Probability Theory

Study Notes Solutions Manual

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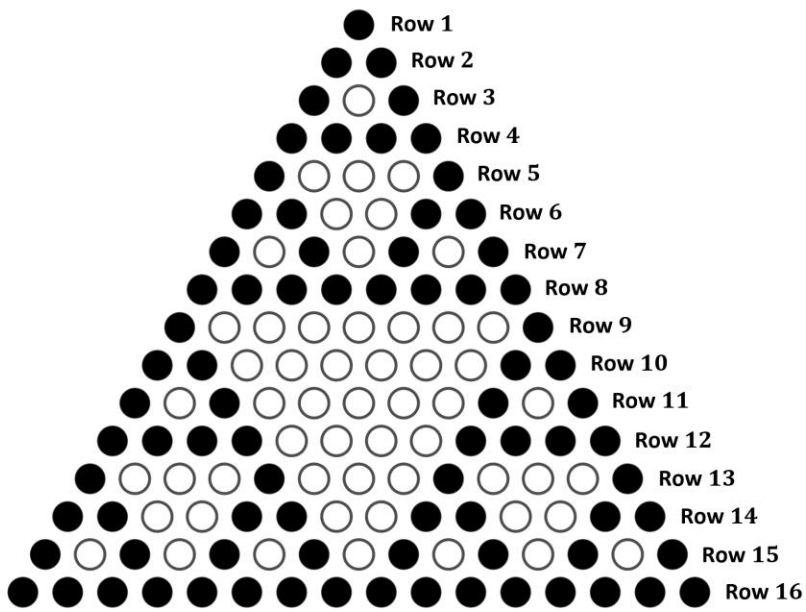

For MN Students Preparing For

High School Mathematics Contests

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Exercises for Chapter 1

1. (5C183) The probability of event A, written as $P(A)$, equals 0.4, and $P(A \text{ and } B) = 0.172$. If A and B are independent events, determine exactly $P(A \text{ or } B)$.

Solution

$$P(A \text{ and } B) = P(A \cap B) = 0.172$$

Because A and B are given to be independent events we know that

$$P(A \cap B) = P(A)P(B) \Longrightarrow P(B) = \frac{P(A \cap B)}{P(A)}.$$

Therefore,

$$\begin{aligned} P(A \text{ or } B) = P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= P(A) + \left(\frac{P(A \cap B)}{P(A)}\right) - P(A \cap B) \\ &= 0.4 + \left(\frac{0.172}{0.4}\right) - 0.172 = \frac{329}{500}. \end{aligned}$$

■

2. If $P(S) = p > 0$, $P(T) = q$ and if S and T are independent events, find $P\big((S \cap T)\big|S\big)$.

Solution

$$P\big((S \cap T)\big|S\big) = \frac{P(S \cap T \cap S)}{P(S)} = \frac{P(S \cap T)}{P(S)} = \frac{P(S)P(T)}{P(S)} = P(T) = q$$

■

3. If $P(S) = p$, $P(T) = q$ and if $0 < p < 1$ and $0 < q < 1$, find $P\big((S \cup T)\big|T\big)$.

Solution

$$P\big((S \cup T)\big|T\big) = \frac{P\big((S \cup T) \cap T\big)}{P(T)} = \frac{P(T)}{P(T)} = 1$$

■

4. Two events A and B are such that $P(A) = 0.2$, $P(B) = 0.3$, and $P(A \cup B) = 0.4$. Find the following:

(a) $P(A \cap B)$

(b) $P(\overline{A} \cup \overline{B})$

(c) $P(\overline{A} \cap \overline{B})$

(d) $P(\overline{A}|B)$

Solution

(a) $P(A \cap B) = P(A) + P(B) - P(A \cup B) = 0.3 + 0.4 - 0.5 = 0.2$

(b) $P(\overline{A} \cup \overline{B}) = 1 - P(A \cap B) = 1 - 0.2 = 0.8$

(c) $P(\overline{A} \cap \overline{B}) = 1 - P(A \cup B) = 1 - 0.5 = 0.5$

(d) $P(\overline{A}|B) = 1 - P(A|B) = 1 - \dfrac{P(A \cap B)}{P(B)} = 1 - \dfrac{0.2}{0.4} = \dfrac{1}{2}.$

■

5. Find $P(A \cup (B' \cup C')')$ if $P(A) = 1/2$, $P(B \cap C) = 1/3$ and $P(A \cap C) = 0$.

Solution

$$\begin{aligned} P(A \cup (B' \cup C')') &= P(A) + P((B' \cup C')') - P(A \cap (B' \cup C')') \\ &= P(A) + P(B \cap C) - P(A \cap (B \cap C)) \\ &= P(A) + P(B \cap C) - P(B \cap (A \cap C)) \\ &= P(A) + P(B \cap C) - P(B \cap \emptyset) \\ &= P(A) + P(B \cap C) - P(\emptyset) \\ &= 1/2 + 1/3. \end{aligned}$$

■

6. Find $P\left((A \cap \overline{B}) \cup (\overline{A} \cap B)\right)$ if A and B are independent events and if $P(A) = 2/3$ and if $P(B) = 1/4$.

Solution

You can immediately see that $(A \cap \overline{B}) \subset A$ and $(\overline{A} \cap B) \subset \overline{A}$ are mutually exclusive. Therefore,

$$\begin{aligned} P\left((A \cap \overline{B}) \cup (\overline{A} \cap B)\right) &= P(A \cap \overline{B}) + P(\overline{A} \cap B) \\ &= P(A)P(\overline{B}) + P(\overline{A})P(B) \\ &= \left(\frac{2}{3}\right)\left(1 - \frac{1}{4}\right) + \left(1 - \frac{2}{3}\right)\left(\frac{1}{4}\right) \\ &= \frac{1}{2} + \frac{1}{12} = \frac{7}{12}. \end{aligned}$$

■

7. Suppose $P(A) = 0.5$ and $P(A \cup B) = 0.6$.

   (a) Find $P(B)$ if A and B are mutually exclusive

   (b) Find $P(B)$ if A and B are independent

   (c) Find $P(B)$ if $P(A|B) = 0.4$.

Solution

(a)

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = P(A) + P(B) - 0$$

$$\begin{aligned} \Rightarrow P(B) &= P(A \cup B) - P(A) \\ &= 0.6 - 0.5 = \frac{1}{10} \end{aligned}$$

(b)

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = P(A) + P(B) - P(A)P(B)$$

$$\Rightarrow P(B) = \frac{P(A \cup B) - P(A)}{1 - P(A)} = \frac{0.6 - 0.5}{1 - 0.5} = \frac{2}{10}$$

(c)

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = P(A) + P(B) - P(A|B)P(A)$$

$$\begin{aligned} \Rightarrow P(B) &= P(A \cup B) + P(A|B)P(A) - P(A) \\ &= 0.6 + (0.4)(0.5) - 0.5 = \frac{3}{10}. \end{aligned}$$

■

8. If $P(C) > 0$, $P(A|C) = 0.7$, $P(B|C) = 0.4$ and $P\big((A \cup B)\big|C\big) = 0.8$, find $P\big((A \cap B)\big|C\big)$.

Solution

$$P\big((A \cap B)\big|C\big) = P(A|C) + P(B|C) - P\big((A \cup B)\big|C\big)$$
$$= 0.7 + 0.4 - 0.8 = 0.3.$$

■

9. If $P(\overline{A}) = 0.3$, $P(B) = 0.4$, and $P(A \cap \overline{B}) = 0.5$, then find $P(A \cap B)$ and $P(A \cup B)$.

Solution

$$P(A \cap B) = P(A) - P(A \cap \overline{B}) = 0.7 - 0.5 = 0.2$$
$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.7 + 0.4 - 0.2 = 0.9$$

■

10. Find $P\big(A \cup (B^c \cup C^c)^c\big)$ if A, B and C are mutually exclusive events and $P(A) = 3/7$.

Solution

$$A \cup (B^c \cup C^c)^c = A \cup (B \cap C) = A \cup \emptyset = A$$
$$P\big(A \cup (B^c \cup C^c)^c\big) = P(A) = 3/7.$$

■

11. Find $P\left(A \cup \left(B^c \cup C^c\right)^c\right)$ if $P\big(A^c \cap (B^c \cup C^c)\big) = 0.65$.

Solution

$$P\left(A \cup \left(B^c \cup C^c\right)^c\right) = 1 - P\left(\left(A \cup (B^c \cup C^c)^c\right)^c\right)$$
$$= 1 - P\big(A^c \cap ((B^c \cup C^c)^c)^c\big)$$
$$= 1 - P\big(A^c \cap (B^c \cup C^c)\big)$$
$$= 1 - 0.65 = 0.35.$$

■

12. Suppose that A and B are mutually exclusive events for which $P(A) = 0.3$ and $P(B) = 0.5$ What is the probability that

    (a) either A or B occurs?

    (b) A occurs but B does not?

Solution

(a) $P(A \text{ or } B) = P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.3 + 0.5 - 0 = 0.8$

(b) $P(A \text{ and } B') = P(A \cap B') = P(A) - P(A \cap B) = 0.3 - 0 = 0.3$

■

Exercises for Chapter 2

1. (5C191) Jamie flips four fair coins. Determine exactly the probability that she gets more heads than tails.

Solution

Binomial, $n = 4, p = 1/2$, success ≡ heads, $x = 3$ or 4.

$$\binom{4}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^1 + \binom{4}{4}\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^0 = \frac{4+1}{16} = \frac{5}{16}.$$

■

2. (5C194) Amy and Ben each have biased coins but they are unfair in different ways: Amy's coin comes up <u>heads</u> only $1/3$ of the time and Ben's coin comes up <u>tails</u> only $1/3$ of the times. They both flip their respective coins twice. Determine exactly the probability that they get the same number of heads.

Solution

Amy, binomial, $n = 2, p = 1/3$, success ≡ heads, $x = 0,1$ or 2.

Ben, binomial, $n = 2, p = 2/3$, success ≡ heads, $x = 0,1$ or 2.

Amy and Ben separate, independent binomial experiments.

$$P(A_0) \cdot P(B_0) + P(A_1) \cdot P(B_1) + P(A_2) \cdot P(B_2)$$

$$= \left[\binom{2}{0}\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^2 \cdot \binom{2}{0}\left(\frac{2}{3}\right)^0\left(\frac{1}{3}\right)^2\right] + \left[\binom{2}{1}\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^1 \cdot \binom{2}{1}\left(\frac{2}{3}\right)^1\left(\frac{1}{3}\right)^1\right]$$

$$+ \left[\binom{2}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^0 \cdot \binom{2}{2}\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^0\right]$$

$$= \left[\frac{4}{9} \cdot \frac{1}{9}\right] + \left[\frac{4}{9} \cdot \frac{4}{9}\right] + \left[\frac{1}{9} \cdot \frac{4}{9}\right] = \frac{24}{81} = \frac{8}{27}.$$

■

3. (5C174) A server noted that when diners have cherry pie for dessert, 3 out of 5 will leave a big tip. If there are 6 diners who have cherry pie, what is the probability that a big tip is left by exactly 4 of them?

Solution

Binomial, $n = 6, p = 3/5$, success = big tip, $x = 4$.

$$\binom{6}{4}\left(\frac{3}{5}\right)^4\left(\frac{2}{5}\right)^2 = 15 \cdot \left(\frac{81}{625}\right) \cdot \left(\frac{4}{25}\right) = \frac{972}{3125}.$$

■

4. (5C153) Three quarters and three dimes are tossed in the air. Determine exactly the probability that the same number of quarters and dimes turn up heads.

Solution

Quarters, binomial, $n = 3, p = 1/2, x = 0,1,2,3$.

Dimes, binomial, $n = 3, p = 1/2, x = 0,1,2,3$.

Let A_j be the event that j quarters turn up heads and let B_j be the event that j dines turn up heads.

$$P(A_0)P(B_0) + P(A_1)P(B_1) + P(A_2)P(B_2) + P(A_3)P(B_3)$$

$$= \left[\binom{3}{0}\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^3 \cdot \binom{3}{0}\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^3\right] + \left[\binom{3}{1}\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^2 \cdot \binom{3}{1}\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^2\right]$$

$$+ \left[\binom{3}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^1 \cdot \binom{3}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^1\right] + \left[\binom{3}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^0 \cdot \binom{3}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^0\right]$$

$$= \left(\frac{1}{2}\right)^6\left(\binom{3}{0}\binom{3}{0} + \binom{3}{1}\binom{3}{1} + \binom{3}{2}\binom{3}{2} + \binom{3}{3}\binom{3}{3}\right) = \left(\frac{1}{2}\right)^6(1 + 9 + 9 + 1)$$

$$= \frac{20}{64} = \frac{5}{16}.$$

■

5. (TT156) An unfair coin lands tails with a probability of $1/5$. When tossed n times, the probability of exactly three tails is the same as the probability of exactly 4 heads. What is the value of n?

Solution

(Binomial, n unknown, $p = 1/5$, success $=$ tails, $x = 3$ equated to Binomial, same n unknown, $p = 4/5$, success $=$ heads, $x = 4$)

Solve for n if

$$P(\text{exactly 3 tails}) = \binom{n}{3}\left(\frac{1}{5}\right)^3\left(\frac{4}{5}\right)^{n-3} = \binom{n}{4}\left(\frac{4}{5}\right)^4\left(\frac{1}{5}\right)^{n-4} = P(\text{exactly 4 heads}).$$

$$\frac{n(n-1)(n-2)}{6}\left(\frac{4^{n-3}}{5^n}\right) = \frac{n(n-1)(n-2)(n-3)}{24}\left(\frac{4^4}{5^n}\right)$$

$$4^{n-3} = \frac{1}{4}(4^4)(n-3)$$

$$n-3 = 4^{n-6}.$$

So, we have a linear term $(n-3)$ equal to an exponential term (4^{n-6}). Unfortunately, there is no elementary way of finding the set of *all* solutions. But since we are only looking for positive integer solutions we can use a calculator to make a table of values of the function $a_n = n - 3 - 4^{n-6}$ to find an positive integer value where this function equals 0. Note that we only need to search over the values of n where $n-3$ is an integer greater than three and $n-3$ is a power of 4. That is, $n-3 \in \{4^0, 4^1, 4^2, 4^3, \ldots\}$.

But you probably can save time and just notice by inspection $n-3 = 4^0$ or $n = 7$ is a solution because

$$\mathbf{7} - 3 = 4^{\mathbf{7}-6}.$$

■

6. (5C141) What is the probability that a fair coin, flipped three times, will land all tails?

Solution

(Binomial, $n = 3, p = 1/2$, success = tails, $x = 3$)

$$\binom{3}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^0 = \left(\frac{1}{2}\right)^3 = 1/8.$$

■

7. (5C142) Determine exactly the probability that a student, by randomly guessing, achieves a score of 3 out of 5 on a pop quiz whose questions are multiple choice with four choices each.

Solution

(Binomial, $n = 5, p = 1/5$, success = guess correctly, $x = 3$)

$$\binom{5}{3}\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^2 = \frac{90}{1024} = \frac{45}{512}.$$

■

8. (MB0610) A die is rolled six times. What is the probability of getting either a 1 or a 6 on at least three rolls?

Solution

Thinking of rolling 1 or 6 as "success" and rolling a 2 or 3 or 4 or 5 as "failure". Then the probability of getting a success on any trial equals $2/6 = 1/3$.

Let X equal the number of successes in $n = 6$ trials where $p = P(\text{success}) = 1/3$. Then

$$P(X = k) = \binom{6}{k}\left(\frac{1}{3}\right)^k\left(1 - \frac{1}{3}\right)^{6-k}, \quad k = 0,1,2,\dots,6$$

and

$$P(X \geq 3) = \sum_{k=3}^{6} P(X = k) = \sum_{k=3}^{6}\binom{6}{k}\left(\frac{1}{3}\right)^k\left(1 - \frac{1}{3}\right)^{6-k}$$

$$= \left(\frac{1}{3}\right)^6 \sum_{k=3}^{6}\binom{6}{k} 2^{6-k} = \frac{233}{729}.$$

■

9. (5A054) (a) Amy, Beth, and Christine toss a coin 15,16 and 17 times respectively. Which girl is least likely to get more heads than tails?

Solution

Let X equal the number of heads in n tosses of a fair coin. Then for $k = 0,1,2,\dots,n$

$$P(X = k) = \binom{n}{k}\left(\frac{1}{2}\right)^k\left(1 - \frac{1}{2}\right)^{n-k} = \binom{n}{k}\left(\frac{1}{2}\right)^k\left(\frac{1}{2}\right)^{n-k} = \frac{\binom{n}{k}}{2^n}.$$

For Amy $n = 15$, for Beth $n = 16$, and for Christine $n = 17$.

$$P(\text{Amy gets more heads than tails}) = P(X \geq 8) = \frac{1}{2^{15}} \sum_{k=8}^{15} \binom{15}{k}$$

$$P(\text{Beth gets more heads than tails}) = P(X \geq 9) = \frac{1}{2^{16}} \sum_{k=9}^{16} \binom{16}{k}$$

$$P(\text{Christine gets more heads than tails}) = P(X \geq 9) = \frac{1}{2^{17}} \sum_{k=9}^{17} \binom{17}{k}.$$

The question asks for the smallest of these three probabilities. If you are allowed a programmable calculator, just evaluate these three probabilities directly. If not, we know from the binomial theorem that

$$2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k}$$

If n is even then

$$2^n = \sum_{k=0}^{n} \binom{n}{k} = \sum_{k=0}^{(n/2)-1} \binom{n}{k} + \binom{n}{n/2} + \sum_{k=(n/2)+1}^{n} \binom{n}{k}$$

$$= \sum_{k=(n/2)+1}^{n} \binom{n}{k} + \binom{n}{n/2} + \sum_{k=(n/2)+1}^{n} \binom{n}{k}.$$

(To see why, remember that $\binom{n}{0} = \binom{n}{n}$, $\binom{n}{1} = \binom{n}{n-1}$, $\binom{n}{2} = \binom{n}{n-2}$, etc.)

So, for n even

$$\sum_{k=(n/2)+1}^{n} \binom{n}{k} = \frac{1}{2}\left(2^n - \binom{n}{n/2}\right) = 2^{n-1} - \frac{1}{2}\binom{n}{n/2}.$$

By the same type of reasoning, if n is odd then

$$2^n = \sum_{k=0}^{n} \binom{n}{k} = \sum_{k=0}^{(n-1)/2} \binom{n}{k} + \sum_{k=(n+1)/2}^{n} \binom{n}{k} = 2 \sum_{k=(n+1)/2}^{n} \binom{n}{k}.$$

So, for n odd

$$\sum_{k=(n+1)/2}^{n} \binom{n}{k} = 2^{n-1}.$$

$$P(\text{Amy gets more heads than tails}) = \frac{1}{2^{15}} \sum_{k=8}^{15} \binom{15}{k} = \frac{1}{2^{15}} (2^{15-1}) = \frac{1}{2}.$$

$$P(\text{Beth gets more heads than tails}) = P(X \geq 9) = \frac{1}{2^{16}} \sum_{k=9}^{16} \binom{16}{k} = \frac{1}{2^{16}} \left(2^{16-1} - \frac{1}{2} \binom{16}{8} \right)$$
$$= \frac{1}{2} - \frac{1}{2^{17}} \binom{16}{8}.$$

$$P(\text{Christine gets more heads than tails}) = P(X \geq 9) = \frac{1}{2^{17}} \sum_{k=9}^{17} \binom{17}{k} = \frac{1}{2^{17}} (2^{17-1}) = \frac{1}{2}.$$

So, Beth is the least likely to get more heads than tails because

$$\frac{1}{2} - \frac{1}{2^{17}} \binom{16}{8} < \frac{1}{2}.$$

■

10. (5A054) (b) Amy, Beth and Christine toss a coin 18,19 and 20 times respectively. Which girl is least likely to get more heads than tails?

Solution

Now suppose that $n = 18$ for Amy, $n = 19$ for Beth and $n = 20$ for Christine.

$$P(\text{Amy gets more heads than tails}) = P(X \geq 10) = \frac{1}{2^{18}} \sum_{k=10}^{18} \binom{18}{k} = \frac{1}{2} - \frac{1}{2^{19}} \binom{18}{8}$$

$$P(\text{Beth gets more heads than tails}) = P(X \geq 10) = \frac{1}{2^{19}} \sum_{k=10}^{19} \binom{19}{k} = \frac{1}{2}$$

$$P(\text{Christine gets more heads than tails}) = P(X \geq 11) = \frac{1}{2^{20}} \sum_{k=11}^{20} \binom{20}{k} = \frac{1}{2} - \frac{1}{2^{21}} \binom{20}{10}.$$

So, it comes down to comparing

$$\frac{\binom{18}{9}}{2^{19}} \text{ to } \frac{\binom{20}{10}}{2^{21}}.$$

If we continue to assume we do not have a calculator, then we the best approach is to **look at the ratio r** of these two numbers.

$$r = \frac{\frac{\binom{18}{9}}{2^{19}}}{\frac{\binom{20}{10}}{2^{21}}} = \frac{2^{21}}{2^{19}} \cdot \frac{\binom{18}{9}}{\binom{20}{10}}.$$

If $r < 1$ then the numerator is smaller than the denominator.

If $r = 1$ then the numerator and the denominator are equal

If $r > 1$ then the numerator is larger than the denominator.

Simplifying the above expression for r we find

$$r = \frac{2^{21}}{2^{19}} \cdot \frac{\binom{18}{9}}{\binom{20}{10}} = 4 \cdot \frac{\frac{18!}{9!\,9!}}{\frac{20!}{10!\,10!}} = 4 \cdot \frac{10!\,10!}{9!\,9!} \cdot \frac{18!}{20!}$$

$$= 4 \cdot (10 \cdot 10) \cdot \left(\frac{1}{19 \cdot 20}\right) = \frac{400}{380} > 1.$$

This means that

$$\frac{\binom{18}{9}}{2^{19}} > \frac{\binom{20}{10}}{2^{21}}$$

and hence

$$\frac{1}{2} - \frac{\binom{18}{9}}{2^{19}} < \frac{1}{2} - \frac{\binom{20}{10}}{2^{21}}.$$

So in this part of the problem Amy has the smallest probability of getting more heads than tails.

