Poisson Randomization (Multinomial Allocation)
July 21, 2002

Theorem 1. Poisson Randomization Theorem

Define $\mathbb{S}^n$ to be the product space $\{0,1,\dots\} \times \cdots \times \{0,1,\dots\}$ and let $\mathbb{S}_t^n$ be the set of all vectors $(s_1, \dots, s_n)$ in $\mathbb{S}^n$ such that $s_1 + \dots + s_n = t$.

Suppose that $(X_{1,t}, \dots, X_{n,t})$ is a multinomial random vector. That is, for all $(s_1, \dots, s_n) \in \mathbb{S}_t^n$,

$$P(X_{1,t} = s_1, \dots, X_{n,t} = s_n) = \frac{t!}{s_1!\, s_2! \cdots s_n!} (p_1)^{s_1} (p_2)^{s_2} \cdots (p_n)^{s_n}$$

where $p_1 + \cdots + p_n = 1$ and $p_j \geq 0$ for $j = 1,\dots,n$.

Suppose that $(Y_1, \dots, Y_n)$ is a vector of independent Poisson random variables and that Y_j has parameter λp_j, $j = 1,\dots,n$. That is, suppose that for all $(s_1, \dots, s_n) \in \mathbb{S}^n$,

$$\begin{aligned} P(Y_1 = s_1, \dots, Y_n = s_n) &= \prod_{i=1}^n P(Y_i = s_i) \\ &= \prod_{i=1}^n \frac{e^{-\lambda p_i} (\lambda p_i)^{s_i}}{s_i!} \\ &= \frac{e^{-\lambda} \lambda^{s_1 + \cdots + s_n}}{t!} \frac{t!}{s_1! \cdots s_n!} (p_1)^{s_1} (p_2)^{s_2} \cdots (p_n)^{s_n} \end{aligned}$$

Let $\mathcal{A} \subseteq \mathbb{S}^n$ and define $\mathcal{A}_t = \mathcal{A} \cap \mathbb{S}_t^n$. Then for $t \geq 0$,

$$P\left((X_{1,t}, \dots, X_{n,t}) \in \mathcal{A}_t\right) = \frac{\mathrm{d}^t}{\mathrm{d}\lambda^t} \left(e^{\lambda} P\left((Y_1, Y_2, \dots, Y_n) \in \mathcal{A}\right) \right) \Big|_{\lambda = 0}$$

and

$$\left(g_t\left(X_{1,t}, \dots, X_{n,t}\right) \right) = \frac{\mathrm{d}^t}{\mathrm{d}\lambda^t} \left(e^{\lambda} \left(g(Y_1, Y_2, \dots, Y_n) \right) \right) \Big|_{\lambda = 0}$$

$Note$: need to fill in definition of g_t and g]

Theorem 2. Factorial Moments via Poisson Randomization

Define the random vectors $(X_{1,t}, \dots, X_{n,t})$ and $(Y_1, \dots, Y_n)$ as in Theorem 1.

Define random variables $W_x = g(X_{1,t}, \dots, X_{n,t})$ and $W_y = g(Y_1, \dots, Y_n)$ with support on $\mathbb{S} \subseteq \{0,1,\dots\}$. Define $\mathcal{A}_w$ to be the event that $W_y = w$ and define $\mathcal{A}_{t,w} = \mathcal{A}_w \cap \mathbb{S}_t^n$ so that

$$P\left((X_{1,t}, \dots, X_{n,t}) \in \mathcal{A}_{t,w}\right) = P(W_x = w).$$

Then,

$$\mathsf{E}_t\left((W_x)_{[r]}\right) = \frac{\mathrm{d}^t}{\mathrm{d}\lambda^t}\frac{\mathrm{d}^r}{\mathrm{d}\theta^r}\left(e^{\lambda}\sum_{w=0}^{\infty} P\left((Y_1, \dots, Y_n) \in \mathcal{A}_w\right)\theta^w\right)\Bigg|_{\theta=1}\Bigg|_{\lambda=0}.$$

Theorem 3. Expected Waiting Time to Meet Given Quotas

Suppose we distribute balls independently into m distinguishable boxes such that the probability that a ball is distributed into Box j on any given trial is p_j. When $q_i \geq 1$ balls have been distributed into Box i we will say Box i has reached its quota.

Let $W_{(r:m:q_1,q_2,\dots,q_m)} \equiv W_{r:m:Q}$ represent the waiting time until exactly r different boxes have reached their quota.

Let $\mathsf{E}\left(W_{(r:m:q_1,q_2,\dots,q_m)}^{[k]}\right)$ represent the k^{th} ascending moment of $W_{r:m:Q}$. That is,

$$\mathsf{E}\left(W_{r:m:Q}^{[k]}\right) = \left((W_{r:m:Q} + 0)(W_{r:m:Q} + 1)\cdots(W_{r:m:Q} + k - 1)\right)$$

Define $N_{(q_1,q_2,\dots,q_m)}(t) \equiv N_{m:Q}(t)$ to be the number of boxes that have <u>not</u> reached their quota after t balls have been distributed.

We allow for the possibility that $\sum_{j=1}^{m} p_j < 1$ so as to include the case where our interest is limited to a particular set of m boxes out of a larger set (say $m + v$) of possible boxes.

Then the k^{th} ascending moment of $W_{r:m:Q}$ is given by

$$\mathsf{E}\left(W_{r:m:Q}^{[k]}\right) = k \sum_{j=m-k+1}^{m} \sum_{\mathbb{C}_j} \sum_{v_{t_1}=0}^{q_{t_1}-1} \cdots \sum_{v_{t_j}=0}^{q_{t_j}-1} (-1)^{j-(m-k+1)} \binom{j-1}{m-k}$$

$$\times \left(\frac{\left(p_{t_1}\right)^{v_{t_1}} \cdots \left(p_{t_j}\right)^{v_{t_j}}}{v_{t_1}! \cdots v_{t_j}!} \right) \left(\frac{\left(v_{t_1} + \cdots + v_{t_j} + k - 1\right)!}{\left(p_{t_1} + \cdots + p_{t_j}\right)^{v_{t_1}+\cdots+v_{t_j}+k}} \right)$$

where $\mathbb{C}_j$ is the set of all samples of size j drawn without replacement from $\{1,2,\ldots,m\}$, where the order of sampling is not considered important.

Applications

Problem 1.

Suppose m distinguishable balls are distributed independently among $n+r$ distinguishable urns and that all urns are equally likely. Suppose that n of the $n+r$ urns are marked. Let T equal the number of urns among these n marked urns which are occupied by at least one ball. Let W_i equal the number of urns among these n marked urns which are occupied by exactly i balls. We note that without loss of generality we can assume that the n marked urns are the first n urns when we arrange the $n+r$ urns in a row.

(a) Show

$$P(T=k) = n_{[k]}(n+r)^{-m} \left(\frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (r+j)^m \right)$$

(b) Show

$$E\left(T_{[v]}\right) = \frac{n!}{(n-v)!}(n+r)^{-m} \left(\sum_{s=0}^{v} (-1)^{v-s} \binom{v}{s} (r+s+n-v)^m \right)$$

provided $n \geq v$.

(c) Show

$$P(W_i = k) = \frac{\binom{n}{k} m!}{(n+r)^m} \sum_{j=0}^{n-k} (-1)^{n-k-j} \binom{n-k}{j} \frac{(r+j)^{(m-i(n-j))}}{(i!)^{n-j}\big(m-i(n-j)\big)!}$$

provided $m \geq i \cdot n$.

(d) Show

$$E\big((W_i)_{[v]}\big) = \frac{\binom{n}{n-v} v!\, m!\, (n+r-v)^{m-iv}}{(i!)^v (m-iv)!\, (n+r)^m}$$

provided $m \geq iv$ and $v \leq n$.

A Unified Derivation of Occupancy and Sequential Occupancy Distributions, Ch. A. Charalambides, Advances in Combinatorial Methods and Applications to Probability and Statistics, pages 259-273.

Problem 2.

Suppose n balls are independently distributed into m distinguishable boxes and that a ball is equally likely to land in any of the m boxes. Let, N_j =the number of boxes containing j balls, $j = 1, \dots, n$.

(a) Show

$$P\big((N_0, \dots, N_n) = (a_0, \dots, a_n)\big) = \frac{m!\, n!}{m^n ((0!)^{a_0} (1!)^{a_1} \cdots (n!)^{a_n})(a_0!\, a_1! \cdots a_n!)}$$

where

(i) $a_j \geq 0$ for $0 \leq j \leq n$ and $a_j = 0$ for all $j > n$

(ii) $a_0 + \cdots + a_n =$ the number of boxes $= m$, and

(iii) $(0 \cdot a_0) + \cdots + (n \cdot a_n) =$ the number of balls $= n$.

Note that is not necessary to specify whether the n balls are distinguishable or not because the required probability would be the same in either case.

(b) Show that the number of ways to distribute n distinguishable balls into m distinguishable boxes so that exactly a_r boxes contain exactly r balls equals

$$\sum_{j=a_r}^{\min\left(m,\left\lfloor\frac{n}{r}\right\rfloor\right)} (-1)^{j-a_r} \frac{\binom{m}{a_r}\binom{m-a_r}{j-a_r}\, n!\,(m-j)^{n-rj}}{(r!)^j (n-rj)!}$$

provided $0 \le a_r \le \min\left(m, \left\lfloor\frac{n}{r}\right\rfloor\right)$.

The special case $r = 1$ is considered in, A recursive solution to an occupancy problem resulting from TDM radio communication application, Jyh-Horng Wen, Jee-Wey Wang, Applied Mathematics and Computation, Vol. 101, 1999, no. 1, pages 1-3. They do not find a closed form solution but rather derive a recurrence relation for the case $r = 1$.

(c) Show that

$$P(N_0 = m - k) = \frac{m_{(k)}}{m^n} S(n,k)$$

provided $\max(0, m-n) \le k \le m-1$ and where $S(n,k)$ is the Stirling Number of the Second Kind.

Problem 3.

Special cases illustrating Theorem 3.

(1) Coupon Collector's Problems

(need references)

$\boldsymbol{q_1 = q_2 = \cdots = q_m = 1}$

$$E\left(W_{r:m:Q}^{[k]}\right) = k! \sum_{j=m-r+1}^{m} \sum_{\mathbb{C}_j} (-1)^{j-(m-r+1)} \binom{j-1}{m-r} \left(p_{t_1} + \cdots + p_{t_j}\right)^{-k}$$

$\boldsymbol{p_1 = p_2 = \cdots = p_m = p}$ and $\boldsymbol{q_1 = q_2 = \cdots = q_m = 1}$

$$E\left(W_{r:m:Q}^{[k]}\right) = \frac{k!}{p^k} \sum_{j=m-r+1}^{m} (-1)^{j-(m-r+1)} \binom{m}{j} \binom{j-1}{m-r} \left(\frac{1}{j}\right)^k$$

which agrees with Charalambides,page 271,$\left(\text{with} p = \frac{1}{m+v}\right)$ after appropriate notational adjustments. Furthermore, in this case

$$E\left(W_{r:m:Q}\right) = \frac{1}{p} \sum_{j=m-r+1}^{m} (-1)^{j-(m-r+1)} \binom{m}{j} \binom{j-1}{m-r} \left(\frac{1}{j}\right)$$

$$= \frac{1}{p} \sum_{j=m-r+1}^{m} \left(\frac{1}{j}\right)$$

(needs proof. relate to sum of indep. but not identically dist. geometric random vars.)

$\boldsymbol{p_1 = p_2 = \cdots = p_m = p}$ and $\boldsymbol{q_1 = q_2 = \cdots = q_m} = 2$

Newman, Donald and Shepp, Lawrence; “Double Dixie Cup Problem”, American Mathematical Monthly, Vol. 67, 1960, pages 58-61.

Holst, Lars; “On Birthday, Collector’s, Occupancy and Other Classical Urn Problems”, International Statistical Review, 1986, 54, 1, pp. 15-27.

(Two children collecting cards in cooperation.)

$$E\left(W_{r:m:2}^{[k]}\right) = k \quad \sum_{j=m-r+1}^{m} \sum_{\mathbb{C}_j} \sum_{v_{t_1}=0}^{1} \cdots \sum_{v_{t_j}=0}^{1} (-1)^{j-(m-r+1)} \binom{j-1}{m-r}$$

$$\times (p)^{v_{t_1}+\cdots+v_{t_j}} \left(\frac{\left(v_{t_1}+\cdots+v_{t_j}+k-1\right)!}{(jp)^{v_{t_1}+\cdots+v_{t_j}+k}} \right)$$

$$= \frac{k}{p^k} \sum_{j=m-r+1}^{m} \sum_{i=0}^{j} (-1)^{j-(m-r+1)} \binom{m}{j} \binom{j-1}{m-r} \binom{j}{i} \frac{(i+k-1)!}{j^{i+k}}$$

$\boldsymbol{r=1}$

(Birthday problem. Waiting for the first person to meet their quota.)

$$E\left(W_{1:m:q}^{[k]}\right) = k \sum_{v_1=0}^{q_1-1} \cdots \sum_{v_m=0}^{q_m-1} \left(\frac{(p_1)^{v_1} \cdots (p_m)^{v_m}}{v_1! \cdots v_m!} \right) \left(\frac{(v_1+\cdots+v_m+k-1)!}{(p_1+\cdots+p_m)^{v_1+\cdots+v_m+k}} \right)$$

$$= k \sum_{v_1=0}^{q_1-1} \cdots \sum_{v_m=0}^{q_m-1} \frac{(v_1+\cdots+v_m+k-1)!}{v_1! \cdots v_m!} \left(\frac{(p_1)^{v_1} \cdots (p_m)^{v_m}}{(p_1+\cdots+p_m)^{v_1+\cdots+v_m+k}} \right)$$

$\boldsymbol{p_1=p_2=\cdots=p_m=1/m}$, $\boldsymbol{q_1=q_2=\cdots=q_m=q}$ and $\boldsymbol{r=1}$

$$E\left(W_{1:m:q}^{[k]}\right) = k \sum_{v_1=0}^{q-1} \cdots \sum_{v_m=0}^{q-1} \frac{(v_1+\cdots+v_m+k-1)!}{v_1! \cdots v_m!} \left(\frac{1}{m}\right)^{v_1+\cdots+v_m}$$

$\boldsymbol{p_1=p_2=\cdots=p_m=1/m}$, $\boldsymbol{q_1=q_2=\cdots=q_m=2}$ and $\boldsymbol{r=1}$

$$E\left(W_{1:m:2}^{[k]}\right) = k \sum_{v_1=0}^{1} \cdots \sum_{v_m=0}^{1} (v_1+\cdots+v_m+k-1)! \left(\frac{1}{m}\right)^{v_1+\cdots+v_m}$$

$$= k \sum_{i=0}^{m} \binom{m}{i} (i + k - 1)! \left(\frac{1}{m}\right)^{i}.$$

(2) Knock'm Down

Benjamin, Arthur and Fluet, Matthew, "The Best Way to Knock'm Down", The UMAP Journal, 20.1, 1999, pages 11-20.

