Solution

35 Answer: (79497)

Clearly, m and n are both 5-digit numbers.

Next, it would be helpful that we know $mn = 2 \times 3^5 \times 5 \times 7 \times 11^2 \times 19 \times 37$.

Now since the last digit of mn is 0, we may assume $5 \mid m$ and $2 \mid n$. But the first digit of mn is 1 $\Rightarrow$ Last digit of m is 5 (not 0) and last digit of n is 2 (not 4, 6 or 8).

Also, $3^5 \mid mn$, so 9 divides at least one of m and n. On the other hand, $9 \mid m \Rightarrow 9 \mid n$. Similarly $11 \mid m \Rightarrow 11 \mid n$.

Set $n = 198k$. Then the last digit of k is 4 or 9.

Since the remaining factors 3, 7, 19, 37 are odd, the last digit of k must be 9.
We have only the following combinations: $k = 7 \times 37$ or $3 \times 7 \times 19$ or $3 \times 19 \times 37$.

Recall that the first digit of n is 5, so $50000 \le 198k < 60000 \Rightarrow k = 7 \times 37$.

Hence $n = 198k = 51282$ and $m = 28215 \Rightarrow m + n = 79497$.

■

AMC 1990 Problem #19

For how many integers N between 1 and 1990 is the improper fraction

$$\frac{N^2 + 7}{N + 4}$$

not in lowest terms?

(A) 0	(B) 86	(C) 90	(D) 104	(E) 105

Solution

■

Mu Alpha Theta Florida State Convention 2007, Number Theory Test, Problem #25

25. Find the largest integer n such that $n+10$ divides n^3+2007 evenly.

A. 997 B. 1997 C. 2006 D. No maximum exists E. NOTA

Solution

■

3.7 Divisibility Properties

Pg. 39, Theorem 2.1 (Divisibility Properties)

(i) If $a \neq 0$, then $a|0$ and $a|a$
(ii) $1|b$ for any b
(iii) If $a|b$, then $a|bc$ for any c
(iv) If $a|b$ and $b|c$, then $a|c$
(v) If $a|b$ and $a|c$, then $a|(bx+cy)$ for any x and y.

- Let $a, b \in \mathbb{Z}$. If $a|b$ then $a^n|b^n$ for every positive integer n.

- Let $n \in \mathbb{Z}$ with $n > 0$. Then $n|((n+1)^n - 1)$ for every positive integer n.

- Let a, m and n be positive integers with $a > 1$. Then, $a^m - 1|a^n - 1$ if and only if $m|n$.

3.8 List of all Positive Divisors

Theorem 2.23 makes it extremely easy to write down **all the positive divisors** of a positive integer once its canonical representation has been obtained.
For example, consider $72 = 2^3 \cdot 3^2$. From Theorem 2.23 we know that every positive divisor b of 72 must have a canonical representation of the form $b = 2^{b_1} \cdot 3^{b_2}$ with $0 \leq b_1 \leq 3$ and $0 \leq b_2 \leq 2$. The list of the twelve positive divisors of 72 follows:

$$
\begin{array}{lll}
2^0 \cdot 3^0 = 1 & 2^0 \cdot 3^1 = 3 & 2^0 \cdot 3^2 = 9 \\
2^1 \cdot 3^0 = 2 & 2^1 \cdot 3^1 = 6 & 2^1 \cdot 3^2 = 18 \\
2^2 \cdot 3^0 = 4 & 2^2 \cdot 3^1 = 12 & 2^2 \cdot 3^2 = 36 \\
2^3 \cdot 3^0 = 8 & 2^3 \cdot 3^1 = 24 & 2^3 \cdot 3^2 = 72.
\end{array}
$$

We could have determined that there would be twelve positive divisors without writing out the entire list by noting that for each of the four numbers $2^0, 2^1, 2^2, 2^3$ there will be three numbers $3^0, 3^1, 3^2$ to pair them with. This produces $4 \times 3 = 12$ possibilities.

3.9 Number of Positive Divisors

This idea generalizes to yield a convenient formula for the **number of positive divisors of an integer $\boldsymbol{a}$** which we represent by $\boldsymbol{\tau(a)}$.

Pg. 57, Theorem 2.24 Part(a) (Number of Positive Divisors of a)

If $a = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r} = \prod_{i=1}^{r} p_i^{a_i}$ with $a_i > 0$ for each i is the canonical representation of a, then

$$\tau(a) = (a_1 + 1)(a_2 + 1) \cdots (a_r + 1) = \prod_{i=1}^{r} (a_i + 1).$$

2016 Lehigh University High School Math Contest, Problem #10

What is the minimal number of positive divisors of x if $x > 1$ and $x, x^{5/6}$, and $x^{7/8}$ are all integers?

Solution

Central to understanding this problem is the result that for integral $n > 1$ and $k > 1$, $\boldsymbol{n^{1/k}}$ **is an integer if and only if $\boldsymbol{n = N^k}$** for some positive integer positive. (The proof is an extension of the proof that $\sqrt{2}$ is irrational.)

It follows from this result that for integral $x > 1$, $x^{5/6}$ and $x^{7/8}$ are both integers if and only if x is an integer of the form N^k for some positive integers N and k such that both $k(5/6)$ and $k(7/8)$ are integers.

But this requires that k is a multiple of $\text{lcm}(6,8) = 24$.

That is, $x = N^{24j}$ for some positive integers N and j. So the original question has come down to finding positive integers N and j such that $x = N^{24j}$ has the minimum number of positive divisors.

We know that the number of positive divisors of $x = N^{24j}$ increases with the number of prime divisors of N. That is, to minimize the number of positive divisors we must take N to be some prime number p (the number of positive divisors for all prime p).

But $x = p^{24j}$ has $(24j + 1)$ positive divisors for any prime p which is clearly minimized when we take $j = 1$.
That is, $x = p^{24}$ for some prime p will meet the requirement that $x^{5/6}$ and $x^{7/8}$ are both integers and have the minimum number of positive divisors.
And in this case x will have $(24 + 1) = 25$ positive divisors.

■

Pg. 58, Exercise 9
Show that the number of positive divisors of a positive integer a is odd if and only if a is the square of an integer (*i.e.* a perfect square).

Furthermore,

$\tau(a) = 2$ if and only if a is a prime number

$\tau(a) = 3$ if and only if $a = p^2$ for some prime number p.

$\tau(a) = 4$ if and only if $a = p^3$ for some prime number p or $a = p_i^1 \cdot p_j^1 = p_i \cdot p_j$ for distinct (different) primes p_i and p_j

(1T881)
1. There are 60 distinct positive integers, including 1, 9, and N itself that divide $N = 2^a 3^b 7^2 \cdot 11$, a and b being positive integers. What is N ?
Solution

The number of factors of N is
$(a+1)(b+1)(3)(2) = 60$
$(a+1)(b+1) = 10$, and we know $b \geq 2$
$\therefore\ b+1 = 5$; $b = 4$, $a = 1$
$N = 2 \cdot 3^4 \cdot 7^2 \cdot 11 = 87{,}318$

■

Mathematics Teacher, Calendar Problems, Number 14, September 1988
How many positive integers less than 1000 have an odd number of positive integral divisors?
Solution
31. Only the number 1 and all perfect squares have an odd number of factors; 31^2 is the largest perfect square less than 1000.

■

(1T156) Let $N = p^{2017} - 4p^{2016} + 4p^{2015}$, where N is a positive number. If p is a prime number, determine the least possible number of factors of N.
Solution
Rewrite N as $N = p^{2017} - 4p^{2016} + 4p^{2015} = p^{2015}\left(p^2 - 4p + 4\right) = p^{2015}\left(p-2\right)^2$. *The value of p that will give us the fewest number of factors is 3. This means that* $N = 3^{2015}\left(3-2\right)^2 = 3^{2015}$, *which has* $\boxed{2016}$ *factors.*

■

2008 Mu Alpha Theta National Convention **Open Number Theory**

7. If y has 25 factors, y^2 can have z factors. Find the sum of all possible values of z.

A. 52 B. 130 C. 154 D. Infinity E. NOTA

Solution

7. **B**. Since y has 25 factors, either $y = a^4b^4$ for primes a, b or $y = c^{24}$ for some prime c. This means $y^2 = a^8b^8$ or $y^2 = c^{48}$ so y^2 has 81 or 49 factors, so 130.

■

***Mathematics Teacher*, Calendar Problems, Number 28, November 1988**

For positive integer k, the number $1984 \cdot k$ has exactly 21 divisors. Compute all possible values of k.
Solution
21 is an odd number, hence $1984 \cdot k$ must be a perfect square (perfect squares are the only numbers with an odd number of divisors). The prime factorization of 1984 is
$$1984 = 2^6 \cdot 31.$$
For $1984 \cdot k = 2^6 \cdot 31 \cdot k$ to be a perfect square all primes must occur to an even power. Therefore k must be divisible by $31 \cdot 31^t$ for some even integer t. Furthermore all other primes in the prime factorization of k must occur an even number of times.
That is,
$$k = 31 \cdot 2^{2a}3^{2b}5^{2c} \cdots$$
for some integers $a \geq 0, b \geq 0, c \geq 0, \ldots$ and the number of factors of
$$1984 \cdot k = 2^5 \cdot 31 \cdot 31 \cdot 2^{2a}3^{2b}5^{2c} \cdots = 2^5 \cdot 31^2 \cdot 2^{2a}3^{2b}5^{2c} \cdots$$
equals
$$(5+1)(2+1)(2a+1)(2b+1)\cdots = 21 \cdot (2a+1)(2b+1)(2c+1)\cdots.$$

The only way this product equals 21 is for $0 = a = b = c = \cdots$. That is,
$$k = 31 \cdot 2^{2(0)}3^{2(0)}5^{2(0)} \cdots = 31.$$

■

***Mathematics Teacher*, Calendar Problem 27, February 2007**

Find the number of odd divisors of 7!.

Solution

27. 12. The prime factorization of 7! is $2^4 \cdot 3^2 \cdot 5 \cdot 7$. Each odd factor of 7! is a product of odd prime factors, and there are $3 \cdot 2 \cdot 2 = 12$ ways to combine the powers of the odd prime factors (3, 5, and 7) to generate odd factors.

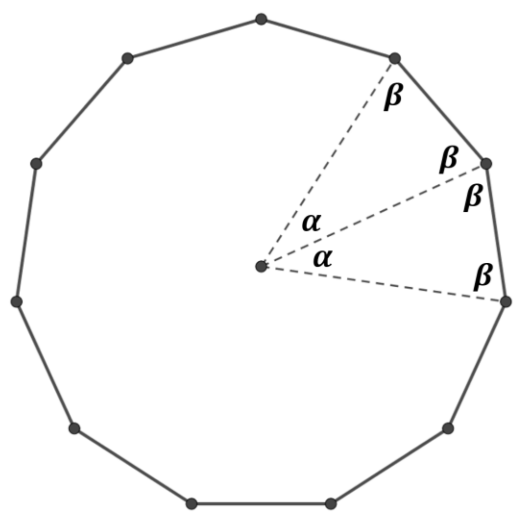
■

AMC 1993 Problem #15

For how many values of n will an n-sided regular polygon have interior angles with integral degree measures?

(A) 16	(B) 18	(C) 20	(D) 22	(E) 24

Solution



In a regular n-sided polygon, $\alpha = 360°/n$. But we also know that $\alpha + 2\beta = 180°$ (sum of the angles in any triangle equals 180°). Therefore

$$2\beta = 180° - \alpha = 180°\left(1 - \frac{2}{n}\right) = 180°\left(\frac{n-2}{n}\right).$$

But 2β is also the measure of each interior angle of the n-sided regular polygon. That is, each interior angle of a regular n-sided polygon has measure

$$180°\left(\frac{n-2}{n}\right).$$

The question is to find the values of n where

$$180°\left(\frac{n-2}{n}\right) = 180° - \frac{360°}{n}$$

is an integer. But this angle measure is an integer if and only if

$$\frac{360°}{n}$$

Is an integer. That is, when n divides $360°$. The prime factorization of 360 is

$$360 = 2^3 \cdot 3^2 \cdot 5.$$

So 360 has $(3+1)(2+1)(1+1) = 24$ divisors, including 1 and 2. But the smallest n can be is 3. (A polygon has a minimum of 3 sides.) So, there are $24 - 2 = 22$ values of n such that

$$2\beta = 180°\left(\frac{n-2}{n}\right) = 180° - \frac{360°}{n}$$

is an integer.

■

How many positive integral factors does 91091 have?

Solution

$$\begin{aligned} 91091 &= 91 \cdot 1000 + 91 = 91(1001) \\ &= (7 \cdot 13) \cdot (7 \cdot 11 \cdot 13) \\ &= 7^2 \cdot 11^2 \cdot 13^2 \end{aligned}$$

Therefore 91091 has

$$(2+1)(2+1)(2+1) = 3^3 = 27$$

positive integral factors.

■

3.10 Greatest number of factors that a positive integer less than n can have

(1T905)

5. The integers 8, 12, and 231 all have three prime factors. There is a maximum number m of prime factors that an integer N can have if N < 1990. Find the sum of all integers less than 1990 that have m prime factors.

Solution

Since $2^{10} = 1024$, the maximum number of prime factors of $N \le 1990$ is ten. Note

$$2^9 \cdot 3 = 1536 < 1990$$
$$2^8 \cdot 3^2 = 2304 > 1990 \quad \text{(too big)}$$

Sum $= 1536 + 1024 = 2560$

■

What is the least composite number with exactly 8 factors?

Answer

$$24$$

Solution

There are 3 ways for $(a+1) \times (b+1) \times (c+1) \times (d+1) \times (e+1) \times (f+1) \times \ldots$ to equal 8 that we need to consider: $(2 \times 2 \times 2 \text{ or } a = 1, b = 1, c = 1)$, $(4 \times 2 \text{ or } a = 3, b = 1)$ and $(8 \text{ or } a = 7)$.

$(a = 1, b = 1, c = 1)$ yields $2^1 \times 3^1 \times 5^1 = 30$.
$(a = 3, b = 1)$ yields $2^3 \times 3^1 = 24$.
$(a = 7)$ yields $2^7 = 128$.