■

11. (5C054) Sarah was sent to get 8 cans of soda to have on hand for the study session. When she got to the machine, she found that she had six choices. Remembering that she had a die in her purse for some homework for her probability class, she decided to roll it 8 times, choosing the first flavor if a 1 came up, etc. for all six possibilities. Using this scheme, what is the probability that she gets exactly four diet cokes?

Solution

Binomial, $n = 8, p = P(\text{diet Coke}) = 1/6, X = \#$ diet Cokes.

$$P(X = 4) = \binom{8}{4}\left(\frac{1}{6}\right)^4\left(\frac{5}{6}\right)^{8-4} = \frac{21875}{839808} = 0.026$$

■

12. (TI0512) John flips a fair coin 12 times. What is the probability that he gets more heads than tails?

Solution

Binomial, $n = 12, p = P(\text{heads}) = 1/2, X = \#$ heads. Using the results derived in Problem MSHSML 5A054, we have

$$P(X \geq 7) = \sum_{k=7}^{12}\binom{12}{k}\left(\frac{1}{2}\right)^k\left(\frac{1}{2}\right)^{12-k} = \frac{1}{2^{12}}\sum_{k=7}^{12}\binom{12}{k} = \frac{1}{2} - \frac{1}{2^{13}}\binom{12}{6} = \frac{793}{2048} \approx 0.387.$$

(Note: There is a typo (in my copy at least) of the MSHSML solutions for this problem where at one point they took $2^{12} = 2048$ instead of the correct $2^{12} = 4096$.)

■

13. (MB0510) A die is rolled six times. What is the probability of getting either a 1 or a 6 on at least three rolls?

Solution

Binomial, $n = 6$, success ≡ rolling either a 1 or a 6. Let X equal the number of successes, $p = 2/6 = 1/3$.

$$P(X \geq 3) = \sum_{k=3}^{6} \binom{6}{k} \left(\frac{1}{3}\right)^{k} \left(\frac{2}{3}\right)^{6-k}$$

$$= \binom{6}{3}\left(\frac{1}{3}\right)^{3}\left(\frac{2}{3}\right)^{6-3} + \binom{6}{4}\left(\frac{1}{3}\right)^{4}\left(\frac{2}{3}\right)^{6-4} + \binom{6}{5}\left(\frac{1}{3}\right)^{5}\left(\frac{2}{3}\right)^{6-5} + \binom{6}{6}\left(\frac{1}{3}\right)^{6}\left(\frac{2}{3}\right)^{6-6}$$

$$= \frac{20(2^3) + 15(2^2) + 6(2) + 1}{3^6} = \frac{233}{729} \approx 0.320$$

■

14. (TI008) If the probability of A beating B is $3/5$ (so the probability of losing to B is $2/5$), find the probability of A winning exactly 8 of 12 games the teams play during a season.

Solution

Binomial, $n = 12$, success ≡ A wins, $p = 3/5$, $X =$ # games A wins

$$P(X = 8) = \binom{12}{8}\left(\frac{3}{5}\right)^{8}\left(\frac{2}{5}\right)^{4} \approx 0.2128$$

■

15. (5C974) In the game of Zonk, one throws six dice on each turn. What is the probability that on a random throw, exactly three dice will be 2's?

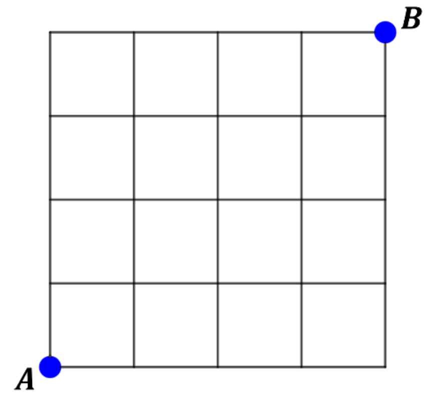
Solution

Binomial, $n = 6$, success ≡ roll a 2, $p = 1/6$, $X =$ # two's

$$P(X = 3) = \binom{6}{3}\left(\frac{1}{6}\right)^{3}\left(\frac{5}{6}\right)^{3} \approx 0.0534$$

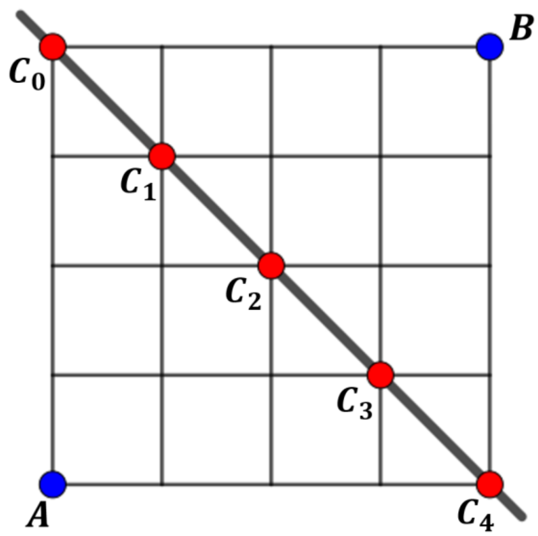
■

16. (TT974) John starts at A and walks toward B (eight blocks), moving only to the right or up (east or north). Jane starts at B and walks toward A (eight blocks), moving only to the left or down (west or south). If they start at the same time and walk at the same rate, and they each choose their direction at intersections (when they have a choice) in a random manner, what is the probability they will meet along the way?



Solution

After John has walked four blocks he will be at one of the points C_0, C_1, C_2, C_3 or C_4 as seen in the diagram below.



The same is true for Jane.

If John and Jane are going to meet it will have to be at the time when they reach one of these five points because at every other moment in time they must be on opposite sides of this diagonal.

For $j = 0,1,2,3,4$ define the events

A_j : John goes through point C_j on his walk from point A to point B

B_j : Jane goes through point C_j on her walk from point B to point A.

Then

$$P(\text{John and Jane meet during their walks})$$
$$= P\Big((A_0 \cap B_0) \cup (A_1 \cap B_1) \cup (A_2 \cap B_2) \cup (A_3 \cap B_3) \cap (A_4 \cap B_4)\Big)$$
$$= P(A_0 \cap B_0) + P(A_1 \cap B_1) + P(A_2 \cap B_2) + P(A_3 \cap B_3) + P(A_4 \cap B_4)$$
$$= P(A_0)P(B_0) + P(A_1)P(B_1) + P(A_2)P(B_2) + P(A_3)P(B_3) + P(A_4)P(B_4).$$

All of these probabilities can be handled similarly so let's just focus on $P(A_3)$ as an example to illustrate the process.

For John to go through the point C_3 he has to decide to go east on exactly three of the first four intersections he stands at (counting his starting point at A as the first intersection he stands at). The story tells us that John chooses his direction at random at each intersection where he has a choice of direction. And John does have a choice of direction at each of the first four intersections he stands at.

Saying that John chooses his direction at random at an intersection (when he has a choice to make) is equivalent to saying that John flips an imaginary fair coin and he will go east if the coin lands heads and he will go north if the coin lands tails. Thinking this way, John will go through the point C_3 if and only if this imaginary coin lands heads in exactly 3 of 4 flips. So,

$$P(\text{John goes through the point } C_3) = P(\text{get heads on 3 of 4 flips of a fair coin}).$$

But the number of heads in a fixed number of flips of a coin is a binomial random variable. If we let X equal the number of heads in $n = 4$ flips of a coin that lands heads with probability $p = 1/2$, then

$$P(A_3) = P(\text{John goes through the point } C_3)$$
$$= P(X = 3) = \binom{4}{3}\left(\frac{1}{2}\right)^3\left(1 - \frac{1}{2}\right)^{4-3} = \binom{4}{3}\left(\frac{1}{2}\right)^4.$$

By the same reasoning,

$$P(A_j) = P(X = j) = \binom{4}{j}\left(\frac{1}{2}\right)^4, \ j = 0,1,2,3,4.$$

We can make the same argument for Jane by simply replacing the word "east" with "south" and "north" with "west".

For example, Jane will go through the point C_3 if she decides to go south on exactly three of the first four intersections she stands at (counting her starting point at B as the first intersection she stands at).

If we let Y equal the number of heads (heads south) in $n = 4$ flips of a coin that lands heads with probability $p = 1/2$, then

$$P(B_j) = \binom{4}{j}\left(\frac{1}{2}\right)^4 = P(A_j),\ \ j = 0,1,2,3,4.$$

Therefore,

$$P(\text{John and Jane meet during their walks})$$
$$= P(A_0)P(B_0) + P(A_1)P(B_1) + P(A_2)P(B_2) + P(A_3)P(B_3) + P(A_4)P(B_4).$$
$$= \big(P(A_0)\big)^2 + \big(P(A_1)\big)^2 + \big(P(A_2)\big)^2 + \big(P(A_3)\big)^2 + \big(P(A_4)\big)^2$$
$$= \left(\binom{4}{0}\left(\frac{1}{2}\right)^4\right)^2 + \left(\binom{4}{1}\left(\frac{1}{2}\right)^4\right)^2 + \left(\binom{4}{2}\left(\frac{1}{2}\right)^4\right)^2 + \left(\binom{4}{3}\left(\frac{1}{2}\right)^4\right)^2 + \left(\binom{4}{4}\left(\frac{1}{2}\right)^4\right)^2$$
$$= \frac{1}{2^8}\left(\binom{4}{0}^2 + \binom{4}{1}^2 + \binom{4}{2}^2 + \binom{4}{3}^2 + \binom{4}{4}^2\right)$$
$$= \frac{1}{2^8}(1^2 + 4^2 + 6^2 + 4^2 + 1^2)$$
$$= \frac{70}{256} = \frac{35}{128}.$$

■

17. (TC942) A fair coin is flipped 8 times. What is the probability of obtaining exactly 3 heads?

Solution

Binomial, $n = 8$, success ≡ heads, $p = 1/2$, $X = \#$ heads

$$P(X = 3) = \binom{8}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^5 = \frac{56}{256} = \frac{7}{32}.$$

■

18. (TI9213) A pitcher throws a sequence of pitches and a blind umpire calls each one a ball or strike at random, meaning that independent of previous calls, the probability that a

particular pitch will be called a strike is $1/2$. The following questions assume that the batter does not swing at any pitch (*i.e.* is taking).

(a) What is the probability that the first three pitches will be called strikes?

(b) What is the probability that after two pitches, the count will be 1 ball, 1 strike?

(c) What is the probability that the batter will be called out (have 3 strikes called) before a fifth pitch has been thrown? (No more pitches will be thrown to a batter once three strikes have been called.)

(d) What is the probability that the umpire will call 3 strikes before calling 4 balls.

Solution

Parts (a) and (b)

Let X equal the number of strikes after n pitches. Then X follows the binomial distribution with $p = P(\text{success}) = 1/2$. Therefore,

$$P(X = k) = \binom{n}{k}\left(\frac{1}{2}\right)^k\left(1-\frac{1}{2}\right)^{n-k} = \frac{\binom{n}{k}}{2^n}.$$

(a) $n = 3, P(X = 3) = \dfrac{\binom{3}{3}}{2^3} = \dfrac{1}{8}$

(b) $n = 2, P(X = 1) = \dfrac{\binom{2}{1}}{2^2} = \dfrac{2}{4} = \dfrac{1}{2}$

Part (c) we need to remember that if a batter is called out after k pitches, then the k^{th} pitch ***has to be the third strike*** (because no more pitches will be thrown to a batter once three strikes have been called.)

Let X equal the number of the pitches required for the pitcher to throw three strikes.

As long as the third strike occurs before the fourth ball (the baseball term for a “non-strike”) then the batter will be called out.

Then X follows a **negative binomial distribution** because the number of successes (strikes) is fixed and the number of trials (pitches) is random. [Recall that for the binomial distribution it’s the other way around - the number of trials is fixed and the number of successes is random.]

From the negative binomial distribution function, if we let X equal the number of (Bernoulli) trials required in order to get a total of r successes. Then

$$P(X=k)=\begin{cases}\binom{k-1}{r-1}p^r(1-p)^{k-r} & k\in\{r,r+1,r+2,\dots\}\\ 0 & k\notin\{r,r+1,r+2,\dots\}.\end{cases}$$

(c) The problem asks for $P(X<5)$ where X follows the *negative binomial distribution* with $r=3$ and $p=1/2$.

$$P(X<5)=P(X=3)+P(X=4)$$

$$=\binom{3-1}{3-1}\left(\frac{1}{2}\right)^3\left(1-\frac{1}{2}\right)^{3-3}+\binom{4-1}{3-1}\left(\frac{1}{2}\right)^3\left(1-\frac{1}{2}\right)^{4-3}$$

$$=\left(\frac{1}{2}\right)^3+3\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^1=\frac{1}{8}+\frac{3}{16}=\frac{5}{16}.$$

(d) The umpire will have called 3 strikes before calling 4 balls if and only if the third strike occurs on the third, fourth, fifth or sixth pitch. (If the third strike occurs on the seventh pitch this would mean that the batter already had four balls (non-strikes) when the third strike was thrown, and this goes against the rules for baseball.)

So, the problem asks for $P(X\le 6)$ where X follows the *negative binomial distribution* with $r=3$ and $p=1/2$.

$$P(X\le 6)=P(X=3)+P(X=4)+P(X=5)+P(X=6)$$

$$=\binom{3-1}{3-1}\left(\frac{1}{2}\right)^3\left(1-\frac{1}{2}\right)^{3-3}+\binom{4-1}{3-1}\left(\frac{1}{2}\right)^3\left(1-\frac{1}{2}\right)^{4-3}$$

$$+\binom{5-1}{3-1}\left(\frac{1}{2}\right)^3\left(1-\frac{1}{2}\right)^{5-3}+\binom{6-1}{3-1}\left(\frac{1}{2}\right)^3\left(1-\frac{1}{2}\right)^{6-3}$$

$$=\left(\frac{1}{2}\right)^3+3\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^1+6\left(\frac{1}{2}\right)^5+10\left(\frac{1}{2}\right)^6$$

$$=\frac{1}{8}+\frac{3}{16}+\frac{6}{32}+\frac{10}{64}=\frac{42}{64}=\frac{21}{32}.$$

■

19. (TI8912) It is rumored that Eric Vee, Math Leaguer from Moose Lake, beats his father Stan, Math League coach at Barnum, about $2/3$ of the time in chess. Assuming this to be true, what is the probability that Eric will beat his father in a best of three match (which, in the usual fashion, is over as soon as someone wins two games)?

Solution

Let X equal the games required for Eric to reach his second win.

As long as Eric's second win occurs before the fourth game then Eric would have won the best of 3 chess match (by the rules, the games stop when either player reaches their second win).

Then X follows a **negative binomial distribution** because the number of successes (wins for Eric) is fixed and the number of trials (games played) is random. [Recall that for the binomial distribution it's the other way around - the number of trials is fixed and the number of successes is random.]

From the negative binomial distribution function, if we let X equal the number of (Bernoulli) trials required in order to get a total of r successes. Then

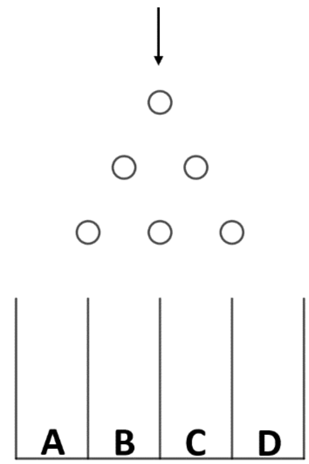
$$P(X = k) = \begin{cases} \binom{k-1}{r-1} p^r (1-p)^{k-r} & k \in \{r, r+1, r+2, \dots\} \\ 0 & k \notin \{r, r+1, r+2, \dots\}. \end{cases}$$

The problem asks for $P(X \leq 3)$ where X follows the *negative binomial distribution* with $r = 2$ and $p = 2/3$.

$$\begin{aligned} P(X \leq 3) &= P(X = 2) + P(X = 3) \\ &= \binom{2-1}{2-1}\left(\frac{2}{3}\right)^2\left(1-\frac{2}{3}\right)^{2-2} + \binom{3-1}{2-1}\left(\frac{2}{3}\right)^2\left(1-\frac{2}{3}\right)^{3-2} \\ &= \left(\frac{2}{3}\right)^2 + 2\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^1 \\ &= \frac{4}{9} + \frac{8}{27} = \frac{20}{27}. \end{aligned}$$

■

20. (5D852) A ball in a certain pinball machine bounces to your left with probability $1/3$ and to your right with probability $2/3$ whenever it comes to a bumper. It falls through the configuration shown, hitting a bumper at each level, finally coming to rest in one of the slots A, B, C or D. What is the probability that the ball will come to rest in slot C?



Solution

In order for a ball to land in slot C a ball has to bounce right at exactly 2 of the 3 bumpers it reaches as it falls down. We can model this as a binomial random variable where X equals the number of success (bounces right) in a total of $n = 3$ trials (bumpers). We are given that $p = P(\text{success}) = P(\text{ball bounces right at a bumper}) = 2/3$.

$$P(\text{ball lands in slot } C) = P(X = 2) = \binom{3}{2}\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^1 = \frac{12}{27} = \frac{4}{9}.$$

■

21. (TI8414) Oddsmakers believe the probability of the Tigers beating the Cubs in a single game is 0.6. The Tigers and the Cubs are to play a series of up to seven games, the first team to win four games being declared the champion.

(i) What probability should they assign to the outcome of the Cubs winning the series in exactly 6 games?

(ii) What probability should they assign to the outcome that the Tigers will win the series in less than 7 games?

Solution

Let X equal the games required for the Cubs to reach their fourth win.

As long as the Cub's fourth win occurs before the eighth game then the Cub's will be the champion.

Then X follows a **negative binomial distribution** because the number of successes (wins for the Cubs's) is fixed at four and the number of trials (games played) is random. [Recall that for the binomial distribution it's the other way around - the number of trials is fixed and the number of successes is random.]

From the negative binomial distribution function, if we let X equal the number of (Bernoulli) trials required in order to get a total of r successes. Then

$$P(X=k)=\begin{cases}\binom{k-1}{r-1}p^r(1-p)^{k-r} & k\in\{r,r+1,r+2,\dots\}\\ 0 & k\notin\{r,r+1,r+2,\dots\}.\end{cases}$$

(i) The problem asks for $P(X=6)$ where X follows the *negative binomial distribution* with $r=4$ and $p=1-0.6=0.4$.

$$P(X=6)=\binom{6-1}{4-1}(0.4)^4(0.6)^{6-4}=10\left(\frac{2}{5}\right)^4\left(\frac{3}{5}\right)^2=\frac{288}{3125}=0.09216$$

(ii) The problem switches to the perspective of the Tigers where $p=0.6$ and asks for $P(X<7)$ where X follows the *negative binomial distribution* with $r=4$ and $p=0.6$.

$$P(X<7)=P(X=4)+P(X=5)+P(X=6)$$

$$=\binom{4-1}{4-1}(0.6)^4(0.4)^{4-4}+\binom{5-1}{4-1}(0.6)^4(0.4)^{5-4}+\binom{6-1}{4-1}(0.6)^4(0.4)^{6-4}$$

$$=\left(\frac{3}{5}\right)^4+4\left(\frac{3}{5}\right)^4\left(\frac{2}{5}\right)^1+10\left(\frac{3}{5}\right)^4\left(\frac{2}{5}\right)^2=\frac{1701}{3125}=0.54432$$

■

22. (3T836) Rodney Roller is down to his last turn in a Yahtzee game. He has two 6's on the table, but he needs one more. He gets to roll three dice. What is the probability that Rodney will get at least one 6 when he rolls the three dice?

Solution

Let X equal the number of six's Rodney gets in three rolls of a fair die. Then X follows a binomial distribution with $n=3$ and $p=1/6$.

The problem asks for

$$P(X\geq 1)=1-P(X=0)=1-\binom{3}{0}\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^3$$

$$= 1 - \left(\frac{5}{6}\right)^3 = \frac{91}{216}.$$

■

23. (5C182) When tossing two 8-sided dice, the sides of each die being numbered 1 through 8, determine the probability of rolling two numbers a and b, such that $|a-b| < 3$.