Mauer, Stephen, "The King Chicken Theorems", Mathematics Magazine, Vol 53, 1980, pages 67-80.

The game of Knock'm Down is played by two players, each of whom is given a 6-sided die, n tokens, and a card with the numbers 2 through 12. Each player allocates tokens among the eleven numbers on their card. It is permissible to allocate multiple tokens to the same number of a card. The players roll the dice together and each removes one token from their board on the value equal to the sum of the dice. If a player does not have any tokens on their board on the value equal to the sum of dice, that player's board is unchanged during that turn. Play continues until a player has removed all tokens from their board. The first player to remove all tokens from their board is the winner. If both players remove their last token on the same roll, then the game is a draw.

How should a player allocate their tokens?

Strategy 1. Determine the allocation which has the smallest expected number of turns to remove all tokens.

Strategy 2. Determine the allocation (if it exists) which wins at least as frequently as it loses no matter what allocation your opponent has chosen.

As it will be seen these two strategies can lead to different conclusions. First consider Strategy 1. We will illustrate our notation for the allocation (quotas) through two examples.

If $n = 3$ and we allocate one of the three tokens to the number 6 and the remaining two tokens to the number 7, then we will define $r = m = 2$, $q_1 = 1$and $q_2 = 2$, and $p_1 = P(\text{sumoftwodice} = 6) = \frac{5}{36}$, and $p_2 = P(\text{sumoftwodice} = 7) = \frac{6}{36}$.

If $n = 3$ and we allocate one of the three tokens to the number 6, one to the number 7 and one to the number 8, then we will define $r = m = 3$, $q_1 = 1$, $q_2 = 1$, and $q_3 = 1$, and $p_1 =$

$P(\text{sumoftwodice} = 6) = \frac{5}{36}$, $p_2 = P(\text{sumoftwodice} = 7) = \frac{6}{36}$, $p_3 = P(\text{sumoftwodice} = 8) = \frac{5}{36}$.

In this way for any allocation Q we have

$$E\left(W_{m:m:Q}\right) = \text{expectednumberofturnstoremovealltokens(satisfyallquotas)}$$

$$= \sum_{j=1}^{m} \sum_{\mathbb{C}_j} \sum_{v_{t_1}=0}^{q_{t_1}-1} \cdots \sum_{v_{t_j}=0}^{q_{t_j}-1} (-1)^{j-1}$$

$$\times \left(\frac{\left(v_{t_1} + \cdots + v_{t_j}\right)!}{v_{t_1}! \cdots v_{t_j}!} \right) \left(\frac{\left(p_{t_1}\right)^{v_{t_1}} \cdots \left(p_{t_j}\right)^{v_{t_j}}}{\left(p_{t_1} + \cdots + p_{t_j}\right)^{v_{t_1}+\cdots+v_{t_j}+1}} \right)$$

Strategy 1 reduces to the task of determining an allocation Q which minimizes the above expectation.

Exercises.

1.) Write a computer program which accepts the number of tokens you choose to allocate to each number 2 through 12 as input and returns $\left(W_{m:m:Q}\right)$ as output.

2.) Verify that on average it takes 15.476 turns to remove four tokens if one token is allocated to each of the numbers five through eight. Verify that this is optimal in the sense of Strategy 1. Be sure to exploit symmetry and intuitive conditions given in Benjamin and Fluet's paper to reduce the search space.

Proof (Theorem 1)

$$
\begin{aligned}
P(Y_1 + \cdots + Y_n = t) &= \sum_{\mathbb{S}_t^n} P(Y_1 = s_1, Y_2 = s_2, \ldots, Y_n = s_n) \\
&= \sum_{\mathbb{S}_t^n} \frac{e^{-\lambda}\lambda^{s_1+\cdots+s_n}}{t!} \frac{t!}{s_1!\, s_2! \cdots s_n!} (p_1)^{s_1}(p_2)^{s_2} \cdots (p_n)^{s_n} \\
&= \frac{e^{-\lambda}\lambda^{t}}{t!} \sum_{\mathbb{S}_t^n} \frac{t!}{s_1!\, s_2! \cdots s_n!} (p_1)^{s_1}(p_2)^{s_2} \cdots (p_n)^{s_n} \\
&= \frac{e^{-\lambda}\lambda^{t}}{t!}
\end{aligned}
$$

Thus,

$$P((Y_1, \ldots, Y_n)) \in \mathcal{A} | Y_1 + \cdots + Y_n = t)$$

$$= \frac{P\big((Y_1, \ldots, Y_n) \in \mathcal{A} \text{and} Y_1 + \cdots + Y_n = t\big)}{P(Y_1 + \cdots + Y_n = t)}$$

$$= \frac{P\big((Y_1, \ldots, Y_n) \in \mathcal{A}_t\big)}{P(Y_1 + \cdots + Y_n = t)}$$

$$= \frac{\sum_{\mathcal{A}_t} \frac{e^{-\lambda}\lambda^{s_1+\cdots+s_n}}{t!} \frac{t!}{s_1!\, s_2! \cdots s_n!} (p_1)^{s_1}(p_2)^{s_2} \cdots (p_n)^{s_n}}{\frac{\mathrm{e}^{-\lambda}\lambda^{t}}{t!}}$$

$$= \frac{\sum_{\mathcal{A}_t} \frac{e^{-\lambda}\lambda^{t}}{t!} \frac{t!}{s_1!\, s_2! \cdots s_n!} (p_1)^{s_1}(p_2)^{s_2} \cdots (p_n)^{s_n}}{\frac{\mathrm{e}^{-\lambda}\lambda^{t}}{t!}}$$

$$= \sum_{\mathcal{A}_t} \frac{t!}{s_1!\, s_2! \cdots s_n!} (p_1)^{s_1}(p_2)^{s_2} \cdots (p_n)^{s_n}$$

$$= P\left(\left(X_{1,t}, \ldots, X_{n,t}\right) \in \mathcal{A}_t\right)$$

Therefore,

$$P\big((Y_1, Y_2, \dots, Y_n) \in \mathcal{A}\big)$$

$$= \sum_{t=0}^{\infty} P\left((Y_1, \dots, Y_n) \in \mathcal{A} \Big| \sum Y_i = t\right) P\left(\sum Y_i = t\right)$$

$$= \sum_{t=0}^{\infty} P\left(\big(X_{1,t}, \dots, X_{n,t}\big) \in \mathcal{A}_t\right) \frac{e^{-\lambda}\lambda^t}{t!}$$

and

$$e^{\lambda} P\big((Y_1, Y_2, \dots, Y_n) \in \mathcal{A}\big) = \sum_{t=0}^{\infty} P\left(\big(X_{1,t}, \dots, X_{n,t}\big) \in \mathcal{A}_t\right) \frac{1}{t!}\lambda^t$$

It follows that

$$\frac{\mathrm{d}^r}{\mathrm{d}\lambda^r}\Big(e^{\lambda} P\big((Y_1, Y_2, \dots, Y_n) \in \mathcal{A}\big)\Big)\Big|_{\lambda=0}$$

$$= \frac{\mathrm{d}^r}{\mathrm{d}\lambda^r}\Big(\sum\nolimits_{t=0}^{\infty} P\left(\big(X_{1,t}, \dots, X_{n,t}\big) \in \mathcal{A}_t\right) \frac{1}{t!}\lambda^t\Big)\Big|_{\lambda=0}$$

$$= \sum_{t=0}^{\infty} P\left(\big(X_{1,t}, \dots, X_{n,t}\big) \in \mathcal{A}_t\right) \frac{1}{t!}\Big(\frac{\mathrm{d}^r}{\mathrm{d}\lambda^r}\lambda^t\Big|_{\lambda=0}\Big)$$

$$= \sum_{t=0}^{\infty} P\left(\big(X_{1,t}, \dots, X_{n,t}\big) \in \mathcal{A}_t\right) \mathbb{I}_{\{r\}}(t)$$

$$= P\left(\big(X_{1,r}, \dots, X_{n,r}\big) \in \mathcal{A}_r\right).$$

Thus for $r \geq 0$,

$$P\left(\big(X_{1,r}, \dots, X_{n,r}\big) \in \mathcal{A}_r\right) = \frac{\mathrm{d}^r}{\mathrm{d}\lambda^r}\Big(e^{\lambda} P\big((Y_1, Y_2, \dots, Y_n) \in \mathcal{A}\big)\Big)\Big|_{\lambda=0}.$$

Proof (Theorem 2)

$$\mathrm{E}_S\big((W_x)_{[r]}\big) = \mathrm{E}\big((W_x)(W_x-1)\cdots(W_x-r+1)\big)$$

$$= \sum_{w=0}^{\infty} w(w-1)\cdots(w-r+1)P(W_x = w)$$

$$= \sum_{w=0}^{\infty} w_{[r]}P(W_x = w)$$

$$= \sum_{w=0}^{\infty} w_{[r]}P\left((X_1,\dots,X_n)\in\mathcal{A}_{s,w}\right)$$

$$= \sum_{w=0}^{\infty}\left(\frac{d^r}{d\theta^r}\theta^w\Big|_{\theta=1}\right)P\left((X_1,\dots,X_n)\in\mathcal{A}_{s,w}\right)$$

$$= \left(\frac{d^r}{d\theta^r}\sum_{w=0}^{\infty}\theta^w P\left((X_1,\dots,X_n)\in\mathcal{A}_{s,w}\right)\right)\Bigg|_{\theta=1}.$$

Therefore,

$$\sum_{s=0}^{\infty}\frac{\mathrm{E}_s\big((W_x)_{[r]}\big)\lambda^s}{s!} = \sum_{s=0}^{\infty}\left(\frac{d^r}{d\theta^r}\sum_{w=0}^{\infty}\theta^w P\left((X_1,\dots,X_n)\in\mathcal{A}_{t,w}\right)\right)\Bigg|_{\theta=1}\frac{\lambda^s}{s!}$$

$$= \left(\frac{d^r}{d\theta^r}\sum_{w=0}^{\infty}\theta^w\left(\sum_{s=0}^{\infty}P\left((X_1,\dots,X_n)\in\mathcal{A}_{s,w}\right)\frac{\lambda^s}{s!}\right)\right)\Bigg|_{\theta=1}$$

$$= \left(\frac{d^r}{d\theta^r}\sum_{w=0}^{\infty}\theta^w\big(P\big((Y_1,\dots,Y_n)\in\mathcal{A}_w\big)e^{\lambda}\big)\right)\Bigg|_{\theta=1}.$$

It follows that,

$$\mathsf{E}_t\big((W_x)_{[r]}\big) = \frac{d^t}{d\lambda^t}\left(\left(\frac{d^r}{d\theta^r}\sum_{w=0}^{\infty}\theta^w\big(P\big((Y_1,\ldots,Y_n)\in\mathcal{A}_w\big)e^{\lambda}\big)\right)\Bigg|_{\theta=1}\right)\Bigg|_{\lambda=0}$$

$$= \frac{d^t}{d\lambda^t}\frac{d^r}{d\theta^r}\left(\left(e^{\lambda}\sum_{w=0}^{\infty}P\big((Y_1,\ldots,Y_n)\in\mathcal{A}_w\big)\theta^w\right)\Bigg|_{\theta=1}\right)\Bigg|_{\lambda=0}.$$

Proof (Theorem 3)

Within the proof of the Poisson Randomization Theorem, we showed

$$P\big((Y_1,Y_2,\ldots,Y_m)\in\mathbb{A}\big) = \sum_{n=0}^{\infty}P\big((X_1,X_2,\ldots,X_m)\in\mathbb{A}_n\big)\frac{e^{-\lambda}\lambda^n}{\mathrm{n!}}.$$

It follows that

$$P\big(N_Q^P(t) > m-r\big) = \sum_{n=0}^{\infty}P\big(N_{m:Q}(t) > m-r\big)\frac{e^{-\lambda}\lambda^n}{\mathrm{n!}}.$$

However,

$$W_{r:m:Q} > n \quad\Leftrightarrow\quad N_{m:Q}(n) > m-r.$$

Therefore,

$$P\big(W_{r:m:Q} > n\big) = P\big(N_{m:Q}(t) > m-r\big)$$

and

$$P\big(N_Q^P(t) > m-r\big) = \sum_{n=0}^{\infty}P\big(W_{r:m:Q} > n\big)\frac{e^{-\lambda}\lambda^n}{n!}.$$

Thus,

$$\int_0^{\infty}\lambda^{k-1}\Big(P\big(N_Q^P(t) > m-r\big)\Big)\,\mathrm{d}\lambda$$

$$= \int_0^{\infty}\lambda^{k-1}\left(\sum_{n=0}^{\infty}P\big(W_{r:m:Q} > n\big)\frac{e^{-\lambda}\lambda^n}{n!}\right)\mathrm{d}\lambda$$

$$= \sum_{n=0}^{\infty} P\left(W_{r:m:Q} > n\right)\left(\frac{1}{n!}\int_0^{\infty} e^{-\lambda}\lambda^{n+k-1}\mathrm{d}\lambda\right)$$

$$= \sum_{n=0}^{\infty} P\left(W_{r:m:Q} > n\right)\left(\frac{(n+k-1)!}{n!}\right)$$

$$= \sum_{n=0}^{\infty} P\left(W_{r:m:Q} > n\right)(n+k-1)_{[k-1]}$$

$$= \sum_{n=0}^{\infty} P\left(W_{r:m:Q} + k - 1 > n\right)n_{[k-1]}$$

$$= \frac{1}{k}\mathsf{E}\left(\left(W_{r:m:Q} + k - 1\right)_{[k]}\right)$$

(see Problem ???).