The numbers $30, 24$, and 128 each have 8 factors but the least of these is 24.

Note: We can construct "lots and lots" of other numbers with 8 factors by considering cases such as $(a = 0, b = 3, c = 1)$, $(a = 1, b = 3)$, $(a = 0, b = 7)$, etc. We didn't even consider these because they clearly lead to unnecessarily large numbers. For example, $(a = 0, b = 3, c = 1)$ corresponds to $2^0 \times 3^3 \times 5^1$ and clearly this is bigger than $2^3 \times 3^1 \times 5^0$.

■

(see file: "Greatest number of factors that a positive integer less than 100 can have")

Consequently, when searching for the integer less than n with the most factors, we don't need to waste time considering any number whose prime factorization is <u>not</u> of the type

$$2^a \times 3^b \times 5^c \times 7^d \times 11^e \times 13^f \times \ldots$$

with $\boldsymbol{a} \geq \boldsymbol{b} \geq \boldsymbol{c} \geq \boldsymbol{d} \geq \boldsymbol{e} \geq \boldsymbol{f} \geq \cdots$ because there will necessarily be a smaller number with the same number of factors.

Illustrative Exercises:

Find a positive integer less than the given number that has the same number of factors as the given number.

(a) $20{,}244{,}510 = 2^1 \times 3^2 \times 5^1 \times 7^0 \times 11^3 \times 13^2$.

Answer: $2^3 \times 3^2 \times 5^2 \times 7^1 \times 11^1 \times 13^0 = 138{,}600$.

Clearly

$$138{,}600 < 20{,}244{,}510$$

and they both have

$$(3+1)(2+1)(2+1)(1+1)(1+1)(0+1) = 144$$

factors.

(b) $1{,}050 = 2^1 \times 3^0 \times 5^2 \times 7^1$.

Answer: $2^2 \times 3^1 \times 5^1 \times 7^0 = 60$.

Clearly

$$60 < 1050$$

and they both have

$$(2+1)(1+1)(1+1)(0+1) = 12$$

factors.

Define k to be that positive integer such that $2^k \leq n < 2^{k+1}$.

For example, if $n = 575$, then $k = 9$ because

$$2^9 = 512 \leq 575 < 1024 = 2^{10}.$$

Then $a + b + c + d + e + f + \cdots \leq k$ for the prime factorization

$$2^a \times 3^b \times 5^c \times 7^d \times 11^e \times 13^f \times \ldots$$

of every positive integer less than for equal to n.

Illustrative Exercises:

When looking for the greatest number of factors that a positive integer less than n can have, what is the maximum possible value of $a + b + c + d + \cdots$ in a prime factorization

$$2^a \times 3^b \times 5^c \times 7^d \times 11^e \times 13^f \times \ldots ?$$

you need to consider?

(a) Let $n = 100$.

Answer: 6 because $2^6 = 64 \leq 100 < 128 = 2^7$.

(b) Let $n = 300$.

Answer: 7 because $2^7 = 128 \leq 300 < 512 = 2^8$.

(c) Let $n = 3322$.

Answer: 11 because $2^{11} = 2048 \leq 3322 < 4096 = 2^{12}$.

By way of example, suppose that

$$2 \times 3 \times 5 \times 7 \times 11 \leq n < 2 \times 3 \times 5 \times 7 \times 11 \times 13.$$

Then when looking for number less than n which has the greatest number of factors it is not necessary to consider any integer with a prime factor greater than 11.

Illustrative Exercises:

When looking for the greatest number of factors that a positive integer less than n can have, what is the maximum prime factor you need to consider?

(a) Let $n = 100$.

Answer: 5 because $2 \cdot 3 \cdot 5 = 60 \leq 100 < 210 = 2 \cdot 3 \cdot 5 \cdot 7$.

(b) Let $n = 300$.

Answer: 7 because $2 \cdot 3 \cdot 5 \cdot 7 = 210 \leq 300 < 2310 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$.

(c) Let $n = 3322$.

Answer: 11 because $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 = 2310 \leq 3322 < 30030 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$.

Now let's put these ideas to use.

What is the greatest number of factors that a positive integer less than $n = 100$ can have?

We are looking for a prime factorization

$$2^a \times 3^b \times 5^c \times 7^d \times 11^e \times 13^f \times \ldots < 100$$

where

$$(a+1) \times (b+1) \times (c+1) \times (d+1) \times (e+1) \times (f+1) \times \ldots$$

is as large as possible.

By #1, we only need to consider the cases where $a \geq b \geq c \geq d \geq \cdots$

By #2, we only need to consider the cases where $a + b + c + d + \cdots \leq 6$ because

$$2^6 = 64 \leq 100 < 128 = 2^7.$$

By #3, the largest prime factor we need to include is 5 because

$$2 \cdot 3 \cdot 5 = 60 \leq 100 < 210 = 2 \cdot 3 \cdot 5 \cdot 7$$

Putting this all together, we are looking for a prime factorization

$$2^a \times 3^b \times 5^c < 100$$

where $a \geq b \geq c \geq 0$, $a + b + c \leq 6$ and

$$(a+1) \times (b+1) \times (c+1)$$

is as large as possible.

Candidates:

$(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$	$\mathbf{2^a \times 3^b \times 5^c}$	**Number of Factors**
$(6,0,0)$	$2^6 \times 3^0 \times 5^0 = 64 < 100$	$(6+1)(0+1)(0+1) = 7$
$(5,1,0)$	$2^5 \times 3^1 \times 5^0 = 96 < 100$	$(5+1)(1+1)(0+1) = 12$
$(4,1,1)$	$2^4 \times 3^1 \times 5^1 = 240 > 100$	

$(4,2,0)$	$2^4 \times 3^2 \times 5^0 = 144 > 100$	
$(3,2,0)$	$2^3 \times 3^2 \times 5^0 = 72 < 100$	$(3+1)(2+1)(0+1) = 12$
$(3,1,1)$	$2^3 \times 3^1 \times 5^1 = 120 > 100$	
$(2,1,1)$	$2^2 \times 3^1 \times 5^1 = 60 < 100$	$(2+1)(1+1)(1+1) = 12$

I don't see any way around this "organized brute force" approach.

Highly Composite Numbers (sometimes called *antiprimes*)
https://mathworld.wolfram.com/HighlyCompositeNumber.html

If

$$N = 2^{a_2}\, 3^{a_3} \cdots p^{a_p}$$

is the prime factorization of a highly composite number, then

1. The primes 2, 3, ..., p form a string of consecutive primes,

2. The exponents are nonincreasing, so $a_2 \geq a_3 \geq \ldots \geq a_p$, and

3. The final exponent a_p is always 1, except for the two cases $N = 4 = 2^2$ and $N = 36 = 2^2 \cdot 3^2$, where it is 2.

Mu Alpha Theta National Convention, 2001, Number Theory Test, Alpha Division, Problem # 4
What is the smallest counting number with exactly 32 positive integer factors?
Solution

4. Using the same method of factor counting as in problem #1, we can arrange 32 as the product of exponents (+1) in several ways. 32 = (4)(2)(2)(2) = (4)(4)(2), etc. We can make the problem easier by noting that we can produce a smaller number with four factors using only powers of 2 than with a single factor of 2 and a prime greater than 2 x 2 (for instance, 8 is less than 10 or 14). Noting such relationships (we could test 3 to the third vs. 3 x 7 or 3 x 11) we can see that 32 = (4)(2)(2)(2) produces the smallest possible integer using the smallest primes (2, 3, 5, and 7). (2)(2)(2)(3)(5)(7) = 840.

■

3.11 Sum of the Positive Divisors

Pg. 57, Sum of the Positive Divisors of a

Sometimes you are asked to find the sum of all positive divisors of a number a. Consider again the example of $a = 72$. In this case we would want to calculate:

$$(2^0 3^0 + 2^0 3^1 + 2^0 3^2) + (2^1 3^0 + 2^1 3^1 + 2^1 3^2) + (2^2 3^0 + 2^2 3^1 + 2^2 3^2) + (2^3 3^0 + 2^3 3^1 + 2^3 3^2).$$

Fortunately, we can factor this expression to simplify the calculation.

$$\begin{aligned}
&(2^0 3^0 + 2^0 3^1 + 2^0 3^2) + (2^1 3^0 + 2^1 3^1 + 2^1 3^2) + (2^2 3^0 + 2^2 3^1 + 2^2 3^2) \\
&\quad + (2^3 3^0 + 2^3 3^1 + 2^3 3^2) \\
&= 2^0(3^0 + 3^1 + 3^2) + 2^1(3^0 + 3^1 + 3^2) + 2^2(3^0 + 3^1 + 3^2) + 2^3(3^0 + 3^1 + 3^2) \\
&= (2^0 + 2^1 + 2^2 + 2^3)(3^0 + 3^1 + 3^2).
\end{aligned}$$

This idea generalizes to yield a convenient formula for the **sum of all of positive divisors of an integer $\boldsymbol{a}$** which we represent by $\boldsymbol{\sigma(a)}$.

Pg. 57, Theorem 2.24 Part(b) (Sum of the Positive Divisors of a)

If $a = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r} = \prod_{i=1}^{r} p_i^{a_i}$ with $a_i > 0$ for each i is the canonical representation of a, then

$$\sigma(a) = \left(p_1^0 + p_1^1 + \cdots + p_1^{a_1}\right) \cdots \left(p_r^0 + p_r^1 + \cdots + p_r^{a_r}\right) = \prod_{i=1}^{r} \left(p_i^0 + p_i^1 + \cdots + p_i^{a_i}\right).$$

Flashback **Geometric Series** We can simplify the formula for $\sigma(a)$ another step by remembering the following summation rule for geometric series. $$b^0 + b^1 + \cdots + b^k = \frac{b^{k+1} - 1}{b - 1}.$$

So, it follows from the above formula for a geometric series that

$$\sigma(a) = \left(\frac{p_1^{a_1+1} - 1}{p_1 - 1}\right) \cdots \left(\frac{p_r^{a_r+1} - 1}{p_r - 1}\right) = \prod_{i=1}^{r} \frac{p_i^{a_i+1} - 1}{p_i - 1}.$$

Example
Make a list of all of the divisors of 4459 and then find $\tau(4459)$, the number of positive divisors of 4459 and $\sigma(4459)$, the sum of the positive divisors of 4459, given that it's canonical representation is $4459 = 7^3 \cdot 13$.
Solution

$$\begin{array}{rr} 7^0 \cdot 13^0 = \quad 1 & 7^0 \cdot 13^1 = \quad 13 \\ 7^1 \cdot 13^0 = \quad 7 & 7^1 \cdot 13^1 = \quad 91 \\ 7^2 \cdot 13^0 = \ \ 49 & 7^2 \cdot 13^1 = \ \ 637 \\ 7^3 \cdot 13^0 = 343 & 7^3 \cdot 13^1 = 4459. \end{array}$$

$$\tau(4459) = \tau(7^3 \cdot 13^1) = (3+1)(1+1) = 4 \cdot 2 = 8$$

and

$$\sigma(4459) = \sigma(7^3 \cdot 13^1) = \left(\frac{7^{3+1}-1}{7-1}\right)\left(\frac{13^{1+1}-1}{13-1}\right) = 400 \cdot 14 = 5600.$$

Just as a check we can sum the positive divisors "by hand" to verify that

$$\sigma(4459) = 1 + 13 + 7 + 91 + 49 + 637 + 343 + 4459 = 5600.$$

You might also note that the real bottleneck in making a list or counting or summing up all positive divisors a number the size of 4459 would have been determining the prime factorization of 4459 had that not been given.

■

Mu Alpha Theta National Convention, 1991, Number Theory Test, Alpha Division, Problem # 18
Find the sum of the positive divisors (including 1 and 360) of 360.
Solution
The prime factorization of 360 is $360 = 2^3 3^2 5^1$. Therefore

$$\sigma(360) = \prod_{i=1}^{r} \frac{p_i^{a_i+1} - 1}{p_i - 1} = \left(\frac{2^{3+1}-1}{2-1}\right)\left(\frac{3^{2+1}-1}{3-1}\right)\left(\frac{5^{1+1}-1}{5-1}\right)$$

$$= \left(\frac{15}{1}\right)\left(\frac{26}{2}\right)\left(\frac{24}{4}\right) = 15 \cdot 13 \cdot 6$$

$$= 1170.$$

■

Mu Alpha Theta National Convention, 2001, Number Theory Test, Alpha Division, Problem # 2

Find the sum of the positive proper integral factors of 512.
Solution
Be careful to not overlook the term "proper" as used in the problem statement. By definition, 512 is not considered to be a proper factor of 512.
The easiest way to deal with this is to first find the sum of *all* positive integral factors and then just subtract out the extra 512 at the end.
We know $512 = 2^9$. Therefore, the sum of *all* factors (including the improper 512) equals

$$\sigma(512) = \prod_{i=1}^{r} \frac{p_i^{a_i+1} - 1}{p_i - 1} = \frac{2^{10} - 1}{2 - 1} = \frac{1024 - 1}{1} = 1023.$$

Now we need to subtract out the extra 512 from this sum.

$$1023 - 512 = 511.$$

■

3.12 Sum of the Reciprocals of the Positive Divisors

(see file: "Sum of reciprocals of divisors given the sum of divisors")

Generally the sum of the reciprocals of the divisors of n is equal to $\frac{\sigma(n)}{n}$ where σ is the sum of divisors function. This quantity is sometimes referred to as the abundancy ratio or abundancy index of n. It can be used to tell whether n is abundant, deficient, or perfect.

Saint Mary's College Mathematics Contest Problems
397. If the sum of the divisors of a number N including the number itself is 3 times the number, what is the sum of the reciprocals of the divisors?
Solution

■

3.13 Product of the Positive Divisors

Problem 4.

Find the formula $\pi(n)$ for the product of the divisors of the positive integer n.