Solution

Let X_j equal the number of times the number j is rolled when rolling $n = 2$ (fair) eight-sided dice. Then $(X_1, X_2, \ldots, X_8)$ follows a multinomial distribution where

$$P(X_1 = x_1, \ldots, X_8 = x_8) = \begin{cases} \dfrac{2!}{x_1! \cdots x_8!}\left(\dfrac{1}{8}\right)^{x_1} \cdots \left(\dfrac{1}{8}\right)^{x_8} & \begin{array}{c} x_j \in \{0,1,2\}, j = 1, \ldots, 8 \\ x_1 + x_2 + \cdots + x_8 = 2 \end{array} \\ 0 & \text{else.} \end{cases}$$

It follows that

$$P\big(|a-b| < 3\big) = P\big(|a-b| = 0\big) + P\big(|a-b| = 1\big) + P\big(|a-b| = 2\big)$$

$$= \left(\frac{2!}{2!\,0!\,0!\,0!\,0!\,0!\,0!\,0!}\left(\frac{1}{8}\right)^2\left(\frac{1}{8}\right)^0\left(\frac{1}{8}\right)^0\left(\frac{1}{8}\right)^0\left(\frac{1}{8}\right)^0\left(\frac{1}{8}\right)^0\left(\frac{1}{8}\right)^0\left(\frac{1}{8}\right)^0 \right.$$

$$\left. + \cdots + \frac{2!}{0!\,0!\,0!\,0!\,0!\,0!\,0!\,2!}\left(\frac{1}{8}\right)^0\left(\frac{1}{8}\right)^0\left(\frac{1}{8}\right)^0\left(\frac{1}{8}\right)^0\left(\frac{1}{8}\right)^0\left(\frac{1}{8}\right)^0\left(\frac{1}{8}\right)^0\left(\frac{1}{8}\right)^2 \right)$$

$$+ \left(\frac{2!}{1!\,1!\,0!\,0!\,0!\,0!\,0!\,0!}\left(\frac{1}{8}\right)^1\left(\frac{1}{8}\right)^1\left(\frac{1}{8}\right)^0\left(\frac{1}{8}\right)^0\left(\frac{1}{8}\right)^0\left(\frac{1}{8}\right)^0\left(\frac{1}{8}\right)^0\left(\frac{1}{8}\right)^0 \right.$$

$$\left. + \cdots + \frac{2!}{0!\,0!\,0!\,0!\,0!\,0!\,0!\,0!}\left(\frac{1}{8}\right)^0\left(\frac{1}{8}\right)^0\left(\frac{1}{8}\right)^0\left(\frac{1}{8}\right)^0\left(\frac{1}{8}\right)^0\left(\frac{1}{8}\right)^0\left(\frac{1}{8}\right)^1\left(\frac{1}{8}\right)^1 \right)$$

$$+\left(\frac{2!}{1!\,0!\,1!\,0!\,0!\,0!\,0!\,0!}\left(\frac{1}{8}\right)^1\left(\frac{1}{8}\right)^0\left(\frac{1}{8}\right)^1\left(\frac{1}{8}\right)^0\left(\frac{1}{8}\right)^0\left(\frac{1}{8}\right)^0\left(\frac{1}{8}\right)^0\left(\frac{1}{8}\right)^0\right.$$

$$\left.+\cdots+\frac{2!}{0!\,0!\,0!\,0!\,0!\,1!\,0!\,1!}\left(\frac{1}{8}\right)^0\left(\frac{1}{8}\right)^0\left(\frac{1}{8}\right)^0\left(\frac{1}{8}\right)^0\left(\frac{1}{8}\right)^0\left(\frac{1}{8}\right)^1\left(\frac{1}{8}\right)^0\left(\frac{1}{8}\right)^1\right)$$

$$= 8\cdot\left(\frac{1}{8}\right)^2 + 7\cdot 2\cdot\left(\frac{1}{8}\right)^2 + 6\cdot 2\cdot\left(\frac{1}{8}\right)^2 = \frac{8+14+12}{64} = \frac{34}{64} = \frac{17}{32}.$$

■

24. (TC894) A fair die is rolled three times. What is the probability that two of the three rolls, but not all three will be equal?

Solution

Let X_j equal the number of times the number j is rolled when rolling $n = 3$ (fair) six-sided dice. Then $(X_1, X_2, \ldots, X_6)$ follows a multinomial distribution where

$$P(X_1 = x_1, \ldots, X_8 = x_6) = \begin{cases} \frac{3!}{x_1!\cdots x_6!}\left(\frac{1}{6}\right)^{x_1}\cdots\left(\frac{1}{6}\right)^{x_6} & x_j \in \{0,1,2,3\}, j = 1, \ldots, 6 \\ & x_1 + x_2 + \cdots + x_6 = 3 \\ 0 & \text{else.} \end{cases}$$

It follows that

$$P(\text{two of the three rolls, but not all three will be equal})$$

$$= P(\text{the number 1 is rolled twice}) + \cdots + P(\text{the number 6 is rolled twice})$$

$$= \Big(P(X_1 = 2, X_2 = 1, X_3 = 0, \ldots, X_6 = 0)$$

$$+ \cdots + P(X_1 = 2, X_2 = 0, X_3 = 0, \ldots, X_6 = 1)\Big)$$

$$+ \cdots + \Big(P(X_1 = 1, X_2 = 0, X_3 = 0, \ldots, X_6 = 2)$$

$$+\cdots+P(X_1=0,X_2=0,\dots X_5=1,X_6=2)\Big)$$

$$=\left(\frac{3!}{2!\,1!\,0!\,0!\,0!\,0!}\left(\frac{1}{6}\right)^2\left(\frac{1}{6}\right)^1\left(\frac{1}{6}\right)^0\left(\frac{1}{6}\right)^0\left(\frac{1}{6}\right)^0\left(\frac{1}{6}\right)^0\right.$$

$$\left.+\cdots+\frac{3!}{2!\,0!\,0!\,0!\,0!\,1!}\left(\frac{1}{6}\right)^2\left(\frac{1}{6}\right)^0\left(\frac{1}{6}\right)^0\left(\frac{1}{6}\right)^0\left(\frac{1}{6}\right)^0\left(\frac{1}{6}\right)^1\right)$$

$$+\cdots+\left(\frac{3!}{1!\,0!\,0!\,0!\,0!\,2!}\left(\frac{1}{6}\right)^1\left(\frac{1}{6}\right)^0\left(\frac{1}{6}\right)^0\left(\frac{1}{6}\right)^0\left(\frac{1}{6}\right)^0\left(\frac{1}{6}\right)^2\right.$$

$$\left.+\cdots+\frac{3!}{0!\,0!\,0!\,0!\,1!\,2!}\left(\frac{1}{6}\right)^0\left(\frac{1}{6}\right)^0\left(\frac{1}{6}\right)^0\left(\frac{1}{6}\right)^0\left(\frac{1}{6}\right)^1\left(\frac{1}{6}\right)^2\right)$$

$$=5\left(\frac{3!}{2!\,1!}\right)\left(\frac{1}{6}\right)^3+\cdots+5\left(\frac{3!}{2!\,1!}\right)\left(\frac{1}{6}\right)^3$$

$$=6\cdot 5\cdot 3\cdot\left(\frac{1}{6}\right)^3$$

$$=\frac{5}{12}.$$

■

25. Every hospital has backup generators for critical systems should the electricity go out. Independent but identical backup generators are installed so that the probability that at least one system will operate correctly when called upon is no less than 0.99. Let n denote the number of backup generators in a hospital. How large must n be to achieve the specified probability of at least one generator operating, if the probability that any backup generator will work correctly is 0.95?

Solution

■

26. You flipped a coin 10 times and got 8 heads and this made you wonder if this was a “fair” coin (i.e. $50/50$ chance of heads/tails). To find out you decide to run an experiment consisting of 5 replications of flipping this coin 10 times (in each replication) in a controlled manner. If this really is a fair coin, what is the probability of observing at least one replication where you observe at least 8 heads?

Solution

■

27. A tire maker knows from past experience that 20% of their top-of-the-line brand tire will not satisfy the conditions of their warranty. The tire maker also knows from experience that only 10% of their customers of this top-of-the-line brand tire will bother to make a claim on their warranty when they could. What is the probability that a particular tire store selling this top-of the-line brand tire will not see any appropriate claims against the warranty from their next 30 customers buying this brand of tire?

Solution

■

28. Consider a multiple-choice examination with 10 questions, each of which has 4 possible answers. If a student knows the correct answer with probability 0.8 and guesses with probability 0.2, what is the probability that this student will score at least an 80? Assume each question is worth 10 points and no partial credit is given. Also assume that the questions are answered independently, that is, whether this student answers question #3 correct or incorrect will not influence whether they answer question #7 correct or incorrect?

Solution

■

29. A fair die is rolled n times. What is the smallest value of n such that the probability of getting at least one six in these n rolls is 0.95 or higher?

Solution

■

30. Suppose you know from experience that 1% of the parts coming off an assembly line at a local manufacturing plant are defective.

    (i) What is the probability that a lot of 500 will have less than 3 defectives in it?

    (ii) Suppose you cannot tell by just looking whether a part is defective but rather have to subject the part to a test in order to tell. What is the probability that you will have to test more than 150 parts before you find a defective one?

Solution

■

31. An airline knows from experience that 10% of the people holding reservations on a given flight will not appear. The plane holds 90 people. If 95 reservations have been sold, what is the probability that the airline will be able to accommodate everyone appearing for the flight?

Solution

■

32. Suppose that a four-engine plan can fly if at least two engines work and suppose that a two-engine plan can fly if at least one engine works. Would you rather fly on a four-engine or two-engine plan?

Solution

■

33. During the 1978 baseball season, Pete Rose of the Cincinnati Reds set a National League record by hitting safely in 44 consecutive games. Assume that Rose is a 300 hitter, (i.e. the probability he hits safely on any given time at bat is .300) and assume that he comes to bat four times each game. If each at bat is assumed to be an independent event, what is the probability of hitting safely in 44 consecutive games?

Solution

■

34. A coin is altered so that the probability that it lands on heads is less than $1/2$ and then the coin is flipped four times, the probability of an equal number of heads and tails is $1/6$. What is the probability that the coin lands on heads? (2010 AMC 12 A Problem 15)

Solution

■

35. Coin A is flipped three times and coin B is flipped four times. What is the probability that the number of heads obtained from flipping the two fair coins is the same? (2004 AMC 10a Problem 10)

Solution

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35. A coin with an unknown probability of landing heads is tossed ten times and lands on heads exactly three times. Find the conditional probability that the first toss landed on heads. (Source: https://math.la.asu.edu/~jtaylor/teaching/Spring2017/STP421/problems/ps2-solutions.pdf)

Solution

What is p? Does it matter? How to take advantage of this.

Let p equal the unknown probability that this coin will on heads on any toss. Let E be the event of getting exactly 3 heads in 10 tosses. By the binomial distribution we have

$$P(E) = \binom{10}{3} p^3 (1-p)^7.$$

Let H_1 be the event that the first toss landed on heads. Then

$$P(H_1|E) = \frac{P(E|H_1)P(H_1)}{P(E)}.$$

The distribution of E conditional on the information that the first toss landed on heads is the same as the unconditional probability of getting 2 heads in 9 tosses. But this is again modeled by the binomial distribution. That is,

$$P(E|H_1) = \binom{9}{2} p^2 (1-p)^7.$$

Also, we know that $P(H_1) = p$. Therefore,

$$P(H_1|E) = \frac{P(E|H_1)P(H_1)}{P(E)} = \frac{\binom{9}{2} p^2 (1-p)^7 \cdot p}{\binom{10}{3} p^3 (1-p)^7}$$

$$= \frac{\binom{9}{2}}{\binom{10}{3}} = \frac{9!}{10!} \cdot \frac{3!}{2!} = \frac{3}{10}.$$

■

Exercises for Chapter 3

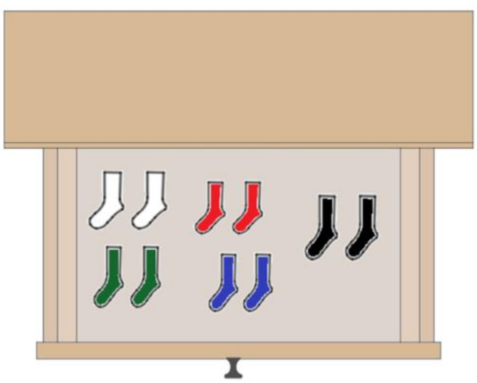
<u>Read</u> each of the following problems which were all taken from old MSHSML tests and consider how each of them depends in various ways on a hypergeometric random variable. (*i.e.* Identify what is playing the role of the white balls and the black walls, identify how is it made clear we are sampling *without* replacement and identify how many balls we are sampling in total).

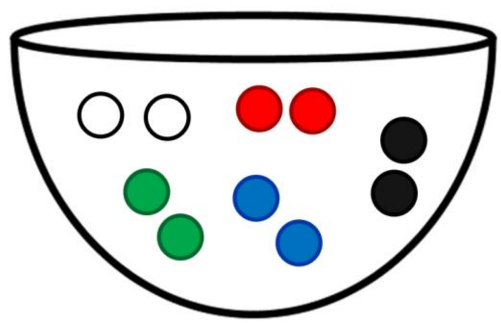
1. (5T194) Johnny has a drawer containing two green socks, two red socks, two black socks, two white socks and two blue socks. Johnny likes to wear matching socks, but he is color blind and cannot distinguish red from green. Johnny randomly pulls our four socks from his drawer. Determine exactly the probability that Johnny does not get a pair of socks of the same color, but *thinks* he does.

Solution

Multivariate Hypergeometric

A drawer (urn) contains two green socks (balls), two red balls, two black balls, two white balls and two blue balls. We take out 4 of the 10 balls without replacement.





1 of the two green		1 of the two green		1 of the two green
1 of the two red		1 of the two red		1 of the two red
1 of the two black	or	1 of the two black	or	0 of the two blacks
1 of the two blue		0 of the two blues		1 of the two blue
0 of the two whites		1 of the two white		1 of the two white

$$\frac{\binom{2}{1}\binom{2}{1}\binom{2}{1}\binom{2}{1}\binom{2}{0}}{\binom{10}{4}}+\frac{\binom{2}{1}\binom{2}{1}\binom{2}{1}\binom{2}{0}\binom{2}{1}}{\binom{10}{4}}+\frac{\binom{2}{1}\binom{2}{1}\binom{2}{0}\binom{2}{1}\binom{2}{1}}{\binom{10}{4}}=\frac{16+16+16}{210}=\frac{8}{35}.$$

■

2. (TI199) I have a bag with 21 beads of various colors, the majority being green. If I remove two beads (without replacement), the probability I will end up with one green and one non-green is $3/7$. How many green beads were in the bag originally?

Solution

Hypergeometric

Suppose there are g green and $21-g$ non-green beads in the bag, with $g > 21-g$.

$$\frac{\binom{g}{1}\binom{21-g}{1}}{\binom{21}{2}}=\frac{3}{7}$$

$$\frac{g(21-g)}{\frac{21(20)}{2}}=\frac{3}{7}$$

$$g^2-21g+90=0$$

$$(g-15)(g-6)=0$$

$$g=15 \text{ or } 6.$$

But g is a majority, so $g=15$.

■

3. (5C171) Sal's drawer contains 4 black socks and 2 red socks. If he chooses two socks at random without replacement, what is the probability he chooses two socks of the same color? Express your answer as a quotient of two relatively prime integers.

Solution

Hypergeometric

$$\frac{\binom{4}{2}\binom{2}{0}}{\binom{6}{2}} + \frac{\binom{4}{0}\binom{2}{2}}{\binom{6}{2}} = \frac{6+1}{15} = \frac{7}{15}.$$

■

4. (5C102) An ordinary deck of 52 playing cards is shuffled and 2 cards are dealt face up. Calculate the probability that at least one of these is a spade. *[NYCC, Fall 1984]*

Solution

$$P(\text{at least 1 spade}) = 1 - P(\text{no spades})$$
$$= 1 - \frac{\binom{13}{0}\binom{39}{2}}{\binom{52}{2}} = 1 - \frac{19}{34} = \frac{15}{34}.$$

■

5. (MB097) Ten playing cards are laying face down on a table. Exactly three of them are aces. You get to turn over two cards. What is the probability that at least one of them is an ace?

Solution

$$P(\text{at least 1 ace}) = 1 - P(\text{no aces})$$
$$= 1 - \frac{\binom{3}{0}\binom{7}{2}}{\binom{10}{2}} = 1 - \frac{7}{15} = \frac{8}{15}.$$

■

6. (TI086) A hand of 5 cards is drawn from a standard deck of 52 cards. Find the probability of getting exactly one ace.

Solution

Hypergeometric: 4 white balls (aces), 48 black balls (non-aces).

Let X equal the number of white ball (aces) in the sample of size 5.

$$P(X=1)=\frac{\binom{4}{1}\binom{48}{4}}{\binom{52}{5}}=0.299474.$$

■

7. (TT082) A hand of 5 cards is drawn from a standard deck of 52 cards. Successful participants in this morning's Invitational found that the probability of getting exactly one ace is 0.299474. You get one point for finding the probabilities (again to six decimal places) of each of the other possibilities.

Solution

Hypergeometric: 4 white balls (aces), 48 black balls (non-aces).

Let X equal the number of white ball (aces) in the sample of size 5.

$$P(X=x)=\frac{\binom{4}{x}\binom{48}{5-x}}{\binom{52}{5}},\ \ x=0,1,2,3,4.$$

Therefore,

$$P(0\text{ aces})=\ P(X=0)=\frac{\binom{4}{0}\binom{48}{5-0}}{\binom{52}{5}}=0.658842$$

$$P(2\text{ aces})=\ P(X=2)=\frac{\binom{4}{2}\binom{48}{5-2}}{\binom{52}{5}}=0.039930$$

$$P(3\text{ aces})=\ P(X=3)=\frac{\binom{4}{3}\binom{48}{5-3}}{\binom{52}{5}}=0.001736$$

$$P(4\text{ aces})=\ P(X=2)=\frac{\binom{4}{4}\binom{48}{5-4}}{\binom{52}{5}}=0.000018.$$

■

8. (TC063) A cooler contains 4 bottles of carbonated mineral water and 6 bottles of non-carbonated spring water. Not realizing there is a choice, Alicia and Beth each plunge their

hands into the ice and grab a bottle. What is the probability that both girls get the same kind of water?

Solution

Hypergeometric: 4 white balls, 6 black balls, sample of size 2.

$$\frac{\binom{4}{2}\binom{6}{0}+\binom{4}{0}\binom{6}{2}}{\binom{10}{2}}=\frac{6+15}{45}=\frac{21}{45}=\frac{7}{15}$$

■

9. (5C052) If you draw three cards from an ordinary 52 card deck, what is the probability of drawing at least one of the twelve face cards (*i.e.* a Jack, Queen, or King)?

Solution

Hypergeometric: 12 white balls, 40 black balls, sample of size $n=3$.

$$\begin{aligned} P(\text{at least one face card}) &= 1-P(0\text{ face cards}) \\ &= 1-\frac{\binom{12}{0}\binom{40}{3}}{\binom{52}{3}}=\frac{47}{85}. \end{aligned}$$

■

10. (5C012) Nine playing cards are lying face down on a table. Exactly two of them are Aces. You pick up two cards. What is the probability that at least one of them will be an Ace?

Solution

$$\begin{aligned} P(\text{at least 1 ace}) &= 1-P(\text{no aces}) \\ &= 1-\frac{\binom{2}{0}\binom{7}{2}}{\binom{9}{2}}=1-\frac{7}{12}=\frac{5}{12}. \end{aligned}$$

■

11. (5T003) A box contains eleven balls numbered $1,2,3,\dots,11$. If six balls are drawn simultaneously at random, what is the probability that the sum of the numbers on the balls drawn is odd? [1984 AHSME, prob. 19]

Solution

The sum is odd if and only if there are an odd number of odd numbered balls chosen.

There are six odd numbers and five even numbers in $1,2,3,\dots,11$.

$\{1,3,5,7,9,11\}\,\{2,4,6,8,10\}$

$$P(\text{sum is odd}) = \frac{\binom{6}{1}\binom{5}{5}+\binom{6}{3}\binom{5}{3}+\binom{6}{5}\binom{5}{1}}{\binom{11}{6}}$$

$$= \frac{6+200+30}{462} = \frac{118}{231}.$$

■

12. (TI0010) Alicia grabs two bagels from a bag containing 10 whole wheat and 6 pumpernickel bagels. What is the probability that she gets 1 of each?

Solution

Hypergeometric (10 white balls, 6 black balls, sample of size 2)

$$\frac{\binom{10}{1}\binom{6}{1}}{\binom{16}{2}} = \frac{60}{\frac{16(15)}{2}} = \frac{120}{240} = \frac{1}{2}.$$

■

13. (5C991) Cards are turned over one at a time (without replacement) from an ordinary well-shuffled decks (52 cards). What is the probability that the first two cards are aces? (There are four aces in a deck.)

Solution

Hypergeometric: 4 white balls (aces), 48 black ball (non-aces), sample of size 2

$$\frac{\binom{4}{2}\binom{48}{0}}{\binom{52}{2}} = \frac{4}{52}\cdot\frac{3}{51} = \frac{1}{221}.$$

■

14. (TC994) A piggy bank contains 1 quarter, 3 dimes, 2 nickels, and 1 cent. Needing 46 cents, Aaron shakes out four coins. Assuming the coins are equally likely to be shaken out, what is the probability that he has shaken out exactly 46 cents?

Solution

The only combination of four of these coins that will equal 46 cents is a quarter, two dimes and a penny.

Multivariate Hypergeometric (1 quarter, 3 dimes, 2 nickels, 1 penny)

$$\frac{\binom{1}{1}\binom{3}{2}\binom{2}{0}\binom{1}{1}}{\binom{7}{4}} = \frac{3}{35}.$$

■

15. (TI9911) A bag contains white marbles and black marbles. If you were to reach in without looking and pull out one marble, the probability that it would be white is $1/4$. However if you were to pull out two marbles, the probability that both would be white is $1/18$. How many white marbles are in the bag?

Solution

Hypergeometric

Assume there are w white and b black marbles in the bag. Let $t = w + b$. Then

$$\frac{\binom{w}{1}\binom{b}{0}}{\binom{t}{1}} = \frac{1}{4} \Leftrightarrow t = 4w$$

and

$$\begin{aligned}\frac{\binom{w}{2}\binom{b}{0}}{\binom{t}{2}} = \frac{1}{18} &\Leftrightarrow 18w(w-1) = t(t-1) \\ &\Leftrightarrow 18(w-1) = 4(4w-1) \\ &\Leftrightarrow 2w = 14 \\ &\Leftrightarrow w = 7.\end{aligned}$$

■

16. (5T985) Three cards are drawn at random from a standard deck of 52 playing cards. Calculate the probability (rounded to four decimal places) that at least one of these three cards is a face card. (Here, a face cards is a jack, queen, or king. The motivation for this problem is the question of whether or not one would give even odds that at least one of the three cards is a face card.)