But

$$\left(W_{r:m:Q} + k - 1\right)_{[k]} \equiv W_{r:m:Q}^{[k]}.$$

Thus

$$\int_0^{\infty} \lambda^{k-1}\left(P\left(N_Q^P(t) > m - r\right)\right)\mathrm{d}\lambda = \frac{1}{k}\mathsf{E}\left(W_{r:m:Q}^{[k]}\right)$$

and

$$\mathsf{E}\left(W_{r:m:Q}^{[k]}\right) = k\int_0^{\infty} \lambda^{k-1}P\left(N_Q^P(t) > m - r\right)\mathrm{d}\lambda.$$

Now let A_i be the event that $Y_i < q_i$. From the General Probability Theorem, we have

$$P\left(N_Q^P(t) > m - r\right)$$

$$= P(\text{atleast}m - r + 1\text{ofthe}m\text{events}A_1, A_2, \ldots, A_m\text{occurs})$$

$$= \sum_{j=0}^{m-(m-r+1)} (-1)^j \binom{j + (m-r+1) - 1}{(m-r+1) - 1}\mathbb{S}_{j+(m-r+1)}$$

$$= \sum_{j=m-r+1}^{m} (-1)^{j-(m-r+1)} \binom{j-1}{m-r} \mathbb{S}_j$$

where

$$\mathbb{S}_j = \begin{cases} \sum_{(t_1,\ldots,t_j)\in\mathbb{C}_j} P\left(A_{t_1} \cap \cdots \cap A_{t_j}\right) & 1 \le j \le m \\ 1 & j = 0 \end{cases}$$

and where $\mathbb{C}_j$ is the set of all samples of size j drawn without replacement from $\{1,\ldots,m\}$, when the order of sampling is considered unimportant.

In the Poisson model,

$$P(A_j) = P(Y_j < q_j) = \sum_{v=0}^{q_j-1} \frac{e^{-(\lambda p_j)}(\lambda p_j)^v}{v!}.$$

Also, as the Y_i's are independently,

$$P\left(A_{t_1} \cap \cdots \cap A_{t_j}\right) = P(A_{t_1})P(A_{t_2}) \cdots P\left(A_{t_j}\right)$$

$$= \sum_{v_{t_1}=0}^{q_{t_1}-1} \cdots \sum_{v_{t_j}=0}^{q_{t_j}-1} \frac{e^{-(\lambda p_{t_1})}(\lambda p_{t_1})^{v_{t_1}}}{v_{t_1}!} \cdots \frac{e^{-\left(\lambda p_{t_j}\right)}\left(\lambda p_{t_j}\right)^{v_{t_j}}}{v_{t_j}!}$$

$$= \sum_{v_{t_1}=0}^{q_{t_1}-1} \cdots \sum_{v_{t_j}=0}^{q_{t_j}-1} \frac{e^{-\lambda\left(p_{t_1}+\cdots+p_{t_j}\right)}\lambda^{\left(v_{t_1}+\cdots+v_{t_j}\right)}\left(p_{t_1}\right)^{v_{t_1}} \cdots \left(p_{t_j}\right)^{v_{t_j}}}{v_{t_1}! \cdots v_{t_j}!}$$

Therefore,

$$\mathbb{S}_j = \sum_{(t_1,\ldots,t_j)\varepsilon\mathbb{C}_j} \left(\sum_{v_{t_1}=0}^{q_{t_1}-1} \cdots \sum_{v_{t_j}=0}^{q_{t_j}-1} \frac{e^{-\lambda\left(p_{t_1}+\cdots+p_{t_j}\right)}\lambda^{\left(v_{t_1}+\cdots+v_{t_j}\right)}\left(p_{t_1}\right)^{v_{t_1}} \cdots \left(p_{t_j}\right)^{v_{t_j}}}{v_{t_1}! \cdots v_{t_j}!} \right)$$

$$= \sum_{\mathbb{C}_j} \left(\sum_{v_{t_1}=0}^{q_{t_1}-1} \cdots \sum_{v_{t_j}=0}^{q_{t_j}-1} \frac{e^{-\lambda\left(p_{t_1}+\cdots+p_{t_j}\right)} \lambda^{\left(v_{t_1}+\cdots+v_{t_j}\right)} \left(p_{t_1}\right)^{v_{t_1}} \cdots \left(p_{t_j}\right)^{v_{t_j}}}{v_{t_1}! \cdots v_{t_j}!} \right)$$

and

$$P\left(N_Q^P(t) > m - r\right)$$

$$= \sum_{j=m-r+1}^{m} (-1)^{j-(m-r+1)} \binom{j-1}{m-r} \sum_{\mathbb{C}_j}$$

$$\left(\sum_{v_{t_1}=0}^{q_{t_1}-1} \cdots \sum_{v_{t_j}=0}^{q_{t_j}-1} \frac{e^{-\lambda\left(p_{t_1}+\cdots+p_{t_j}\right)} \lambda^{\left(v_{t_1}+\cdots+v_{t_j}\right)} \left(p_{t_1}\right)^{v_{t_1}} \cdots \left(p_{t_j}\right)^{v_{t_j}}}{v_{t_1}! \cdots v_{t_j}!} \right)$$

$$= \sum_{j=m-r+1}^{m} \sum_{\mathbb{C}_j} (-1)^{j-(m-r+1)} \binom{j-1}{m-r}$$

$$\left(\sum_{v_{t_1}=0}^{q_{t_1}-1} \cdots \sum_{v_{t_j}=0}^{q_{t_j}-1} \frac{e^{-\lambda\left(p_{t_1}+\cdots+p_{t_j}\right)} \lambda^{\left(v_{t_1}+\cdots+v_{t_j}\right)} \left(p_{t_1}\right)^{v_{t_1}} \cdots \left(p_{t_j}\right)^{v_{t_j}}}{v_{t_1}! \cdots v_{t_j}!} \right)$$

and

$$\begin{aligned}
\left(W_{r:m:Q}^{[k]}\right) &= k\int_0^\infty \lambda^{k-1} P\left(N_Q^P(t) > m-r\right)\mathrm{d}\lambda \\
&= k\int_0^\infty \lambda^{k-1} \sum_{j=m-r+1}^{m} \sum_{\mathbb{C}_j} (-1)^{j-(m-r+1)}(j-1m-r) \\
&\qquad \times \left(\sum_{v_{t_1}=0}^{q_{t_1}-1} \cdots \sum_{v_{t_j}=0}^{q_{t_j}-1} \frac{e^{-\lambda\left(p_{t_1}+\cdots+p_{t_j}\right)}\lambda^{\left(v_{t_1}+\cdots+v_{t_j}\right)}\left(p_{t_1}\right)^{v_{t_1}}\cdots\left(p_{t_j}\right)^{v_{t_j}}}{v_{t_1}!\cdots v_{t_j}!} \right)\mathrm{d}\lambda \\
&= k\sum_{j=m-r+1}^{m} \sum_{\mathbb{C}_j} \sum_{v_{t_1}=0}^{q_{t_1}-1} \cdots \sum_{v_{t_j}=0}^{q_{t_j}-1} (-1)^{j-(m-r+1)}(\mathrm{j}-1m-r) \\
&\qquad \times \left(\frac{\left(p_{t_1}\right)^{v_{t_1}}\cdots\left(p_{t_j}\right)^{v_{t_j}}}{v_{t_1}!\cdots v_{t_j}!} \right)\left(\int_0^\infty e^{-\lambda\left(p_{t_1}+\cdots+p_{t_j}\right)}\lambda^{\left(v_{t_1}+\cdots+v_{t_j}+k-1\right)}\mathrm{d}\lambda \right) \\
&= k\sum_{j=m-r+1}^{m} \sum_{\mathbb{C}_j} \sum_{v_{t_1}=0}^{q_{t_1}-1} \cdots \sum_{v_{t_j}=0}^{q_{t_j}-1} (-1)^{j-(m-r+1)}(j-1m-r) \\
&\qquad \times \left(\frac{\left(p_{t_1}\right)^{v_{t_1}}\cdots\left(p_{t_j}\right)^{v_{t_j}}}{v_{t_1}!\cdots v_{t_j}!} \right)\left(\frac{\left(v_{t_1}+\cdots+v_{t_j}+k-1\right)!}{\left(p_{t_1}+\cdots+p_{t_j}\right)^{v_{t_1}+\cdots+v_{t_j}+k}} \right)
\end{aligned}$$

Solutions

Problem 1(a)

Let X_j equal the number of balls that go into the j^{th} urn. Then $(X_1, \ldots, X_{n+r})$ is a multinomial random vector. That is, for all $(s_1,\ldots,s_{n+r}) \in \mathbb{S}_m^{n+r}$,

$$\begin{aligned}
P(X_1 = s_1, \ldots, X_{n+r} = s_{n+r}) &= \frac{m!}{s_1!\cdots s_{n+r}!}\left(\frac{1}{n+r}\right)^{s_1}\cdots\left(\frac{1}{n+r}\right)^{s_{n+r}} \\
&= \frac{m!}{s_1!\cdots s_{n+r}!}\left(\frac{1}{n+r}\right)^{m}
\end{aligned}$$

Therefore we can use Theorem 1 with $\mathcal{A}$ defined as that subset of $\mathbb{S}^{n+r}$ where exactly k of $(Y_1, \ldots, Y_n)$ are greater than 0 and the random variables $(Y_{n+1}, \ldots, Y_{n+r})$ can be anything.

Therefore,

$$
\begin{aligned}
P((Y_1, \ldots, Y_{n+r}) \in \mathcal{A}) &= \binom{n}{k}\left(1 - \frac{e^{-\left(\frac{\lambda}{n+r}\right)}\left(\frac{\lambda}{n+r}\right)^0}{0!}\right)^k \left(\frac{e^{-\left(\frac{\lambda}{n+r}\right)}\left(\frac{\lambda}{n+r}\right)^0}{0!}\right)^{n-k} \\
&= \binom{n}{k}\left(e^{-\left(\frac{\lambda}{n+r}\right)}\right)^{n-k}\left(1 - e^{-\left(\frac{\lambda}{n+r}\right)}\right)^k \\
&= \binom{n}{k}\left(e^{-\left(\frac{\lambda}{n+r}\right)}\right)^{n-k}\left(\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}e^{-\left(\frac{\lambda}{n+r}\right)(k-j)}\right) \\
&= \binom{n}{k}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}e^{-\left(\frac{\lambda}{n+r}\right)(n-k+k-j)} \\
&= \binom{n}{k}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}e^{-\left(\frac{\lambda}{n+r}\right)(n-j)}
\end{aligned}
$$

and

$$
\begin{aligned}
e^{\lambda}P((Y_1, \ldots, Y_{n+r}) \in \mathcal{A}) &= e^{\lambda}\binom{n}{k}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}e^{-\left(\frac{\lambda}{n+r}\right)(n-j)} \\
&= \binom{n}{k}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}e^{-\left(\frac{\lambda}{n+r}\right)(n-j-n-r)} \\
&= \binom{n}{k}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}e^{\left(\frac{r+j}{n+r}\right)\lambda}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
P(T=k) &= P((X_1,\ldots,X_{n+r}) \in \mathcal{A}_m) \\
&= \left(\frac{\mathrm{d}^m}{\mathrm{d}\lambda^m} \left(e^{\lambda} P((Y_1,\ldots,Y_{n+r}) \in \mathcal{A}) \right) \right) \Bigg|_{\lambda=0} \\
&= \left(\frac{\mathrm{d}^m}{\mathrm{d}\lambda^m} \left(\binom{n}{k} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} e^{\left(\frac{r+j}{n+r}\right)\lambda} \right) \right) \Bigg|_{\lambda=0} \\
&= \binom{n}{k} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \left(\frac{\mathrm{d}^m}{\mathrm{d}\lambda^m} e^{\left(\frac{r+j}{n+r}\right)\lambda} \right) \Bigg|_{\lambda=0} \\
&= \binom{n}{k} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \left(\frac{r+j}{n+r} \right)^m \\
&= n_{[k]} (n+r)^{-m} \left(\frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (r+j)^m \right)
\end{aligned}
$$

which agrees with Charlambides, page 265, and with Barton and David, “Contagious Occupancy”, Journal of the Royal Statistical Society, Series B, 21, 120-133.

Problem 1(b).