Answer:

$$\pi(n) = n^{\tau(n)/2}$$

If $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ is the unique factorization of n into a product of distinct primes, then

We see that $30 = 2^1 \cdot 3^1 \cdot 5^1 = 2 \cdot 3 \cdot 5$ and hence

Factors of $n = 30$,

$$\begin{gathered} 1 = 2^0 3^0 5^0 \\ 2 = 2^1 3^0 5^0 \\ 3 = 2^0 3^1 5^0 \\ 5 = 2^0 3^0 5^1 \\ 6 = 2^1 3^1 5^0 \\ 10 = 2^1 3^0 5^1 \\ 15 = 2^0 3^1 5^1 \\ 30 = 2^1 3^1 5^1 \end{gathered}$$

$\pi(n)$, the product of the factors, is

$$\pi(30) = \; 2^4 3^4 5^4.$$

More generally, let's consider how many times the prime p_1 shows up in this product.

$$(p_1^0)^{\frac{\tau(n)}{a_1+1}} \cdot (p_1^1)^{\frac{\tau(n)}{a_1+1}} \cdot (p_1^2)^{\frac{\tau(n)}{a_1+1}} \cdots \cdot \left(p_1^{a_1}\right)^{\frac{\tau(n)}{a_1+1}}$$

$$= (p_1)^{\left(\frac{\tau(n)}{a_1+1}\right)(0+1+2+\cdots+a_1)}$$

$$= (p_1)^{\left(\frac{\tau(n)}{a_1+1}\right)\left(\frac{(a_1)(a_1+1)}{2}\right)}$$

$$= (p_1)^{\left(\frac{\tau(n)}{2}\right)(a_1)}$$

$$= \left(p_1^{a_1}\right)^{\left(\frac{\tau(n)}{2}\right)}$$

Repeating this for each prime can see that the product of all the factors of $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ equals

$$\pi(n) = \left(p_1^{a_1}\right)^{\left(\frac{\tau(n)}{2}\right)} \cdot \left(p_2^{a_2}\right)^{\left(\frac{\tau(n)}{2}\right)} \cdots \left(p_k^{a_k}\right)^{\left(\frac{\tau(n)}{2}\right)}$$

$$= \left(p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}\right)^{\left(\frac{\tau(n)}{2}\right)}$$

$$= (n)^{\left(\frac{\tau(n)}{2}\right)}$$

Another example, $n = 4$. The factors are $\{1,2,4\}$ and the product of the factors equals 8. For $n = 4$, $\tau(4) = 3$ and

$$\pi(4) = 4^{(\tau(4)/2)} = 4^{3/2} = 2^3 = 8.$$

3.14 Expressing n as the Product of Two Integers

Consider the very simple example of expressing $n = 6$ as the product of two integers. To get the "correct" count for the number of ways this can be done you need to look carefully at the wording of the problem.

Issue 1. Should $2 \times 3 = 6$ be counted as distinct from $3 \times 2 = 6$?

Issue 2. Should $(-2) \times (-3) = 6$ be counted?

You may have to consider the wording of the problem carefully to sort out these two issues.

Consider the case where we do not want to treat $a \times b = n$ and $b \times a = n$ as different products and where a and b must be positive integers

In this case let $\boldsymbol{l(n)}$ equal the number of ways to express the positive integer n as the product of two positive integers (including $n \times 1$).

Then there are two cases to consider in finding a formula for $l(n)$, the number of ways to express n as the product of two distinct positive integers.

Theorem

Formula for $\boldsymbol{l(n)}$.

Case 1. n is a perfect square (equivalently, if $\tau(n)$ is odd). In this case,

$$l(n) = \frac{\tau(n) + 1}{2}.$$

Case 2. n is not a perfect square. In this case,

$$l(n) = \frac{\tau(n)}{2}.$$

In both cases we use $\tau(n)$ to represent the number of positive divisors of n.

■

The next two examples will help to clarify where this formula for $l(n)$ comes from.

Example

How many ways are there to write 10800 as the product of two distinct positive integers (including 10800×1)? Assume we do not want to treat $a \times b = 10800$ and $b \times a = 10800$ as different products.

Solution

The prime factorization of 10800 is $10800 = 2^4 \cdot 3^3 \cdot 5^2$. Therefore $\tau(10800)$, the number of divisors (factors) of 10800 equals $(4 + 1)(3 + 1)(2 + 1) = 60$.

The set $\mathcal{D}$ of all 60 divisors would look something like this:

$$\mathcal{D} = \{1,2,3,4,5,6, \dots ,2700,3600,5400,10800\}.$$

Clearly if a and b are positive integers such that $a \times b = 10800$ then a and b are both elements of $\mathcal{D}$.

Furthermore, each divisor in this list of 60 numbers can be paired with a *different* divisor in this list to form a product of two positive integers equaling 10800.

For example, the divisor 12 can be paired with the divisor $10800/12 = 900$. This gives us $\tau(10600) = 60$ pairs of divisors whose product equals 10600.

But this double counts each possible pair! That is, this method would count both 12×900 as well as 900×12 – which we do not want to do.

So, there are only $\tau(10600)/2 = 60/2 = 30$ ways to write 10800 as the product of two positive integers.

■

Example

How many ways are there to write 36 as the product of two positive integers (including 36×1)? Assume we do not want to treat $a \times b = 10800$ and $b \times a = 10800$ as different products.

Solution

The prime factorization of 36 is $36 = 2^2 \cdot 3^2$. Therefore $\tau(36)$, the number of divisors (factors) of 36 equals $(2 + 1)(2 + 1) = 9$.

The complete set $\mathcal{D}$ of all 9 divisors would be:

$$\mathcal{D} = \{1,2,3,4,6,9,12,18,36\}.$$

Looking back at the first example of this section, we saw that each divisor in the list of 60 could be paired with a *different* divisor to form a product of 10800 to form $60/2 = 30$ distinct pairs.

Is that true in this example? No. We can $(1,36), (2,18), (3,12), (4,9)$ but that leaves us trying to pair 6 with itself – which we don't want to do.

So, in this example where are $(\tau(36) - 1)/2 = (9 - 1)/2 = 4$ ways to express 36 as the product of two distinct positive integers.

■

What was the critical difference between the previous two examples?

Whenever $\tau(n)$ is even, such as the case $\tau(10600) = 60$, then every number in the list of divisors can be paired with a distinct divisor from that list to form a product equaling n.

But when $\tau(n)$ is odd, such as the case $\tau(36) = 9$, then the median number in the list of divisors <u>cannot</u> be paired with a distinct divisor from that list to form a product equaling n.

So, the critical point in finding a formula for $l(n)$, the number of ways to express n as the product of two distinct positive integers, was whether $\tau(n)$ is even or odd. But we know that $\tau(n)$ is odd if and only if n is a perfect square.

Example

How many ways can $n = 30$ be written as a product of two positive integers, including 30 and 1?

Solution

We see that $30 = 2^1 \cdot 3^1 \cdot 5^1 = 2 \cdot 3 \cdot 5$ and hence

$$\tau(30) = (1+1)(1+1)(1+1) = 8.$$

Furthermore, 30 is not a perfect square, so

$$l(30) = \frac{\tau(30)}{2} = \frac{8}{2} = 4.$$

So there are exactly 4 ways to write 30 as the product of two positive integers. They are

$$30 = 1 \times 30,\ \ 30 = 2 \times 15,\ \ 30 = 3 \times 10,\ \text{and}\ \ 30 = 5 \times 6.$$

■

AMC 2019 10B Problem #19

Let S be the set of all positive integer divisors of 100,000. How many numbers are the product of two distinct elements of S ?

(A) 98	(B) 100	(C) 117	(D) 119	(E) 121

<u>Solution</u>

■

Mu Alpha Theta National Convention, 2001, Number Theory Test, Alpha Division, Problem # 1
For how many ordered pairs of integers (m, n) does m multiplied by n equal 120?
<u>Solution</u>
Ordered pairs translates to counting (a, b) as distinct from (b, a) whenever $a \neq b$. Also notice that the problem does not specify that a and b must be positive. In mathematics, the custom is to assume the most general case unless it has been specifically ruled out.
By this custom, we should count (a, b) as well as $(-a, -b)$.
Because 120 is not a perfect square, the count for *unordered pairs* with positive factors is

$$l(n) = \frac{\tau(n)}{2}.$$

To allow for *ordered pairs* we need to double this number. To allow for both positive and negative factors we need to double this number a second time.

So, the correct count becomes

$$\left(\frac{\tau(120)}{2}\right) \cdot 2 \cdot 2 = 2 \cdot \tau(2^3 \cdot 3 \cdot 5) = 2(3+1)(1+1)(1+1) = 32.$$

■

3.15 The Sum of the Squares of the Positive Divisors

Pg. 58, Exercise 7

If $a = \displaystyle\prod_{i=1}^{r} p_i^{a_i}$ with $a_i > 0$ for each i is the canonical representation of a, deduce a formula for the sum of the squares of the positive divisors of a.

Pg. 58, Exercise 8

Let $a = \displaystyle\prod_{i=1}^{r} p_i^{a_i}$ with $a_i > 0$ for each i be the canonical representation of a. Prove that a is the square of an integer if and only if a_i is even for each i.

3.16 Extra Divisibility Problems

Math Wrangle Problems, American Mathematics Competitions, December 3, 2020, Problem #1

Find the sum of all positive two-digit integers that are divisible by each of their digits.
Solution
Let $\overline{ab} = 10a + b$ with $a \in \{1,2,\dots,9\}$ and $b \in \{0,1,2,\dots,9\}$. We are given the information that $a|(10a+b)$ and $b|(10a+b)$.
This tells us that $a|b$ and $b|10a$.
But $a|b \Leftrightarrow b = ak$ for some positive integer k. By substitution, this means $ak|10a$.
But for general $c \neq 0$, $a|b \Leftrightarrow ac|bc$. Therefore, $ak|10a \Leftrightarrow k|10 \Leftrightarrow k = 1,2$ or 5.
Now enumerate all possible cases.

$$k = 1, a = b \Leftrightarrow \overline{ab} \in \{11,22,\dots,99\}$$
$$k = 2, 2a = b \Leftrightarrow \overline{ab} \in \{12, 24,36,48\}$$
$$k = 5, 5a = b \Leftrightarrow \overline{ab} \in \{15\}.$$

Hence the sum is

$$11(1+2+\cdots+9) + 12(1+2+3+4) + 15 = 630.$$

■

Find all integers n such that $(n-3) \,\big|\, (n^3-3)$.
Solution
Let $k = n-3$ and note that the set of all integers n such that $(n-3) \,\big|\, (n^3-3)$ is the same as the set of all $k+3$ such that $k \,\big|\, \left((k+3)^3-3\right)$. We see that

$$k \,\big|\, \left((k+3)^3-3\right) \Leftrightarrow k \,\big|\, \left(k^3+9k^2+27k+24\right) \Leftrightarrow k \,\big|\, 24$$

so

$$\begin{aligned} k \,\big|\, \left((k+3)^3-3\right) &\Leftrightarrow k \in \{\pm1,\pm2,\pm3,\pm4,\pm6,\pm8,\pm12,\pm24\} \\ &\Leftrightarrow n = k+3 \in \{3\pm1, 3\pm2, 3\pm3, 3\pm4, 3\pm6, 3\pm8, 3\pm12, 3\pm24\} \\ &\Leftrightarrow n \in \{-21,-9,-3,-5,-3,-1,0,1,2,4,5,6,7,9,11,15,27\}. \end{aligned}$$

■

Mu Alpha Theta National Convention 2007, Mu Division, Number Theory Test, Problem #3
Determine the value of a for any nonnegative integer n such that $(10^n) \equiv a \bmod(9)$.
Solution
$\Big(\underbrace{999\cdots999}_{n\,9's}\Big) \bmod(9) = 0$ for all nonnegative integer n. Therefore,

$$\begin{aligned} (10^n) \bmod(9) &= \Big(\underbrace{999\cdots999}_{n\,9's} + 1\Big) \bmod(9) \\ &= \Big(\underbrace{999\cdots999}_{n\,9's}\Big) \bmod(9) + (1) \bmod(9) \\ &= 0 + 1 = 1 \end{aligned}$$

for all nonnegative integer n.

■

Mu Alpha Theta 1995 National Convention, Number Theory Topic Test, Problem #5

What is the sum of all the odd three-digit numbers that are divisible by 5?
Solution
The problem is asking for the sum
$$s = 105 + 115 + 125 + \cdots + 985 + 995.$$

We recognize that $105, 115, 125, \dots, 985, 995$ is an *arithmetic sequence* with common difference $d = 10$.
Recall that the sum of the arithmetic sequence $a_1, a_2, \dots, a_n$ equals
$$\sum_{i=1}^{n} a_i = \frac{n(a_1 + a_n)}{2}$$
where n, the number of terms in the sequence, equals
$$n = \frac{\text{Last term} - \text{First term}}{d} + 1.$$
Applied to this problem
$$s = \frac{n(105 + 995)}{2}$$
with
$$n = \frac{995 - 105}{10} + 1 = 90.$$
Thus,
$$s = \frac{90(105 + 995)}{2} = 45(1100) = 49500.$$

■

Chapter 4. GCD's and LCM's

4.1 Common Divisor

Pg. 41, Definition (Common Divisor)
If $d|a$ and $d|b$, then d is said to be a ***common divisor*** of a and b.

Pg. 41, Definition 2.1 (Greatest Common Divisor)
If d is the largest common divisor of a and b, it is called the *greatest common divisor* of a and b and is denoted by (a, b).
Note: (a, b) is necessarily positive because if for some $c > 0$, $-c$ divides both a and b, then c also must divide both a and b. And clearly $c > -c$ for $c > 0$.

Pg. 41, Theorem 2.4 (Strayer)
If a and b are not both zero and if $d = (a, b)$, then d is the least element in the set of all positive integers of the form $ax + by$.
Proof
https://math.stackexchange.com/questions/219941/is-greatest-common-divisor-of-two-numbers-really-their-smallest-linear-combinati

Let e be the smallest positive linear combination $as + bt$ of a and b, where s and t are integers. Suppose in particular that $e = ax + by$.

Let $d = \gcd(a, b)$. Then d divides a and b, so it divides $ax + by$. Thus d divides e, and therefore in particular $d \leq e$.

We show that in fact e is a common divisor of a and b, which will imply that $e \leq d$. That, together with our earlier $d \leq e$, will imply that $d = e$.

So, it remains to show that e divides a and e divides b. We show that e divides a. The proof that e divides b is essentially the same.