Solution

Hypergeometric (12 white, 40 non-white, sample of size 3)

$$
\begin{aligned}
P(\text{at least one face card}) &= 1 - P(\text{no face cards}) \\
&= 1 - \frac{\binom{12}{0}\binom{40}{3}}{\binom{52}{3}} \\
&= 1 - \frac{\binom{12}{0}\binom{40}{3}}{\binom{52}{3}} \\
&= 1 - 0.447059 = 0.5529.
\end{aligned}
$$

■

17. (TT983) A bag contains 16 billiard balls, some black and the remainder white. Two balls are drawn at the same time. It is equally likely that the two balls will be the same color as different colors. How are the balls divided within the bag?

Solution

Let b equal the number of black balls and let w equal the number of white balls in this bag. We are given that

$$\frac{\binom{b}{2}\binom{w}{0}}{\binom{16}{2}} + \frac{\binom{b}{0}\binom{w}{2}}{\binom{16}{2}} = \frac{\binom{b}{1}\binom{w}{1}}{\binom{16}{2}}$$

and

$$b + w = 16.$$

On simplifying,

$$\frac{b(b-1)+w(w-1)}{2}=bw$$

$$b(b-1)+w(w-1)=2bw$$

$$b(b-1)+(16-b)(15-b)=2b(16-b)$$

$$4b^2-64b+240=0$$

$$b^2-16b+60=0$$

$$(b-10)(b-6)=0$$

$$b=6 \text{ or } b=10.$$

So, the split is either (6 black and 10 white) or (10 white and 6 black).

■

18. (5C972) In your drawer there is a pair of green gloves and a pair of brown gloves. You reach in with your eyes closed and pull out two gloves at random. What is the probability that you have a matched pair?

Solution

$$\frac{\binom{2}{2}\binom{2}{0}}{\binom{4}{2}}+\frac{\binom{2}{0}\binom{2}{2}}{\binom{4}{2}}=\frac{1+1}{6}=\frac{1}{3}.$$

■

19. (5T913) I have n nuts in my hand, m of which will fit a bolt on my bicycle. If I select 2 of the nuts at random, there is an even chance (*i.e.* the probability is $1/2$) that both will fit the bolt. What is the smallest value of n?

Solution

$$\frac{\binom{m}{2}\binom{n-m}{0}}{\binom{n}{2}}=\frac{1}{2}$$

$$\frac{m(m-1)}{n(n-1)} = \frac{1}{2}$$

$$2m(m-1) = n(n-1)$$

for positive integers m and n.

First note that $2m(m-1) = n(n-1)$ is impossible unless $n > m \geq 2$. The problem is to find the smallest integer n so that $2m(m-1) = n(n-1)$ has a solution for some integer $m \geq 2$.

The easiest approach is to guess and check.

$n = 3,\ n(n-1) = 6$	$2m(m-1) = 6$ or $m^2 - m - 3 = 0$	No positive integer solutions.
$n = 4,\ n(n-1) = 12$	$2m(m-1) = 12$ or $m^2 - m - 6 = 0$	$m = 3$ is a solution.

So $n = 4, m = 3$ works. As a check we note that

$$\frac{m(m-1)}{n(n-1)} = \frac{3(2)}{4(3)} = \frac{1}{2}$$

as required.

■

20. (5C902) There are 4 black socks and 6 brown socks all mixed together in a drawer in a dark room. If Pat grabs two socks from the drawer and takes them into the light, what is the probability that they will match?

Solution

$$\frac{\binom{4}{2}\binom{6}{0}}{\binom{10}{2}} + \frac{\binom{4}{0}\binom{6}{2}}{\binom{10}{2}} = \frac{6+15}{45} = \frac{21}{45} = \frac{7}{15}.$$

■

21. (5C903) There are 4 black socks and 6 brown socks all mixed together in a drawer in a dark room. If Pat grabs three socks from the drawer and takes them into the light,

a) what is the probability of a match?

b) what is the probability of at least two black socks?

Solution

a) Something a trick question, with three socks there will always be a match. So the probability of a match equals 1.

b)

$$\frac{\binom{4}{2}\binom{6}{1}}{\binom{10}{3}} + \frac{\binom{4}{3}\binom{6}{0}}{\binom{10}{3}} = \frac{36+2}{120} = \frac{40}{120} = \frac{1}{3}.$$

■

22. (5T881) Amy, who puts only nickels and dimes in her bank, currently has an even number of nickels. When two coins are drawn at random from this bank, the probability that both are dimes is $1/2$. What is the smallest amount of money that Amy might have?

Solution

Let n equal the number of nickels and let d equal the number of dimes that Amy has in her bank. We are given the information that $n = 2k$ for some nonnegative integer k and

$$P(\text{both dimes}) = \frac{\binom{n}{0}\binom{d}{2}}{\binom{n+d}{2}} = \frac{1}{2}.$$

$$\frac{d(d-1)}{(n+d)(n+d-1)} = \frac{1}{2}.$$

On simplification we have

$$d^2 - (2n+1)d - (n^2 - n) = 0.$$

Treating this as a quadratic equation in d, we have from the quadratic formula that

$$d = \frac{-(2n+1) \pm \sqrt{\left(-(2n+1)\right)^2 - 4(-1)(n^2-n)}}{2}$$

$$= \frac{(2n+1) \pm \sqrt{(4n^2+4n+1)+4(n^2-n)}}{2}$$

$$= \frac{(2n+1) \pm \sqrt{8n^2+1}}{2}$$

In order for d to be an integer it is necessary that $8n^2+1$ is a perfect square, say m^2. The equation $8n^2+1=m^2$ or alternatively $m^2-8n^2=1$ is of the form called Pell's Equation. There are results in number theory for finding an expression for all possible solutions but it is beyond what we can introduce here.

First note that $8n^2+1$ is odd, which means that $m^2=8n^2+1$ is odd. Therefore m must be odd.

This implies that $(2n+1) \pm \sqrt{8n^2+1}$ is even for all perfect squares $8n^2+1$. Therefore,

$$d = \frac{(2n+1) \pm \sqrt{8n^2+1}}{2}$$

will be an integer for all perfect squares $8n^2+1$. Hence, it suffices to find the smallest even integer $n \geq 2$ for which $8n^2+1$ is a perfect square. (I imagine the restriction that n is even is to rule out the overly simple solution $n=1, d=3$.)

The easiest approach is guess and check.

$n=2$	$8n^2+1=8(2^2)+1=33$
$n=4$	$8n^2+1=8(4^2)+1=129$
$n=6$	$8n^2+1=8(6^2)+1=289=17^2$

Actually, we can rule out the possibility

$$d = \frac{(2n+1) - \sqrt{8n^2+1}}{2}$$

because in this case $d<0$ for all $n \geq 2$. Solving for

$$d = \frac{(2n+1) + \sqrt{8n^2+1}}{2}$$

when $n=6$ gives us

$$d = \frac{(2(6)+1)+17}{2} = 15.$$

Therefore, the least amount of money that Amy can have is $(n = 6, d = 15)$ which comes to $30 + 150 = 180¢$

■

23. (TD883) A box contains nine balls numbered 1 through 9. If five balls are drawn at random and without replacement, what is the probability that the sum of the numbers on the balls will be odd?

Solution

The sum will be odd as long as you have an odd number of odd numbers. Imagine an urn with $n = 5$ white balls (one for each of the odd numbers 1,3,5,7,9) and $b = 4$ black balls (one for each of the even numbers 2,4,6,8).

$$P(\text{sum is odd}) = \frac{\binom{5}{1}\binom{4}{4}}{\binom{9}{5}} + \frac{\binom{5}{3}\binom{4}{2}}{\binom{9}{5}} + \frac{\binom{5}{5}\binom{4}{0}}{\binom{9}{5}} = \frac{5+60+1}{126} = \frac{11}{21}.$$

■

24. Suppose that 100 cards marked $1,2,\dots,100$ are randomly arranged in a line. What is the probability that exactly 8 of the first 20 cards in this line are marked with an even number?

Solution

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25. Suppose you play a lottery where you choose five <u>different</u> numbers from the integers $1, 2, \dots, 45$. Then you choose one number from $1, 2, \dots, 45$. This last number is often called the *powerball* and <u>can</u> (but does not have to) repeat one of the first five numbers you chose. Lottery officials choose five numbers and the powerball in the same way. What is the probability that you match 3 of the first 5 but do not match the powerball?

Solution

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26. 15 women and 9 men were eligible for promotion at a university and from this group 2 women and 4 men were selected for promotion. Is there reason to suspect gender bias?

Solution

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27. An urn contains 3 red, 5 blue and 7 yellow balls. You pull out 4 balls (without replacement). What is the probability you get exactly 2 yellow balls?

Solution

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28. A Hypergeometric Problem Nested Inside a Binomial Problem

    Imagine a game that consists of two stages. In the first stage you have to draw 3 balls at random and without replacement from an urn containing 4 black and 8 white balls. According to the rules, in order to be able to proceed to the second stage of this game you have to get at least one black ball in this sample of size three during the first phase.

    (a) Out of ten players (all starting with the same urn of 4 black and 8 white balls), what is the probability that five or more will make it to the second stage?

    (b) How many out of these ten players do you expect to make it to the second stage?

Solution

■

29. Suppose there are two identical urns and that each urn contains N balls numbered from 1 to N. One of these urns is given to Mary and the other is given to Bob. Mary reaches into her urn and randomly selects n balls without replacement and makes a list of the numbers she got. Bob reaches into his urn and randomly selects k balls without replacement and makes a list of the numbers he got.

    Let X equal how many numbers are on both Mary's and Bob's list. Find $P(X = x)$.

Solution

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30. In the game of Texas Hold'em, players are each dealt two private cards, and five community cards are dealt face-up on the table. Each player makes the best 5-card hand they can with their two private cards and the five community cards. What is the probability that a particular player can make a flush of spades (i.e. 5 spades)?

    https://brilliant.org/wiki/hypergeometric-distribution/

Solution

■

Exercises for Chapter 4

(5T184) Two teams, A and B, are playing in a tournament. They will play until one team wins four games (no ties allowed). The probability of either team winning the first game is 50%. For both teams the probability of winning the very next game after winning one is 60%, winning a third game after winning two in a row 70%, and winning a fourth game in a row is 75%. Determine the probability team A wins in exactly 5 games.

Solution

The solution involves repeated application of the general multiplication (or product) rule.

$$P(A \cap B \cap C \cap D) = P(A)P(B|A)P\big(C\big|(A \cap B)\big)P\big(D\big|(A \cap B \cap C)\big).$$

$$P(BAAAA) = P(B)P(A|B)P(A|BA)P(A|BAA)P(A|BAAA) = (0.5)(0.4)(0.6)(0.7)(0.75)$$
$$P(ABAAA) = P(A)P(B|A)P(A|AB)P(A|ABA)P(A|ABAA) = (0.5)(0.4)(0.4)(0.6)(0.7)$$
$$P(AABAA) = P(A)P(A|A)P(B|AA)P(A|AAB)P(A|AABA) = (0.5)(0.6)(0.3)(0.4)(0.6)$$
$$P(AAABA) = P(A)P(A|A)P(A|AA)P(B|AAA)P(A|AAAB) = (0.5)(0.6)(0.7)(0.25)(0.4)$$

$$P(\text{Team } A \text{ wins in exactly 5 games}) = P(BAAAA) + P(ABAAA) + P(AABAA) + P(AAABA)$$

$$= 0.063 + 0.0336 + 0.0216 + 0.021 = 0.1392$$

■

(5A164) Two people play a game, taking turns drawing a ball from an urn randomly, without replacing them. The urn starts out with 3 black balls and 4 white balls. If a player draws a black ball, their turn is complete. If a player draws a white ball, they must also (nonrandomly) take two black balls from the urn before they complete their turn. A player loses when they cannot complete their turn. If you draw first, determine the exact probability you will win the game.

Solution

Begin by listing out the complete sample space (the list of all ways this game could proceed from start to finish).

Notation: Let, WB, for example, represent that on your first turn you draw a white ball and on your friend's first turn they draw a black ball. Similarly, let BBB represent the sequence, you draw a black, then your friend draws a black and then you draw black on your second turn.

Note that WB represents a win for your friend because the initial $(black, white) = (3,4)$ count would look like

$$(3,4) \longrightarrow (1,3) \longrightarrow (0,3)$$

and you could not complete your second turn because the urn would contain 3 white and 0 black which guarantees you would draw a white. But then you could not complete your turn because there are no blacks left so you cannot remove two blacks as required.

(WB) $(3,4) \longrightarrow (1,3) \longrightarrow (0,3)$	Friend wins.
(WW) $(3,4) \longrightarrow (1,3)$	**You win** (because your friend could only remove 1 black ball after they drew a white and hence could not complete their turn).
(BW) $(3,4) \longrightarrow (2,4) \longrightarrow (0,3)$	Friend wins (because you would necessarily draw a white on your second turn and would not be able to remove two black balls at that point).
(BBB) $(3,4) \longrightarrow (2,4) \longrightarrow (1,4) \longrightarrow (0,4)$	**You win** (because your friend would necessarily draw a white on their second turn and would not be able to remove two black balls at that point).
(BBW) $(3,4) \longrightarrow (2,4) \longrightarrow (1,4)$	Friend wins (because you could only remove 1 black ball after you drew a white on your second draw).

Now we can determine the probability of each of these four possible outcomes.

$$P(WB) = P(W) \cdot P(B|W) = \left(\frac{4}{7}\right)\left(\frac{1}{4}\right) = \frac{1}{7}$$

$$P(WW) = P(W) \cdot P(W|W) = \left(\frac{4}{7}\right)\left(\frac{3}{4}\right) = \frac{3}{7}$$

$$P(BW) = P(B) \cdot P(W|B) = \left(\frac{3}{7}\right)\left(\frac{4}{6}\right) = \frac{2}{7}$$

$$P(BBB) = P(B) \cdot P(B|B) \cdot P(B|BB) = \left(\frac{3}{7}\right)\left(\frac{2}{6}\right)\left(\frac{1}{5}\right) = \frac{1}{35}$$

$$P(BBW) = P(B) \cdot P(B|B) \cdot P(W|BB) = \left(\frac{3}{7}\right)\left(\frac{2}{6}\right)\left(\frac{4}{5}\right) = \frac{4}{35}.$$

Note that the sum of the probabilities of these five possible outcomes equals 1 as it must.

$$\frac{1}{7}+\frac{3}{7}+\frac{2}{7}+\frac{1}{35}+\frac{4}{35}$$
$$=\frac{5}{35}+\frac{15}{35}+\frac{10}{35}+\frac{1}{35}+\frac{4}{35}$$
$$=\frac{35}{35}=1.$$

The sum of the probabilities of the two outcomes where you win equal

$$\frac{3}{7}+\frac{1}{35}=\frac{16}{35}.$$

■

(5C122) Kathy flips a fair coin until she gets three heads in a row. If it is known that she stopped after seven flips, what is the probability her first flip was heads?

Solution

We are given the information that Kathy stopped on the 7^{th} flip. That is, the last three flips had to be HHH. But it also means that the 4^{th} flip could not also be H because in that case Kathy would have stopped on the 6^{th} flip.

But it tells us one more thing. If she stopped on the 7^{th} flip then the first three flips could not be HHH.

So, the question asked is to find

$$P\big(\text{first flip } H \big| (\text{first three flips not } HHH \text{ and last 4 flips are } THHH)\big).$$

Now define the events

A : first flip is a H

B : first three flips are not HHH

C : last 4 flips are $THHH$

so that are question can be written as $P(A|(B \cap C))$. A problem with trying to simplify this probability is that the event A is not independent of the event $(B \cap C)$ because both events involve the result of the first flip.

However, we can express $P(A|(B \cap C))$ in a different form.

$$P\big(A\big|(B \cap C)\big)=\frac{P(A \cap B \cap C)}{P(B \cap C)}$$

$$
\begin{aligned}
&= \frac{P(B \cap A \cap C)}{P(B \cap C)} \\
&= \frac{P(B)P(A|B)P\big(C\big|(A \cap B)\big)}{P(B)P(C|B)} \\
&= \frac{P(A|B)P\big(C\big|(A \cap B)\big)}{P(C|B)}.
\end{aligned}
$$

We can assume that the outcomes of Kathy's seven flips of her coin are mutually independent (not just pairwise independent).

Hence the following general result about mutually independent events applies:

> If $E_1, \dots E_k, E_{k+1}, \dots, E_n$ are mutually independent events then for general set functions f and g, $f(E_1, \dots, E_k)$ is independent of $g(E_{k+1}, \dots, E_n)$.

We note that C is a set function of the last four flips and both A and $A \cap B$ are set functions of the first three flips. Hence, the event C is independent of both B and $A \cap B$. Consequently,

$$P(C|B) = P(C) \text{ and } P\big(C\big|(A \cap B)\big) = P(C).$$

Therefore,

$$
\begin{aligned}
P\big(A\big|(B \cap C)\big) &= \frac{P(A|B)P\big(C\big|(A \cap B)\big)}{P(C|B)} \\
&= \frac{P(A|B)P(C)}{P(C)} \\
&= P(A|B).
\end{aligned}
$$

Finding $P(A|B) = P(\text{first flip } H|\text{first three flips not } HHH)$

Our original sample space for the first three flips consists of the eight outcomes $\mathbb{S} = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$ and because it is a fair coin each of these eight outcomes are equally likely.

The conditional sample space consists of that subset $\mathbb{A} \subset \mathbb{S}$ where the first three flips are not HHH. That is $\mathbb{A}$ consists of the seven outcomes $\{$~~HHH~~$, HHT, HTH, THH, HTT, THT, TTH, TTT\}$.

Because the original 8 (unconditional) outcomes were equally likely we know (by the method of scaling) that the 7 outcomes which are still possible after conditioning continue to remain equally likely. Hence each of the above seven outcomes in the condition sample space

$$\mathbb{A} = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$$

now has the conditional probability of $1/7$.

Which of the outcomes in the conditional sample space $\mathbb{A}$ make up the event where the first flip is H? $\boldsymbol{HHT}, \boldsymbol{HTH}, \boldsymbol{HTT}$

So,

$$\begin{aligned} &P(\text{first flip } H|\text{first three flips not } HHH) \\ &\quad = P(HHT \text{ or } HTH \text{ or } HTT|\text{first three flips not } HHH) \\ &\quad = P(HHT|\text{first three flips not } HHH) + P(HTH|\text{first three flips not } HHH) \\ &\qquad\qquad + P(HTT|\text{first three flips not } HHH) \\ &\quad = \frac{1}{7} + \frac{1}{7} + \frac{1}{7} = \frac{3}{7}. \end{aligned}$$

■

(5C102) An ordinary deck of 52 playing cards is shuffled and 2 cards are dealt face up. Calculate the probability that at least one of these is a spade. ***[NYCC, FALL 1984]***

Solution

$$\begin{aligned} P(\text{at least card 1 is a spade}) &= 1 - P(\text{neither card is a spade}) \\ &= 1 - P(1^{st} \text{ card is not a spade and } 2^{nd} \text{ card is not a spade}) \\ &= 1 - P(1^{st} \text{ card is not a spade}) \times \\ &\qquad P(2^{nd} \text{ card is not a spade}|1^{st} \text{ card is not a spade}) \\ &= 1 - \left(\frac{39}{52}\right)\left(\frac{38}{51}\right) = 1 - \frac{19}{34} = \frac{15}{34}. \end{aligned}$$

■

(5C083) At a carnival game, a black velvet pouch contains either a gold or silver coin, with equal probability of each. You win if you can correctly guess the color of the coin. While you are contemplating your choice, the game operator says, “Let me help you.” He drops in two gold coins, shakes the pouch, then reaches in and draws out two gold coins. What is the probability the remaining coin is gold? (Variant of one of Lewis' Carroll's famous "*Pillow Problems*".)

Solution

There are 2 random aspects to this problem.

(1) whether the original coin in the pouch was gold or silver

(2) which two coins the operator pulls out of the bag after dropping in two gold coins.

Make sure your sample space covers both random aspects.

Label the three coins as A, B and C, with A being the coin that was originally in the pouch.

Notation: Let (G, A, B), for example, represent the outcome where the initial coin was Gold and the game operator pulled out coins A and B.

In this notation, the sample space consists of the following 6 equally likely outcomes.

$$(G, A, B), (S, A, B)$$

$$(G, A, C), (S, A, C)$$

$$(G, B, C), (S, B, C).$$

Which of these six (unconditional) outcomes remain possible once we see that the two coins the game operator has pulled out are both gold?

$$(G, A, B), \cancel{(S, A, B)}$$

$$(G, A, C), \cancel{(S, A, C)}$$

$$(G, B, C), (S, B, C).$$

Because the original six (unconditional) outcomes were equally likely, the remaining four outcomes after conditioning on the information that both coins pulled out of the pouch were gold remain equally likely. Hence these four remaining outcomes each has probability $1/4$.

Which of these four remaining outcomes leaves a gold coin in the pouch after pulling out two gold coins? This could only be the cases with three gold coins. That is, the three outcomes

$$(G, A, B), (G, A, C), (G, B, C).$$

So in three of the four remaining equally likely outcomes in the conditional sample space, the coin left in the pouch is gold. That is,

$$P(\text{coin left in pouch is gold}|\text{two coins removed from the pouch were both gold}) = \frac{3}{4}.$$

■

Pillow Problems Thought Out During Wakeful Hours is a collection of 72 problems written by Charles L. Dodgson and was first published in 1895 as Part 2 of his book *Curiosa Mathematica*. Later editions were published under his more familiar pen name Lewis Carroll. The book is still available as a Dover edition but you can also get a free copy in many places on the web. Problems 5,10,16,19,23,27,38,41,45,50, 58 and 66 all involve probability. The last problem in the book, No. 72, is a tongue-in-check situation where Dodgson "proves" that if two balls are in a bag and if each is equally likely to be black or white and if their colors are determined independently then it will always be the case that one is black and the other is white. It is a problem of the same type you can find in problem books where you are given a proof that $1 = 2$ and the question is to spot the mistake in the proof. His Problem 72 is famous because the mistake in the proof is very well hidden.