From Theorem 2

$$
\begin{aligned}
\left(T_{[v]}\right) &= \frac{d^m}{d\lambda^m}\frac{d^v}{d\theta^v}\left(\sum_{k=0}^{\infty} e^{\lambda} P\left((Y_1,\dots,Y_{n+r})\in\mathcal{A}\right)\theta^k\right)\Bigg|_{\theta=1}\Bigg|_{\lambda=0} \\
&= \frac{d^m}{d\lambda^m}\frac{d^v}{d\theta^v}\left(\sum_{k=0}^{\infty}\left(\binom{n}{k}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}e^{\left(\frac{r+j}{n+r}\right)\lambda}\right)\theta^k\right)\Bigg|_{\theta=1}\Bigg|_{\lambda=0} \\
&= \frac{d^m}{d\lambda^m}\frac{d^v}{d\theta^v}\left(\sum_{k=0}^{n}\left(\binom{n}{k}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}e^{\left(\frac{r+j}{n+r}\right)\lambda}\right)\theta^k\right)\Bigg|_{\theta=1}\Bigg|_{\lambda=0} \\
&= \frac{d^m}{d\lambda^m}\frac{d^v}{d\theta^v}\left(\sum_{j=0}^{n}\sum_{k=j}^{n}\binom{n}{j}(-1)^{k-j}\binom{n-j}{k-j}e^{\left(\frac{r+j}{n+r}\right)\lambda}\theta^k\right)\Bigg|_{\theta=1}\Bigg|_{\lambda=0} \\
&= \frac{d^m}{d\lambda^m}\frac{d^v}{d\theta^v}\left(\sum_{j=0}^{n}\sum_{l=0}^{n-j}\binom{n}{j}(-1)^{l}\binom{n-j}{l}e^{\left(\frac{r+j}{n+r}\right)\lambda}\theta^{l+j}\right)\Bigg|_{\theta=1}\Bigg|_{\lambda=0} \\
&= \frac{d^m}{d\lambda^m}\frac{d^v}{d\theta^v}\left(\sum_{j=0}^{n}\left(\sum_{l=0}^{n-j}\binom{n-j}{l}(-\theta)^{l}\right)\binom{n}{j}e^{\left(\frac{r+j}{n+r}\right)\lambda}\theta^j\right)\Bigg|_{\theta=1}\Bigg|_{\lambda=0} \\
&= \frac{d^m}{d\lambda^m}\frac{d^v}{d\theta^v}\left(\sum_{j=0}^{n}\left((-1)^{n-j}(\theta-1)^{n-j}\right)\binom{n}{j}e^{\left(\frac{r+j}{n+r}\right)\lambda}\theta^j\right)\Bigg|_{\theta=1}\Bigg|_{\lambda=0} \\
&= \frac{d^m}{d\lambda^m}\left(\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}e^{\left(\frac{r+j}{n+r}\right)\lambda}\left(\frac{d^v}{d\theta^v}(\theta-1)^{n-j}\theta^j\right)\Big|_{\theta=1}\right)\Bigg|_{\lambda=0}
\end{aligned}
$$

We have simplified the initial expression in such a way so as to simplify the process of taking the derivatives with respect to θ and λ and evaluating the resulting expression at $\theta = 1$ and $\lambda = 0$. In particular,

$$
\begin{aligned}
\left(T_{[v]}\right) &= \frac{d^m}{d\lambda^m}\left(\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}e^{\left(\frac{r+j}{n+r}\right)\lambda}\left(\sum_{i=0}^{v}\binom{v}{i}\left(\frac{d^i}{d\theta^i}(\theta-1)^{n-j}\right)\left(\frac{d^{v-i}}{d\theta^{v-i}}\theta^j\right)\right)\Bigg|_{\theta=1}\right)\Bigg|_{\lambda=0} \\
&= \frac{d^m}{d\lambda^m}\left(\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}e^{\left(\frac{r+j}{n+r}\right)\lambda}\left(\sum_{i=0}^{v}\binom{v}{i}\left((n-j)!\,\mathrm{I}_{\{n-j\}}(i)\right)\left(\frac{j!}{(j-v+i)!}\mathrm{I}_{\{v-i,\dots\}}(j)\right)\right)\right)\Bigg|_{\lambda=0} \\
&= \frac{d^m}{d\lambda^m}\left(\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}e^{\left(\frac{r+j}{n+r}\right)\lambda}\left(\binom{v}{n-j}(n-j)!\frac{j!}{\big(j-v+(n-j)\big)!}\mathrm{I}_{\{n-v,\dots\}}(j)\mathrm{I}_{\{n\geq v\}}\right)\right)\Bigg|_{\lambda=0} \\
&= \frac{d^m}{d\lambda^m}\left(\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}e^{\left(\frac{r+j}{n+r}\right)\lambda}\left(\binom{v}{n-j}(n-j)!\frac{j!}{(n-v)!}\mathrm{I}_{\{n-v,\dots\}}(j)\mathrm{I}_{\{n\geq v\}}\right)\right)\Bigg|_{\lambda=0} \\
&= \sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\left(\frac{r+j}{n+r}\right)^m\binom{v}{n-j}(n-j)!\frac{j!}{(n-v)!}\mathrm{I}_{\{n-v,\dots\}}(j)\mathrm{I}_{\{n\geq v\}} \\
&= \sum_{j=n-v}^{n}(-1)^{n-j}\binom{n}{j}\left(\frac{r+j}{n+r}\right)^m\binom{v}{n-j}(n-j)!\frac{j!}{(n-v)!}\mathrm{I}_{\{n\geq v\}} \\
&= \sum_{s=0}^{v}(-1)^{v-s}\binom{n}{v-s}\binom{v}{v-s}(v-s)!\frac{(s+n-v)!}{(n-v)!}\left(\frac{r+s+n-v}{n+r}\right)^m\mathrm{I}_{\{n\geq v\}} \\
&= \frac{n!}{(n-v)!}(n+r)^{-m}\left(\sum_{s=0}^{v}(-1)^{v-s}\binom{v}{s}(r+s+n-v)^m\right)\mathrm{I}_{\{n\geq v\}}
\end{aligned}
$$

We note that there is a misprint in Charlambides, page 265, where he has omitted a constant factor of $j!$ (in his notation) in his statement of the above result.

Problem 1(c).

Define $\mathcal{A}$ to be that subset of $\mathbb{S}^{n+r}$ where exactly k of $(Y_1, \dots, Y_n)$ equal i and the random variables $(Y_{n+1}, \dots, Y_{n+r})$ can be anything.

Hence,

$$
\begin{aligned}
P((Y_1,\dots,Y_{n+r})\in\mathcal{A}) &= \binom{n}{k}\left(1-\frac{e^{-\left(\frac{\lambda}{n+r}\right)}\left(\frac{\lambda}{n+r}\right)^i}{i!}\right)^{n-k}\left(\frac{e^{-\left(\frac{\lambda}{n+r}\right)}\left(\frac{\lambda}{n+r}\right)^i}{i!}\right)^k \\
&= \sum_{j=0}^{n-k}\frac{\binom{n}{k}\binom{n-k}{j}(-1)^{n-k-j}}{(i!)^k(i!)^{n-k-j}(n+r)^{ik+i(n-k-j)}}e^{\lambda\left(\frac{-k-(n-k-j)}{n+r}\right)}\lambda^{ik+i(n-k-j)} \\
&= e^{-\lambda}\sum_{j=0}^{n-k}\frac{\binom{n}{k}\binom{n-k}{j}(-1)^{n-k-j}}{(i!)^{n-j}(n+r)^{i(n-j)}}e^{\lambda\left(\frac{r+j}{n+r}\right)}\lambda^{i(n-j)}
\end{aligned}
$$

and

$$
e^{\lambda}P((Y_1,\dots,Y_{n+r})\in\mathcal{A}) = \sum_{j=0}^{n-k}\frac{\binom{n}{k}\binom{n-k}{j}(-1)^{n-k-j}}{(i!)^{n-j}(n+r)^{i(n-j)}}e^{\lambda\left(\frac{r+j}{n+r}\right)}\lambda^{i(n-j)}
$$

Therefore,

$$
\begin{aligned}
P(W_i=k) &= P((X_1,\dots,X_{n+r})\in\mathcal{A}_m) \\
&= \left(\frac{\mathrm{d}^m}{\mathrm{d}\lambda^m}\left(e^{\lambda}P((Y_1,\dots,Y_{n+r})\in\mathcal{A})\right)\right)\Bigg|_{\lambda=0} \\
&= \left(\frac{\mathrm{d}^m}{\mathrm{d}\lambda^m}\left(\sum_{j=0}^{n-k}\frac{\binom{n}{k}\binom{n-k}{j}(-1)^{n-k-j}}{(i!)^{n-j}(n+r)^{i(n-j)}}e^{\lambda\left(\frac{r+j}{n+r}\right)}\lambda^{i(n-j)}\right)\right)\Bigg|_{\lambda=0} \\
&= \sum_{j=0}^{n-k}\frac{\binom{n}{k}\binom{n-k}{j}(-1)^{n-k-j}}{(i!)^{n-j}(n+r)^{i(n-j)}}\left(\frac{\mathrm{d}^m}{\mathrm{d}\lambda^m}\left(e^{\lambda\left(\frac{r+j}{n+r}\right)}\lambda^{i(n-j)}\right)\right)\Bigg|_{\lambda=0}
\end{aligned}
$$

However for integers a and c,

$$
\left(\frac{\mathrm{d}^a}{\mathrm{d}\lambda^a}\left(e^{b\lambda}\lambda^c\right)\right)\Bigg|_{\lambda=0} = \begin{cases}\dfrac{b^{a-c}a!}{(a-c)!} & a\in\{c,c+1,\dots\} \\ 0 & a\in\{0,1,\dots,c-1\}\end{cases}
$$

It follows that,

$$
\begin{aligned}
P(W_i = k) &= \sum_{j=0}^{n-k} \frac{\binom{n}{k}\binom{n-k}{j}(-1)^{n-k-j}}{(i!)^{n-j}(n+r)^{i(n-j)}} \left(\frac{\mathrm{d}^m}{\mathrm{d}\lambda^m} \left(e^{\lambda\left(\frac{r+j}{n+r}\right)} \lambda^{i(n-j)} \right) \right) \Bigg|_{\lambda=0} \\
&= \sum_{j=0}^{n-k} \frac{\binom{n}{k}\binom{n-k}{j}(-1)^{n-k-j}}{(i!)^{n-j}(n+r)^{i(n-j)}} \left(\left(\frac{r+j}{n+r} \right)^{(m-i(n-j))} \frac{m!}{\big(m-i(n-j)\big)!} \mathrm{I}_{\{m \geq i(n-j)\}} \right) \\
&= \frac{\binom{n}{k} m!}{(n+r)^m} \sum_{j=0}^{n-k} (-1)^{n-k-j} \binom{n-k}{j} \frac{(r+j)^{(m-i(n-j))}}{(i!)^{n-j}\big(m-i(n-j)\big)!} \mathrm{I}_{\{m \geq i(n-j)\}} \\
&= \frac{\binom{n}{k} m!}{(n+r)^m} \sum_{j=0}^{n-k} (-1)^{n-k-j} \binom{n-k}{j} \frac{(r+j)^{(m-i(n-j))}}{(i!)^{n-j}\big(m-i(n-j)\big)!}
\end{aligned}
$$

provided $m \geq i \cdot n$.

This results agrees with Charlambides, page 265.

Problem 1(d).

$$
\begin{aligned}
\left((W_i)_{[v]}\right) &= \frac{d^m}{d\lambda^m}\frac{d^v}{d\theta^v}\left(\sum_{k=0}^{\infty} e^{\lambda} P\left((Y_1,\ldots,Y_{n+r})\in\mathcal{A}\right)\theta^k\right)\Bigg|_{\theta=1}\Bigg|_{\lambda=0} \\
&= \frac{d^m}{d\lambda^m}\frac{d^v}{d\theta^v}\left(\sum_{k=0}^{\infty}\left(\sum_{j=0}^{n-k}\frac{\binom{n}{k}\binom{n-k}{j}(-1)^{n-k-j}}{(i!)^{n-j}(n+r)^{i(n-j)}} e^{\lambda\left(\frac{r+j}{n+r}\right)}\lambda^{i(n-j)}\right)\theta^k\right)\Bigg|_{\theta=1}\Bigg|_{\lambda=0} \\
&= \frac{d^m}{d\lambda^m}\frac{d^v}{d\theta^v}\left(\sum_{k=0}^{n}\left(\sum_{j=0}^{n-k}\frac{\binom{n}{k}\binom{n-k}{j}(-1)^{n-k-j}}{(i!)^{n-j}(n+r)^{i(n-j)}} e^{\lambda\left(\frac{r+j}{n+r}\right)}\lambda^{i(n-j)}\right)\theta^k\right)\Bigg|_{\theta=1}\Bigg|_{\lambda=0} \\
&= \frac{d^m}{d\lambda^m}\frac{d^v}{d\theta^v}\left(\sum_{j=0}^{n}\sum_{k=0}^{n-j}\frac{\binom{n}{k}\binom{n-k}{j}(-1)^{n-k-j}}{(i!)^{n-j}(n+r)^{i(n-j)}} e^{\lambda\left(\frac{r+j}{n+r}\right)}\lambda^{i(n-j)}\theta^k\right)\Bigg|_{\theta=1}\Bigg|_{\lambda=0} \\
&= \frac{d^m}{d\lambda^m}\frac{d^v}{d\theta^v}\left(\sum_{j=0}^{n}\sum_{k=0}^{n-j}\frac{\binom{n}{j}\binom{n-j}{k}(-1)^{n-k-j}}{(i!)^{n-j}(n+r)^{i(n-j)}} e^{\lambda\left(\frac{r+j}{n+r}\right)}\lambda^{i(n-j)}\theta^k\right)\Bigg|_{\theta=1}\Bigg|_{\lambda=0} \\
&= \frac{d^m}{d\lambda^m}\frac{d^v}{d\theta^v}\left(\sum_{j=0}^{n}\frac{\binom{n}{j}}{(i!)^{n-j}(n+r)^{i(n-j)}} e^{\lambda\left(\frac{r+j}{n+r}\right)}\lambda^{i(n-j)}\left(\sum_{k=0}^{n-j}\binom{n-j}{k}(-1)^{n-j-k}\theta^k\right)\right)\Bigg|_{\theta=1}\Bigg|_{\lambda=0} \\
&= \frac{d^m}{d\lambda^m}\frac{d^v}{d\theta^v}\left(\sum_{j=0}^{n}\frac{\binom{n}{j}}{(i!)^{n-j}(n+r)^{i(n-j)}} e^{\lambda\left(\frac{r+j}{n+r}\right)}\lambda^{i(n-j)}(\theta-1)^{n-j}\right)\Bigg|_{\theta=1}\Bigg|_{\lambda=0} \\
&= \frac{d^m}{d\lambda^m}\left(\sum_{j=0}^{n}\frac{\binom{n}{j}}{(i!)^{n-j}(n+r)^{i(n-j)}} e^{\lambda\left(\frac{r+j}{n+r}\right)}\lambda^{i(n-j)}\left(\frac{d^v}{d\theta^v}(\theta-1)^{n-j}\right)\Big|_{\theta=1}\right)\Bigg|_{\lambda=0} \\
&= \frac{d^m}{d\lambda^m}\left(\frac{\binom{n}{n-v}}{(i!)^v(n+r)^{iv}} e^{\lambda\left(\frac{r+n-v}{n+r}\right)}\lambda^{iv} v!\,\mathrm{I}_{\{0,1,\ldots,n\}}(v)\right)\Bigg|_{\lambda=0} \\
&= \frac{\binom{n}{n-v} v!}{(i!)^v(n+r)^{iv}}\left(\frac{r+n-v}{n+r}\right)^{m-iv}\frac{m!}{(m-iv)!}\mathrm{I}_{\{m\geq iv\}}\mathrm{I}_{\{0,1,\ldots,n\}}(v) \\
&= \frac{\binom{n}{n-v} v!\, m!\,(n+r-v)^{m-iv}}{(i!)^v(m-iv)!\,(n+r)^m}
\end{aligned}
$$

provided $m \geq iv$ and $v \leq n$.