Suppose to the contrary that e **does not** divide a. Then when we try to divide a by e, we get a positive remainder. More precisely,

$$a = qe + r,$$

where $0 < r < e$. Then

$$r = a - qe = a - q(ax + by) = a(1 - qx) + b(-qy).$$

This means that r is a linear combination of a and b, and is positive and less than e. This contradicts the fact that e is the *smallest* positive linear combination of a and b.

■

What is the smallest positive integer that can be written as a linear combination of 2191 and 1351?

Solution

By what we just proved the answer equals $\gcd(2191, 1351) = \gcd(7 \cdot 313, 7 \cdot 193) = 7$.

■

2. What is the smallest positive rational number that can be expressed in the form $x/30 + y/36$ with x and y integers?

Solution

Let $x/30 + y/36 = r$. Then $36x + 30y = (30 \cdot 36)r$. To make r positive and as small as possible, we need to make $36x + 30y$ positive and as small as possible. But we know from Proposition 1.11 that the $\gcd(36,30) = 6$ is the minimum positive value of $36x + 30y$.

So we can find the minimum positive value of r by solving the equation

$$6 = (30 \cdot 36)r.$$

Therefore, the minimum positive value of $r = 6/(30 \cdot 36) = 1/180$.

Note: You can use the same approach to show that the smallest positive value of

$$\frac{x}{a} + \frac{y}{b}$$

is $(1/N)$ where $N = \text{lcm}(a, b)$.

■

4.1.1 Euclidean Algorithm

Pg. 42, Euclidean Algorithm (Note: It's actually a theorem)

Let $a \geq b$ be positive integers and let r_1 be the remainder when a is divided by b where $0 \leq r_1 < b$. Then $(a, b) = (b, r_1)$.

If $r_1 > 0$, continue in the same way and let r_2 be the remainder when b is divided r_1 where

$0 \leq r_2 < r_1$. Then $(a, b) = (b, r_1) = (r_1, r_2)$.

If $r_2 > 0$, continue in the same way and let r_3 be the remainder when r_1 is divided by r_2 where $0 \leq r_3 < r_2$. Then $(a, b) = (b, r_1) = (r_1, r_2) = (r_2, r_3)$.

Continue in this same way until some remainder $r_k = 0$ is reached. Then
$(a, b) = (b, r_1) = \cdots = (r_{k-1}, r_k) = (r_{k-1}, 0) = r_{k-1}$.

Example

Find $(120,28)$.

Solution

$$
\begin{aligned}
120 &= 4 \cdot 28 + 8 \Longrightarrow (120,28) = (28,8) \\
28 &= 3 \cdot 8 + 4 \Longrightarrow (28,8) = (8,4) \\
8 &= 2 \cdot 4 + 0 \Longrightarrow (8,4) = (4,0) = 4
\end{aligned}
$$

Therefore,

$$(120,28) = (28,8) = (8,4) = (4,0) = 4.$$

■

Find $(288,51)$.

Solution

$$
\begin{aligned}
288 &= 51 \cdot 5 + 33 \\
51 &= 33 \cdot 1 + 18 \\
33 &= 18 \cdot 1 + 15 \\
18 &= 15 \cdot 1 + 3 \\
15 &= 3 \cdot 5 + 0
\end{aligned}
$$

Therefore,

$$(288,51) = 3.$$

■

Find $(223,163)$.

Solution

$$
\begin{aligned}
223 &= 163 \cdot 1 + 60 \\
163 &= 60 \cdot 2 + 43 \\
60 &= 43 \cdot 1 + 17 \\
43 &= 17 \cdot 2 + 9 \\
17 &= 9 \cdot 1 + 8 \\
9 &= 8 \cdot 1 + 1 \\
8 &= 1 \cdot 8 + 0
\end{aligned}
$$

Therefore,

$$(233,163) = 1.$$

■

Find the $\gcd(3248,214)$.

Solution

There are two main approaches for determining $\gcd(a, b)$.

Method 1. Use the prime factorizations of a and b.
Method 2. Use the Euclidean Algorithm.

Method 1 is quicker if you are provided or can easily find the prime factorizations of a and b.

Method 2 is quicker if a and b are "large" and their prime factorization are not obvious to you.

I will illustrate both approaches in these notes but if this problem was on a timed test I would have opted for the Euclidean Algorithm approach. Finding the prime factorization *can* be very time consuming. It all depends on how the numbers work out.

Method 1: Prime Factorization Approach

$$\begin{aligned} 3248 &= 2(1624) \\ 1624 &= 2(812) \\ 812 &= 2(406) \\ 406 &= 2(203) \\ 203 &= 7(29) \text{ and 29 is prime} \end{aligned}$$

Therefore,

$$3248 = 2^4 \cdot 7^1 \cdot 29^1.$$

$$214 = 2(107) \text{ and 107 is prime}$$

Therefore,

$$214 = 2^1 \cdot 107^1.$$

$$\gcd(3248,214) = 2^{\min(4,1)} \cdot 7^{\min(1,0)} \cdot 29^{\min(1,0)} \cdot 107^{\min(0,1)}$$

$$= 2^1 \cdot 7^0 \cdot 29^0 \cdot 107^0 = 2 \cdot 1 \cdot 1 \cdot 1 = 2.$$

Method 2: Euclidean Algorithm Approach

$$\begin{aligned} 3248 &= 214(15) + 38 \\ 214 &= 38(5) + 24 \\ 38 &= 24(1) + 14 \\ 24 &= 14(1) + 10 \\ 14 &= 10(1) + 4 \end{aligned}$$

$$10 = 4(2) + \mathbf{2}$$
$$4 = 2(2) + 0$$

Therefore, $\gcd(3248{,}214) = \mathbf{2}$, the last non-zero remainder.

■

4.2 Properties of GCD

Pg. 45, Definition 2.2 (Relatively Prime)
If $(a, b) = 1$, then a and b are said to be ***relatively prime***. More generally, if $\left(a_i, a_j\right) = 1$ for $i \neq j$, $1 \leq i \leq r$, $1 \leq j \leq r$, the integers $a_1, a_2, \dots, a_r$ are said to be ***pairwise relatively prime***.

Pg. 45, Theorem 2.8
If $a|bc$ and $(a, b) = 1$, then $a|c$.

Pg. 46, Theorem 2.13
If $a|c$, $b|c$, and $(a, b) = 1$, then $ab|c$.

Pg. 48, Ex. 9
If $c > 0$, prove that $(ac, bc) = c(a, b)$.

Pg. 49, Ex. 22
If $(a, b) = 1$, and $a > b > 0$, prove that
$$(a^m - b^m, a^n - b^n) = a^{(m,n)} - b^{(m,n)}$$
for any positive integers m and n.

- $(0{,}0)$ is undefined.

- $(a, b) = d \Longrightarrow (-a, b) = (a, -b) = (-a, -b) = d$.

- $(a, b) = d \Longrightarrow d|a$ and $d|b$.

Proposition 1.10:

Let $a, b \in \mathbb{Z}$ with $(a, b) = d$. Then $(a/d\,, b/d) = 1$.

Noted results from Exercise Set 1.3

- $(a, 0) = |a|$ provided $a \neq 0$.

- Let $a, b \in \mathbb{Z}$ with a and b not both zero and let c be a nonzero integer. Then

$$(ca, cb) = |c| \cdot (a, b).$$

- Let $a, b, c \in \mathbb{Z}$ with $(a, b) = 1$ and $c | a + b$. Then

$$(a, c) = 1 \text{ and } (b, c) = 1.$$

- Let $a, b, c \in \mathbb{Z}$ with $(a, b) = 1$. Then

$$a | c \text{ and } b | c \implies ab | c.$$

- Let $a, b, c \in \mathbb{Z}$ with $(a, b) = 1$. Then

$$a | bc \implies a | c.$$

- Let $a, b \in \mathbb{Z}$ and let m and n be positive integers. Then

$$(a, b) = 1 \Leftrightarrow (a^m, b^n) = 1.$$

Proposition 1.14 (Elementary properties of the gcd)**.** *Let $a, b \in \mathbf{Z}$, with a and b not both 0.*

(i) $(a, b) = (-a, b) = (a, -b) = (-a, -b)$.

(ii) $(a, b) = (a + bn, b) = (a, b + am)$ *for any* $n, m \in \mathbf{Z}$.

(iii) $(ma, mb) = m(a, b)$ *for any* $m \in \mathbf{N}$.

(iv) *If* $d = (a, b)$, *then* $(a/d, b/d) = 1$.

(v) *Let* $d \in \mathbf{N}$. *Then* $d \mid (a, b)$ *holds if and only if* $d \mid a$ *and* $d \mid b$.

THEOREM 2. Let m be a common divisor of a and b; then

$$\left(\frac{a}{m}, \frac{b}{m}\right) = \frac{(a, b)}{m}.$$

THEOREM 3. If several numbers $a, b, c, \ldots, k$ are multiplied by the same integer m, then

$$(ma, mb, mc, \ldots, mk) = m(a, b, c, \ldots, k).$$

THEOREM 4. Let m be a common divisor of several integers $a, b, c, \ldots, k$; then

$$\left(\frac{a}{m}, \frac{b}{m}, \frac{c}{m}, \ldots, \frac{k}{m}\right) = \frac{(a, b, c, \ldots, k)}{m}.$$

COROLLARY. Several numbers are said to be *without common divisor* (meaning without common divisor > 1) if their g.c.d. $= 1$. From Theorem 4 it follows that, on dividing several numbers by their g.c.d., the resulting quotients will be numbers without common divisors.

THEOREM 8. If a and b are relatively prime positive integers and their product is an exact power c^n of an integer, then a and b themselves are exact nth powers.

Theorem

$\gcd(a, a+1) = 1$

<u>Solution</u>

Let $\gcd(a, a+1) = d$. Then $d|a$ and $d|a+1$. So d divides all linear combinations of a and $a+1$ (Prop. 1.2) including $(-1)a + (1)(a+1) = 1$. So $d|1$ which means that $1 = d \cdot k$ for some $k \in \mathbb{Z}$. But remember that $d \in \mathbb{Z}$ and d is (by definition) positive. From which it follows that k must also be positive (if k negative then $d \cdot k$ would be negative but we know $d \cdot k = 1$). Therefore, $d = 1$ (and also $k = 1$). So $d = \gcd(a, a+1) = 1$.

(Strayer)

(33d) Find gcd $(3a + 5, 7a + 12)$.

<u>Solution</u>

Let $\gcd(3a+5, 7a+12) = d$. Then $d|3a+5$ and $d|7a+12$. So $d|\big(3(7a+12) - 7(3a+5)\big)$. That is, $d|1$. By the argument of part (b) above, $d = 1$.

4.3 Least Common Multiple

Pg. 49, Definition 2.3 (Least Common Multiple)
If m is the smallest positive common multiple of a and b, it is called the *least common multiple* of a and b and is denoted by $[a, b]$.

Pg. 50, Theorem 2.19
If $ab \neq 0$, then

$$[a,b] = \left|\frac{ab}{(a,b)}\right|.$$

Example
Find the following Least Common Multiples using our previous examples on Greatest Common Divisors.

$$[120,28] = \left|\frac{120 \cdot 28}{(120,28)}\right| = \left|\frac{120 \cdot 28}{4}\right| = 840$$
$$[288,51] = \left|\frac{288 \cdot 51}{(288,51)}\right| = \left|\frac{288 \cdot 51}{3}\right| = 4896$$
$$[233,163] = \left|\frac{233 \cdot 163}{(233,163)}\right| = \left|\frac{233 \cdot 163}{1}\right| = 37{,}979.$$

Corollary 1.20: Let $a, b \in \mathbb{Z}$ with $a, b > 0$. Then $[a,b] = ab$ if and only if $(a,b) = 1$.

• Let a, b be positive integers. Then

$$(a,b)|[a,b].$$

1A973

Three good friends dine in the same restaurant. All are eating there today. However, they do not eat there every day. The first eats there every twelfth day, the second every fourteenth day, and the third every twenty-first day. How many days from today will they next all meet in the restaurant?

71. Find $a \in \mathbb{Z}, a > 0$ such that $\operatorname{lcm}(a, a+1) = 240$.

<u>Solution</u>

$$\gcd(a, a+1) = 1 \implies \operatorname{lcm}(a, a+1) = a(a+1).$$

Therefore, $a(a+1) = 240$ and $a = 15$ because $15 \cdot 16 = 240$.

4.4 Properties of LCM's

Let a, b, c be positive integers. Then

$$[ca, cb] = c \cdot [a, b]$$

If $c|a$ and $c|b$, then

$$\left[\frac{a}{c}, \frac{b}{c}\right] = \frac{[a, b]}{c}$$

$$\max\{a, b\} \leq [a, b] \leq ab$$

Proposition 1.18 (Elementary properties of the lcm). *Let a, b be nonzero integers.*

(i) $[a, b] = [-a, b] = [a, -b] = [-a, -b]$.

(ii) $[ma, mb] = m[a, b]$ *for any* $m \in \mathbf{N}$.

(iii) $[a, b] = \dfrac{|ab|}{(a, b)}$.

(iv) *Let* $m \in \mathbf{N}$. *Then* $[a, b] \mid m$ *holds if and only if* $a \mid m$ *and* $b \mid m$.

4.5 GCD and LCM for More Than Two Integers

The calculation of the greatest common divisor and least common multiple of more than two integers can be accomplished in successive steps in accordance with the following theorems.

Pg. 51, Theorem 2.20
If none of $a_1, a_2, \dots, a_r$ is zero, then

$$(a_1, a_2, \dots, a_r) = \big((a_1, a_2, \dots, a_{r-1}), a_r\big).$$

Pg. 51, Theorem 2.21

If none of $a_1, a_2, \dots, a_r$ is zero, then

$$[a_1, a_2, \dots, a_r] = \big[[a_1, a_2, \dots, a_{r-1}], a_r\big].$$

Example

Find $(120,28,6)$.

Solution

We showed in the previous example that $(120,28) = 4$. Therefore,

$$(120,28,6) = \big((120,28), 6\big) = (4,6) = (6,4).$$

By inspection we can see that $(6,4) = 2$. Therefore, $(120,28,6) = 2$.

$\blacksquare$

Find $[120,28,6]$.