Problem (5C083) is a variant of Dodgson's *Pillow Problem* #5.

(5C063) A bag contains 4 red and 8 white marbles, well mixed. One marble is removed and replaced by two marbles of the other color. Again, after thorough mixing, a marble is draw. What is the probability that this last marble to be removed is red? [original source AHSME 1995 #20]

Solution

Let R_1 represent the event that the first marble draw is red and let R_2 represent the event that the second marble draw is red.

Similarly, define W_1 as the event that the first marble draw is white and W_2 as the event that the second marble draw is white.

The problem is asking for $P(R_2)$. By the law of total probability

$$P(R_2) = P(R_2|R_1)P(R_1) + P(R_2|W_1)P(W_1).$$

From the given information we know that

$$P(R_1) = \frac{4}{12} \quad \text{and} \quad P(W_1) = \frac{8}{12}.$$

If the first marble draw is red, then we don't return that red back into the bag, but we add two white marbles to the bag. That would leave us with a bag containing, 3 red and 10 white.

It follows that,

$$P(R_2|R_1) = \frac{3}{13}.$$

If the first marble draw is white, then we don't return that white back into the bag, but we add two red marbles to the bag. That would leave us with a bag containing, 6 red and 7 white.

It follows that,

$$P(R_2|W_1) = \frac{6}{13}.$$

Therefore,

$P(R_2)$

$= P(R_2|R_1)P(R_1) + P(R_2|W_1)P(W_1)$

$= \frac{3}{13}\cdot\frac{4}{12} + \frac{6}{13}\cdot\frac{8}{12}$

$= \frac{12+48}{12\cdot 13} = \frac{60}{12\cdot 13} = \frac{5}{13}.$

■

(MB062) A bag contains ten balls numbered $1,2,\dots,10$, that are thoroughly mixed. Alice draws a ball at random, looks at the number, and replaces the ball in the bag, which is then mixed again. Beth then draws a ball at random. What is the probability that the girls will have drawn the same number?

Solution

Let A_i be the event that Alice draws the ball numbered i and let B_i be the event that Beth draw the ball numbered i.

$P(\text{girls will draw the same number})$

$= P\big((A_1 \text{ and } B_1) \text{ or } (A_2 \text{ and } B_2) \text{ or } \cdots \text{ or } (A_{10} \text{ and } B_{10})\big)$

$= P(A_1 \text{ and } B_1) + P(A_2 \text{ and } B_2) + \cdots + P(A_{10} \text{ and } B_{10})$

$= P(A_1)P(B_1) + P(A_2)P(B_2) + \cdots + P(A_{10})P(B_{10})$

$= \left(\frac{1}{10}\right)^2 + \left(\frac{1}{10}\right)^2 + \cdots + \left(\frac{1}{10}\right)^2 = \frac{1}{10}.$

■

(5C991) Cards are turned over one at a time (without replacement) from an ordinary well-shuffled deck (52 cards). What is the probability that the first two cards are aces? (There are four aces in a deck.)

Solution

Let A be the event that the first card is an ace and let B be the event that the second card is an ace. Then the probable asks for $P(A \cap B)$.

$$P(A \cap B) = P(A)P(B|A) = \frac{4}{52}\left(\frac{3}{51}\right) = \frac{1}{221}.$$

■

(5C981) A woman goes to visit the house of some friends whom she has not seen in many years. She knows that, besides the two married adults in the household, there are two children of different ages. But she does not know their genders. When she knocks on the door of the house, a boy answers. What is the probability that the other child is a boy?

Solution

This problem is known as the "Two Children Problem" and shows up in many forms on puzzle blogs. This problem is notorious for tripping up problem solvers intuition and is also notorious for tripping up problem writers for creating ambiguity in the wording.

To make the solution provided here complete the underlying assumptions being used must be stated explicitly.

Assumptions:

(1) the gender determination of all children in a family are independent events

(2) $P(\text{boy}) = P(\text{girl}) = 1/2$ for all children in a family

(3) $P(\text{younger child answers the door}) = P(\text{older child answers the door}) = 1/2$.

The unconditional sample space must account for the gender of the younger child, the gender of the older child and which child (younger or older) opens the door.

Younger	**Older**	**Opens Door**	**Probability**
Male	Male	Younger	$1/8$
Male	Male	Older	$1/8$
Male	Female	Younger	$1/8$
Male	Female	Older	$1/8$
Female	Male	Younger	$1/8$
Female	Male	Older	$1/8$
Female	Female	Younger	$1/8$
Female	Female	Older	$1/8$

Which of these eight outcomes are **not** possible once we see that a boy answers the door? Answer: (Male, Female, Older), (Female, Male, Younger), (Female, Female, Younger), (Female, Female, Older)

Removing these four outcomes and rescaling the remaining possibilities leaves us with our conditional sample space and their probabilities.

Younger	**Older**	**Opens Door**	**Conditional Probability**
Male	Male	Younger	$1/4$
Male	Male	Older	$1/4$
Male	Female	Younger	$1/4$
Female	Male	Older	$1/4$

Now we can easily find $P(\text{other child is boy}|\text{boy opens the door})$. We just need to sum the conditional probabilities where both children are boys.

$$P(\text{other child is boy}|\text{boy opens the door}) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

■

The solution provided by MSHSML for this problem was $1/3$. This answer is correct if we amend the storyline as follows: "Suppose the visitor knows there are two children in the family they are visiting and when the visitor knocks the mother answers the door. The visitor then asks the mother if she has at least one boy and the mother informs her that she does have at least one boy but does not elaborate further. Based on this information what is the probability that the mother has two boys?"

Solution

Let A be the event that the younger child is a boy and let B be the event that the older child is a boy. Then the question in this amended form becomes

$$P(\text{two boys}|\text{at least one boy})$$

$$= P\big((A \cap B)\big|(A \cup B)\big)$$

$$= \frac{P\big((A \cap B) \cap (A \cup B)\big)}{P(A \cup B)}$$

$$= \frac{P(A \cap B)}{P(A \cup B)}$$

$$= \frac{P(A \cap B)}{P(A) + P(B) - P(A \cap B)}$$

$$= \frac{P(A)P(B)}{P(A) + P(B) - P(A)P(B)}$$

$$= \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)}{\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)}$$

$$= \frac{\frac{1}{4}}{1 - \frac{1}{4}} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}.$$

■

Notice that in both the original and the amended storylines the visitor learns that the family they are visiting has at least one son. The difference is in ***how*** this information is learned. In the original version this information is learned as the outcome of a random event (namely, which child answers the door). In contrast, in the amended version this information is learned as "given information" and not through any random event.

(5D942) A bag of popping corn contains $2/3$ white kernels and $1/3$ yellow kernels. Only $1/2$ of the white will pop, whereas $2/3$ of the yellow will pop. After the corn is popped, you select sight unseen a single popped piece from a well-mixed bag. What is the probability that it will be yellow?

Solution

Let A be the event a kernel is yellow and let B be the event that a kernel successfully popped. Using this notation the problem is asking for $P(A|B)$.

By Bayes Theorem,

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A')P(A')}$$

$$= \frac{(2/3)(1/3)}{(2/3)(1/3) + (1/2)(2/3)} = \frac{(2/9)}{(2/9) + (1/3)} = \frac{2}{5}.$$

■

(5C902) There are 4 black socks and 6 brown socks all mixed together in a drawer in a dark room. If Pat grabs two socks from the drawer and takes them into the light, what is the probability that they will match?

Solution

$P\Big((1^{st}\text{ black and }2^{nd}\text{ black})\text{ or }(1^{st}\text{ brown and }2^{nd}\text{ brown})\Big)$

$= P(1^{st}\text{ black and }2^{nd}\text{ black}) + P(1^{st}\text{ brown and }2^{nd}\text{ brown})$

$= P(1^{st}\text{ black})P(2^{nd}\text{ black}|1^{st}\text{ black}) + P(1^{st}\text{ brown})P(2^{nd}\text{ brown}|1^{st}\text{ brown})$

$= \left(\frac{4}{10}\right)\left(\frac{3}{9}\right) + \left(\frac{6}{10}\right)\left(\frac{5}{9}\right)$

$= \frac{42}{90} = \frac{7}{15}.$

■

(5T894) Assume that a certain test for cancer is 98% accurate, by which we mean that if the test is administered to 100 people with cancer, it will detect the cancer in 98 of them; and if it is administered to 100 people without cancer, it will show 98 of them to be free of cancer. If this test is given to $10{,}000$ people $(1/2)\%$ of whom actually have cancer, and if Mr. Casetest is one of the people who is told that he has cancer, what is the probability that Mr. Casetest really does have cancer? [Adapted from Innumeracy by John Paulas.]

Solution

Let A be the event that a random person has cancer.

Let B be the event that a test is positive for cancer.

$$P(A) = 0.005$$

$$P(B|A) = 0.98$$

$$P(B'|A') = 0.98$$

Find $P(A|B)$. By Bayes rule,

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A')P(A')}$$

$$= \frac{P(B|A)P(A)}{P(B|A)P(A) + \left(1 - P(B'|A')\right)\left(1 - P(A)\right)}$$

$$= \frac{(0.98)(0.005)}{(0.98)(0.005) + (1 - 0.98)(1 - 0.005)}$$

$$= \frac{49}{49 + 199} = \frac{49}{248}.$$

■

Homework:

Dodgson's *Pillow Problem* #16.

Imagine two urns where the first urn is known to contain jone ball and that ball is equally likely to be either black or white. The second urn is known to contain three balls, two black and one white. A white ball is added to Urn 1 and afterwards a ball is randomly selected and removed from Urn 1. This ball is seen to be white. This ball is not replaced back into Urn 1. After this, an urn is randomly selected from these two urns and a ball is randomly selected from that urn. What is the probability that this last selected ball is white?

Solution

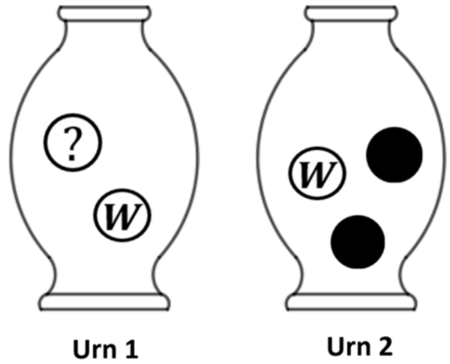
We can simplify the story a bit and say we start with two urns. Urn 1 is known to contain one white ball and Urn 2 is known to contain one black and one white ball. At this point

Stage 1. A ball that is equally likely to be either black or white is put into Urn 1.

Stage 2. A ball is randomly selected and removed from Urn 1.

Stage 3. An urn is selected at random between Urns 1 and 2.

Stage 4. A ball is randomly selected from the urn selected in Stage 3.



The situation after Stage 1

Let A be the event that the ball is Stage 1 is White.

Let B be the event that the ball removed from Urn 1 in Stage 2 is White.

Let C be the event that Urn 1 is selected in Stage 3.

Let D be the event that the ball taken from the urn selected in Stage 3 is White.

Using this notation, the problem is asking for the conditional probability $P(D|B)$.

We will start the problem by enumerating all sixteen possible outcomes with their probabilities of the four stages of this experiment in order to determine the complete *unconditional* sample space with their probabilities.

Stage 1. A ball that is equally likely to be either black or white is put into Urn 1.

Stage 2. A ball is randomly selected and removed from Urn 1.

Stage 3. An urn is selected at random between Urns 1 and 2.

Stage 4. A ball is randomly selected from the urn selected in Stage 3.

Stage 1	Stage 2	Stage 3	Stage 4	Probability
White	White	Urn 1	White	$1/4$
White	White	Urn 1	Black	0
White	White	Urn 2	White	$1/12$
White	White	Urn 2	Black	$1/6$
White	Black	Urn 1	White	0
White	Black	Urn 1	Black	0

White	Black	Urn 2	White	0
White	Black	Urn 2	Black	0
Black	White	Urn 1	White	0
Black	White	Urn 1	Black	$1/8$
Black	Black	Urn 1	White	$1/8$
Black	Black	Urn 1	Black	0
Black	White	Urn 2	White	$1/24$
Black	White	Urn 2	Black	$1/12$
Black	Black	Urn 2	White	$1/24$
Black	Black	Urn 2	Black	$1/12$

As an important check on ourselves we note that the sum of all the probabilities in our unconditional sample space equals 1 as it must. It is not necessary to include outcomes with probability 0 as members of the sample space but I have done so here to make it explicit what is not possible.

To form our conditional sample space we need to remove all the outcomes among the sixteen unconditional outcomes where the color of the ball in Stage 2 is Black.

Stage 1	**Stage 2**	**Stage 3**	**Stage 4**	**Probability**
White	White	Urn 1	White	$1/4$
White	White	Urn 1	Black	0
White	White	Urn 2	White	$1/12$
White	White	Urn 2	Black	$1/6$
Black	White	Urn 1	White	0
Black	White	Urn 1	Black	$1/8$
Black	White	Urn 2	White	$1/24$
Black	White	Urn 2	Black	$1/12$

The sum of the unconditional probabilities of the remaining six outcomes is

$$\frac{1}{4}+\frac{1}{12}+\frac{1}{6}+\frac{1}{8}+\frac{1}{24}+\frac{1}{12}=\frac{3}{4}.$$

So, to form the appropriate conditional probabilities of these remaining outcomes we need to scale each of these unconditional probabilities by the factor $4/3$.

Stage 1	**Stage 2**	**Stage 3**	**Stage 4**	**Conditional Probability**
White	White	Urn 1	White	$1/3$

White	White	Urn 1	Black	0
White	White	Urn 2	White	$1/9$
White	White	Urn 2	Black	$2/9$
Black	White	Urn 1	White	0
Black	White	Urn 1	Black	$1/6$
Black	White	Urn 2	White	$1/18$
Black	White	Urn 2	Black	$1/9$

We again note that the sum of all the conditional probabilities equals 1 which it must. Now we can easily find $P(D|B)$. We just need to sum the conditional probabilities where the color of the ball in Stage 4 is white.

$$P(D|B) = \frac{1}{3} + \frac{1}{9} + \frac{1}{18} = \frac{9}{18} = \frac{1}{2}.$$

■

Consider a deck of 4 cards consisting of the Ace of Hearts, Ace of Diamonds, King of Hearts, and the Queen of Hearts. Suppose these 4 cards are shuffled and then you draw two cards from this deck and set them apart. Consider the four different scenarios.

(a) Your friend looks at one of the two cards you selected and then tells you that your selection contains the Ace of Hearts.

(b) Your friend looks at one of the two cards you selected and then tells you that your selection contains an Ace.

(c) Your friend looks at both of the cards you selected and then tells you that your selection contains the Ace of Hearts.

(d) Your friend looks at both of the cards you selected and then tells you that your selection contains an Ace.

In all four scenarios, find the conditional probability that you selected both Aces.

Green and Blue Taxi Cabs

We already observed that, for certain problems, it may be more convenient to use the Bayes' ratio to evaluate comparative odds of two events than Bayes' theorem itself. For three events A, B, C

$$\frac{p(A|C)}{p(B|C)} = \frac{p(C|A)}{p(C|B)} \cdot \frac{p(A)}{p(B)}$$

A. Tversky and D. Kahneman give a dramatic example that makes use of the latter formula:

A cab was involved in a hit and run accident at night. Two cab companies, the Green and the Blue, operate in the city. You are given the following data:

85% of the cabs in the city are Green and 15% are Blue.

A witness identified the cab as Blue. The court tested the reliability of the witness under the same circumstances that existed on the night of the accident and concluded that the witness correctly identified each one of the two colors 80% of the time and failed 20% of the time. What is the probability that the cab involved in the accident was Blue rather than Green?

Solution

Let G be the event of the delinquent being Green. Let B be the event of the delinquent being Blue. Finally, let W be the witness' report. Clearly,

$$\frac{p(B|W)}{p(G|W)} = \frac{p(W|B)}{p(W|G)} \cdot \frac{p(B)}{p(G)}$$

$$= \frac{0.8}{0.2} \cdot \frac{0.15}{0.85}$$

$$= \frac{12}{17}.$$

Because $p(G|W) + p(B|W) = 1$, it follows that

$$p(B|W) = \frac{12}{12 + 17} \cong 0.41.$$

This shows that, in spite of the witness testimony, the hit-and-run cab is more likely to be Green than Blue.

A. Tversky, D. Kahneman, Evidential impact of base rates, in *Judgement under uncertainty: Heuristics and biases*, D. Kahneman, P. Slovic, A. Tversky (editors), Cambridge University Press, 1982.

Science 27 Sep 1974.

https://www.cut-the-knot.org/Probability/RedBlueTaxicabs.shtml

Probability, James R. Gray, 1967

Page 33, Problem 22

The probability that A can solve a certain problem is $2/5$ and that B can solve it is $1/3$. If they both try it, independently, what is the probability that it is solved?

Ans: $3/5$

Solution

Let E_A be the event that A solves the problem and let E_B be the event that B solves the problem. Then

$$P(\text{problem is solved}) = P(E_A \text{ or } E_B) = P(E_A) + P(E_B) - P(E_A \cap E_B)$$
$$= P(E_A) + P(E_B) - P(E_A)P(E_B)$$
$$= \frac{2}{5} + \frac{1}{3} - \frac{2}{15} = \frac{6+5-2}{15} = \frac{9}{15} = \frac{3}{5}.$$

■

Probability, James R. Gray, 1967

Page 37, Problem 38

A man draws a card at random from a pack of 52 playing cards. He then draws as many cards from the remainder of the pack as the number on the card already drawn, ace counting one and king, queen and jack each counting ten. What is the probability that the ace of spades is among the cards drawn?

Ans: $5/34$

Solution

Let A be the event that the first card drawn is the ace of spades

Let B be the event that the ace of spades is among the cards drawn

Let C_i be the event that the number on the card first drawn equals i, $1 \leq i \leq 10$

$$P(B) = P(B|A)P(A) + P(B \cap A')$$

$$= 1\left(\frac{1}{52}\right) + P(B \cap A')$$

$$P(B \cap A') = \sum_{i=1}^{10} P(B \cap A' \cap C_i)$$

$$= \sum_{i=1}^{10} P(B|A' \cap C_i)P(A'|C_i)P(C_i)$$

$$= P(B|A' \cap C_1)P(A'|C_1)P(C_1) + \sum_{i=1}^{10} P(B|A' \cap C_i)P(A'|C_i)P(C_i)$$

$$+P(B|A' \cap C_{10})P(A'|C_{10})P(C_{10})$$

$$= \left(\frac{1}{51}\right)\left(\frac{3}{4}\right)\left(\frac{4}{52}\right) + \sum_{i=2}^{9}\left(\frac{i}{51}\right)(1)\left(\frac{4}{52}\right)$$

$$+ \left(\frac{10}{51}\right)(1)\left(\frac{16}{52}\right)$$

$$= \left(\frac{1}{51}\right)\left(\frac{3}{52}\right) + \left(\frac{10}{51}\right)\left(\frac{16}{52}\right) + \left(\frac{1}{51}\right)\left(\frac{4}{52}\right)\sum_{i=2}^{9} i$$

$$= \left(\frac{1}{51}\right)\left(\frac{3}{52}\right) + \left(\frac{10}{51}\right)\left(\frac{16}{52}\right) + \left(\frac{1}{51}\right)\left(\frac{4}{52}\right)\left(\frac{9 \cdot 10}{2} - 1\right)$$

$$= \left(\frac{1}{51}\right)\left(\frac{3}{52}\right) + \left(\frac{10}{51}\right)\left(\frac{16}{52}\right) + \left(\frac{1}{51}\right)\left(\frac{4}{52}\right)(44)$$

$$= \frac{3 + 160 + 176}{51 \cdot 52}$$

$$P(B) = P(B|A)P(A) + P(B \cap A')$$
$$= \frac{1}{52} + P(B \cap A')$$
$$= \frac{1}{52} + \frac{3 + 160 + 176}{51 \cdot 52} = \frac{5}{34}.$$

■

Probability, James R. Gray, 1967

Page 39, Problem 45

A man chooses a painting from a group containing eight originals and two copies. He consults an expert whose chance of judging either an original or a copy correctly is $5/6$.

If the expert considers that the chosen painting is an original, what is the probability that this is so?

Ans: $20/21$

<u>Solution</u>

Let A be the event that the man chose an original. Let B be the event that the expert considers the choice an original.

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A')P(A')}$$
$$= \frac{\left(\frac{5}{6}\right)\left(\frac{8}{10}\right)}{\left(\frac{5}{6}\right)\left(\frac{8}{10}\right) + \left(\frac{1}{6}\right)\left(\frac{2}{10}\right)}$$
$$= \frac{40}{42} = \frac{20}{21}.$$

■

If the expert considers the painting is a copy and the man returns it and choses another painting at random from the other nine, what is the probability that this second painting is an original?

Ans: $68/81$

Solution

We again let A be the event that the first painting the man chose is an original and let B be the event that the expert considers the first painting the man chose to an original.

Let C be the event that the second painting is an original.

$$P(C|B') = \frac{P(C \cap B')}{P(B')} = \frac{P(C \cap B' \cap A) + P(C \cap B' \cap A')}{P(B')}$$

$$= \frac{P\big(C\big|(B' \cap A)\big)P(B'|A)P(A) + P\big(C\big|(B' \cap A')\big)P(B'|A')P(A')}{P(B'|A)P(A) + P(B'|A')P(A')}$$

$$= \frac{\left(\frac{7}{9}\right)\left(\frac{1}{6}\right)\left(\frac{8}{10}\right) + \left(\frac{8}{9}\right)\left(\frac{5}{6}\right)\left(\frac{2}{10}\right)}{\left(\frac{1}{6}\right)\left(\frac{8}{10}\right) + \left(\frac{5}{6}\right)\left(\frac{2}{10}\right)}$$

$$= \frac{56 + 80}{72 + 90} = \frac{136}{162} = \frac{68}{81}.$$

■

Probability, James R. Gray, 1967

Page 40, Problem 48

One hundred bags each contain two balls. In 99 bags one ball is white and one ball is black; the hundredth bag contains two white balls. One bag is selected at random a ball is drawn and a ball is drawn at random from it. What is the probability that this ball is white?