This results agrees with Charlambides, page 265 and with Buoncristiani, Cerosoli (???) who considered the special case $r = 0$.

Solution (2a)

Let X_j equal the number of balls that land in box j, $j = 1,2,\ldots,m$. Define $\mathbb{S}^m$ to be the product space $\{0,1,\ldots\} \times \cdots \times \{0,1,\ldots\}$ and let $\mathbb{S}_n^m$ be the set of all vectors $(c_1, c_2, \ldots, c_m)$ in $\mathbb{S}^m$ such that $c_1 + c_2 + \ldots + c_m = n$.

Let $\mathbb{A}_n$ be that set of all $(c_1, c_2, \ldots, c_m)$ such that

(i) a_j of the m values in the vector $(c_1, c_2, \ldots, c_m)$ equal j, $j = 0,1,2,\ldots,n$ and

(ii) $(c_1, c_2, \ldots, c_m) \in \mathbb{S}_n^m$.

Then,

$$P\big((N_0, N_1, \ldots, N_n) = (a_0, a_1, \ldots, a_n)\big) = P\big((X_1, X_2, \ldots, X_m) \in \mathbb{A}_n\big)$$

where $(X_1, X_2, \ldots, X_m)$ is a multinomial random vector with equal cell probabilities. That is,

$$P(X_1 = c_1, X_2 = c_2, \ldots, X_m = c_m) = \frac{n!}{c_1!\, c_2! \cdots \boldsymbol{c}_m!}\left(\frac{1}{m}\right)^{c_1}\left(\frac{1}{m}\right)^{c_2}\cdots\left(\frac{1}{m}\right)^{c_m}.$$

Now define $\mathbb{A}$ be that set of all $(c_1, c_2, \ldots, c_m)$ such that

(i) a_j of the m values in the vector $(c_1, c_2, \ldots, c_m)$ equal j, $j = 0,1,2,\ldots,n$ and

(ii) $(c_1, c_2, \ldots, c_m) \in \mathbb{S}^m$.

By the Poisson Randomization Theorem we have,

$$P\big((X_1, X_2, \ldots, X_m) \in \mathbb{A}_n\big) = \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\Big(e^{\lambda}P\big((Y_1, Y_2, \ldots, Y_m) \in \mathbb{A}\big)\Big)\Big|_{\lambda=0}$$

where $Y_1, Y_2, \ldots, Y_m$ are independent, identically distributed Poisson random variables with parameter $\frac{\lambda}{m}$. Now define,

N_j^P =the number of Y_i's that equal j for $j = 1,2,\ldots,n$

and

$N_{n_+}^P$ = the number of Y_i's$> n$.

Then,

$$P\big((Y_1, Y_2, \dots, Y_m) \in \mathbb{A}\big) = P\left(\left(N_0^P, N_1^P, \dots, N_n^P, N_{n_+}^P\right) = (a_0, a_1, \dots, a_n, 0)\right)$$

where the vector $\left(N_0^P, N_1^P, \dots, N_n^P, N_{n_+}^P\right)$ follows a multinomial distribution with parameters $\theta_0, \theta_1, \dots, \theta_n$, and θ_{n_+}, with

$$\theta_j = P(Y_1 = j) = \frac{e^{-\left(\frac{\lambda}{m}\right)}\left(\frac{\lambda}{m}\right)^j}{j!} \quad \text{and} \quad \theta_{n_+} = P(Y_1 > n).$$

Thus,

$$P\left(\left(N_0^P, N_1^P, \dots, N_n^P, N_{n_+}^P\right) = (a_0, a_1, \dots, a_n, 0)\right)$$

$$= \frac{m!}{a_0! a_1! \cdots \boldsymbol{a}_n! 0!} \; (\theta_0)^{a_0} \, (\theta_1)^{a_1} \cdots (\theta_n)^{a_n} \left(\theta_{n_+}\right)^0$$

$$= \frac{m!}{a_0! a_1! \cdots \boldsymbol{a}_n! 0!} \cdot \prod_{j=0}^{n} \left(\frac{e^{-\left(\frac{\lambda}{m}\right)}\left(\frac{\lambda}{m}\right)^j}{j!} \right)^{a_j}$$

$$= \frac{m!}{((0!)^{a_0}(1!)^{a_1} \cdots (n!)^{a_n}) \cdot (a_0! a_1! \cdots \boldsymbol{a}_n!)} \boldsymbol{e}^{-\left(\frac{\lambda}{m}\right)(a_0 + a_1 + \cdots + a_n)} \left(\frac{\lambda}{m}\right)^{(0 \cdot \boldsymbol{a}_0 + 1 \cdot \boldsymbol{a}_1 + \cdots + n \cdot \boldsymbol{a}_n)}$$

$$= \frac{m!}{m^n \cdot ((0!)^{a_0}(1!)^{a_1} \cdots (n!)^{a_n}) \cdot (a_0! a_1! \cdots \boldsymbol{a}_n!)} e^{-\lambda} \lambda^n.$$

Therefore,

$$P\big((X_1, X_2, \dots, X_m) \in \mathbb{A}_n\big) = \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\Big(e^{\lambda} P\big((Y_1, Y_2, \dots, Y_m) \in \mathbb{A}\big)\Big)\Big|_{\lambda=0}$$

$$= \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\left(e^{\lambda} \cdot \frac{m!}{m^n \cdot ((0!)^{a_0}(1!)^{a_1} \cdots (n!)^{a_n}) \cdot (a_0! \, a_1! \cdots \boldsymbol{a}_n!)} e^{-\lambda} \lambda^n\right)\Big|_{\lambda=0}$$

$$= \frac{m!}{m^n \cdot ((0!)^{a_0}(1!)^{a_1} \cdots (n!)^{a_n}) \cdot (a_0! a_1! \cdots \boldsymbol{a}_n!)} \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}(\lambda^n)\Big|_{\lambda=0}$$

$$= \frac{m! \cdot n!}{m^n \cdot ((0!)^{a_0}(1!)^{a_1} \cdots (n!)^{a_n}) \cdot (a_0! a_1! \cdots \boldsymbol{a}_n!)}.$$

Solution (**2**b)

We can proceed exactly as in (a) except now define $\mathbb{A}_n$ to be the set of all $(c_1, c_2, \dots, c_m)$ such that

(i)a_r of the m values in the vector $(c_1, c_2, \dots, c_m)$ equal r

(ii)$(c_1, c_2, \dots, c_m) \in \mathbb{S}_n^m$

and define $\mathbb{A}$ to be the set of all $(c_1, c_2, \dots, c_m)$ such that

(i)a_r of the m values in the vector $(c_1, c_2, \dots, c_m)$ equal r

(ii)$(c_1, c_2, \dots, c_m) \in \mathbb{S}^m$.

Then,

$$P(N_r = a_r) = P\big((X_1, X_2, \dots, X_m) \in \mathbb{A}_n\big)$$

and by the Poisson Randomization Theorem we have,

$$P\big((X_1, X_2, \dots, X_m) \in \mathbb{A}_n\big) = \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\Big(P\big((Y_1, Y_2, \dots, Y_m) \in \mathbb{A}\big)\Big)\Big|_{\lambda=0}.$$

However

$$P\big((Y_1, Y_2, \dots, Y_m) \in \mathbb{A}\big) = P(N_r^P = a_r)$$

where N_r^P =the number of Y_i's that equal r and N_r^P follows a binomial distribution with parameter θ_r, with

$$\theta_r = P(Y_1 = r) = \frac{e^{-\left(\frac{\lambda}{m}\right)}\left(\frac{\lambda}{m}\right)^r}{r!}.$$

Thus,

$$P(N_r^P = a_r)$$

$$= \frac{m!}{a_r!(m-a_r)!}\left(\frac{e^{-\left(\frac{\lambda}{m}\right)}\left(\frac{\lambda}{m}\right)^r}{r!}\right)^{a_r}\left(1 - \frac{e^{-\left(\frac{\lambda}{m}\right)}\left(\frac{\lambda}{m}\right)^r}{r!}\right)^{m-a_r}$$

$$= \frac{m!}{a_r!(m-a_r)!}\left(\frac{e^{-\left(\frac{\lambda}{m}\right)}\left(\frac{\lambda}{m}\right)^r}{r!}\right)^{a_r}\left(\sum_{j=0}^{m-a_r}\binom{m-a_r}{j}(-1)^j\left(\frac{e^{-\left(\frac{\lambda}{m}\right)}\left(\frac{\lambda}{m}\right)^r}{r!}\right)^j\right)$$

$$= \sum_{j=0}^{m-a_r}\left(\frac{\binom{m}{a_r}\binom{m-a_r}{j}(-1)^j}{(r!)^{(a_r+j)}\cdot m^{r\cdot(a_r+j)}}\right)\left(e^{-\lambda\left(\frac{m-a_r-j}{m}\right)}\lambda^{r\cdot(a_r+j)}\right)$$

.

Hence,

$$P\big((X_1, X_2, \dots, X_m) \in \mathbb{A}_n\big) = \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\big(P(N_r^P = a_r)\big)\Big|_{\lambda=0}$$

$$= \sum_{j=0}^{m-a_r}\left(\frac{\binom{m}{a_r}\binom{m-a_r}{j}(-1)^j}{(r!)^{(a_r+j)}m^{r(a_r+j)}}\right)\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\left(e^{-\lambda\left(\frac{m-a_r-j}{m}\right)}\lambda^{r(a_r+j)}\right)\Big|_{\lambda=0}$$

$$= \sum_{j=0}^{m-a_r} \left(\frac{\binom{m}{a_r}\binom{m-a_r}{j}(-1)^j}{(r!)^{(a_r+j)}m^n} \right) \frac{n!}{(n-ra_r-rj)!} (m-a_r-j)^{n-ra_r-rj}$$

By a change of variable, this solution can also be expressed in the form :

$$\sum_{j=a_r}^{m} \left(\frac{\binom{m}{a_r}(m-a_rj-a_r)(-1)^{j-a_r}}{(r!)^j m^n} \right) \frac{n!}{(n-rj)!} (m-j)^{n-rj}.$$

Note : $\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\left(e^{-\lambda t}\lambda^r\right)\Big|_{\lambda=0} = \frac{n!}{(n-r)!}t^{n-r}.$

Solution (**2**c)

For $0 \le a_0 \le m$

$$P(N_0 = a_0)$$

$$= \sum_{j=0}^{m-a_0} \left(\frac{\binom{m}{a_0}\binom{m-a_0}{j}(-1)^j}{(0!)^{(a_0+j)} \cdot m^n} \right) \frac{n!}{(n-0\cdot a_0-0\cdot j)!} (m-a_0-j)^{n-0\cdot a_0-0\cdot j}$$

$$= \sum_{j=0}^{m-a_0} \frac{\binom{m}{a_0}\binom{m-a_0}{j}(-1)^j}{m^n} (m-a_0-j)^n$$

$$= \frac{m_{(m-a_0)}}{m^n} \left(\sum_{j=0}^{m-a_0} \frac{1}{(m-a_0)!} \binom{m-a_0}{j} (-1)^j (m-a_0-j)^n \right)$$

$$= \frac{m_{(m-a_0)}}{m^n} S(n, m-a_0).$$

Taking $a_0 = m-k$, we have,

$$P(N_0 = m-k) = \frac{m_{(k)}}{m^n} S(n,k).$$

From the above it is clear that the two probability models are related. However the critical difference should be stressed. $(X_1, X_2, \dots, X_n)$ is a vector of dependent random variables while $(Y_1, Y_2, \dots, Y_n)$ is a vector of independent random variables. It is the purpose of the Poisson Randomization Theorem given below to restructure problems involving the multinomial model into equivalent problems involving the Poisson model. By so doing, we will be able to exploit the independence of the Y_i's.

Theorem

Suppose an experiment consists of t independent trials and that every trial has n distinct outcomes. Let p_j equal the probability of outcome j on any trial and let C_j equal the number of times outcome j occurs in these t (multinomial) trials. Then

.

$$\big(\Psi(C_1,\dots,C_n)\big)=\frac{\mathrm{d}^t}{\mathrm{d}\lambda^t}\Big(e^{\lambda}\big(\Psi(Z_1,\dots,Z_n)\big)\Big)\Big|_{\lambda=0}.$$

where $Z_1,\dots,Z_n$ are independent Poisson random variables such that Z_j has parameter λp_j, $j=1,\dots,n$. That is

$$P(Z_i=z_i)=\frac{e^{-\lambda p_i}(\lambda p_i)^{z_i}}{z_i!}\qquad z_i\in\{0,1,\dots\}.$$

Theorem

Let D_k equal the number of C_j's which equal k, $k\in\{0,1,\dots,t\}$. Then

$D_0+D_1+\cdots+D_t=n$ and $0D_0+1D_1+\cdots+tD_t=t$ and

$$\big(\Psi(D_0,D_1,\dots,D_t,0,0,\dots)\big)=\frac{d^t}{d\lambda^t}\frac{d^n}{d\theta^n}\left(e^{\theta e^{\left(\frac{\lambda}{n}\right)}}\big(\Psi(Z_0,Z_1,\dots)\big)\right)\Bigg|_{\substack{\lambda=0\\ \theta=0}}$$

where $Z_0,Z_1,\dots$ are independent and $Z_j\sim$Poisson$\left(\frac{\theta\lambda^j}{n^j j!}\right)$ $\quad j=0,1,\dots$