Solution

We showed in the previous example that $[120,28] = 840$. Therefore,

$$[120,28,6] = \big[[120,28], 6\big] = [840,6].$$

Now that we have this reduced to just the two integers we can apply Theorem 2.19 to determine that

$$[120,28,6] = [840,6] = \left|\frac{840 \cdot 6}{(840,6)}\right| = \left|\frac{5040}{(840,6)}\right|.$$

At this point we have to use the Euclidean Algorithm to find $(840,6)$ which is easy in this case because $6|840$.

$$840 = 6 \cdot 140 + 0$$

Therefore,

$$(840,6) = 6.$$

This establishes

$$[120,28,6] = \left|\frac{5040}{(840,6)}\right| = \left|\frac{5040}{6}\right| = 840.$$

$\blacksquare$

Find $[540,75,70]$.

Solution

$$[540,75,70] = \big[[540,75], 70\big].$$

Now to find $[540,75]$.

$$[540,75] = \left|\frac{540 \cdot 75}{(540,75)}\right| = \left|\frac{40{,}500}{(540,75)}\right|$$

and

$$540 = 75 \cdot 7 + 15$$
$$75 = 15 \cdot 5 + 0$$

Therefore,

$$(540,75) = 15$$

and

$$[540,75] = \left|\frac{40{,}500}{(540,75)}\right| = \left|\frac{40{,}500}{15}\right| = 2700$$

and

$$[540,75,70] = \big[[540,75],70\big] = [2700,70].$$

Now we need to use Theorem 2.19 again to determine that

$$[540,75,70] = [2700,70] = \left|\frac{2700 \cdot 70}{(2700,70)}\right| = \left|\frac{189{,}000}{(2700,70)}\right|.$$

We see that

$$2700 = 70 \cdot 38 + 40$$
$$70 = 40 \cdot 1 + 30$$
$$40 = 30 \cdot 1 + 10$$
$$30 = 10 \cdot 3 + 0$$

therefore,

$$(2700,70) = 10$$

and

$$[540,75,70] = \left|\frac{189{,}000}{(2700,70)}\right| = \left|\frac{189{,}000}{10}\right| = 18{,}900.$$

■

Pg. 50, Definition (Extending the Term "Relatively Prime" to Cases of More than Two Integers)
If $(a_1, a_2, \dots, a_r) = 1$, then we say that $a_1, a_2, \dots, a_r$ are ***relatively prime***.

Non-Theorem 2.19
We <u>cannot</u> extend Theorem 2.19 to cases with more than two integers. That is,

$$[a,b,c] \neq \left|\frac{abc}{(a,b,c)}\right|$$

and

$$[a,b,\dots,k] \neq \left|\frac{ab\cdots k}{(a,b,\dots,k)}\right|$$

in general.

Pg. 52, Ex. 13

Give an example to show that the equation of Exercise 12 is sometimes true. Can you discover under what conditions the equation is generally true?

4.6 (a, b) and $[a, b]$ in terms of the canonical representations of a and b

Pg. 57, Theorem 2.25

If $a = \prod_{i=1}^{r} p_i^{a_i}$ and $b = \prod_{i=1}^{r} p_i^{b_i}$ with $a_i \geq 0$ and $b_i \geq 0$ for each i are the canonical representations of a and b, then

$$(a, b) = \prod_{i=1}^{r} p_i^{\min(a_i, b_i)} \quad \text{and} \quad [a, b] = \prod_{i=1}^{r} p_i^{\max(a_i, b_i)}.$$

[Note: In this theorem we need to compare the prime factorizations of a and b. Therefore, we include "place holders primes" (primes to the zeroth power) for those primes that are factors of a but not b and vice versa. This is why we only assume $a_i \geq 0$ and $b_i \geq 0$. Compare that with the statement of Theorems 2.24 Part(a) and Part(b) where we assume $a_i > 0$.]

Example

Use Theorem 2.25 to find $(354200, 84942)$ and $[354200, 84942]$ given that their canonical prime factorizations are

$$354200 = 2^3 \cdot 5^2 \cdot 7^1 \cdot 11^1 \cdot 23^1$$
$$84942 = 2^1 \cdot 3^3 \cdot 11^2 \cdot 13^1.$$

Solution

Our first step is to fill in the place holder primes (primes to the zeroth power).

$$354200 = 2^3 \cdot 3^0 \cdot 5^2 \cdot 7^1 \cdot 11^1 \cdot 13^0 \cdot 23^1$$
$$84942 = 2^1 \cdot 3^3 \cdot 5^0 \cdot 7^0 \cdot 11^2 \cdot 13^1 \cdot 23^0.$$

It now follows from Theorem 2.25 that

$$\begin{aligned}
&(354200, 84942) \\
&\quad = 2^{\min(3,1)} \cdot 3^{\min(0,3)} \cdot 5^{\min(2,0)} \cdot 7^{\min(1,0)} \cdot 11^{\min(1,2)} \cdot 13^{\min(0,1)} \cdot 23^{\min(1,0)} \\
&\quad = 2^1 \cdot 3^0 \cdot 5^0 \cdot 7^0 \cdot 11^1 \cdot 13^0 \cdot 23^0 \\
&\quad = 2 \cdot 11 \\
&\quad = 22.
\end{aligned}$$

$[354200, 84942]$
$$= 2^{\max(3,1)} \cdot 3^{\max(0,3)} \cdot 5^{\max(2,0)} \cdot 7^{\max(1,0)} \cdot 11^{\max(1,2)} \cdot 13^{\max(0,1)} \cdot 23^{\max(1,0)}$$
$$= 2^3 \cdot 3^3 \cdot 5^2 \cdot 7^1 \cdot 11^2 \cdot 13^1 \cdot 23^1$$
$$= 1367566200.$$

Here too the real bottleneck would have been determining the prime factorizations of 354200 and 84942 had they not been given. But if you are given or can easily determine the prime factorizations of a and b, then the approach of Theorem 2.25 for finding (a, b) and $[a, b]$ is much faster than using the Euclidean Algorithm followed by an application of Theorem 2.19.

(1A163) If 48 and x have a lowest common multiple of 2640 and a greatest common factor of 12, determine the minimum possible value of x.
Solution

$48 = 2^4 \bullet 3, 12 = 2^2 \bullet 3$ and $2640 = 2^4 \bullet 3 \bullet 5 \bullet 11$. The GCF(48, x) = 12, so x is a multiple of 12. Since LCM(48, x) = 2640, x must have at least one factor of 5 and one factor of 11. Therefore, the minimum possible value for x is $2^2 \bullet 3 \bullet 5 \bullet 11 = 660$.

■

(1T161) The roots of $x^2 + bx + c$ are integers r and s where $r < s < 0$. The greatest common factor of $|r|$, $|s|$, and c is 6 and the lowest common multiple of $|s|$ and b is 84. Find r and s.
Solution

If the greatest common factor of $|r|, |s|$, and c is 6, then r and s must be multiples of 6. Suppose $|s| = 6$, then b =28 or 84 since $84 = 2^2 \bullet 3 \bullet 7$. Since b = –(r + s), $|r| = 22$ or 76. However, neither of these are possible since the greatest common factor of $|r|, |s|$, and c would not be 6. If $|s| = 12$, then b = 28 or 42 or 84. This in turn means $|r| = 16$ or 30 or 72. The only value that works is 30 since the greatest common factor of 6 is required. Other possible values of $|s|$ like 42 and 84 are not possible since r < s < 0. Therefore, r = $\boxed{-30}$ and s = $\boxed{-12}$.

■

4.6.1 (a, b, c) and $[a, b, c]$ in terms of the canonical representations of a, b and c

Pg. 58, Exercise 11

If $a = \prod_{i=1}^{r} p_i^{a_i}$, $b = \prod_{i=1}^{r} p_i^{b_i}$, $c = \prod_{i=1}^{r} p_i^{c_i}$ with $a_i \geq 0, b_i \geq 0, c_i \geq 0$ for each i are the canonical representations of a, b and c (with the necessary place holder primes inserted) then

$$(a,b,c) = \prod_{i=1}^{r} p_i^{\min(a_i,b_i,c_i)} \quad \text{and} \quad [a,b,c] = \prod_{i=1}^{r} p_i^{\max(a_i,b_i,c_i)}.$$

This result could be extended in the same way to more than three integers.

Example

Find $[2,4,5,6,12]$.

Solution

First, we find the prime factorization of 2, 4, 5, 6 and 12 and input the necessary place holder primes.

$$\begin{aligned}
2 &= \quad 2^1 = 2^1 3^0 5^0 \\
4 &= \quad 2^2 = 2^2 3^0 5^0 \\
5 &= \quad 5^1 = 2^0 3^0 5^1 \\
6 &= 2^1 3^1 = 2^1 3^1 5^0 \\
12 &= 2^2 3^1 = 2^2 3^1 5^0
\end{aligned}$$

So,

$$\begin{aligned}
&[2,4,5,6,12] \\
&\quad = [2^1 3^0 5^0, 2^2 3^0 5^0, 2^0 3^0 5^1, 2^1 3^1 5^0, 2^2 3^1 5^0] \\
&\quad = 2^{\max\{1,2,0,1,2\}} \times 3^{\max\{0,0,0,1,1\}} \times 5^{\max\{0,0,1,0,0\}} \\
&\quad = 2^2 \times 3^1 \times 5^1 = 60.
\end{aligned}$$

■

(1A172) Compute

$$\frac{\operatorname{lcm}(20,18)}{\gcd(20,18)}.$$

Solution

$$\frac{lcm(20,18)}{gcd(20,18)} = \frac{lcm(2\cdot 2\cdot 5,\ 2\cdot 3\cdot 3)}{gcd(2\cdot 2\cdot 5,\ 2\cdot 3\cdot 3)} = \frac{2\cdot 2\cdot 3\cdot 3\cdot 5}{2} = 90.$$

■

(1A174) What is the sum of all positive integers n for which $\text{lcm}(10,18,n) = 630$ and $\gcd(10,18,n) = 2$?

<u>Solution</u>

In order to get the lcm to work, n must be of the form $2^a 3^b 5^c 7$ *where a is 0 or 1, b is 0, 1, or 2, and c is 0 or 1. In order for the gcd to work a must be 1. Therefore, there are six choices for n: 14, 42, 70, 126, 210, and 630. Their sum is* 1092.

■

4.7 Solving $ax + by = k$ in integers

Pg. 42, Extended Euclidean Algorithm: Solving $\boldsymbol{ax + by = (a, b)}$

Bezout's Theorem

There exist integers x_0 and y_0 such that $(a, b) = ax_0 + by_0$.

The basic idea for finding x_0 and y_0 is to take the steps of the Euclidean Algorithm for finding (a, b) and work backwards.

To illustrate how this is done, consider the steps of the Euclidean Algorithm for showing $(120,28) = 4$ we went through previously and notice how you can run the algorithm from the bottom up.

$$\begin{aligned} 4 &= 28 - 3 \cdot 8 \\ &= 28 - 3 \cdot (120 - 4 \cdot 28) \\ &= 120(-3) + 28(1 + 12) \\ &= 120(-3) + 28(13) \end{aligned}$$

This shows that

$$120x_0 + 28y_0 = 120(-3) + 28(13) = (120,28) = 4.$$

■

Now take the above steps of the Euclidean Algorithm for showing $(288,51) = 3$ and work backwards to find integers x_0 and y_0 such that $288x_0 + 51y_0 = (288,51) = 3$.

$$\begin{aligned} 3 &= 18 - 15 \cdot 1 \\ &= 18 - (33 - 18 \cdot 1) \cdot 1 \\ &= 33(-1) + 18(2) \\ &= 33(-1) + (51 - 33 \cdot 1)(2) \\ &= 51(2) + 33(-1 - 2) \\ &= 51(2) + 33(-3) \\ &= 51(2) + (288 - 51 \cdot 5)(-3) \end{aligned}$$

$$
\begin{aligned}
&= 288(-3) + 51(2 + 15) \\
&= 288(-3) + 51(17).
\end{aligned}
$$

This shows that

$$288x_0 + 51y_0 = 288(-3) + 51(17) = (288,51) = 3.$$

■

As another example, take the above steps of the Euclidean Algorithm for showing that $(223,163) = 1$ and work backwards to find integers x_0 and y_0 such that $223x_0 + 163y_0 = (223,163) = 1$.

$$
\begin{aligned}
1 &= 9 - 8 \cdot 1 \\
&= 9 - (17 - 9 \cdot 1) \\
&= 17(-1) + 9(2) \\
&= 17(-1) + (43 - 17 \cdot 2)(2) \\
&= 43(2) + 17(-1 - 4) \\
&= 43(2) + 17(-5) \\
&= 43(2) + (60 - 43 \cdot 1)(-5) \\
&= 60(-5) + 43(2 + 5) \\
&= 60(-5) + 43(7) \\
&= 60(-5) + (163 - 60 \cdot 2)(7) \\
&= 163(7) + 60(-5 - 14) \\
&= 163(7) + 60(-19) \\
&= 163(7) + (223 - 163 \cdot 1)(-19) \\
&= 223(-19) + 163(7 + 19) \\
&= 223(-19) + 163(26).
\end{aligned}
$$

This shows that

$$223x_0 + 163y_0 = 223(-19) + 163(26) = (223,163) = 1.$$

■

Example (MSHSML 2006-2007, Test 1A, Problem 4)

If d is the greatest common divisor of 399 and 959, then it is possible to find integers r and s so that $d = 399r + 959s$. Find d, r, and s.

Solution

By the Euclidean Algorithm

$$
\begin{aligned}
959 &= 2(399) + 161 \\
399 &= 2(161) + 77 \\
161 &= 2(77) + 7 \\
77 &= 11(7) + 0
\end{aligned}
$$

Hence $\gcd(959,399) = 7$. By working the Euclidean Algorithm backwards we find

$$
\begin{aligned}
7 &= 161 - 2(77) \\
&= 161 - 2\big(399 - 2(161)\big) \\
&= (5)(161) - 2(399) \\
&= (5)\big(959 - 2(399)\big) - 2(399) \\
&= 5(959) - 12(399)
\end{aligned}
$$

So $399(-12) + 959(5) = 7$ is a solution to $399r + 959s = \gcd(399,959)$. That is, the solution to this problem is $d = 7, r = -12, s = 5$.