Ans: $101/200$

Solution

Let A be the event that the selected bag contains two whites and let D_1 be the event that the ball randomly chosen from the selected bag is white.

$$P(D_1) = P(D_1|A)P(A) + P(D_1|A')P(A')$$
$$= 1\left(\frac{1}{100}\right) + \left(\frac{1}{2}\right)\left(\frac{99}{100}\right) = \frac{101}{200}.$$

■

If the ball selected was found to be white, what is the probability that the other ball in that bag is also white?

Ans: $2/101$

Solution

We again let A be the event that the selected bag contains two whites and let D_1 be the event that the ball randomly chosen from the selected bag is white.

Let C be the event that the other ball in the selected bag is white.

$$P(C|D_1) = \frac{P(C \cap D_1)}{P(D_1)} = \frac{P(A)}{P(D_1)} = \frac{\frac{1}{100}}{\frac{101}{200}} = \frac{2}{101}.$$

■

If the selected white ball was replaced and a second ball drawn at random from the same bag, what is the probability that this second ball is white?

Ans: $103/202$

Solution

We again let A be the event that the selected bag contains two whites and let D_1 be the event that the ball randomly chosen from the selected bag is white.

Let D_2 be the event that the ball randomly drawn from the selected bag is white if the initial white ball drawn from that bag is replaced before the second draw.

$$P(D_2|D_1) = \frac{P(D_2 \cap D_1)}{P(D_1)} = \frac{P(D_2 \cap D_1 \cap A) + P(D_2 \cap D_1 \cap A')}{P(D_1)}$$

$$= \frac{P\big(D_2\big|(D_1 \cap A)\big)P(D_1|A)P(A) + P\big(D_2\big|(D_1 \cap A')\big)P(D_1|A')P(A')}{P(D_1)}$$

$$= \frac{1 \cdot 1 \cdot \frac{1}{100} + \left(\frac{1}{2}\right) \cdot \left(\frac{1}{2}\right) \cdot \left(\frac{99}{100}\right)}{\left(\frac{101}{200}\right)} = \frac{\frac{103}{400}}{\frac{202}{400}} = \frac{103}{202}.$$

■

If the procedure is repeated and it is found that n balls drawn at random and with replacement from the same bag were all white, what is the probability that both balls in the selected bag are white?

Ans: $2^n/(99 + 2^n)$

Solution

Let D_i be the event that the ball randomly drawn (with replacement) from the initially selected bag is white in the i^{th} replication of this process if the initial white ball drawn from that bag is replaced before the second draw.

$$P(A|D_1 \cap D_2 \cap \cdots \cap D_n)$$

$$= \frac{P\big((D_1 \cap D_2 \cap \cdots \cap D_n)\big|A\big)P(A)}{P(D_1 \cap D_2 \cap \cdots \cap D_n)}$$

$$= \frac{P\big((D_1 \cap D_2 \cap \cdots \cap D_n)\big|A\big)P(A)}{P\big((D_1 \cap D_2 \cap \cdots \cap D_n)\big|A\big)P(A) + P\big((D_1 \cap D_2 \cap \cdots \cap D_n)\big|A'\big)P(A')}$$

$$= \frac{1 \cdot \left(\frac{1}{100}\right)}{1 \cdot \left(\frac{1}{100}\right) + \left(\frac{1}{2}\right)^n \left(\frac{99}{100}\right)}$$

$$= \frac{1}{1 + \frac{99}{2^n}} = \frac{2^n}{2^n + 99}.$$

■

Find the smallest value of n for which this exceeds 0.95.

Ans: 11

Solution

$$\frac{2^n}{2^n + 99} \geq 0.95$$

$$2^n \geq 0.95 \cdot 2^n + 0.95 \cdot 99$$

$$2^n(1-0.95) \geq (0.95)(99)$$

$$2^n \geq \frac{(0.95)(99)}{0.05}$$

$$n \cdot \ln(2) \geq \ln(0.95) + \ln(99) - \ln(0.05)$$

$$n \geq \frac{\ln(0.95) + \ln(99) - \ln(0.05)}{\ln(2)} = 10.8772841335232$$

Therefore $n = 11$.

19. A speaks truth 3 times out of 4, and B 7 times out of 10; they both assert that a white ball has been drawn from a bag containing 6 balls all of different colours: find the probability of the truth of the assertion.

Higher Algebra, Hall and Knight, Fourth Edition, Macmillan and Co. (1891), Section 478, example, pages 397-398.

Solution

What is random?

(1) the color of the ball drawn from the bag (white, hue 1, hue 2, hue 3, hue 4, hue 5)

(2) whether A tells the truth or lies (A true, A lies)

(3) whether B tells the truth or lies (B true, B lies)

(4) the color A says the ball is (white, hue 1, hue 2, hue 3, hue 4, hue 5)

(5) the color B says the ball is (white, hue 1, hue 2, hue 3, hue 4, hue 5)

Question: Find $P(\text{ball drawn is white}|A \text{ says the ball is white and } B \text{ says the ball is white})$.

The following table enumerates all the possibilities where A and B both say a white ball was drawn and lists their unconditional probabilities.

Color A says	**Color B says**	**A truth or lie**	**B truth or lie**	**Color of the ball**	**Probability**

White	white	Truth	truth	white	$\left(\frac{1}{6}\right)\left(\frac{3}{4}\right)\left(\frac{7}{10}\right)$
White	white	Truth	lie	$\cdots$	0
White	white	Lie	truth	$\cdots$	0
White	white	lie	lie	white	0
White	white	lie	lie	hue 1	$\left(\frac{1}{6}\right)\left(\frac{1}{4}\right)\left(\frac{3}{10}\right)\left(\frac{1}{5}\right)\left(\frac{1}{5}\right)$
White	white	lie	lie	hue 2	$\left(\frac{1}{6}\right)\left(\frac{1}{4}\right)\left(\frac{3}{10}\right)\left(\frac{1}{5}\right)\left(\frac{1}{5}\right)$
White	white	lie	lie	hue 3	$\left(\frac{1}{6}\right)\left(\frac{1}{4}\right)\left(\frac{3}{10}\right)\left(\frac{1}{5}\right)\left(\frac{1}{5}\right)$
White	white	lie	lie	hue 4	$\left(\frac{1}{6}\right)\left(\frac{1}{4}\right)\left(\frac{3}{10}\right)\left(\frac{1}{5}\right)\left(\frac{1}{5}\right)$
White	white	lie	lie	hue 5	$\left(\frac{1}{6}\right)\left(\frac{1}{4}\right)\left(\frac{3}{10}\right)\left(\frac{1}{5}\right)\left(\frac{1}{5}\right)$

The sum of the unconditional probabilities of all outcomes where A and B both say a white ball was drawn.

$$\left(\frac{1}{6}\right)\left(\frac{3}{4}\right)\left(\frac{7}{10}\right) + 5\cdot\left(\frac{1}{6}\right)\left(\frac{1}{4}\right)\left(\frac{3}{10}\right)\left(\frac{1}{5}\right)\left(\frac{1}{5}\right) = \frac{9}{100}$$

From here we can construct the table of all outcomes given that A and B both say a white ball was drawn and we list their conditional probabilities.

A truth or lie	**B truth or lie**	**Color of the ball**	**Probability Conditional on A saying a white was drawn and B saying a white was drawn**
Truth	Truth	white	$\left(\frac{1}{6}\right)\left(\frac{3}{4}\right)\left(\frac{7}{10}\right)\left(\frac{100}{9}\right) = \frac{35}{36}$
Truth	Lie	$\cdots$	0
Lie	Truth	$\cdots$	0
Lie	Lie	white	0

Lie	Lie	hue 1	$\left(\frac{1}{6}\right)\left(\frac{1}{4}\right)\left(\frac{3}{10}\right)\left(\frac{1}{5}\right)\left(\frac{1}{5}\right)\left(\frac{100}{9}\right)=\frac{1}{180}$
Lie	Lie	hue 2	$\left(\frac{1}{6}\right)\left(\frac{1}{4}\right)\left(\frac{3}{10}\right)\left(\frac{1}{5}\right)\left(\frac{1}{5}\right)\left(\frac{100}{9}\right)=\frac{1}{180}$
Lie	Lie	hue 3	$\left(\frac{1}{6}\right)\left(\frac{1}{4}\right)\left(\frac{3}{10}\right)\left(\frac{1}{5}\right)\left(\frac{1}{5}\right)\left(\frac{100}{9}\right)=\frac{1}{180}$
Lie	Lie	hue 4	$\left(\frac{1}{6}\right)\left(\frac{1}{4}\right)\left(\frac{3}{10}\right)\left(\frac{1}{5}\right)\left(\frac{1}{5}\right)\left(\frac{100}{9}\right)=\frac{1}{180}$
Lie	Lie	hue 5	$\left(\frac{1}{6}\right)\left(\frac{1}{4}\right)\left(\frac{3}{10}\right)\left(\frac{1}{5}\right)\left(\frac{1}{5}\right)\left(\frac{100}{9}\right)=\frac{1}{180}$

The problem asked for

$$P(\text{ball drawn is white}|A \text{ says the ball is white and } B \text{ says the ball is white}).$$

By the Law of Total Probability

$$\begin{aligned}
&P(\text{ball drawn is white}|A \text{ says the ball is white and } B \text{ says the ball is white}) \\
&= P(\text{ball drawn is white and } A \text{ true and } B \text{ true}|A \text{ says white and } B \text{ says white}) \\
&+ P(\text{ball drawn is white and } A \text{ true and } B \text{ lies}|A \text{ says white and } B \text{ says white}) \\
&+ P(\text{ball drawn is white and } A \text{ lies and } B \text{ true}|A \text{ says white and } B \text{ says white}) \\
&+ P(\text{ball drawn is white and } A \text{ lies and } B \text{ lies}|A \text{ says white and } B \text{ says white}) \\
&= \frac{35}{36}+0+0+0=\frac{35}{36}.
\end{aligned}$$

■

20. **Bertrand's Box Problem** There are three boxes. One box contains two gold coins, one box contains one gold and one silver coin, and one box contains two silver coins. You picked a box at random and then selected a coin at random from that box which turns out to be gold. Find the probability that the second coin in the box selected is also gold.
Source: https://en.wikipedia.org/wiki/Bertrand%27s_box_paradox

Solution

Let A_j be the event that the box with (j silver, $2-j$ gold) was chosen, $j=0,1,2$. We are given the information that $P(A_0)=P(A_1)=P(A_2)=1/3$.

Let B be the event that the first coin chosen was gold and let C be the event that the second coin chosen was gold. Then the problem is asking for $P(C|B)$.

Before we opened a box and pulled out a gold coin, the three events A_0, A_1 and A_2 were equally likely. Then we pulled out a gold coin from the box selected which makes event A_2 impossible. Does this mean that the two remaining events A_0 and A_1 must still be equally likely, and hence have a conditional probability of $1/2$ each?

It is easy to jump to this conclusion, *but it isn't true.* Why not? Isn't this what we mean by rescaling to find conditional probabilities?

The underlying premise of solving for $P(A|B)$ by rescaling all outcomes in B and removing all outcomes in $A \cap \overline{B}$ is that we came to the information that the result of the experiment is an outcome in B not by a separate experiment but by some means equivalent to a third party privately looking at the result of the experiment and then telling us that the result is an outcome in B.

As we stated before, how we learn the new information affects the answer.

In this problem we learned the information that the box contains at least one gold coin as the result of the random experiment of picking a box at random and then picking a coin from that box at random.

If we made the mistake of assuming that $P(A_0|B)=P(A_1|B)=1/2$, then we would come to the *false* conclusion that

$$P(C|B)=P(A_0|B)=1/2.$$

Here is the correct approach and correct answer.

$$P(C|B)=\frac{P(B\cap C)}{P(B)}$$

$$=\frac{P\big((B\cap C)\big|A_0\big)P(A_0)+P\big((B\cap C)\big|A_1\big)P(A_1)+P\big((B\cap C)\big|A_2\big)P(A_2)}{P(B|A_0)P(A_0)+P(B|A_1)P(A_1)+P(B|A_2)P(A_2)}$$

$$=\frac{1\left(\frac{1}{3}\right)+0\left(\frac{1}{3}\right)+0\left(\frac{1}{3}\right)}{1\left(\frac{1}{3}\right)+\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)+0\left(\frac{1}{3}\right)}=\frac{\frac{1}{3}}{\frac{3}{6}}=\left(\frac{1}{3}\right)\left(\frac{6}{3}\right)=\frac{6}{9}=\frac{2}{3}.$$

■

21. The Miracle Marble Manufacturing Company manufactures orange marbles and purple marbles. A bag of their marbles may contain any combination of orange and purple marbles (including all orange or all purple) and all combinations are equally likely.

    Henry bought a bag of their marbles and pulled one out at random. It was purple. What is the probability that if he pulled out a second marble at random it would also be purple?

    (Source: *Jim Totten's Problems of the Week*, Problem 318, Editors John McLoughlin, Joseph Khoury, Bruce Shawyer, pages 270-271.)

Solution

Notice that the problem never mentions how many marbles in total are in the bag. The problem never states, "let the total number of marbles in the bag be n". What should you make of that? The problem writers for MSHSML and national contests such as AMC and AIME vet their problems very carefully. It would be safe to assume that this was not an oversight and that the final answer will not depend on how many marbles in total are in the bag.

In a contest setting, I personally would make that assumption and take $n = 2$, the smallest feasible number of balls in the bag if we are able to pull out a second marble. But for $n = 2$ this is just Bertrand's Box problem (we can imagine picking a bag at random from three bags, one with two orange and no purple, one with one orange and one purple, and one with no orange and two purple). This is just another way of saying that all possible combinations of orange and purple marbles are equally likely.

Assuming, in a contest setting, you were already familiar with the frequently cited Bertrand Box problem, you could write down $2/3$ and save your time for other problems.

But suppose your teacher asked you to *prove* that this is the correct answer for all possible values of n, the total number of marbles in the bag, for $n \geq 2$.

As mentioned above for the case $n = 2$, stating that all possible combinations of orange and purple marbles are equally likely is probabilistically equivalent to picking a box at random from a row of $n + 1$ boxes, where the j^{th} box contains j orange and $n - j$ purple marbles, $j = 0,1, \dots, n$.

If we let A_j be the event that the box with (j orange, $n - j$ purple) was chosen, $j = 0,1, \dots, n$, then from the information given, we know that $P(A_0) = P(A_1) = \cdots = P(A_n) = 1/(n + 1)$.

Then let B be the event that the first ball selected was purple and C be the event that the second ball selected was purple.

Following the work in the Bertrand's box problem, we have

$$P(C|B) = \frac{P\big((B\cap C)\big|A_0\big)P(A_0) + P\big((B\cap C)\big|A_1\big)P(A_1) + \cdots + P\big((B\cap C)\big|A_n\big)P(A_n)}{P(B|A_0)P(A_0) + P(B|A_1)P(A_1) + \cdots + P(B|A_n)P(A_n)}$$

$$= \frac{P\big((B\cap C)\big|A_0\big) + P\big((B\cap C)\big|A_1\big) + \cdots + P\big((B\cap C)\big|A_n\big)}{P(B|A_0) + P(B|A_1)P(A_1) + \cdots + P(B|A_n)}$$

$$= \frac{\left(\frac{n}{n}\right)\left(\frac{n-1}{n-1}\right) + \left(\frac{n-1}{n}\right)\left(\frac{n-2}{n-1}\right) + \cdots + \left(\frac{1}{n}\right)\left(\frac{0}{n-1}\right) + \left(\frac{0}{n}\right)\left(\frac{0}{n-1}\right)}{1 + \left(\frac{n-1}{n}\right) + \cdots + \left(\frac{1}{n}\right) + \left(\frac{0}{n}\right)}$$

$$= \left(\frac{n}{n(n-1)}\right)\left(\frac{\sum_{j=1}^{n-1} j(j+1)}{\sum_{j=1}^{n} j}\right)$$

$$= \left(\frac{1}{n-1}\right)\left(\frac{\sum_{j=1}^{n-1} j^2 + \sum_{j=1}^{n-1} j}{\sum_{j=1}^{n} j}\right)$$

$$= \left(\frac{1}{n-1}\right)\left(\frac{\frac{(n-1)n(2n-1)}{6} + \frac{(n-1)n}{2}}{\frac{n(n+1)}{2}}\right)$$

$$= \left(\frac{1}{n-1}\right)\left(\frac{2(n-1)}{3}\right) = \frac{2}{3}.$$

So, as we suspected, the final answer does not depend on n, the total number of marbles in the bag.

■

Exercises for Chapter 5

MSHSML Problems Using Exchangeability

(5C013) A box contains exactly seven chips, four red and three white. Chips are randomly drawn one at a time without replacement until all the red chips are drawn or all the white chips are drawn. What is the probability that the last chip drawn is white?

[Variation, AMC, 2001, No. 11]

Solution

In this problem the last chip drawn is white if and only if there is at least one red chip left in the box after the last white chip is drawn out. Therefore, if drawing continued until all seven chips were drawn, the seventh chip would have to be red.

That is,

$$P(\text{last chip drawn in this story is white})$$
$$= P(\text{seventh chip drawn is red})$$
$$= P(\text{first chip drawn is red})$$
$$= 4/7.$$

■

(TC013) A bag contains 5 red balls and 4 white ones. They are to be drawn one at a time without replacement until all the red balls are drawn, or all the white ones are drawn. What is the probability that the last ball drawn is white?

Solution

Following the same argument as developed in MSHSML 5C013,

$$P(\text{last ball drawn in this story is white})$$
$$= P(\text{ninth ball drawn is red})$$
$$= P(\text{first ball drawn is red})$$
$$= 5/9.$$

■

Problems: Use Exchangeability to Solve the Following Problems.

1. A box contains 6 red balls, 7 green balls, and 9 yellow balls. Eleven balls are chosen at random one after the other without replacement. What is the probability that the 3rd ball chosen was yellow given that the 9th ball chosen was green?

<u>Solution</u>

By the exchangeability of these events

$$P(\text{3rd ball chosen was yellow}|\text{9th ball chosen was green})$$

$$= P(\text{2nd ball chosen was yellow}|\text{1st ball chosen was green})$$

$$= \frac{9}{(6+7+9)-1} = \frac{9}{21} = \frac{3}{7}.$$

■

2. A box contains a white balls and b black balls. Balls are drawn from the box at random and without replacement. What is the probability that all black balls are drawn before the last white ball?

<u>Solution</u>

Just for the moment, consider a slightly different problem. "From a box containing a white balls and b black balls we continue to draw out balls one at a time without replacement until $a + b$ balls have been removed. What is the probability that the last ball taken out is white?"

I claim that the two probabilities have the same answer.

In the original scenario, if we have drawn out all the black balls before the last white ball then at moment all of the balls remaining in the box must be white. So, in this case, if we did continue taking balls out of the box, the last ball taken out must be white.

On the other hand, if we consider the alternative scenario where we remove all $a + b$ balls and if in that scenario the last ball taken out was white, then it must be the case that all of the black balls were drawn out before all of the white balls were.

In short, all of the black balls are drawn before the last white ball in the original scenario is true **if and only if** the last ball drawn in the alternative scenario is white.

But in the alternative scenario, the probability that the last ball is white is the same as the probability that the first ball drawn is white (by exchangeability) which is clearly $a/(a+b)$.

■

3. You play a game with a friend where you each start with a shuffled deck of 52 cards turned face down. At each turn you both flip over the top card in your deck. If the cards are the same color, you win that turn. If the cards are not the same color, your friend wins that turn. On the next turn you repeat the process but with just the cards remaining in your decks.

What is the probability that you win the fifth turn (the cards are both red or both black on the fifth turn) of this game?

Solution

By exchangeability,

$$P(\text{you win the fifth turn})$$
$$= P(\text{you win the first turn})$$
$$= P(\text{both first cards are red}) + P(\text{both first cards are black})$$
$$= \left(\frac{26}{52}\right)\left(\frac{26}{52}\right) + \left(\frac{26}{52}\right)\left(\frac{26}{52}\right) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

■

4. Four balls are drawn at random and without replacement from a box containing 10 red balls and 8 white balls and then discarded without noting the color of any of these four balls. At this point four more balls are drawn out at random and without replacement. What is the probability that exactly two of this second set of four balls are red?

Solution

By exchangeability, the probability that exactly two of the second set of four balls are red equals the probability that exactly two of the first set of four balls (the four previously discarded balls) were red.

But we recognize that this question fits the hypergeometric model and equals

$$\frac{\binom{10}{2}\binom{8}{2}}{\binom{18}{4}}.$$

■

Probability, James R. Gray, 1967

Pages 34-35, Problem 29

Three bags A_1, A_2, A_3 each contain r red balls and g green balls. A ball is drawn at random from each of A_1 and A_2. These two balls are interchanged and replaced. A ball is then drawn at random from each of A_2 and A_3; these balls are then interchanged and replaced. If a ball is now drawn at random from A_3, what is the probability that it is red?

Ans: $r/(r+g)$

Solution

The probability of drawing a red ball from bag A_1 in the initial set up is $r/(r+g)$.

Now let's show that the probability of drawing a red ball from bag A_2 after the first interchange remains $r/(r+g)$.

Let E_1 be the event that we draw a red ball from bag A_1 and a green ball from bag A_2 in the first interchange.

Let E_2 be the event that we draw a green ball from bag A_1 and a red ball from bag A_2 in the first interchange.

Let E_3 be the event that we draw a red ball from both of bags A_1 and A_2 or a green ball from both of bags A_1 and A_2 in the first interchange.