Proof

Let $Z_j\sim$Poisson$\left(\frac{\theta\lambda^j}{n^j j!}\right)$ $\quad j=0,1,\dots$

$$P(Z_0=z_0,Z_1=z_1,\dots)$$

$$=e^{-\left(\frac{\theta\lambda^0}{n^0 0!}+\frac{\theta\lambda^1}{n^1 1!}+\cdots\right)}\frac{\left(\frac{\theta\lambda^0}{n^0 0!}\right)^{z_0}\left(\frac{\theta\lambda^1}{n^1 1!}\right)^{z_1}\cdots}{z_0!\,z_1!\cdots}$$

$$=\left(e^{-\theta e^{\left(\frac{\lambda}{n}\right)}}\right)\frac{\left(\frac{1}{n^0 0!}\right)^{z_0}\left(\frac{1}{n^1 1!}\right)^{z_1}\cdots}{z_0!\;z_1!\cdots}(\theta^{z_0+z_1+\cdots})(\lambda^{0\cdot z_0+1\cdot z_1+\cdots})$$

$$P(Z_0 = z_0, Z_1 = z_1, \dots | z_0 + z_1 + \cdots = r \quad \text{and} \quad 0 \cdot z_0 + 1 \cdot z_1 + \cdots = m)$$

$$= \frac{\frac{\left(\frac{1}{n^0 0!}\right)^{z_0}\left(\frac{1}{n^1 1!}\right)^{z_1}\cdots}{z_0!\ z_1!\cdots}}{\sum_{(z_0,z_1,\dots)\ni z_0+z_1+\cdots=r} \cdots \sum_{0\cdot z_0+1\cdot z_1+\cdots=m} \frac{\left(\frac{1}{n^0 0!}\right)^{z_0}\left(\frac{1}{n^1 1!}\right)^{z_1}\cdots}{z_0!\ z_1!\cdots}}$$

$$= \frac{\left(\frac{1}{n^{0\cdot z_0+1\cdot z_1+\cdots}}\right)\frac{\left(\frac{1}{0!}\right)^{z_0}\left(\frac{1}{1!}\right)^{z_1}\cdots}{z_0!\ z_1!\cdots}}{\sum_{(z_0,z_1,\dots)\ni z_0+z_1+\cdots=r} \cdots \sum_{0\cdot z_0+1\cdot z_1+\cdots=m} \left(\frac{1}{n^{0\cdot z_0+1\cdot z_1+\cdots}}\right)\frac{\left(\frac{1}{0!}\right)^{z_0}\left(\frac{1}{1!}\right)^{z_1}\cdots}{z_0!\ z_1!\cdots}}$$

$$= \frac{\left(\frac{1}{n^m}\right)\frac{\left(\frac{1}{0!}\right)^{z_0}\left(\frac{1}{1!}\right)^{z_1}\cdots}{z_0!\ z_1!\cdots}}{\sum_{(z_0,z_1,\dots)\ni z_0+z_1+\cdots=r} \cdots \sum_{0\cdot z_0+1\cdot z_1+\cdots=m} \left(\frac{1}{n^m}\right)\frac{\left(\frac{1}{0!}\right)^{z_0}\left(\frac{1}{1!}\right)^{z_1}\cdots}{z_0!\ z_1!\cdots}}$$

$$= \frac{\frac{1}{((0!)^{z_0}(1!)^{z_1}\cdots)\ z_0!\ z_1!\cdots}}{\sum_{(z_0,z_1,\dots)\ni z_0+z_1+\cdots=r} \cdots \sum_{0\cdot z_0+1\cdot z_1+\cdots=m} \frac{1}{((0!)^{z_0}(1!)^{z_1}\cdots)\ z_0!\ z_1!\cdots}}$$

provided $z_0 + z_1 + \cdots = r$ and $0 \cdot z_0 + 1 \cdot z_1 + \cdots = m$

$$= \frac{\frac{m!r!}{r^m((0!)^{z_0}(1!)^{z_1}\cdots)\ z_0!\ z_1!\cdots}}{\sum_{(z_0,z_1,\dots)\ni z_0+z_1+\cdots=r} \cdots \sum_{0\cdot z_0+1\cdot z_1+\cdots=m} \frac{m!r!}{r^m((0!)^{z_0}(1!)^{z_1}\cdots)\ z_0!\ z_1!\cdots}}$$

provided $z_0 + z_1 + \cdots = r$ and $0 \cdot z_0 + 1 \cdot z_1 + \cdots = m$

$$= \frac{m!r!}{r^m((0!)^{z_0}(1!)^{z_1}\cdots)\ z_0!\ z_1!\cdots}$$

provided $z_0 + z_1 + \cdots = r$ and $0 \cdot z_0 + 1 \cdot z_1 + \cdots = m$

$$\big(\Psi(Z_0, Z_1, \dots)\big)$$

$$= \sum_m \sum_r (\Psi(Z_0, Z_1, \dots) | Z_0 + Z_1 + \cdots = r 0 Z_0 + 1 Z_1 + \cdots = m) P \begin{pmatrix} Z_0 + Z_1 + \cdots = r \\ 0 Z_0 + 1 Z_1 + \cdots = m \end{pmatrix}$$

$$= \sum_m \sum_r \big(\Psi(D_0, D_1, \dots, D_m, 0, 0, \dots)\big) P \begin{pmatrix} Z_0 + Z_1 + \cdots = r \\ 0 Z_0 + 1 Z_1 + \cdots = m \end{pmatrix}$$

$$= \sum_m \sum_r \big(\Psi(D_0, D_1, \dots, D_m, 0, 0, \dots)\big) \left(e^{-\theta e^{\left(\frac{\lambda}{n}\right)}}\right) \left(\frac{1}{m!\ r!}\right) (\lambda^m)(\theta^r)$$

$$\frac{d^t}{d\lambda^t}\frac{d^n}{d\theta^n}\left(e^{\theta e^{\left(\frac{\lambda}{n}\right)}}\left(\Psi(Z_0, Z_1, \dots)\right)\right)\Bigg|_{\substack{\lambda=0\\ \theta=0}}$$

$$= \sum\nolimits_m \sum\nolimits_r \left(\Psi(D_0, D_1, \dots, D_m, 0,0, \dots)\right)\left(\frac{1}{m!\, r!}\right)\left(\frac{d^t}{d\lambda^t}\frac{d^n}{d\theta^n}\left((\lambda^m)(\theta^r)\right)\Big|_{\substack{\lambda=0\\ \theta=0}}\right)$$

$$= \left(\Psi(D_0, D_1, \dots, D_t, 0,0, \dots)\right)$$

$$\left(\Psi(D_0, D_1, \dots, D_t, 0,0, \dots)\right) = \frac{d^t}{d\lambda^t}\frac{d^n}{d\theta^n}\left(e^{\theta e^{\left(\frac{\lambda}{n}\right)}}\left(\Psi(Z_0, Z_1, \dots)\right)\right)\Bigg|_{\substack{\lambda=0\\ \theta=0}}$$

$$P\begin{pmatrix} Z_0 + Z_1 + \cdots = r \\ 0Z_0 + 1Z_1 + \cdots = m \end{pmatrix}$$

$$= \left(e^{-\theta e^{\left(\frac{\lambda}{n}\right)}}\right)\left(\frac{1}{m!\, r!}\right)(\lambda^m)(\theta^r) \sum_{(z_0, z_1, \dots)\ni z_0 + z_1 + \cdots = r 0 \cdot z_0 + 1 \cdot z_1 + \cdots = m} \cdots \sum \frac{m!\, r!}{r^m((0!)^{z_0}(1!)^{z_1}\cdots)\; z_0!\; z_1! \cdots}$$

$$= \left(e^{-\theta e^{\left(\frac{\lambda}{n}\right)}}\right)\left(\frac{1}{m!\, r!}\right)(\lambda^m)(\theta^r)$$

$$e^{\theta e^{\left(\frac{\lambda}{n}\right)}}\left(\Psi(Z_0, Z_1, \dots)\right)$$

$$= \sum\nolimits_m \sum\nolimits_r \left(\Psi(D_0, D_1, \dots, D_m, 0,0, \dots)\right)\left(\frac{1}{m!\, n!}\right)(\theta^m)$$

Suppose identical balls are independently distributed into n equally likely boxes (i.e. a multinomial model or classical allocation scheme) until any k of the n boxes have at least m balls each. Let M_v denote the number of boxes containing exactly v balls when the distribution of balls into boxes stops. Let $Z \sim$Poisson(θ). For $v = m$

$$P(M_v - 1 = s) = \int_0^\infty \frac{n!}{(m-1)!(n-k)!s!(k-s-1)!}\left(P(Z \le m-1)\right)^{n-k}$$

$$\times \left(P(Z=m)\right)^{s}\left(P(Z\geq m+1)\right)^{k-s-1}\,\theta^{m-1}e^{-\theta}d\theta$$

$s \in \{0,\dots,k-1\}$ and equals 0 else. In the case $v=m$

$$\left((M_v-1)_{(r)}\right)=\int_0^{\infty}\frac{n!}{(v-1)!(n-k)!(k-r-1)!}\left(P(Z\leq v-1)\right)^{n-k}$$

$$\times \left(P(Z=v)\right)^{r}\left(P(Z\geq v)\right)^{k-r-1}\,\theta^{v-1}e^{-\theta}d\theta.$$

To illustrate we note the special case $v=m=1$ simplifies to

$$P(M_1=s)=\sum_{j=0}^{k-s-1}\sum_{i=0}^{j}(-1)^j\binom{k-s-1}{j}\binom{j}{i}$$

$$\times\frac{n!(s+i)!}{(n-k)!s!(k-s-1)!}\left(\frac{1}{n-k+s+j+1}\right)^{s+i+1}$$

and

$$\left((M_1-1)_{(r)}\right)=\sum_{j=0}^{k-r-1}(-1)^j\binom{k-r-1}{j}$$

$$\times\frac{n!r!}{(n-k)!(k-r-1)!}\left(\frac{1}{n-k+r+j+1}\right)^{r+1}.$$

Holst ， considers the special case $v=m=1$ and additionally assumes $k=n$ but the form of the solutions given in these papers is complicated.

Proof

For $v = m$

$$P(M_v - 1 = s) = \sum_{j=1}^{n} P(M_v - 1 = s \text{and} j^{th} \text{box is last box to receive a ball})$$

$$= nP(M_v - 1 = s \text{and} 1^{st} \text{box is last box to receive a ball})$$

$$= nP(\mathcal{A} \cap 1^{st} \text{box is last box to receive a ball})$$

where event $\mathcal{A}$ is

$\mathcal{A}$: exactly $k - 1$ of boxes 2 through n have at least m balls and exactly s of boxes 2 through n have exactly m balls.

Stated in this way we can view the problem as asking for the probability of event $\mathcal{A}$ where the stopping rule is to stop when the 1^{st} box has exactly m balls. The purpose of casting the problem in this way is to make Theorem 4 appropriate. Let C_j equal the number of balls distributed into box $j + 1, j \in \{1, \dots, n - 1\}$, before the trials stop. Then define

$$\Psi(C_1, \dots, C_{n-1}) = \begin{cases} 1 & \text{exactly } k - 1 \text{ of } C_1, \dots, C_{n-1} \text{ are} \geq m \text{ and exactly } s \text{ of } C_1, \dots, C_{n-1} \text{ equal } m \\ 0 & \text{else.} \end{cases}$$

It follows from Theorem 4 that

$$P(M_v - 1 = s) = n \int_0^\infty \frac{1}{(m-1)!} P\begin{pmatrix} \text{exactly } k - 1 \text{ of } Z_1, \dots, Z_{n-1} \text{ are} \geq m \\ \text{and exactly } s \text{ of } Z_1, \dots, Z_{n-1} \text{ equal } m \end{pmatrix} \theta^{k-1} e^{-\theta} d\theta$$

where $Z_1, \ldots, Z_n$ are iid Poisson(θ). The final form follows on recognizing that exactly $k-1$ of $Z_1,\ldots,Z_{n-1}$ are at least m and exactly s of $Z_1,\ldots,Z_{n-1}$ equal m if and only if exactly $n-k$ of $Z_1,\ldots,Z_{n-1}$ are less than m, exactly s of $Z_1,\ldots,Z_{n-1}$ equal m, and exactly $k-s-1$ of $Z_1,\ldots,Z_{n-1}$ exceed m.

The factorial moment result follows from definition. We have

$$\left((M_v-1)_{(r)}\right) = \sum_{s=r}^{k-1} s_{(r)} P(M_v - 1 = s)$$

$$= \int_0^\infty \left(\frac{n!}{(m-1)!(n-k)!}\left(P(Z \le m-1)\right)^{n-k} \theta^{m-1} e^{-\theta}\right.$$

$$\left. \times \sum_{s=r}^{k-1} \frac{1}{(s-r)!(k-s-1)!}\left(P(Z=m)\right)^{s}\left(P(Z \ge m+1)\right)^{k-s-1}\right) d\theta$$

and the final form follows on applying the binomial theorem to remove the sum.