■

Pg. 44, Definition (Linear Combination)
An expression of the form $ax + by$ is called a ***linear combination*** of a and b.

Pg. 44, Theorem
If $d = ax + by$ for some integers a, b, d, x and y, then $d|(a,b)$. That is, the only possible values for $ax + by$ in integers are multiples of (a,b).

Theorem
If a and b are positive integers and if $ax_0 + by_0 = (a,b)$, then $x_0 y_0 \le 0$.

Finding a solution of $\boldsymbol{ax + by = k \cdot}$ gcd($\boldsymbol{a,b}$)
In the previous section we learned how to reverse the steps of the Euclidean Algorithm when finding $\gcd(a,b)$.

Suppose $x = x_0$, $y = y_0$ is a solution to $ax + by = \gcd(a,b)$. Then by multiplying both sides of this equation by the constant k we get

$$a(kx_0) + b(ky_0) = k \cdot \gcd(a,b).$$

That is, if $(x,y) = (x_0, y_0)$ is a solution to $ax + by = \gcd(a,b)$, then $(x,y) = (kx_0, ky_0)$ is a solution to $ax + by = k \cdot \gcd(a,b)$.

■

Theorem Finding all solutions of $\boldsymbol{ax + by = k \cdot}$ gcd($\boldsymbol{a,b}$)

If $(x,y) = (x_0, y_0)$ is any particular solution to $ax + by = k \cdot \gcd(a,b)$ then

$$(x,y) = \left(x_0 + \left(\frac{b}{\gcd(a,b)}\right)n,\ \ y_0 - \left(\frac{a}{\gcd(a,b)}\right)n \right), \quad n = 0, \pm 1, \pm 2, \ldots$$

gives the set of all possible solutions to $ax + by = k \cdot \gcd(a,b)$.

■

Pay close attention to the form of the answer in the theorem above. There are two things about this formula that you need to be careful to notice. First is that the location of a and b in

$$\left(x_0 + \left(\frac{\color{red}{\boldsymbol{b}}}{\gcd(a,b)}\right)n,\ \ y_0 - \left(\frac{\color{red}{\boldsymbol{a}}}{\gcd(a,b)}\right)n\right)$$

might seem backwards, but this is correct. Secondly, notice that we ***add*** the extra term to x_0 but we ***subtract*** the extra term from y_0.

$$\left(x_0 \color{red}{+} \left(\frac{b}{\gcd(a,b)}\right)n,\ \ y_0 \color{red}{-} \left(\frac{a}{\gcd(a,b)}\right)n\right).$$

Example

Find an expression for x and y that shows *all possible* integer solutions x and y such that $288x + 51y = \gcd(288{,}51)$.

Solution

We found in a previous example that $\gcd(288{,}51) = 3$ and that $x_0 = -3$ and $y_0 = 17$ is a solution to $288x_0 + 51y_0 = \gcd(288{,}51) = 3$.

It follows from this previous example and the above theorem that

$$x = x_0 + \left(\frac{51}{\gcd(288{,}51)}\right)n$$

and

$$y = y_0 - \left(\frac{288}{\gcd(288{,}51)}\right)n$$

with $n = 0, \pm1, \pm2, \pm3, \ldots$ will be the set of all possible solutions to $288x + 51y = \gcd(288{,}51) = 3$.

Therefore,

$$x = -3 + \left(\frac{51}{3}\right)n = -3 + 17n$$

and

$$y = 17 - \left(\frac{288}{3}\right)n = 17 - 96n$$

with $n = 0, \pm1, \pm2, \ldots$ is the set of all possible solutions to $288x + 51y = \gcd(288{,}51) = 3$.

■

Exercise (MSHSML 2001-2002, Test 1A, Problem 4)

(a) Find an integer solution to $13x + 29y = 48$.

(b) Find an expression for all solutions to $13x + 29y = 48$.

(c) Find the three lattice points (points with integer coordinates) closest to the origin that satisfy $13x + 29y = 48$.

Solution

(a)

$$29 = 2(13) + 3$$
$$13 = 4(3) + 1$$
$$3 = 3(1) + 0$$

Therefore, $\gcd(29{,}13) = 1$. We can reverse the above steps to find a solution to $13x + 29y = \gcd(29{,}13) = 1$.

$$\begin{aligned} 1 &= 13 - 4(3) \\ &= 13 - 4\big(29 - 2(13)\big) \\ &= 13(9) - 29(4). \end{aligned}$$

That is, $x = 9$ and $y = -4$ is an integer solution to $13x + 29y = \gcd(29{,}13) = 1$. Therefore,

$$13(9 \cdot 48) - 29(4 \cdot 48) = 1 \cdot 48.$$

That is, $x = 9(48) = 432$ and $y = -4(48) = -192$ is an integer solution to $13x + 29y = 48$.

(b) Recall that if (x_0, y_0) is a solution to $13x + 29y = 48$ then

$$(x, y) = \left(x_0 + \left(\frac{b}{\gcd(a,b)}\right)n,\ \ y_0 - \left(\frac{a}{\gcd(a,b)}\right)n \right), \quad n = 0, \pm1, \pm2, \dots$$

is the set of all possible solutions to $13x + 29y = 48$. Therefore,

$$(x, y) = \left(432 + \left(\frac{29}{1}\right)n, -192 - \left(\frac{13}{1}\right)n\right) = (432 + 29n, -192 - 13n)$$

with $n = 0, \pm1, \pm2, \dots$ gives us the set of all possible solutions to $13x + 29y = 48$.

(c) Part (c) is asking for are lattice points (both coordinates are integers) that are close to the origin. That is, we want to find integer values of n (positive or negative) such that $(432 + 29n, -192 - 13n) \approx (0,0)$.

To get a sense of where to start let's see what non-integer value of n makes the x-coordinate exactly zero.

$$432 + 29n = 0 \Leftrightarrow n = -\frac{432}{29} \approx -14.9.$$

Plugging the nearest integer (namely, $n = -15$) into $(432 + 29n, -192 - 13n)$ we generate the lattice point $\big(432 + 29(-15), -192 - 13(-15)\big) = (-3,3)$.

What happens to x and y if we increase n by one unit?

$$\big(432 + 29(n+1), -192 - 13(n+1)\big) = (432 + 29n, -192 - 13n) + (29, -13)$$

Increasing n by one unit will increase x by 29 units and will increase y by -13 units. Similarly, decreasing n by one unit will decrease x by 29 units and will decrease y by -13 units.

That is, some of the lattice points closest to $(-3,3)$ are

$$\big(-3 + 29, 3 + (-13)\big) = (26, -10)$$
$$\big(-3 + 2(29), 3 + 2(-13)\big) = (55, -23)$$
$$\big(-3 - 29, 3 - (-13)\big) = (-32, 16)$$
$$\big(-3 - 2(29), 3 - 2(-13)\big) = (-61, 29)$$

From here we can identify by inspection the three lattice points closest to $(0,0)$. They are $(-3,3), (26,-10)$ and $(-32,16)$.

But to be precise and leave no room for error we can apply the distance formula to each and see which of these lattice points minimize the distance to the origin.

Lattice Point (x, y)	**Distance to the origin** $= \sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + y^2}$
$(55, -23)$	59.6
$(26, -10)$	27.9

$(-3,3)$	4.2
$(-32,16)$	35.8
$(-61,29)$	67.6

So this confirms that the three lattice points on the line $13x + 29y = 48$ that are closest to the origin are $(-3,3), (26,-10)$ and $(-32,16)$.

■

(3T001)

1. On a forty question examination, 7 points are given for each correct answer, 2 points are deducted for each wrong answer, and 0 points are given for questions not answered. How many questions were left unanswered by a student who got a score of 207?

Solution

Let r = # right, w = # wrong

$7r - 2w = 207$

Reduce everything mod 7

$5w \equiv 4 \mod 7$

w	5	12	---
r	31	33	---

But since $w + r \leq 40$,

$w = 5$, $r = 31$, and 4 are unanswered.

(If you don't know modular arithmetic, more trial and error is necessary.)

■

4.8 Blankenship's Algorithm

Blankenship's Algorithm for finding integers m and n such that

$$ma + nb = (a, b)$$

for the case $a > b > 0$ starts with the matrix

$$\begin{bmatrix} a & 1 & 0 \\ b & 0 & 1 \end{bmatrix}$$

and then proceeds as illustrated in the following three examples.

Example 1.

$$\begin{bmatrix} 42 & 1 & 0 \\ 15 & 0 & 1 \end{bmatrix}$$

15 goes into 42 two times. Multiply Row 2 by -2 and add the result to Row 1.

$$\begin{bmatrix} 12 & 1 & -2 \\ 15 & 0 & 1 \end{bmatrix}$$

12 goes into 15 one time. Multiply Row 1 by -1 and add the result to Row 2.

$$\begin{bmatrix} 12 & 1 & -2 \\ 3 & -1 & 3 \end{bmatrix}$$

3 goes into 12 four times. Multiply Row 2 by -4 and add the result to Row 1.

$$\begin{bmatrix} 0 & 5 & -14 \\ 3 & -1 & 3 \end{bmatrix}$$

So $\gcd(42,15) = 3$ and $3 = (-1)42 + (3)15$.

Example 2.

$$\begin{bmatrix} 35 & 1 & 0 \\ 15 & 0 & 1 \end{bmatrix}$$

15 goes into 35 two times. Multiply Row 2 by -2 and add the result to Row 1.

$$\begin{bmatrix} 5 & 1 & -2 \\ 15 & 0 & 1 \end{bmatrix}$$

5 goes into 15 three times. Multiply Row 1 by -3 and add the result to Row 2.

$$\begin{bmatrix} 5 & 1 & -2 \\ 0 & -3 & 7 \end{bmatrix}$$

So $\gcd(35,15) = 5$ and $5 = (1)35 + (-2)15$

Example 3.

$$\begin{bmatrix} 1876 & 1 & 0 \\ 365 & 0 & 1 \end{bmatrix}$$

365 goes into 1876 five times. Multiply Row 2 by -5 and add the result to Row 1.

$$\begin{bmatrix} 51 & 1 & -5 \\ 365 & 0 & 1 \end{bmatrix}$$

51 goes into 365 seven times. Multiply Row 1 by -7 and add the result to Row 2.

$$\begin{bmatrix} 51 & 1 & -5 \\ 8 & -7 & 36 \end{bmatrix}$$

8 goes into 51 six times. Multiply Row 2 by -6 and add the result to Row 1.

$$\begin{bmatrix} 3 & 43 & -221 \\ 8 & -7 & 36 \end{bmatrix}$$

3 goes into 8 two times. Multiply Row 1 by -2 and add the result to Row 2.

$$\begin{bmatrix} 3 & 43 & -221 \\ 2 & -93 & 478 \end{bmatrix}$$

2 goes into 3 one time. Multiply Row 2 by -1 and add the result to Row 1.

$$\begin{bmatrix} 1 & 136 & -699 \\ 2 & -93 & 478 \end{bmatrix}$$

1 goes into 2 two times. Multiply Row 1 by -2 and add the result to Row 2.

$$\begin{bmatrix} 1 & 136 & -699 \\ 0 & -365 & 1876 \end{bmatrix}$$

So $\gcd(1876,365) = 1$ and $1 = (136) \cdot (1876) + (-699) \cdot (365)$

Reference: W.A. Blankinship, A New Version of the Euclidean Algorithm, *The American Mathematical Monthly*, Vol. 70, No. 7, August-September 1965, pages 742-745.

4.9 Solving $ax + by + cz = k$ in integers

Theorem (Extended Euclidean Algorithm for More Than Two Integers)
There exist integers x_0, y_0 and z_0 such that $(a,b,c) = ax_0 + by_0 + cz_0$. You can use the results of the Extended Euclidean Algorithm for solving

$$aw_0 + bw_1 = (a,b)$$

and

$$(a,b)w_2 + cw_3 = \big((a,b),c\big)$$

to see that

$$\begin{aligned}(a,b,c) &= \big((a,b),c\big)\\ &= (a,b)w_2 + cw_3\\ &= (aw_0 + bw_1)\cdot w_2 + cw_3\\ &= a(w_0\cdot w_2) + b(w_1\cdot w_2) + c(w_3)\\ &= ax_0 + by_0 + cz_0.\end{aligned}$$

Example
Find integers x_0, y_0, z_0 such that

$$(288,51,8) = 288x_0 + 51y_0 + 8z_0.$$

<u>Solution</u>
We previously showed that $(288,51) = 3$ and that $288(-3) + 51(17) = (288,51) = 3$. Also, by Theorem 2.20 we know that $(288,51,8) = \big((288,51),8\big) = (3,8)$.
Even though it is obvious that $(3,8) = (8,3) = 1$ let's go through the Euclidean Algorithm to verify this and go through the Extended Euclidean Algorithm to find integers x_0 and y_0 such that $8x_0 + 3y_0 = (8,3)$.

$$\begin{aligned}8 &= 3\cdot 2 + 2\\ 3 &= 2\cdot 1 + 1\\ 2 &= 1\cdot 2 + 0.\end{aligned}$$

Therefore,

$$(8,3) = 1$$

and

$$(288,51,8) = \big((288,51),8\big) = (3,8) = (8,3) = 1.$$

Now we can work backwards to find integers x_0 and y_0 such that $8x_0 + 3y_0 = (8,3) = 1$.

$$\begin{aligned}1 &= 3 - 2\cdot 1\\ &= 3 - (8 - 3\cdot 2)\cdot 1\\ &= 8(-1) + 3(1+2)\\ &= 8(-1) + 3(3).\end{aligned}$$

This shows that
$$8x_0 + 3y_0 = 8(-1) + 3(3) = (8,3) = 1.$$

Now we can immediately combine the two results
$$288(-3) + 51(17) = (288,51) = 3$$
and
$$8(-1) + 3(3) = (8,3) = 1$$
to find integers x_0, y_0, z_0 such that
$$1 = (288,51,8) = 288x_0 + 51y_0 + 8z_0.$$
We have
$$\begin{aligned} 1 &= 8(-1) + 3(3) \\ &= 8(-1) + \big(288(-3) + 51(17)\big) \cdot 3 \\ &= 288(-3 \cdot 3) + 51(17 \cdot 3) + 8(-1) \\ &= 288(-9) + 51(51) + 8(-1). \end{aligned}$$
This shows that
$$288x_0 + 51y_0 + 8z_0 = 288(-9) + 51(51) + 8(-1) = (288,51,8) = 1.$$
■

Example
Find integers x_0, y_0, z_0 such that
$$(36,30,15) = 36x_0 + 30y_0 + 15z_0.$$
Solution
Applying the Euclidean Algorithm twice we find:
$$\begin{aligned} 36 &= 30 \cdot 1 + 6 \\ 30 &= 6 \cdot 5 + 0. \end{aligned}$$
Therefore,
$$(36,30) = 6$$
and
$$\begin{aligned} 15 &= 6 \cdot 2 + 3 \\ 6 &= 3 \cdot 2 + 0. \end{aligned}$$
Therefore,
$$(15,6) = 3.$$
Hence,
$$(36,30,15) = \big((36,30), 15\big) = (6,15) = (15,6) = 3.$$
Applying the Extended Euclidean Algorithm twice we find:
$$\begin{aligned} 6 &= 36(1) + 30(-1) \\ 3 &= 15(1) + 6(-2). \end{aligned}$$
Combining these we see that
$$3 = 15(1) + 6(-2) = 15(1) + \big(36(1) + 30(-1)\big)(-2) = 36(-2) + 30(2) + 15(1).$$
This shows that
$$36x_0 + 30y_0 + 15z_0 = 36(-2) + 30(2) + 15(1) = (36,30,15) = 3.$$

■

4.10 Nonnegative Solutions to $ax + by = c$

Uspensky, pg. 59

7. Nonnegative Solutions of Linear Indeterminate Equations. The problem of finding nonnegative solutions of an equation

$$ax + by = c,$$

in which a, b, c are positive integers, can be attacked in a different way.