Notice that $P(E_1) = P(E_2) = rg/(r+g)^2$. If we let $q = P(E_1) = P(E_2)$ then $P(E_3) = 1 - P(E_1) - P(E_2) = 1 - 2q$.

Let F be the event that we draw a red ball from bag A_2 after the first interchange.

Then

$$\begin{aligned} P(F) &= P(F|E_1)P(E_1) + P(F|E_2)P(E_2) + P(F|E_3)P(E_3) \\ &= \left(\frac{r+1}{r+g}\right)q + \left(\frac{r-1}{r+g}\right)q + \left(\frac{r}{r+g}\right)(1-2q) \\ &= \frac{rq + q + rq - q + r - 2rq}{r+g} = \frac{r}{r+g}. \end{aligned}$$

Now let's show that the probability of drawing a red ball from bag A_3 after the second interchange is once again $r/(r+g)$.

Let H_1 be the event that we draw a red ball from bag A_2 and a green ball from bag A_3 in the second interchange.

Let H_2 be the event that we draw a green ball from bag A_2 and a red ball from bag A_3 in the second interchange.

Let H_3 be the event that we draw a red ball from both of bags A_2 and A_3 or a green ball from both of bags A_2 and A_3 in the second interchange.

Notice that in the second interchange

$$\begin{aligned} P(H_1) &= P(\text{draw a red ball from bag } A_2)P(\text{draw a green ball from bag } A_3) \\ &= P(F)P(\text{draw a green ball from bag } A_3) \\ &= \left(\frac{r}{r+g}\right)\left(\frac{g}{r+g}\right) = q \end{aligned}$$

and

$$\begin{aligned} P(H_2) &= P(\text{draw a green ball from bag } A_2)P(\text{draw a red ball from bag } A_3) \\ &= \big(1-P(F)\big)P(\text{draw a red ball from bag } A_3) \\ &= \left(\frac{g}{r+g}\right)\left(\frac{r}{r+g}\right) = q. \end{aligned}$$

Hence, $P(H_3) = 1 - P(H_1) - P(H_2) = 1 - 2q$.

Let T be the event that we draw a red ball from bag A_3 after the second interchange.

Then

$$\begin{aligned} P(T) &= P(T|H_1)P(H_1) + P(T|H_2)P(H_2) + P(T|H_3)P(H_3) \\ &= \left(\frac{r+1}{r+g}\right)q + \left(\frac{r-1}{r+g}\right)q + \left(\frac{r}{r+g}\right)(1-2q) \\ &= \frac{rq+q+rq-q+r-2rq}{r+g} = \frac{r}{r+g}. \end{aligned}$$

■

Project 5 Shout Stop Whenever You're Ready

What would be your strategy for deciding when to yell "Stop" and what would be your probability of winning with this strategy?

At whatever point you yell "Stop", it follows from symmetry that whatever the probability that the very next card in the deck is red must be the same for every other card left in the deck – including the last card in the deck.

So, from a probability point of view the problem is equivalent to deciding when to yell "Stop" by whatever strategy you like but change the rule to say that you are a winner if the *last* card in the deck is red (instead of the *next* card in the deck is red).

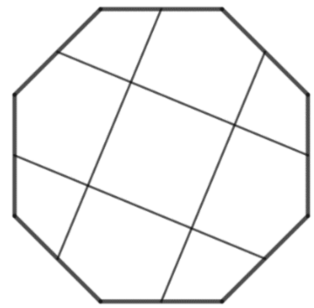
Thus, your probability of winning is "baked in the deck", so to speak, before you ever start turning over any cards. Regardless of your strategy and when you shout "stop", your probability of winning is just the probability that the last card in the deck is red. And the probability that the last card in the deck is red is the same as the probability that the first card in the deck is red, namely, $1/2$. No strategy will alter that you have a probability of $1/2$ of winning this game.

This result is discussed in more general terms, with references, in *Problems and Snapshots from the World of Probability*, Blom, Holst and Sandell, Springer-Verlag, 1994, Section 16.1 Exchangeability III, page 199 ff . The version given here is a variation on Problem 7.6, page 353 in *Stochastic Processes*, Second Edition, Sheldon Ross, Chapter 7.

Exercises for Chapter 6

MSHSML Geometric Probability Problems

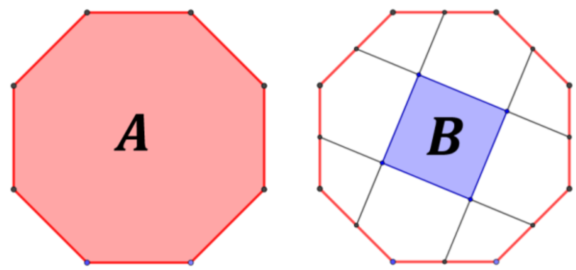
(5T174) Four segments are drawn from the midpoints of the sides of a regular octagon, creating a square, four congruent pentagons, and four congruent kites, as shown in the figure. If a point is chosen at random inside the octagon, determine exactly the probability that the point lies inside the square.



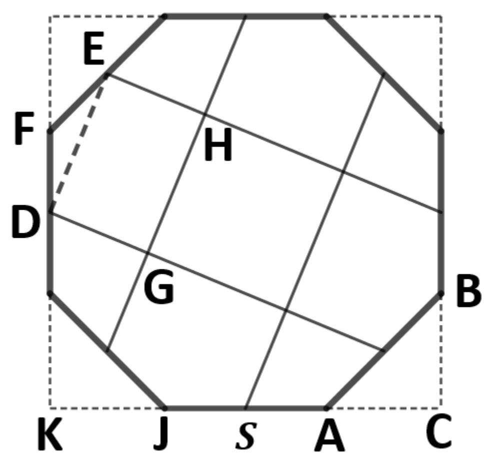
Solution

If a point is chosen at random from a region A (*i.e.* all points in region A are equally likely to be picked) and if region B is a subset of region B, then

$$P(\text{point picked is in region } B) = \frac{\text{Area}(B)}{\text{Area}(A)}.$$



Note: If this were a homework problem, I would have expected my students to justify that the inner region labeled ***B*** in the above diagram was indeed a square. But in a contest setting you would not want to waste time on a result that is stated as a fact in the statement of the problem.



To find the area of the octagon, inscribe it in a square. Let the side of the octagon $\overline{JA}$ have length s.

Note: **In a contest setting you could safely anticipate that the variable s must drop out of the final answer** because they did not give a name for the length of that side. If s was not going to drop out of the final answer then everyone would have different answers depending upon how they chose to label that side length. And you should know from experience that the MSHSML tests has never set a problem in that way. And if you know s will drop out in the final answer, it will save time to take s to be some convenient number such as $s = 1$. This would save you a little bit of time.

In the proof that follows I keep s in the calculations just to show you that s does indeed drop out of the final answer. **But in a contest setting I personally would have taken the shortcut of letting $s = 1$ right from the start.**

Then $AB = JA = s$ and triangle ACB is an isosceles right triangle with sides in the ratio $s/\sqrt{2}$, $s/\sqrt{2}$ and s. That is, $AC = s/\sqrt{2}$. Thus

$$Area(\Delta ACB) = \frac{1}{2}(AC)(AB) = \frac{1}{2}\left(\frac{s^2}{2}\right) = \frac{s^2}{4}$$

and

$$Area(\text{Exscribed Square}) = (KC)^2 = \left(\frac{s}{\sqrt{2}} + s + \frac{s}{\sqrt{2}}\right)^2 = s^2\left(1+\sqrt{2}\right)^2 = s^2\left(3+2\sqrt{2}\right).$$

It follows that

$$Area(\text{Octagon}) = Area(\text{Exscribed Square}) - 4 \cdot Area(\Delta ACB)$$

$$= s^2\left(3+2\sqrt{2}\right) - 4\left(\frac{s^2}{4}\right) = s^2\left(2+2\sqrt{2}\right).$$

To find the area of the interior square, draw in $\overline{DE}$. The area of the interior square equals

$$(GH)^2 = (DE)^2.$$

We know each of the interior angles in a regular n-gon equals $(n-2)180°/n$. Therefore,

$$m\angle DFE = (6 \cdot 180°)/8 = 135°$$

and we know that $DF = EF = s/2$. Therefore, by the Law of Cosines applied to ΔDFE

$$(DE)^2 = \left(\frac{s}{2}\right)^2 + \left(\frac{s}{2}\right)^2 - 2\left(\frac{s}{2}\right)\left(\frac{s}{2}\right)\cos(135°) = \frac{s^2}{4} + \frac{s^2}{4} - \frac{s^2}{2}\left(-\frac{\sqrt{2}}{2}\right) = s^2\left(\frac{1}{2} + \frac{\sqrt{2}}{4}\right).$$

Therefore

$$P(\text{point picked is in region } B) = \frac{\text{Area}(B)}{\text{Area}(A)} = \frac{s^2\left(\frac{1}{2} + \frac{\sqrt{2}}{4}\right)}{s^2\left(2 + 2\sqrt{2}\right)}$$

$$= \frac{1}{8}\left(\frac{2 + \sqrt{2}}{1 + \sqrt{2}}\right) = \frac{1}{8}\left(\frac{\left(2 + \sqrt{2}\right)\left(1 - \sqrt{2}\right)}{\left(1 + \sqrt{2}\right)\left(1 - \sqrt{2}\right)}\right) = \frac{1}{8}\left(\frac{2 + \sqrt{2} - 2\sqrt{2} - 2}{1 - 2}\right) = \frac{\sqrt{2}}{8}.$$

■

(5T115)

(5D104)

(5T104)

(MB072)

(5D024)

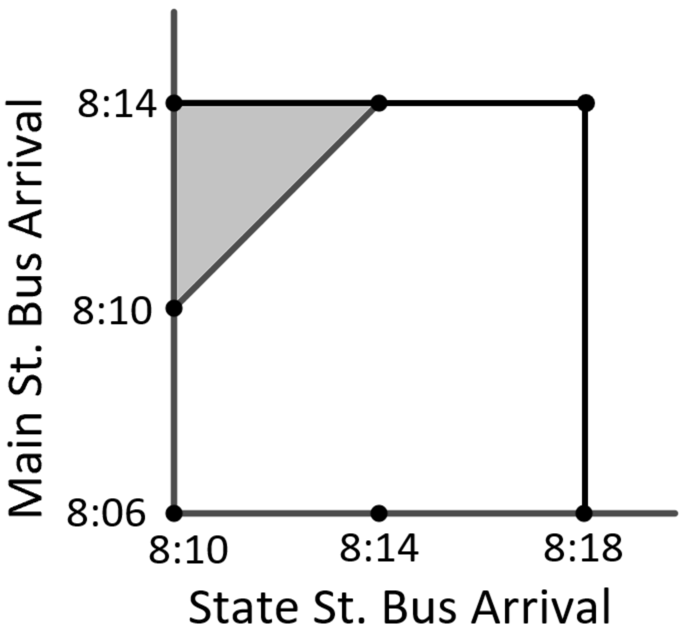
(5C011)

(5T993)

(5D914) The westbound Main Street bus on which I ride is scheduled to arrive at State Street at 8:10, but it actually arrives randomly within 4 minutes on either side of 8:10. The north bound State Street bus that I hope to catch is scheduled to arrive at Main Street at 8:14, but again its arrival is randomly distributed within 4 minutes either side of 8:14. What is the probability of my catching the 8:14 bus?

Solution

You should anticipate that the question writer meant for you to assume that the arrival time is "random distributed" on each axis is intended to mean that all points in the 8×8 square shown in the figure below are equally likely (equally probable). Hence this is a problem in *geometric probability*.



The shaded area represents the region where the question writer will miss the bus. Therefore,

$$P(\text{miss the bus}) = \frac{Area(\text{shaded triangle})}{Area(\text{square})} = \frac{(\frac{1}{2})(4)(4)}{(8)(8)} = \frac{1}{8}.$$

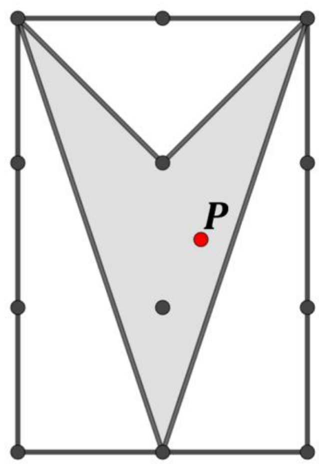
Hence

$$P(\text{not miss the bus}) = 1 - P(\text{miss the bus}) = 1 - \frac{1}{8} = \frac{7}{8}.$$

■

Source: December 2012 Calendar Problem #20, *The Mathematics Teacher*

A rectangle and an arrowhead are drawn on a regularly spaced grid of lattice points. If a point P is chosen at random in the rectangle, what is the probability that P will be in the shaded arrowhead, as shown in the figure below? (Note: The point P can be *anywhere* in or on the rectangle and is not limited to the lattice points.)



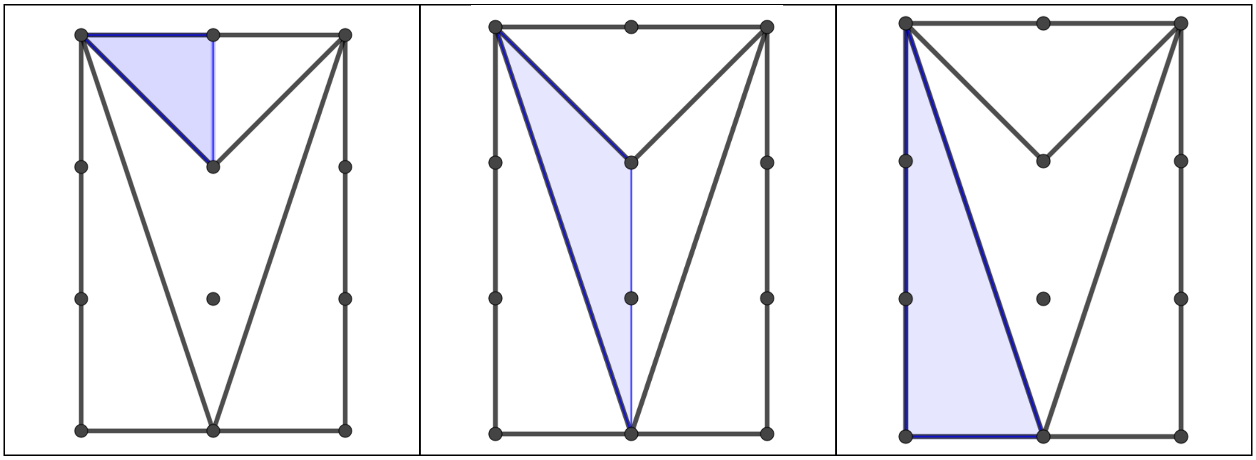
Solution

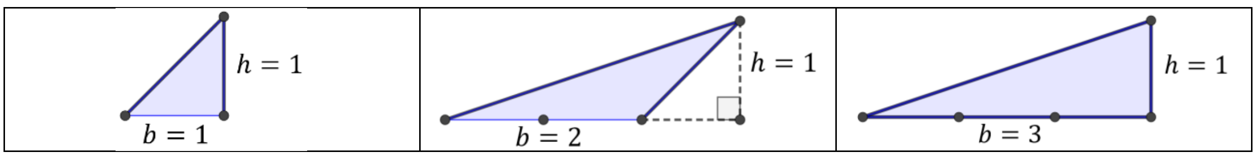
Method 1

By the method of geometric probability, the probability of the chosen point being in the shaded arrowhead equals

$$\frac{Area(\text{Arrowhead})}{Area(\text{Rectangle})}.$$

Compare the following three triangles which together make up half of the rectangle.





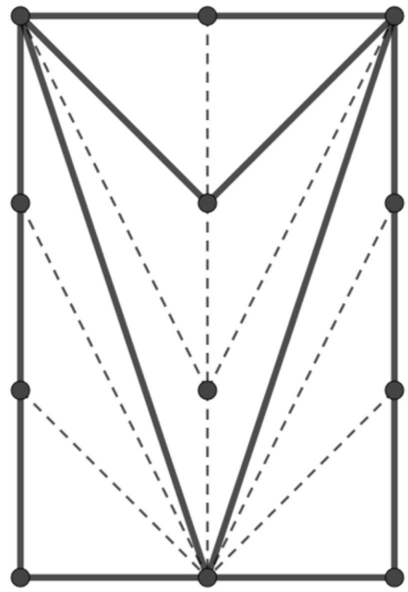
Let A_1, A_2 and A_3 be the areas of these three triangles. We notice that these triangles have the same height and the bases are respectively 1 unit, 2 units and 3 units. Hence $A_2 = 2A_1$ and $A_3 = 3A_1$. Therefore

$$\frac{Area(\text{Arrowhead})}{Area(\text{Rectangle})} = \frac{2A_2}{2A_1 + 2A_2 + 2A_3} = \frac{A_2}{A_1 + A_2 + A_3}$$

$$= \frac{2A_1}{A_1 + 2A_1 + 3A_1} = \frac{2}{6} = \frac{1}{3}.$$

Method 2

Any polygon drawn on a grid (*i.e.* where the corner points of the polygon are grid points) can be "fully triangulated." Fully triangulated means that extra line segments connecting grid points can be added so that the polygon consists of disjoint (non-overlapping) triangles with no grid points in the interior of a triangle and no grid points on the side of a triangle except at the endpoints (vertices) of a triangle. There can be many different ways to "fully triangulate" a polygon on a grid.



Here is one way to triangulate the rectangle in this problem.

<u>Theorem</u> (a variant of Pick's Theorem)

If the distance between grid points is taken as 1 unit, then the area of *every* triangle in a fully triangulated polygon equals $1/2$ square unit.

(I find this variation of Pick's Theorem much easier to remember than the "standard" form of Pick's Theorem. Also, the triangles in a fully triangulated polygon are often called "**fundamental triangles**".)

Now notice that the arrowhead contains four fundamental triangles and the rectangle contains twelve fundamental triangles. Hence,

$$\frac{Area(\text{Arrowhead})}{Area(\text{Rectangle})} = \frac{4(1/2)}{12(1/2)} = \frac{4}{12} = \frac{1}{3}.$$

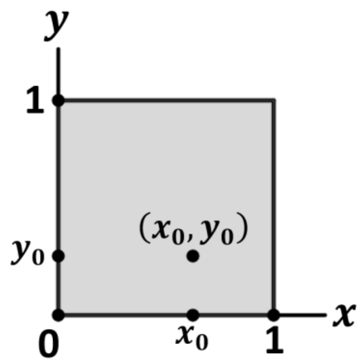
■

Example 4.

Suppose we select two points on a stick of unit length such that the two points are picked independently and such that all points along the stick are equally likely to be picked in both cases. Then suppose we break the stick at these two points and get three pieces of stick. What is the probability that the three pieces can be made into a triangle?

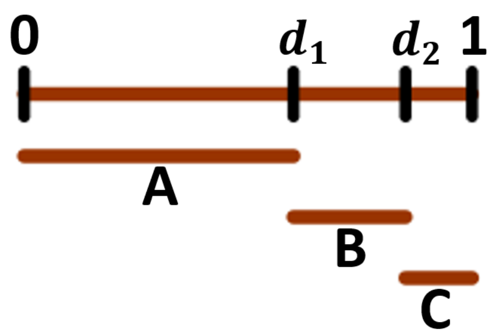
<u>Solution</u>

We can model this as an experiment where we pick a point x_0 at random between 0 and 1 on the x-axis and we pick a point y_0 at random between 0 and 1 on the y-axis. Then the point (x_0, y_0) is equally likely to be any point in the unit square as shown below.



Let $d_1 = \min(x_0, y_0)$ and let $d_2 = \max(x_0, y_0)$.

We would then break our stick at points $d_1 = \min(x_0, y_0)$ and $d_2 = \max(x_0, y_0)$ to get the three pieces A, B and C as shown in the diagram below. In this case the length of piece A is d_1, the length of piece B is $d_2 - d_1$ and the length of piece C is $1 - d_2$.



Recall from geometry that three line segments of lengths a, b, c can be the legs of a triangle if and only if

$$a + b > c$$
$$a + c > b$$
$$b + c > a.$$

Substituting the lengths of the three sticks into these three inequalities, we see that the three sticks can form a triangle if and only if

$$d_1 + (d_2 - d_1) > 1 - d_2$$
$$d_1 + (1 - d_2) > d_2 - d_1$$

$$(d_2 - d_1) + (1 - d_2) > d_1.$$

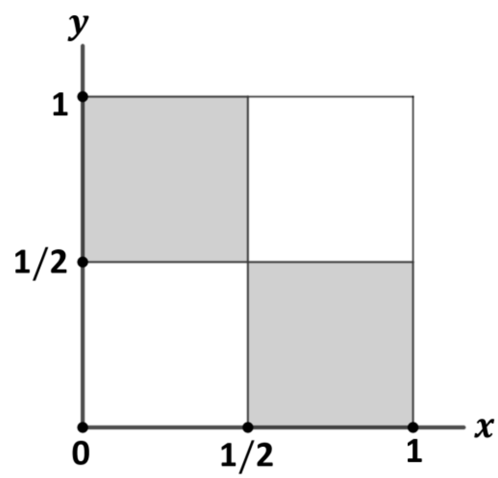
Simplifying these three inequalities shows x and y have to satisfy the following three conditions in order for sticks A, B, C to be capable of forming a triangle:

Condition 1	$d_2 > 1/2$
Condition 2	$d_2 < d_1 + 1/2$
Condition 3	$d_1 < 1/2.$

That is,

Condition 1	$\max(x_0, y_0) > 1/2$
Condition 2	$\max(x_0, y_0) < \min(x_0, y_0) + 1/2$
Condition 3	$\min(x_0, y_0) < 1/2.$

Just to satisfy Conditions 1 and 3 the point (x_0, y_0) has to be in the one of the two shaded regions shown below.



