# Power Series Randomization

We say that the discrete random variable $x$ follows the **Power Series Distribution** if

$$P(X = x) = \frac{a_x \theta^x}{\eta(\theta)} \qquad x = 0, 1, 2, \ldots$$

for any coefficients $a_x \geq 0$ where

$$\eta(\theta) = \sum_{x=0}^{\infty} a_x \theta^x$$

Suppose that $X_1, \ldots, X_n$ are independent and identically distribution **Power Series** random variables. Then for $t = 0, 1, \ldots$

$$\begin{aligned} P(X_1 + \ldots + X_n = t) &= \sum_{\substack{(x_1, \ldots, x_n) \ni \\ x_1 + \ldots + x_n = t}} \cdots \sum \frac{a_{x_1} \cdots a_{x_n} \theta^t}{(\eta(\theta))^n} \\ &= \frac{\theta^t}{(\eta(\theta))^n} \, \mathrm{c}(t, n) \end{aligned}$$

where

$$\mathrm{c}(t, n) = \sum_{\substack{(x_1, \ldots, x_n) \ni \\ x_1 + \ldots + x_n = t}} \cdots \sum a_{x_1} \cdots a_{x_n}$$

Furthermore,

$$\begin{aligned} 1 &= \sum_{t=0}^{\infty} \frac{\theta^t}{(\eta(\theta))^n} \, \mathrm{c}(t, n) \\ &= \frac{1}{(\eta(\theta))^n} \sum_{t=0}^{\infty} \mathrm{c}(t, n) \, \theta^n \end{aligned}$$

Therefore,

$$(\eta(\theta))^n = \sum_{t=0}^{\infty} \mathrm{c}(t, n) \, \theta^n$$

Define $\mathbb{S}^n$ to be the product space $\{0,1,\dots\} \times \cdots \times \{0,1,\dots\}$ and let $\mathbb{S}_t^n$ be the set of all vectors $(s_1, s_2, \dots, s_n)$ in $\mathbb{S}^n$ such that $s_1 + \dots + s_n = t$.

Let $\mathcal{A} \subseteq \mathbb{S}^n$ and define $\mathcal{A}_t = \mathcal{A} \cap \mathbb{S}_t^n$. It follows that

$$\begin{aligned} P((X_1,\dots,X_n) \in \mathcal{A}) &= \sum_{t=0}^{\infty} P\left((X_1,\dots,X_n) \in \mathcal{A} \;\Big|\; \sum_{i=1}^{\infty} X_i = t\right) P\left(\sum_{i=1}^{\infty} X_i = t\right) \\ &= \sum_{t=0}^{\infty} \sum_{\mathcal{A}_t} \left( \frac{\frac{a_{x_1}\cdots a_{x_n}\theta^t}{(\eta(\theta))^n}}{\frac{\theta^t}{(\eta(\theta))^n}\mathrm{c}(t,n)} \right) \frac{\mathrm{c}(t,n)}{(\eta(\theta))^n}\,\theta^t \\ &= \sum_{t=0}^{\infty} \sum_{\mathcal{A}_t} \left( \frac{a_{x_1}\cdots a_{x_n}}{\mathrm{c}(t,n)} \right) \frac{\mathrm{c}(t,n)}{(\eta(\theta))^n}\,\theta^t \end{aligned}$$

Therefore,

$$(\eta(\theta))^n P((X_1,\dots,X_n) \in \mathcal{A}) = \sum_{t=0}^{\infty} P((Y_{1,\mathrm{t}},\dots,Y_{n,\mathrm{t}}) \in \mathcal{A}_t)\,\mathrm{c}(t,n)\,\theta^t$$

where

$$P((Y_{1,\mathrm{t}},\dots,Y_{n,\mathrm{t}}) = (y_1,\dots,y_n)) = \begin{cases} \frac{a_{y_1}\cdots a_{y_n}}{\mathrm{c}(\mathrm{t},n)} & y_1 + \dots + y_n = \mathrm{t}, \quad y_j \geq 0 \;\; \forall j \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mathrm{c}(\mathrm{t},n) = \sum_{\substack{(y_1,\dots,y_n)\ni \\ y_1+\dots+y_n=\mathrm{t}}} \!\!\cdots\!\! \sum a_{y_1}\cdots a_{y_n}$$

Hence,

$$\frac{d^r}{d\theta^r}((\eta(\theta))^n P((X_1,\dots,X_n) \in \mathcal{A}))\Big|_{\theta=0} = \sum_{t=0}^{\infty} P((Y_{1,\mathrm{t}},\dots,Y_{n,\mathrm{t}}) \in \mathcal{A}_t)\,\mathrm{c}(t,n)\,r!\,\mathrm{I}_{\{r\}}(t)$$

Therefore,

$$P((Y_{1,\mathrm{r}},\dots,Y_{n,\mathrm{r}}) \in \mathcal{A}_r) = \frac{1}{\mathrm{c}(r,n)\,r!} \left( \frac{d^r}{d\theta^r}((\eta(\theta))^n P((X_1,\dots,X_n) \in \mathcal{A}))\Big|_{\theta=0} \right)$$

We will refer to this result as the **Power Series Randomization Theorem**.