Example. Let us solve the equation

$$158x + 57y = 20{,}000$$

in nonnegative integers by the method of this section. We have $q = 2$, $R = 1{,}988$. All the solutions of the equation

$$158r + 57s = 1{,}988$$

are given by

$$r = -43{,}736 + 57t, \qquad s = 121{,}268 - 158t.$$

Corresponding to $t = 768$ we have $r = 40$, $s = -76$; consequently we have the second case and the equation

$$158r + 57s = 1{,}988 + 9{,}006$$

is satisfied in the desired manner by $r = 40$, $s = 158 - 76 = 82$. Correspondingly

$$x = 57\xi + 40, \qquad y = 158\eta + 82$$

and

$$\begin{aligned} \xi &= 0, 1 \\ \eta &= 1, 0. \end{aligned}$$

That is, there are two solutions in nonnegative integers

$$\begin{aligned} x &= 40, \qquad y = 240 \\ x &= 97, \qquad y = 82 \end{aligned}$$

as we found before.

1. Solve in positive integers $101x + 753y = 100{,}000$.

Ans. $x = 170$, $y = 110$; and $x = 923$, $y = 9$.

11. Two church bells begin ringing at the same time. The strokes of one follow regularly at intervals of $1\frac{1}{3}$ sec., while the intervals between two strokes of the second are $1\frac{3}{4}$ sec. How many strokes are heard during 15 min. if 2 strokes following each other in an interval of $\frac{1}{2}$ sec. or less are perceived as one sound? *Ans.* 772.

(5T996)

6. You wish to make \$5.55 postage from a sufficiently large supply of 33 cent and 19 cent stamps. Show how it can be done.

Number of 33 cent stamps = ____________, Number of 19 cent stamps = ____________

Solution

Need to solve

$33x + 19y = 555$

which can be done by trial and error with a calculator. (x and y must be positive (or zero))

3	33¢	Stamps	= .99
24	19¢	Stamps	= 4.56
			5.55

This is the only way it can be done.

■

AMC 1967 Problem #24

The number of solution-pairs in positive integers of the equation $3x + 5y = 501$ is:

(A) 33	(B) 34	(C) 35	(D) 100	(E) none of these.

<u>Solution</u>

■

#25

Find all (x, y) pairs which are solutions to the Diophantine equation $21x + 41y = 1867$ and where both x and y are positive.

<u>Solution</u>

$$21x + 41y = 1867$$
$$-y \equiv -2 \pmod{21}$$
$$y \equiv 2 \pmod{21}$$
$$y = 2 + 21k$$
$$21x + 41(2 + 21k) = 1867$$
$$21(x + 41k) = 1785$$
$$x = 85 - 41k$$
$$(x, y) = (85 - 41k, 2 + 21k)$$

x and y both positive

$$k=0:(85,2)$$
$$k=1:(44,23)$$
$$k=2:(3,44).$$

■

4.11 Lattice Point Problems

(2D143)

A *lattice point* is a point on the *xy*-plane whose coordinates are both integers.
How many lattice points lie on the line $4x - 2y = 10$, are within the first quadrant, and have a y-coordinate of at most 2014?

Solution

Figure 3 shows the graph of the line, which has x-intercept 2.5. As the line enters the first quadrant, it passes through the lattice points (3, 1), (4, 3), (5, 5), and so on. The y-coordinates are simply the set of odd numbers! So: we simply count the odd numbers ≤ 2014. This is the same as the # of even numbers ≤ 2014, which is 2014 ÷ 2 = 1007.

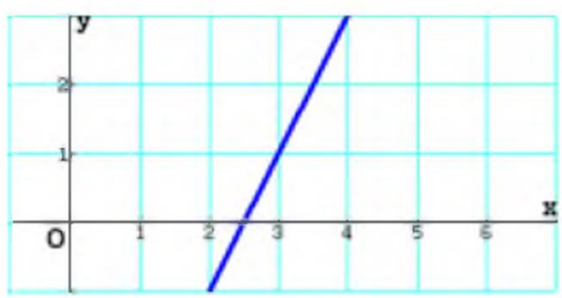


Figure 3

■

(TI119) What lattice point, lying on the line $142x + 103y = 259$, is closest to the origin?
Solution

This is a ***linear Diophantine equation*** *(a linear equation that allows only integer solutions). Use modular arithmetic to simplify the coefficients:* $(142x+103y=259)\bmod 103 \Rightarrow 39x+0\equiv 53 \bmod 103$, *so* $39x=53+103k$ *for some integer k.* $(39x=53+103k)\bmod 39 \Rightarrow 0\equiv 14+25k \bmod 39$, *so* $14+25k=39j$ *for some integer j.* $(14+25k=39j)\bmod 25 \Rightarrow 14+0\equiv 14j \bmod 25 \Rightarrow j\equiv 1 \bmod 25$. *To get (x, y) closest to the origin, choose the smallest possible j > 0:* $j=1 \Rightarrow 14+25k=39(1) \Rightarrow k=1 \Rightarrow 39x=53+103(1) \Rightarrow x=4 \Rightarrow 142(4)+103y=259 \Rightarrow y=-3$. *So (x, y) =* $\boxed{(4,-3)}$.

■

(5D053)

3. A lattice point is a point having integers for coordinates. The point (4,5) is, for instance, a lattice point that lies on the line $7x-11y+27=0$. What is the total number of lattice points on this line that lie in the square

$$S=\{(x,y):0\le x\le 100, 0\le y\le 100\}$$

Solution

$y=\frac{7}{11}x+\frac{27}{11}$ has a slope of $\frac{7}{11}$

The equation can be written $y-5=\frac{7}{11}(x-4)$

from which we see that a parametric form is $x=4+11t$, $y=5+7t$.

$t=0, 1, \ldots, 8$ all give lattice points in S.

■

(TA954)

4. A lattice point is a point in the plane that has integer coordinates. How many lattice points in the first quadrant lie on $3x+8y=110$?

Solution

Those who know modular arithmetic can reduce the equation mod 3 to find that y=1 works, giving x = 34. Others will have to resort to some trial and error to find a first lattice point. Then use the slope of the line = $-\frac{3}{8}$ to find the others.

x	34	26	18	10	2
y	1	4	7	10	13

There are five such points

■

Mu Alpha Theta, Florida State Convention, 1992-1993, Number Theory Topic Test, Number 3 (adapted)

The point $(1,7)$ is a lattice point of the rotated hyperbola $x^2 - xy + x + 2y - 9 = 0$. Find all lattice points of this hyperbola.

Solution

Solving for y in this equation we find

$$x^2 - xy + x + 2y - 9 = 0$$
$$2y - xy = -x^2 - x + 9$$
$$y(2-x) = -x^2 - x + 9$$
$$y = \frac{x^2 + x - 9}{x - 2} = (x+3) - \frac{3}{x-2}$$

So, y will be an integer whenever $x - 2$ divides 3. That is, for $x - 2 \in \{-3, -1, 1, 3\}$ or $x \in \{-1, 1, 3, 5\}$.

Therefore, the set of all lattice points (x, y) is $\{(-1,3), (1,7), (3,3), (5,7)\}$.

■

4.12 Frobenius Numbers

5.3 A theorem of Frobenius

If $(a, b) = d > 1$ then the linear form $ax + by$ skips all non-multiples of d. If $(a, b) = 1$, there is always an integer solution to $ax + by = n$ regardless of the integer n. We will prove the following theorem of Frobenius that tells un when we will find nonnegative solutions to $ax + by = n$.

There is a general question here, known as the *money-changing problem*, the *coin problem* or under a more technical term as the *Frobenius problem*. What is the smallest number $g(a, b)$ such that (2.14) has non-negative solutions for all $n \geq g(a, b)$. In our previous example we have $g(3, 5) = 8$. We have the following general result due to J.J.Sylvester [26].

Theorem 2.4.1. **[Sylvester, 1884]** *The equation (2.14) has non-negative solutions for all $n \geq (a-1)(b-1)$ and no such solution if $n = ab - a - b$. (Hence, $g(a, b) = (a-1)(b-1)$.)*

(see file: “WSU notes and homework on problem of Frobenius”)

Theorem. Problem of Frobenius

Let a and b be relatively prime positive integers. Then the equation

$$ax + by = n$$

(i) is not solvable in nonnegative integers x and y for $n = ab - a - b$

(ii) is solvable in nonnegative integers x and y for all $n > ab - a - b$.

If $\gcd(a, b) = d > 1$ then the equation

$$ax + by = n$$

(iii) is not solvable in nonnegative integers x and y for $n = d\Big((a/d)(b/d) - (a/d) - (b/d)\Big)$

(iv) is solvable in nonnegative integers x and y for n provided $d|n$ and

$$n > d\left((a/d)(b/d) - (a/d) - (b/d)\right)$$

Sylvester, J. J. "Question 7382." *Mathematical Questions from the Educational Times* **41**, 21, 1884.

1) What is the greatest integer not representable in the form $5x + 9y$ for nonnegative x and y?

<u>Solution</u>

We proved in class that if a and b are relatively prime positive integers then the equation

$$ax + by \; = \; n$$

is not solvable in nonnegative integers x and y for $n = ab - a - b$ but is solvable for all $n > ab - a - b$.

So by direct application of this theorem, $n = (5 \cdot 9) - 5 - 9 = 31$ is the greatest integer not representable in the form $5x + 9y$ with nonnegative x and y.

■

Frobenius number

Imagine a country prints stamps in only two denominations. You can stick as many stamps on an envelope as you want, but the question is what is the largest total denomination it is impossible to make?

For example, imagine the two denominations are 5 (pence) and 7 (pence). It turns out the largest number you cannot make in this case is 23 (pence). There is no combination of these stamps which adds up to 23. Sure, you can make some denominations less than this *e.g.* $12 = 5 + 7$, but you can make ANY number larger than this e.g. $24 = 2 \cdot 5 + 2 \cdot 7$.

■

Frobenius with Three Numbers

We can find the L;$\{m, n, p\}$ if any two of the three numbers have greatest common factor k

If the two numbers out of the three have a common factor, then let the 3 numbers be km, kn and p, where $\gcd(m, n) = 1$ Then $akm + bkn + cp = k(am + bn) + cp$. Note that for $am + bn$, all numbers above $mn - m - n$ are expressible.

Now, we let $k(am + bn) + cp = k[(mn - m - n + 1) + d] + cp = k(m - 1)(n - 1) + dk + cp$($d$ is non-negative).

For $dk + cp$, all numbers above $kp - k - p$ are expressible. Hence, all numbers above or equal to $k(m - 1)(n - 1) + kp - k - p + 1$ are expressible.

From here, we can say that the $Largest$; km, kn, p is $k(m - 1)(n - 1) + kp - k - p$.

Mathematics Teacher, Calendar Problems, Number 6, March 1992

6 "Chicken Chunkettes" come in boxes of 6, 9, and 20. What is the largest number of chunkettes you *can't* buy?

<u>Solution</u>

let the 3 numbers be km, kn and p, where $\gcd(m, n) = 1$

all numbers above or equal to $k(m - 1)(n - 1) + kp - k - p + 1$ are expressible.

$$3(2), 3(3), 20 \Rightarrow k = 3, m = 2, n = 3, p = 20$$

$$3(2 - 1)(3 - 1) + 3(20) - 3 - 20 + 1 = 6 + 60 - 22 = 44.$$

So the largest number you *cannot* buy is 43.

■

2008 Mu Alpha Theta National Convention **Open Number Theory**

17. What is the largest integer that **cannot** be written in the form $9x+9y+16z$ where x and y are multiples of 9 and z is a multiple of 16?

A. 25 B. 110 C. 119 D. 145 E. NOTA

Solution

17. ***C***. This question is asking for the largest n that cannot be written as $n = 9x+9y+16z$ for integer x, y, z, but that's equivalent to $n = 9(x+y)+16z$ or $n = 9w+16z$, since $(x+y)$ will always be an integer and can be any integer. From here, it's just a coin problem of order two, so the largest $n = 9*16-16-9 = 119$.

■

Related Results

- Let a and b be relatively prime positive integers. Then there are

$$(a-1)(b-1)/2$$

positive integers n which cannot be represented in the form

$$ax + by = n$$

for nonnegative integers x and y.

(This result is also due to Sylvester.)

e.g.