$$\int_0^\infty \frac{n!}{(v-1)!(n-k)!(k-r-1)!}\left(P(Z \le v-1)\right)^{n-k}$$

$$\times \left(P(Z=v)\right)^{r}\left(P(Z \ge v)\right)^{k-r-1} \theta^{v-1} e^{-\theta} d\theta.$$

$$= n\int_0^\infty \frac{1}{(m-1)!}\frac{(n-1)!}{(n-k)!\,s!\,(k-s-1)!}\left(P(Z \le m-1)\right)^{n-k}\left(P(Z=m)\right)^{s}$$

$$\times \left(P(Z \ge m+1)\right)^{k-s-1} \theta^{m-1} e^{-\theta} d\theta$$

$$= \int_0^\infty \frac{n!}{(m-1)!\,(n-k)!\,s!\,(k-s-1)!}\left(P(Z \le m-1)\right)^{n-k}\left(P(Z=m)\right)^{s}$$

$$\times \left(P(Z \ge m+1)\right)^{k-s-1} \theta^{m-1} e^{-\theta} d\theta$$

and

$$\left((M_v-1)_{(r)}\right)=\sum_{s=r}^{k-1} s_{(r)}P(M_v-1=s)$$

$$=\sum_{s=r}^{k-1} s_{(r)}\int_0^{\infty}\frac{n!}{(m-1)!(n-k)!s!(k-s-1)!}\left(P(Z\le m-1)\right)^{n-k}$$

$$\times\left(P(Z=m)\right)^{s}\left(P(Z\ge m+1)\right)^{k-s-1}\theta^{m-1}e^{-\theta}d\theta$$

$$=\int_0^{\infty}\sum_{s=r}^{k-1}\frac{n!}{(m-1)!(n-k)!(s-r)!(k-s-1)!}\left(P(Z\le m-1)\right)^{n-k}$$

$$\times\left(P(Z=m)\right)^{s}\left(P(Z\ge m+1)\right)^{k-s-1}\theta^{m-1}e^{-\theta}d\theta$$

$$=\int_0^{\infty}\frac{n!}{(m-1)!\,(n-k)!}\left(P(Z\le m-1)\right)^{n-k}\theta^{m-1}e^{-\theta}d\theta$$

$$\times\sum_{s=r}^{k-1}\frac{1}{(s-r)!(k-s-1)!}\left(P(Z=m)\right)^{s}\left(P(Z\ge m+1)\right)^{k-s-1}$$

$$=\int_0^{\infty}\frac{n!}{(m-1)!\,(n-k)!}\left(P(Z\le m-1)\right)^{n-k}\theta^{m-1}e^{-\theta}d\theta$$

$$\times\frac{1}{(k-r-1)!}\sum_{s=r}^{k-1}\frac{(k-r-1)!}{(s-r)!(k-s-1)!}\left(P(Z=m)\right)^{s}\left(P(Z\ge m+1)\right)^{k-s-1}$$

$$=\int_0^{\infty}\frac{n!}{(m-1)!\,(n-k)!}\left(P(Z\le m-1)\right)^{n-k}\theta^{m-1}e^{-\theta}d\theta$$

$$\times\frac{\left(P(Z=m)\right)^{r}}{(k-r-1)!}\left(P(Z\ge m)\right)^{k-r-1}$$

$$=\int_0^{\infty}\frac{n!}{(m-1)!\,(n-k)!\,(k-r-1)!}\left(P(Z\le m-1)\right)^{n-k}\left(P(Z=m)\right)^{r}$$

$$\big(P(Z \geq m)\big)^{k-r-1}\theta^{m-1}e^{-\theta}d\theta$$

For $v < m$ and $s \in \{0,1,\dots,n-k\}$

$$P(M_v = s) = \int_0^\infty \frac{n!}{(m-1)!(n-k-s)!s!(k-1)!}\big(P(Z \leq m-1) - P(Z = v)\big)^{n-k-s}$$

$$\times \big(P(Z = v)\big)^s\big(P(Z \geq m)\big)^{k-1}\,\theta^{m-1}e^{-\theta}d\theta$$

and

$$\big((M_v)_{(r)}\big) = \int_0^\infty \frac{n!}{(m-1)!(n-k-r)!(k-1)!}\big(P(Z \leq m-1)\big)^{n-k-r}$$

$$\times \big(P(Z = v)\big)^r\big(P(Z \geq m)\big)^{k-1}\,\theta^{m-1}e^{-\theta}d\theta.$$

For $v > m$ and $s \in \{0,\dots,k-1\}$

$$P(M_v = s) = \int_0^\infty \frac{n!}{(m-1)!(n-k)!s!(k-s-1)!}\big(P(Z \leq m-1)\big)^{n-k}$$

$$\times \big(P(Z = v)\big)^s\big(P(Z \geq m) - P(Z = v)\big)^{k-s-1}\,\theta^{m-1}e^{-\theta}d\theta$$

and

$$\left((M_v)_{(r)}\right) = \int_0^\infty \frac{n!}{(m-1)!(n-k)!(k-r-1)!}\left(P(Z \le m-1)\right)^{n-k}$$

$$\times \left(P(Z=v)\right)^r \left(P(Z \ge m)\right)^{k-r-1} \theta^{m-1}e^{-\theta}d\theta.$$

Case 1. $v < m$

$$P(M_v = s) = \sum_{j=1}^{n} P(M_v = s \text{and} j^{th} \text{boxislastboxtoreceiveaballbeforestopping})$$

$$= nP(M_v = s \text{and} 1^{st} \text{boxislastboxtoreceiveaballbeforestopping})$$

$= nP(\text{exactly} k-1 \text{ofboxes2through} n \text{ have}$
$\ge\ m\text{ballsexactly}s\text{ofboxes2through}n \text{ haveexactly}v\text{ballsand}1^{st}\text{boxislastboxtoreceiveaballbeforestopping})$

$$= n\int_0^\infty \frac{1}{(m-1)!} P(\text{exactly} k-1 \text{of}(Z_2, \dots, Z_n)$$

$$\ge\ m\text{exactly}s\text{ofof}(Z_2, \dots, Z_n)\text{equal}v)\ \theta^{m-1}e^{-\theta}d\theta$$

$$= \int_0^\infty \frac{n!}{(m-1)!\,(k-1)!\,s!\,(n-k-s)!}\left(P(Z \le m-1) - P(Z=v)\right)^{n-k-s}$$

$$\times \left(P(Z=v)\right)^s \left(P(Z \ge m)\right)^{k-1} \theta^{m-1}e^{-\theta}d\theta$$

and

$$\left((M_v)_{(r)}\right) = \sum_{s=r}^{n-k} s_{(r)} P(M_v = s)$$

$= \sum_{s=r}^{n-k} s_{(r)} \int_0^\infty \frac{n!}{(m-1)!(n-k-s)!s!(k-1)!}\left(P(Z \le m-1) - P(Z=v)\right)^{n-k-s}$

$$\times \left(P(Z=v)\right)^s \left(P(Z \ge m)\right)^{k-1} \theta^{m-1}e^{-\theta}d\theta$$

$= \int_0^\infty \sum_{s=r}^{n-k} \frac{n!}{(m-1)!(n-k-s)!(s-r)!(k-1)!}\left(P(Z \le m-1) - P(Z=v)\right)^{n-k-s}$

$$\times \left(P(Z=v)\right)^s \left(P(Z \ge m)\right)^{k-1} \theta^{m-1}e^{-\theta}d\theta$$

$$= \int_0^\infty \frac{n!}{(m-1)!(k-1)!}\big(P(Z \geq m)\big)^{k-1}\,\theta^{m-1}e^{-\theta}d\theta$$

$$\frac{1}{(n-k-r)!}\sum_{s=r}^{n-k}\binom{n-k-r}{s-r}\big(P(Z \leq m-1) - P(Z=v)\big)^{n-k-s}\big(P(Z=v)\big)^{s}$$

$$= \int_0^\infty \frac{n!}{(m-1)!(k-1)!}\big(P(Z \geq m)\big)^{k-1}\,\theta^{m-1}e^{-\theta}d\theta$$

$$\frac{1}{(n-k-r)!}\big(P(Z=v)\big)^{r}\big(P(Z \leq m-1)\big)^{n-k-r}$$

$$= \int_0^\infty \frac{n!}{(m-1)!\,(n-k-r)!\,(k-1)!}\big(P(Z \leq m-1)\big)^{n-k-r}\big(P(Z=v)\big)^{r}$$

$$\times \big(P(Z \geq m)\big)^{k-1}\,\theta^{m-1}e^{-\theta}d\theta$$

Case 3. $v > m$

$$P(M_v = s) = \sum_{j=1}^{n} P(M_v = s \text{and} j^{th} \text{boxislastboxtoreceiveaballbeforestopping})$$

$$= nP(M_v = s \text{and} 1^{st} \text{boxislastboxtoreceiveaballbeforestopping})$$

$= nP(\text{exactly}k-1\text{ofboxes2through}n\text{ have}$
$\geq\ m\text{ballsexactly}s\text{ofboxes2through}n\text{ haveexactly}v\text{ballsand}1^{st}\text{boxislastboxtoreceiveaballbeforestopping})$

$$= \int_0^\infty \frac{n!}{(m-1)!\,(n-k)!\,s!\,(k-s-1)!}\big(P(Z \leq m-1)\big)^{n-k}\big(P(Z=v)\big)^{s}$$

$$\times \big(P(Z \geq m) - P(Z=v)\big)^{k-s-1}\,\theta^{m-1}e^{-\theta}d\theta$$

and

$$\big((M_v)_{(r)}\big) = \sum_{s=r}^{k-1} s_{(r)}P(M_v = s)$$

$$= \sum_{s=r}^{k-1} s_{(r)} \int_0^\infty \frac{n!}{(m-1)!(n-k)!s!(k-s-1)!}\big(P(Z \leq m-1)\big)^{n-k}$$

$$\times \big(P(Z=v)\big)^{s}\big(P(Z \geq m) - P(Z=v)\big)^{k-s-1}\,\theta^{m-1}e^{-\theta}d\theta$$

$$= \int_0^{\infty} \sum_{s=r}^{k-1} \frac{n!}{(m-1)!(n-k)!(s-r)!(k-s-1)!} \big(P(Z \le m-1)\big)^{n-k}$$

$$\times \big(P(Z=v)\big)^{s} \big(P(Z \ge m) - P(Z=v)\big)^{k-s-1} \theta^{m-1} e^{-\theta} d\theta$$

$$= \int_0^{\infty} \frac{n!}{(m-1)!\,(n-k)!} \big(P(Z \le m-1)\big)^{n-k} \theta^{m-1} e^{-\theta} d\theta$$

$$\times \frac{1}{(k-r-1)!} \sum_{s=r}^{k-1} \frac{(k-r-1)!}{(s-r)!(k-s-1)!} \big(P(Z=v)\big)^{s} \big(P(Z \ge m) - P(Z=v)\big)^{k-s-1}$$

$$= \int_0^{\infty} \frac{n!}{(m-1)!\,(n-k)!} \big(P(Z \le m-1)\big)^{n-k} \theta^{m-1} e^{-\theta} d\theta$$

$$\times \frac{\big(P(Z=v)\big)^{r}}{(k-r-1)!} \big(P(Z \ge m)\big)^{k-r-1}$$

$$= \int_0^{\infty} \frac{n!}{(m-1)!\,(n-k)!\,(k-r-1)!} \big(P(Z \le m-1)\big)^{n-k} \big(P(Z=v)\big)^{r}$$

$$\big(P(Z \ge m)\big)^{k-r-1} \theta^{m-1} e^{-\theta} d\theta$$

Example 5

	"A company hires m people. It declares a holiday on the birthday of any
employee. What value of m maximizes the expected work force per year?" This	

question was posed in Chance, Vol. 13, No. 4, Fall 2000, page 54.

	Assume that people's birthdays are independently determined and that every day of

the year is equally likely to occur as a birthday. Let a year consist of $n + r$ days

where n equals the number of potential work days (days the company will be open

if no employee has a birthday on that day) and r equals the number of

non-potential work days (days the company would not be open even if no

employee has a birthday on that day, e.g. weekends, July 4th, etc.).

	Let T_m equal the number of potential work

	days where the company declares a
holiday because one or more of the m employees has a birthday on that day. Let	
W_m equal the work force per year if the company has m employees.	

	We will define work force per year as the product of the work force per day
(number of employees) and the number of days the company is open per year.	
That is,	

$$W_m = m(n - T_m)$$

	Show that

$$(W_m) = mn\left(\frac{n+r-1}{n+r}\right)^m$$

	and that (W_m) is maximized at $m = n + r$, the total number of days in a year. It
is interesting that the solution depends on n and r only through their sum.	

Proof

	Clearly,
$(W_m) = \big(m(n - T_m)\big) = mn \ - \ m(T_m)$	

	Furthermore, from part (b) of this problem with $v = 1$ (see

	Poisson Randomization
problem 1(b)) we have	

$$(T_m) = \frac{n!}{(n-1)!}(n+r)^{-m}\left(\sum_{s=0}^{1}(-1)^{1-s}\binom{1}{s}(n+r+s-1)^m\right)$$
$$= n(n+r)^{-m}(-(n+r-1)^m + (n+r)^m)$$
$$= n\left(1-\left(\frac{n+r-1}{n+r}\right)^m\right)$$

	Thus,

$$(W_m) = mn - mn\left(1-\left(\frac{n+r-1}{n+r}\right)^m\right) = mn\left(\frac{n+r-1}{n+r}\right)^m$$

	A little calculus shows that (W_m), the expected work force per year, achieves its
maximum at both $m = n + r - 1$ and $m = n + r$.	

Can also easily solve this problem by letting

$$\mathrm{I}_j = \begin{cases} 1 & \text{someemployeehasabirthdayonDay}j \\ 0 & \text{else} \end{cases}$$

Also let p_j =probability a person is born on Day j for any j a potential workday

Then

$$(T_m) = \left(\sum_{j=1}^{n} \mathrm{I}_j\right) = \sum_{j=1}^{n} \left(1 - \left(1 - p_j\right)^m\right)$$

Suppose we perform t multinomial trials with $n + r$ distinct possible outcomes. We will assume that n of the outcomes have been classified as type A outcomes and that the remaining r outcomes as type B outcomes. We will assume that all type A outcomes occur with constant probability p_1 and that all type B outcomes occur with constant probability p_2 so that $np_1 + rp_2 = 1$.

Let C_j equal the number of type A outcomes which occur exactly j times in the t trials and let D_j equal the number of type B outcomes which occur exactly j times in the t trials, $j \in \{0,1, \dots, t\}$.

Then,

$$P(C_i = k) = \sum_{j=0}^{n-k} (-1)^{n-k-j} \frac{\binom{n-k}{j}\binom{n}{k} t!}{(i!)^{n-j}(t-i(n-j))!} (p_1)^{i(n-j)} \left(1 - p_1(n-j)\right)^{t-i(n-j)}$$

and

$$\left((C_0)_{(\alpha_0)} \cdots (C_t)_{(\alpha_t)} (D_0)_{(\beta_0)} \cdots (D_t)_{(\beta_t)}\right)$$

$$= \left(\frac{n!\ \ r!\ \ t!}{(n-\alpha)!\ (r-\beta)!\ \left(t - \Upsilon_\alpha - \Upsilon_\beta\right)!\ (0!)^{\alpha_0+\beta_0} \cdots (t!)^{\alpha_t+\beta_t}}\right) p_1^{\Upsilon_\alpha}\, p_2^{\Upsilon_\beta} (1 - p_1\alpha - p_2\beta)^{t-\Upsilon_\alpha-\Upsilon_\beta}$$

where

$$\alpha = \alpha_0 + \cdots + \alpha_t$$

$$\beta = \beta_0 + \cdots + \beta_t$$

$$\Upsilon_\alpha = 0\alpha_0 + \cdots + t\alpha_t$$

$$\Upsilon_\beta = 0\beta_0 + \cdots + t\beta_t$$

The first result is given in Charalambides . David and Barton give the first result for the special case $i = 0$. Charalambides gives the second result for the special case

$\alpha_k = v,\ \alpha_i = 0$ for $i \neq k$ and $\beta_k = 0$ for all k. We note that $X_{(0)} \equiv 1$ by definition.