Consider the form $3x + 5y = n$. We know from (i) that $n = 3(5) - 3 - 5 = 7$ is the largest integer that cannot be represented in this form with nonnegative integers x and y. The other nonnegative integers not representable in this form are $n = 1{,}2$ and 4.

This makes for a total of 4 positive integers not representable in this form.

We note that

$$(a-1)(b-1)/2 = (3-1)(5-1)/2 = (2 \cdot 4)/2 = 4$$

as it should be.

2) How many positive integers are not representable in the form $5x + 9y$ for nonnegative x and y?

Solution

We stated (but did not prove) in class that if a and b are relatively prime positive integers then there are

$$(a-1)(b-1)/2$$

positive integers n which cannot be represented in the form

$$ax + by = n$$

for nonnegative integers x and y.

So by direct application of this theorem, there are $(5-1)(9-1)/2 = 32/2 = 16$ positive integers which cannot be represented in the form $5x + 9y$ with nonnegative x and y.

■

3) In the game of table football you get 3 points for a field goal and 6 points for a touchdown. There are no other ways to score points (e.g. safeties or extra points). What is the largest score which is a multiple of $\gcd(3,6)$ that cannot be achieved in table football? How many scores which are multiples of $\gcd(3,6)$ are not achieveable? Careful. Remember that $\gcd(3,6) \neq 1$.

Solution

We proved in class that if a and b are positive integers with $\gcd(a, b) = d$ then the equation

$$ax + by = n$$

is not solvable in nonnegative integers x and y for $n = d\left((a/d)(b/d) - (a/d) - (b/d)\right)$ but is solvable for all n if $d|n$ and $n > d\left((a/d)(b/d) - (a/d) - (b/d)\right)$.

We see that $\gcd(3,6) = 3$ and hence by direct application of this theorem the greatest multiple of 3 which is not representable in the form

$$3x + 6y$$

is $n = 3\left((3/3)(6/3) - (3/3) - (6/3)\right) = 3(2 - 1 - 2) = -3$.

This means that all positive multiples of 3 are representable in the form $3x + 6y$ and hence the number of multiples of $\gcd(3,6) = 3$ which are not representable in the form $3x + 6y$ is 0.

■

4) In the online article, "What is a combinatorial game?" it is claimed that 116 is the largest number that cannot be expressed in the form $10x + 14y$ with nonnegative integers x and y. Prove that this is wrong by actually finding nonnegative x and y such that $10x + 14y = 116$ and then find the correct answer. That is, what is the largest multiple of the $\gcd(10,14)$ which cannot be expressed in the form $10x + 14y$?

   Online Article:

   http://www.ams.org/featurecolumn/archive/games2.html

(5A124) The chicken nuggets at DonMickey's restaurants come in packages of $6, 9$, or 20. What is the <u>largest</u> total number of nuggets that <u>cannot</u> be purchased using some combination of these packages?
[Adapted from *Mathematics Teacher*, March 17, 2006.]
<u>Solution</u>
Any multiple of 3 (greater than 3) can be created by packages of 6 and 9 nuggets.
Since 36 = 9 + 9 + 9 + 9, 38 = 20 + 9 + 9, and 40 = 20 + 20, any even number ≥ 36 can be derived by adding groups of 6 to each of these quantities. Also, adding 9 to those three quantities gives us the basis for all odds ≥ 45. Consider the next greatest odd number, 43: it is not a multiple of 3; 43 – 20 = 23 is not a multiple of 3, and 43 – 20 – 20 = 3 is not purchasable. 43 can't be done!

■

Mu Alpha Theta National Convention 2001, Number Theory Test, Alpha Division, Problem # 6
Find the largest integer d for which there are no nonnegative integer solutions (a, b, c) which satisfy the equation $5a + 7b + 11c = d$.
Solution

6. There is a theorem that I have heard called the "Chicken Mcnugget Theorem" which gives a solution for a diophantine equation with only two relatively prime variables (a and b) instead of three. The formula is fairly easy to derive and shows that d = ab – a – b. I leave it as an exercise to the solvers to derive the formula for more variables.

■

4.13 Extra GCD and LCM Problems

Mathematics Teacher, Calendar Problem 6, February 2007
A ray revolves clockwise in jumps of 100-degree increments. The ray first regains its original position in how many jumps?
Solution
After n jumps the ray will have revolved $100n$ degrees. We are looking for the smallest n such that $100n$ is a multiple of 360. Visually, we are looking for the first number that is both of the sets $\{100, 2(100), 3(100), 4(100), \ldots\}$ and $\{360, 2(360), 3(360), 4(360), \ldots\}$. But this is exactly what is meant by the lcm (least common multiple) of 100 and 360.

$$\text{lcm}(100{,}360) = \text{lcm}(2^2 5^2, 2^3 3^2 5^1) = 2^3 3^2 5^2 = 1800 = 100(18).$$

Therefore, 18 jumps of the ray are required to bring the ray back to its original position.

■

(1T923)
3. What is the smallest integer that will divide 183,456 but not 39,312?
Solution

$183{,}456 = 2^5 \cdot 3^2 \cdot 7^2 \cdot 13 = A$

$39{,}312 = 2^4 \cdot 3^3 \cdot 7 \cdot 13 = B$

2^5 will divide A, but not B

■

(1T941)
Let a, b, and c be integers. To add in rational form $\frac{a}{42} + \frac{b}{117} + \frac{c}{728}$, we make use of the lowest common denominator. What is it?

Solution

$$\begin{aligned}\mathrm{lcd}(42{,}117{,}728) &= \mathrm{lcm}(42{,}117{,}728)\\ &= \mathrm{lcm}(2^1 3^1 7^1, 3^2 13^1, 2^3 7^1 13^1)\\ &= 2^{\max(1,0,3)} 3^{\max(1,2,0)} 7^{\max(1,0,1)} 13^{\max(0,1,1)}\\ &= 2^3 3^2 7^1 13^1 = 6552.\end{aligned}$$

■

(1T131) Find the sum of <u>all</u> positive integers n for which $\mathrm{LCM}(2, n) = \mathrm{GCD}(n, 210)$.
Solution

For LCM(2, n) = GCD(n, 210), both quantities must be equal to n. This means n is both a multiple of 2 and a factor of 210. Examining the prime factorization of 210 ($2\cdot3\cdot5\cdot7$), the acceptable values of n must be of the form $2\cdot3^a\cdot5^b\cdot7^c$, where a, b, and c are all either 0 or 1. This yields eight values: 2, 6, 10, 14, 30, 42, 70, and 210. Their sum is $\boxed{384}$.

■

(1A153) Let a and b be positive integers. If the greatest common factor of a and b is 10 and the least common multiple of a and b is 60, determine the minimum possible value of $a + b$.
Solution

Because a and b have a greatest common factor of 10, a and b must both be divisible by 10. This means the units digit of both a and b is 0. Since the LCM(a,b) = 60, a and b can each not be any larger than 60. Thus, the only values a or b can take are 10, 20, 30, or 60. Our possible ordered pairs for (a,b) are (10,60), (60,10), (20,30) and (30,20), so a + b takes a minimum value of 50.

■

(1A144) Let a, b, and c be three positive integers. The greatest common divisor of a and b is 2; the greatest common divisor of b and c is 6, and the least common multiple of a and c is 72. Determine the maximum possible value of $a + c$.
Solution

Because b is divisible by 3, a cannot be (otherwise their GCD would include a factor of 3). Since LCM(a, c) = 72, a can contain a factor of 8, but not a factor of 16. Thus, a can only be 2, 4, or 8. If a = 2 or 4, then c must be 72; if a = 8, then c could be 72, 36, or 18 ***(but not 9 … why?).*** *Our possible ordered pairs for (a, c) are (2, 72), (4, 72), (8, 72), (8, 36), and (8, 18), so a + c takes a maximum value of 80.*

■

(TT142) Three positive integers have a least common multiple of 360 and a greatest common divisor of 2. What are the smallest and largest possible values for the <u>product</u> of the three integers?

Solution

Observe that $360 = 2^3 \cdot 3^2 \cdot 5^1$. At least one of the integers must have 8 as a factor, while the other two will each have only a single factor of 2. Also, at least one of the integers must have factors of 9 and 5. This results in a product of $\boxed{1440}$, achieved for the integers 2, 2, and 360.

To generate the largest possible product, give two integers a factor of 8, with the third integer having only a single factor of 2. Similarly, at least one integer must not be divisible by 3, with the other two divisible by 9, and at least one integer is not divisible by 5, but the other two both can be divisible by 5. This results in the product $\boxed{259200}$, achieved for 2, 360, 360.

■

(1A973)

Three good friends dine in the same restaurant. All are eating three today. However, they do not eat there every day. The first eats there every twelfth day, the second every fourteenth day, and the third every twenty-first day. How many days from today will they next all meet in the restaurant?

Solution

The least common multiple of $12, 14$, and 21 is 84.

■

(1A894)

4. The positive integers m and n each factor into a product of three primes, They have a least common multiple of 441 and a greatest common divisor of 21. Find m + n.

Solution

$$\begin{aligned}
\text{lcm} &= 3^2 \cdot 7^2 \\
\text{gcd} &= 3 \cdot 7 \\
m &= 3 \cdot 7 \cdot 3 = 63 \\
n &= 3 \cdot 7 \cdot 7 = \underline{147} \\
& \qquad\qquad\quad 210
\end{aligned}$$

■

Source: *Number Theory Through Exercises*, Sedrakyan and Sedrakyan
Problem 2.17, page 21

Find $\gcd\left(\underbrace{1\cdots1}_{100\text{ digits}},\ \underbrace{1\cdots1}_{120\text{ digits}}\right)$.

Solution

We begin by stating and proving the following lemma which we will need to apply repeatedly.

Lemma

$$\gcd\left(\underset{a\text{ digits}}{1\cdots1},\ \underset{a+b\text{ digits}}{1\cdots1}\right)=\gcd\left(\underset{a\text{ digits}}{1\cdots1},\underset{b\text{ digits}}{1\cdots1}\right).$$

<u>Proof of Lemma</u>

We will use the following two properties of gcd's in this lemma:

(i) $\gcd(a,b)=\gcd(a,b-a)$

(ii) If $\gcd(a,c)=1$, then $\gcd(a,b\cdot c)=\gcd(a,b)$.

$$\gcd\left(\underset{a\text{ digits}}{1\cdots1},\ \underset{a+b\text{ digits}}{1\cdots1}\right)=\gcd\left(\underset{a\text{ digits}}{1\cdots1},\ \underset{a+b\text{ digits}}{1\cdots1}-\underset{a\text{ digits}}{1\cdots1}\right)\quad\text{from property (i) above}$$

$$=\gcd\left(1\underset{a}{\cdots}1,\ 1\underset{b}{\cdots}10\underset{a}{\cdots}0\right)$$

$$=\gcd\left(1\underset{a}{\cdots}1,\ \left(1\underset{b}{\cdots}1\right)\cdot 2^a5^a\right)$$

$$=\gcd\left(1\underset{a}{\cdots}1,\ 1\underset{b}{\cdots}1\right)\quad\text{from property (ii) above.}$$

■

Now we can apply this lemma repeatedly to establish the required result.

$$\gcd\left(1\underset{100}{\cdots}1,\ 1\underset{120}{\cdots}1\right)=\gcd\left(1\underset{100}{\cdots}1,\ 1\underset{20}{\cdots}1\right)$$

$$=\gcd\left(1\underset{80}{\cdots}1,\ 1\underset{20}{\cdots}1\right)$$

$$=\gcd\left(1\underset{60}{\cdots}1,\ 1\underset{20}{\cdots}1\right)$$

$$\vdots$$

$$=\gcd\left(1\underset{20}{\cdots}1,\ 1\underset{20}{\cdots}1\right)$$

$$=1\underset{20}{\cdots}1.$$

■

Source: *Number Theory Through Exercises*, Sedrakyan and Sedrakyan
Problem 2.15, page 21

Find $\gcd\left(2^{100}-1, 2^{120}-1\right)$.

<u>Solution</u>

We begin by stating and proving the following lemma which we will need to apply repeatedly.

Lemma

$$\gcd\left(2^a-1, 2^{a+b}-1\right) = \gcd\left(2^a-1, 2^b-1\right).$$

<u>Proof of Lemma</u>

We will use the following two properties of gcd's in this lemma:

(i) $\gcd(a,b) = \gcd(a, b-a)$

(ii) If $\gcd(a,c) = 1$, then $\gcd(a, b\cdot c) = \gcd(a,b)$.

$$\begin{aligned}
\gcd\left(2^a-1, 2^{a+b}-1\right) &= \gcd\left(2^a-1, (2^{a+b}-1)-(2^a-1)\right) \quad \text{from property (i) above} \\
&= \gcd\left(2^a-1, 2^{a+b}-2^a\right) \\
&= \gcd\left(2^a-1, 2^a(2^b-1)\right) \\
&= \gcd\left(2^a-1, 2^b-1\right) \quad \text{from property (ii) above.}
\end{aligned}$$

■

Now we can apply this lemma repeatedly to establish the required result.

$$\begin{aligned}
\gcd\left(2^{100}-1, 2^{120}-1\right) &= \gcd\left(2^{100}-1, 2^{20}-1\right) \\
&= \gcd\left(2^{80}-1, 2^{20}-1\right) \\
&= \gcd\left(2^{80}-1, 2^{20}-1\right) \\
&\vdots
\end{aligned}$$

$$= \gcd(2^{20} - 1, 2^{20} - 1)$$
$$= 2^{20} - 1.$$

■

https://testbook.com/objective-questions/mcq-on-number-system--5eea6a1039140f30f369e843

Four bells ring simultaneously at starting and an interval of 6 sec, 12 sec, 15 sec and 20 sec respectively. How many times they ring together in 2 hours?

Solution

LCM of (6, 12, 15, 20) = 60

All 4 bells ring together again after every 60 seconds

Now,

In 2 Hours, they ring together = [(2 × 60 × 60)/60] times + 1 (at the starting) = 121 times

∴ In 2 hours they ring together for 121 times

⚠ Mistake Points

In these type of question we assume that we have started counting the time after first ringing. Due to this when we calculate the LCM it gives us the ringing at 2nd time not the first time. So, we needed to add 1.

■