Proof

For the first result let V_j equal the number of balls in urn j and let

$$\Psi(V_1, \ldots, V_{n+r}) = \begin{cases} 1 & \text{exactly} k \text{of} (V_1, \ldots, V_n) \text{equal} i \\ 0 & \text{else.} \end{cases}$$

By Theorem 1

$$P(C_i = k) = \big(\Psi(V_1, \ldots, V_{n+r})\big)$$

$$= \frac{\mathrm{d}^t}{\mathrm{d}\lambda^t}\Big(e^{\lambda} P(\text{exactly} k \text{of} (Z_1, \ldots, Z_n) \text{equal} i)\Big)\Big|_{\lambda=0}$$

$$= \frac{\mathrm{d}^t}{\mathrm{d}\lambda^t}\left(e^{\lambda}\binom{n}{k}\big(P(Z=i)\big)^k\big(1-P(Z=i)\big)^{n-k}\right)\Big|_{\lambda=0}$$

where $Z \sim$Poisson(λp_1). The final form follows on substituting for $P(Z=i)$ and extracting the coefficient of λ^t.

For the joint factorial moment result, we have by Theorem 5

$$\big((C_0)_{(\alpha_0)} \cdots (C_t)_{(\alpha_t)} (D_0)_{(\beta_0)} \cdots (D_t)_{(\beta_t)}\big)$$

$$= \frac{d^t}{d\lambda^t}\frac{d^n}{d\theta^n}\frac{d^r}{d\tau^r}\left(e^{\theta e^{p_1\lambda} + \tau e^{p_2\lambda}}\big((W_0)_{(\alpha_0)} \cdots (W_t)_{(\alpha_t)} (Z_0)_{(\beta_0)} \cdots (Z_t)_{(\beta_t)}\big)\right)\Big|_{\substack{\lambda=0 \\ \theta=0 \\ \tau=0}}$$

$$= \frac{d^t}{d\lambda^t}\frac{d^n}{d\theta^n}\frac{d^r}{d\tau^r}\left(e^{\theta e^{p_1\lambda} + \tau e^{p_2\lambda}}\big((W_0)_{(\alpha_0)}\big) \cdots \big((W_t)_{(\alpha_t)}\big)\big((Z_0)_{(\beta_0)}\big) \cdots \big((Z_t)_{(\beta_t)}\big)\right)\Big|_{\substack{\lambda=0 \\ \theta=0 \\ \tau=0}}$$

using the independence of all the random variables $(W_0, W_1, \cdots, Z_0, Z_1, \cdots)$ where.

W_i ~Poisson$\left(\frac{\theta(p_1\lambda)^i}{i!}\right)$ and Z_i ~Poisson$\left(\frac{\tau(p_2\lambda)^i}{i!}\right)$.

However $\left(X_{(a)}\right) = \delta^a$ for X ~Poisson(δ) and the final form follows on substituting and extracting the coefficient of $\tau^r\theta^n\lambda^t$

$$= \frac{d^t}{d\lambda^t}\frac{d^n}{d\theta^n}\frac{d^r}{d\tau^r}\left(e^{\theta e^{p_1\lambda}+\tau e^{p_2\lambda}}\left(\frac{\theta(p_1\lambda)^0}{0!}\right)^{\alpha_0}\cdots\left(\frac{\theta(p_1\lambda)^t}{t!}\right)^{\alpha_t}\left(\frac{\tau(p_2\lambda)^0}{0!}\right)^{\beta_0}\cdots\left(\frac{\tau(p_2\lambda)^t}{t!}\right)^{\beta_t}\right)\Bigg|_{\substack{\lambda=0\\ \theta=0\\ \tau=0}}$$

$$= \left(\frac{p_1^{0\alpha_0+\cdots+t\alpha_t}\,p_2^{0\beta_0+\cdots+t\beta_t}}{(0!)^{\alpha_0+\beta_0}\cdots(t!)^{\alpha_t+\beta_t}}\right)\times$$

$$\times\frac{d^t}{d\lambda^t}\frac{d^n}{d\theta^n}\frac{d^r}{d\tau^r}\left(e^{\theta e^{p_1\lambda}+\tau e^{p_2\lambda}}\theta^{\alpha_0+\cdots+\alpha_t}\,\tau^{\beta_0+\cdots+\beta_t}\,\lambda^{0(\alpha_0+\beta_0)+\cdots+t(\alpha_t+\beta_t)}\right)\Bigg|_{\substack{\lambda=0\\ \theta=0\\ \tau=0}}$$

$$= \left(\frac{p_1^{0\alpha_0+\cdots+t\alpha_t}\,p_2^{0\beta_0+\cdots+t\beta_t}}{(0!)^{\alpha_0+\beta_0}\cdots(t!)^{\alpha_t+\beta_t}}\right)\times$$

$$\times\frac{d^t}{d\lambda^t}\frac{d^n}{d\theta^n}\left(e^{\theta e^{p_1\lambda}}\theta^{\alpha_0+\cdots+\alpha_t}\;\lambda^{0(\alpha_0+\beta_0)+\cdots+t(\alpha_t+\beta_t)}\frac{\left(e^{p_2\lambda}\right)^{r-(\beta_0+\cdots+\beta_t)}r!}{\left(r-(\beta_0+\cdots+\beta_t)\right)!}\right)\Bigg|_{\substack{\lambda=0\\ \theta=0}}$$

$$= \left(\frac{p_1^{0\alpha_0+\cdots+t\alpha_t}\,p_2^{0\beta_0+\cdots+t\beta_t}}{(0!)^{\alpha_0+\beta_0}\cdots(t!)^{\alpha_t+\beta_t}}\right)\frac{r!}{\left(r-(\beta_0+\cdots+\beta_t)\right)!}\times$$

$$\times\frac{d^t}{d\lambda^t}\frac{d^n}{d\theta^n}\left(e^{\theta e^{p_1\lambda}}\theta^{\alpha_0+\cdots+\alpha_t}\;\lambda^{0(\alpha_0+\beta_0)+\cdots+t(\alpha_t+\beta_t)}\left(e^{p_2\lambda}\right)^{r-(\beta_0+\cdots+\beta_t)}\right)\Bigg|_{\substack{\lambda=0\\ \theta=0}}$$

$$= \left(\frac{p_1^{0\alpha_0+\cdots+t\alpha_t}\,p_2^{0\beta_0+\cdots+t\beta_t}}{(0!)^{\alpha_0+\beta_0}\cdots(t!)^{\alpha_t+\beta_t}}\right)\frac{r!}{\left(r-(\beta_0+\cdots+\beta_t)\right)!}\times$$

$$\times\frac{d^t}{d\lambda^t}\left(\lambda^{0(\alpha_0+\beta_0)+\cdots+t(\alpha_t+\beta_t)}\left(e^{p_2\lambda}\right)^{r-(\beta_0+\cdots+\beta_t)}\frac{\left(e^{p_1\lambda}\right)^{n-(\alpha_0+\cdots+\alpha_t)}n!}{\left(n-(\alpha_0+\cdots+\alpha_t)\right)!}\right)\Bigg|_{\lambda=0}$$

$$= \left(\frac{p_1^{0\alpha_0+\cdots+t\alpha_t}\,p_2^{0\beta_0+\cdots+t\beta_t}}{(0!)^{\alpha_0+\beta_0}\cdots(t!)^{\alpha_t+\beta_t}}\right)\frac{r!}{\left(r-(\beta_0+\cdots+\beta_t)\right)!}\frac{n!}{\left(n-(\alpha_0+\cdots+\alpha_t)\right)!}\times$$

$$\times\frac{d^t}{d\lambda^t}\left(\lambda^{0(\alpha_0+\beta_0)+\cdots+t(\alpha_t+\beta_t)}\left(e^{p_2\lambda}\right)^{r-(\beta_0+\cdots+\beta_t)}\left(e^{p_1\lambda}\right)^{n-(\alpha_0+\cdots+\alpha_t)}\right)\Bigg|_{\lambda=0}$$

$$= \left(\frac{p_1^{0\alpha_0+\cdots+t\alpha_t} \, p_2^{0\beta_0+\cdots+t\beta_t}}{(0!)^{\alpha_0+\beta_0} \cdots (t!)^{\alpha_t+\beta_t}} \right) \frac{r!}{\big(r-(\beta_0+\cdots+\beta_t)\big)!} \frac{n!}{\big(n-(\alpha_0+\cdots+\alpha_t)\big)!} \times$$

$$\times \frac{d^t}{d\lambda^t} \Big(\lambda^{0(\alpha_0+\beta_0)+\cdots+t(\alpha_t+\beta_t)} \Big(e^{\lambda\left(p_1(n-\alpha_0-\cdots-\alpha_t)+p_2(r-\beta_0-\cdots-\beta_t)\right)} \Big) \Big) \Big|_{\lambda=0}$$

$$= \left(\frac{p_1^{0\alpha_0+\cdots+t\alpha_t} \, p_2^{0\beta_0+\cdots+t\beta_t}}{(0!)^{\alpha_0+\beta_0} \cdots (t!)^{\alpha_t+\beta_t}} \right) \frac{r!}{\big(r-(\beta_0+\cdots+\beta_t)\big)!} \frac{n!}{\big(n-(\alpha_0+\cdots+\alpha_t)\big)!} \times$$

$$\times \frac{\big(p_1(n-\alpha_0-\cdots-\alpha_t)+p_2(r-\beta_0-\cdots-\beta_t)\big)^{t-\left(0(\alpha_0+\beta_0)+\cdots+t(\alpha_t+\beta_t)\right)} t!}{\Big(t-\big(0(\alpha_0+\beta_0)+\cdots+t(\alpha_t+\beta_t)\big)\Big)!}$$

$$= \frac{n!}{\big(n-(\alpha_0+\cdots+\alpha_t)\big)!} \frac{r!}{\big(r-(\beta_0+\cdots+\beta_t)\big)!} \frac{t!}{\Big(t-\big(0(\alpha_0+\beta_0)+\cdots+t(\alpha_t+\beta_t)\big)\Big)!} \left(\frac{p_1^{0\alpha_0+\cdots+t\alpha_t} \, p_2^{0\beta_0+\cdots+t\beta}}{(0!)^{\alpha_0+\beta_0} \cdots (t!)^{\alpha_t+\beta_t}} \right.$$

$\times$

$$\times \big(p_1(n-\alpha_0-\cdots-\alpha_t)+p_2(r-\beta_0-\cdots-\beta_t)\big)^{t-\left(0(\alpha_0+\beta_0)+\cdots+t(\alpha_t+\beta_t)\right)}$$

$$\left(\frac{\mathrm{d}^a}{\mathrm{d}\lambda^a} \left(e^{b\lambda} \lambda^c \right) \right) \Big|_{\lambda=0} = \begin{cases} \dfrac{b^{a-c} a!}{(a-c)!} & a \in \{c, c+1, \dots\} \\ 0 & a \in \{0,1, \dots, c-1\} \end{cases}$$

where $W_j \sim$Poisson$\left(\frac{\theta(p_1\lambda)^j}{j!}\right)$, $Z_j \sim$Poisson$\left(\frac{\tau(p_2\lambda)^j}{j!}\right)$, $np_1 + rp_2 = 1$ and where all random variables are independent.

Proof

Let C_j equal the number of balls in urn j and let

$$\Psi_1(C_1, \dots, C_{n+r}) = \begin{cases} 1 & \text{exactly} k \text{of} (C_1, \dots, C_n) \text{equal} i \\ 0 & \text{else.} \end{cases}$$

and

$$\Psi_2(C_1, \dots, C_{n+r}) = (\mathrm{I}\{C_1 = i\} + \cdots + \mathrm{I}\{C_n = i\})_{(v)}.$$

By Theorem 1

$$P(W_i = k) = \left(\Psi_1(C_1, \dots, C_{n+r})\right)$$

$$= \frac{\mathrm{d}^m}{\mathrm{d}\lambda^m}\left(e^{\lambda} P(\text{exactly} k \text{of}(Z_1, \dots, Z_n) \text{equal} i)\right)\Big|_{\lambda=0}$$

$$= \frac{\mathrm{d}^m}{\mathrm{d}\lambda^m}\left(e^{\lambda}\binom{n}{k}\left(P(Z=i)\right)^k\left(1-P(Z=i)\right)^{n-k}\right)\Big|_{\lambda=0}$$

$$= \frac{\mathrm{d}^m}{\mathrm{d}\lambda^m}\left(e^{\lambda}\binom{n}{k}\left(e^{-\lambda p_1}\frac{(\lambda p_1)^i}{i!}\right)^k\left(1-e^{-\lambda p_1}\frac{(\lambda p_1)^i}{i!}\right)^{n-k}\right)\Big|_{\lambda=0}$$

$$= \sum_{j=0}^{n-k}(-1)^j\binom{n-k}{j}\binom{n}{k}\left(\frac{(p_1)^i}{i!}\right)^{k+j}\frac{\mathrm{d}^m}{\mathrm{d}\lambda^m}\left(\lambda^{i(k+j)}\, e^{\lambda\left(1-p_1(k+j)\right)}\right)\Big|_{\lambda=0}$$

$$= \sum_{j=0}^{n-k}(-1)^j\binom{n-k}{j}\binom{n}{k}\left(\frac{1}{i!}\right)^{k+j}\frac{m!}{\left(m-i(k+j)\right)!}\left(1-p_1(k+j)\right)^{m-i(k+j)}(p_1)^{i(k+j)}$$

and

$$\left((W_i)_{(v)}\right) = \left(\Psi_2(C_1, \dots, C_{n+r})\right)$$

$$= \frac{\mathrm{d}^m}{\mathrm{d}\lambda^m}\left(e^{\lambda}\left((\mathrm{I}\{Z_1=i\}+\cdots+\mathrm{I}\{Z_n=i\})_{(v)}\right)\right)\Big|_{\lambda=0}$$

$$= \frac{\mathrm{d}^m}{\mathrm{d}\lambda^m}\left(e^{\lambda} v!\binom{n}{v}\left(P(Z=i)\right)^v\right)\Big|_{\lambda=0}$$

where Z ~Poisson$\left(\frac{\lambda}{n+r}\right)$. The final form for both problems follows on substituting for $P(Z = i)$ and extracting the coefficient of λ^m.