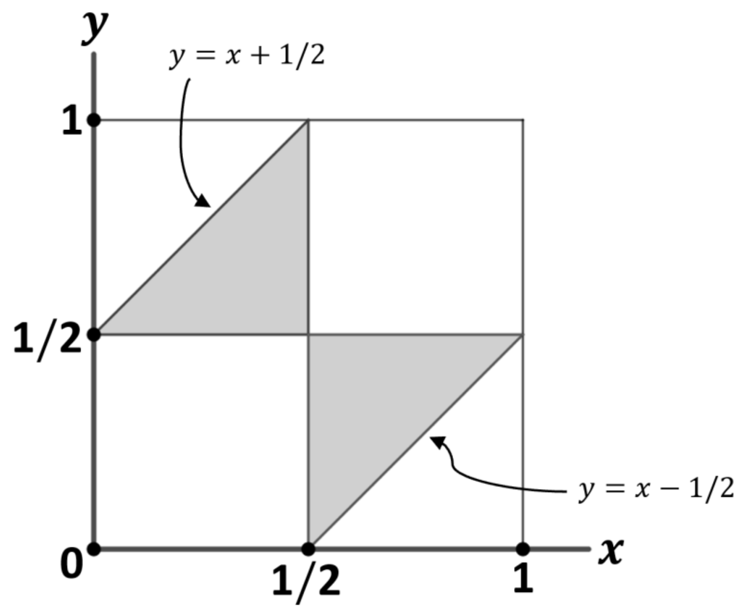
Now let's consider Condition 2.

Case 1. (x_0, y_0) **is in the upper left shaded corner.**

In this case, $\min(x_0, y_0) = x_0$ and $\max(x_0, y_0) = y_0$ and Condition 2 becomes $y_0 < x_0 + 1/2$. That is, (x_0, y_0) has to be below the line $y = x + 1/2$. This line goes through the two points $(0,1/2)$ and $(1/2,1)$. The line segment connecting these two points cuts the upper left shaded corner in half.

Case 2. (x_0, y_0) **is in the lower right shaded corner.**

In this case, $\min(x_0, y_0) = y_0$ and $\max(x_0, y_0) = x_0$ and Condition 2 becomes $y_0 > x_0 - 1/2$. That is, (x_0, y_0) has to be above the line $y = x - 1/2$. This line goes through the two points $(1/2,0)$ and $(1,1/2)$. The line segment connecting these two points cuts the lower right shaded corner in half.



So, to satisfy Conditions 1, 2 and 3 the point (x_0, y_0) has to be in the one of the two shaded triangular regions shown above. Clearly the total area shaded equals $1/4$.

Therefore,

$$P(\text{3 sticks can form a triangle}) = \frac{Area(\text{shaded})}{Area(\text{unit square})} = \frac{1/4}{1} = \frac{1}{4}.$$

■

Example 4.

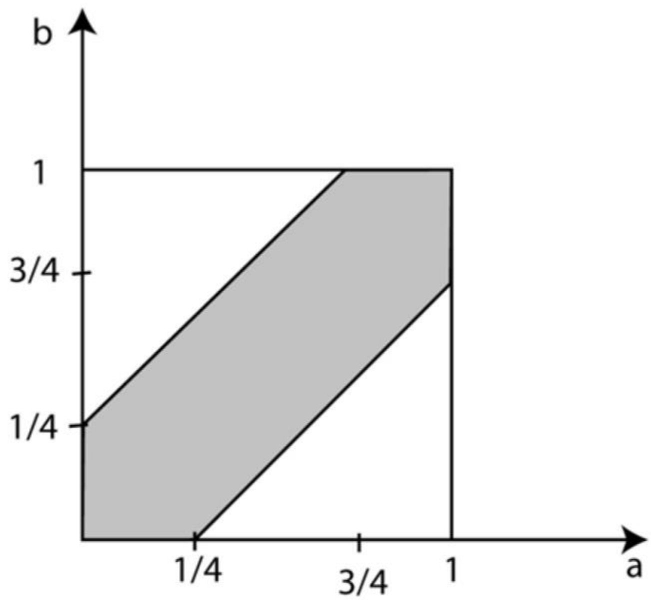
EE178: Probabilistic Systems Analysis, Autumn 2016, David Tse
http://web.stanford.edu/class/archive/ee/ee178/ee178.1172/homework.html

Lunch Date

Alice and Bob agree to try to meet for lunch between 12 and 1pm at their favorite sushi restaurant. Being extremely busy they are unable to specify their arrival times exactly, and can say only that each of them will arrive (independently) at a time that is uniformly distributed within the hour. In order to avoid wasting precious time, if the other person is not there when they arrive they agree to wait exactly fifteen minutes before leaving. What is the probability that they will actually meet for lunch (Hint: draw a picture.)

Solution

Let the random variable A be the time that Alice arrive and random variable B be the time when Bob arrives. Consider the following picture, plotting the space of all outcomes (a, b).



The shaded region is the set of values (a, b) for which Alice and Bob will actually meet for lunch. Since all points in this square are equally likely, the probability they meet is the ratio of the shaded area to the area of the square. If the area of the square is 1, then the area of shaded region is

$$1 - 2\left(\frac{1}{2} \times \left(\frac{3}{4}\right)^2\right) = \frac{7}{16},$$

Since the area of the white triangle on the upper-left is

$$\frac{1}{2} \times \left(\frac{3}{4}\right)^2$$

and the white triangle triangle on the lower-right has the same area. Therefore, the probability that Alice and Bob actually meet is $7/16$.

■

Example 5.

If a point (x_0, y_0, z_0) is picked at random from the unit cube where $0 \leq x \leq 1, 0 \leq y \leq 1$, $0 \leq z \leq 1$, what is the probability that $x_0^2 + y_0^2 + z_0^2 \leq 1$?

Solution

Note 1: The phrase "picked at random from a region" is the common way of expressing that "all points in the region are equally likely to be picked".

Note 2: Quite often a problem like this is worded whereby x_0, y_0 and z_0 are each picked at random from the interval $[0,1]$. It turns out that this is equivalent to picking a point at random from the unit cube where $0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1$. Perhaps this is intuitively obvious but one can often be led astray in probability by believing what is "intuitively obvious". Nevertheless, in this case, what is "intuitively obvious" is also true.

> We will justify why it is true in a later section when we discuss *independent events* more in depth.

It is (at least for me) time consuming to draw a 3D diagram by hand for a problem like this. In a timed contest my instinct is to see what I can do without a diagram. So, let me argue the case without appealing to the diagram below which I've drawn using the free software program Geogebra.

The region described by $x^2 + y^2 + z^2 \leq 1$ is the unit sphere centered at $(0,0,0)$.

The part of the unit sphere that is inside the unit cube where $0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1$ is the *entire octant* of the unit sphere where $0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1$.

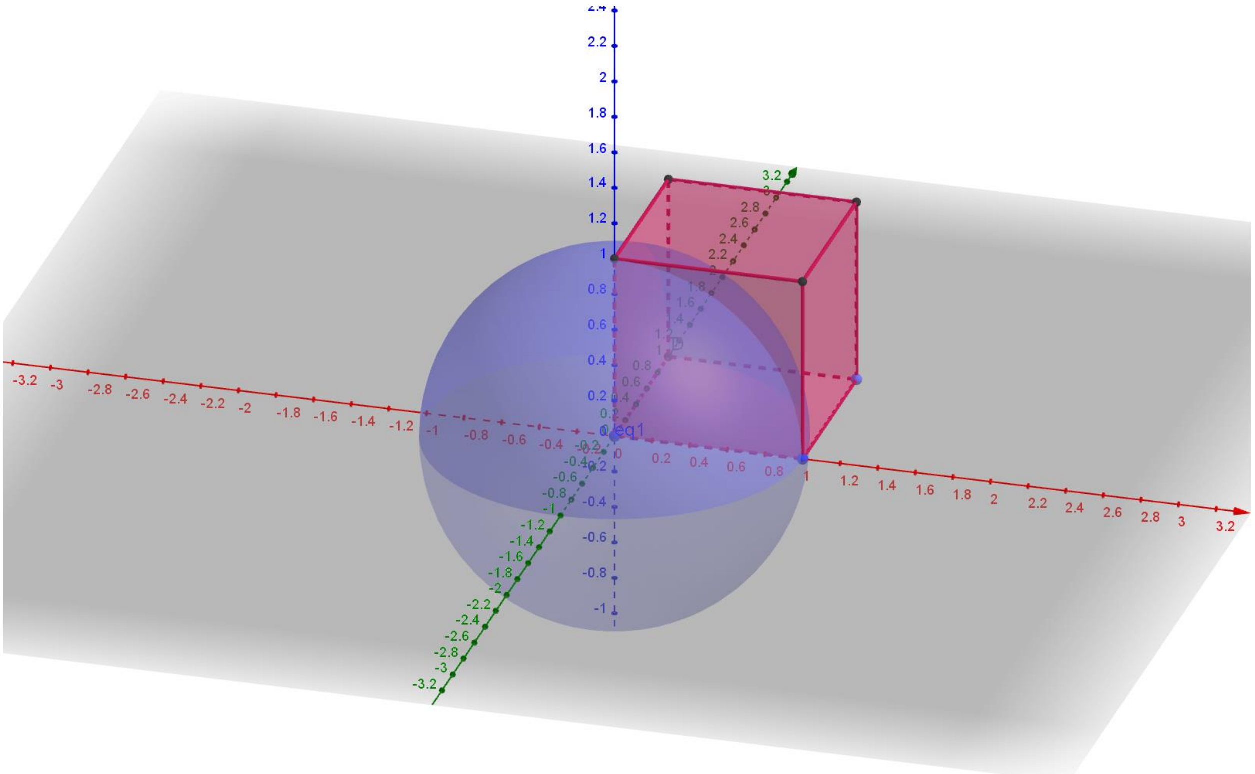
Why? Clearly only those points inside the unit sphere where $x \geq 0, y \geq 0, z \geq 0$ can be in the unit cube described above. But if any of $x > 1, y > 1, z > 1$ were true, then necessarily $x^2 + y^2 + z^2 > 1$ and hence would no longer be in the unit sphere. So, *all points* inside the unit sphere in the octant where $x \geq 0, y \geq 0, z \geq 0$ are also inside the unit cube where $0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1$.

Therefore,

$$P(\text{point picked is inside the unit sphere}) = \frac{Volume(\text{one octant of the unit sphere})}{Volume(\text{unit cube})}$$

$$= \frac{\frac{1}{8} \cdot Volume(\text{unit sphere})}{Volume(\text{unit cube})}$$

$$= \frac{\left(\frac{1}{8}\right)\left(\frac{4}{3}\pi \cdot 1^3\right)}{1^3} = \frac{\pi}{6} \approx 0.52.$$



■

Probability, James R. Gray, 1967

Page 31, Problem 13

The sum of two positive quantities is known. If all pairs of possible values are equally likely, prove that the probability that their product will not be less than five-ninths of the maximum possible product is $2/3$.

Solution

$$x + y = c, x > 0, y > 0$$

$$maximum\ possible\ product\ equals\ (c/2)^2$$

$$P\left(xy \geq \frac{5}{9}\left(\frac{c^2}{4}\right)\right) = \frac{2}{3}.$$

$$P(36xy \geq 5c^2) = \frac{2}{3}$$

$$P(36x(c-x) \geq 5c^2) = \frac{2}{3}$$

$$P(-36x^2 + 36cx - 5c^2 \geq 0) = \frac{2}{3}$$

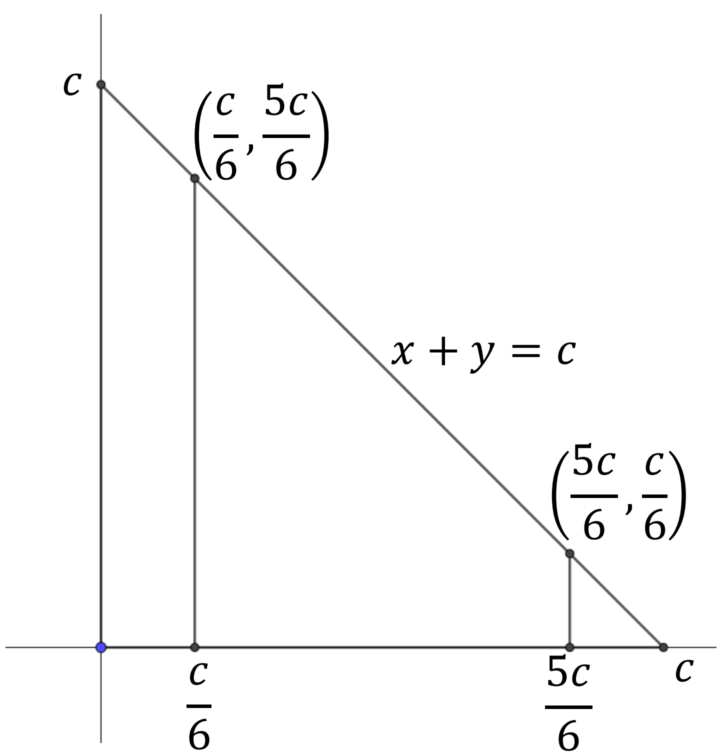
$$P(36x^2 - 36cx + 5c^2 \leq 0) = \frac{2}{3}$$

$$P\big((6x-5c)(6x-c) \leq 0\big) = \frac{2}{3}$$

$$6x - 5c \leq 0, 6x - c \geq 0$$

$$6x \leq 5c, 6x \geq c$$

$$P\left(\frac{c}{6} \leq x \leq \frac{5c}{6}\right) = \frac{2}{3}$$



$$P\left(xy \geq \frac{5}{9}\left(\frac{c^2}{4}\right)\right) = \frac{\frac{1}{2}\left(\frac{5c}{6}\right)\left(\frac{5c}{6}\right) - \frac{1}{2}\left(\frac{c}{6}\right)\left(\frac{c}{6}\right)}{\frac{1}{2}c^2} = \frac{25}{36} - \frac{1}{36} = \frac{24}{36} = \frac{6}{9} = \frac{2}{3}$$

■

Maximum possible product

$$x + y = c$$

$$xy = x(c - x) = cx - x^2 = -\left(x - \frac{c}{2}\right)^2 + \left(\frac{c}{2}\right)^2$$

So $xy \leq \left(\frac{c}{2}\right)^2$ and this happens when $x = y = \frac{c}{2}$.

■

Probability, James R. Gray, 1967

Page 32, Problem 14

If at a certain conference one of the delegates is equally likely to arrive at any time during an hour, find the probability that the greater of the times he was present or absent during that hour is at least n times the smaller.

Ans: $2/(n+1)$

<u>Solution</u>

Suppose the delegate arrives at time x, $0 \leq x \leq 1$. Then the delegate would be present during that hour for a time $1 - x$ and would be absent during that hour for a time x. The problem asks for

$$P(\max(1 - x, x) \geq n \cdot \min(1 - x, x))$$

For what values of x in $[0,1/2]$ is $\max(1 - x, x) \geq n \cdot \min(1 - x, x)$?

For $x \in [0,1/2]$, $\max(1 - x, x) = 1 - x$ and $\min(1 - x, x) = x$. In this case

$$\max(1-x,x) \geq n \cdot \min(1-x,x)$$
$$\Leftrightarrow 1-x \geq nx$$
$$\Leftrightarrow 1 \geq (n+1)x$$
$$\Leftrightarrow x \leq \frac{1}{n+1}$$

For $x \in [1/2,1]$, $\max(1-x,x) = x$ and $\min(1-x,x) = 1-x$. In this case

$$\max(1-x,x) \geq n \cdot \min(1-x,x)$$
$$\Leftrightarrow x \geq n(1-x)$$
$$\Leftrightarrow x + nx \geq n$$
$$\Leftrightarrow (n+1)x \geq n$$
$$\Leftrightarrow x \geq \frac{n}{n+1}$$

So,

$$P(\max(1-x,x) \geq n \cdot \min(1-x,x))$$
$$= P\left(\left(0 \leq x \leq \frac{1}{(n+1)} \text{ and } 0 \leq x \leq \frac{1}{2}\right) \text{or} \left(\frac{n}{n+1} \leq x \leq 1 \text{ and } \frac{1}{2} \leq x \leq 1\right)\right)$$
$$= \frac{Length\left(\left[0,\frac{1}{n+1}\right] \cup \left[\frac{n}{n+1},1\right]\right)}{Length([0,1])}$$
$$= \frac{\frac{2}{n+1}}{1} = \frac{2}{n+1}.$$

■

If a second delegate is equally likely to arrive, independently, at any time during the same hour, what is the probability that the arrivals are separated by at least forty minutes?

Ans: 1/9

Solution

Let x be the time that the first delegate arrives and let y be the time that the second delegate arrives. Then all (x, y) in the unit square are equally likely. The problem asks for

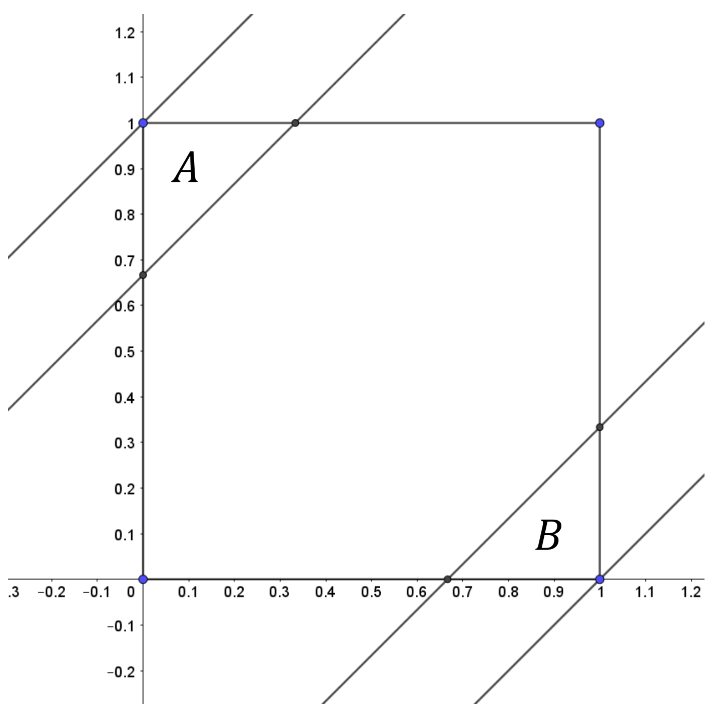
$$P\left(\frac{40}{60} \leq |x - y| \leq \frac{60}{60}\right)$$

$$= P\left(\left(x \geq y \text{ and } \frac{40}{60} \leq x - y \leq \frac{60}{60}\right) \text{ or } \left(x \leq y \text{ and } \frac{40}{60} \leq y - x \leq \frac{60}{60}\right)\right)$$

$$= P\left(\left(x \geq y \text{ and } \frac{40}{60} \leq x - y \text{ and } x - y \leq \frac{60}{60}\right) \text{ or } \left(x \leq y \text{ and } \frac{40}{60} \leq y - x \text{ and } y - x \leq \frac{60}{60}\right)\right)$$

$$= P\left(\left(y \leq x \text{ and } y \leq x - \frac{40}{60} \text{ and } y \geq x - \frac{60}{60}\right) \text{ or } \left(y \geq x \text{ and } y \geq x + \frac{40}{60} \text{ and } y \leq x + \frac{60}{60}\right)\right)$$

$$= P\left(\left(x - 1 \leq y \leq x - \frac{2}{3}\right) \text{ or } \left(x + \frac{2}{3} \leq y \leq x + 1\right)\right)$$



$$P\left(\frac{40}{60} \leq |x-y| \leq \frac{60}{60}\right) = P\big((x,y) \in (A \cup B)\big)$$

$$= \frac{Area(A) + Area(B)}{Area(Unit\ Square)} = \frac{\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)\left(\frac{1}{3}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)}{1} = \frac{1}{9}.$$

■

Probability, James R. Gray, 1967

Page 33, Problem 20

A straight line is divided at random into three parts. What is the probability that an acute-angled triangle can be formed by those three parts?

Ans: $3\log_e(2) - 2$

Solution

Side lengths $0 \leq a \leq b \leq c$ can be formed into an acute triangle if and only if $(i)\ a + b > c$ and $(ii)\ a^2 + b^2 > c^2$.

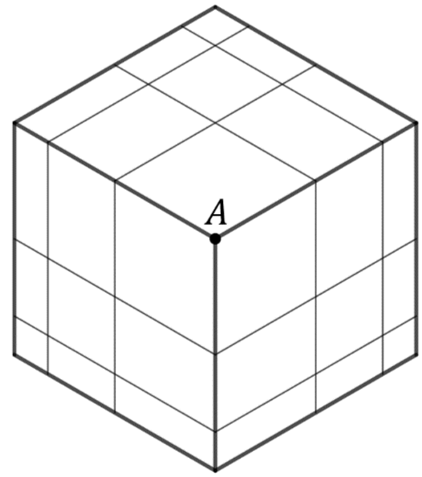
https://hectorpefo.github.io/2017-09-16-Broken-Sticks/

(Solution (almost) without calculus)

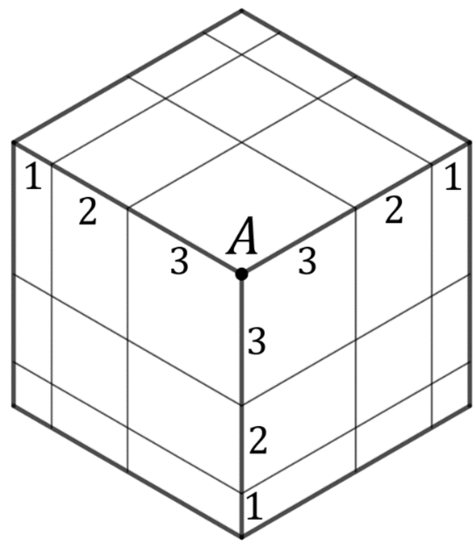
https://fivethirtyeight.com/features/will-you-be-a-ghostbuster-or-a-world-destroyer/

https://laurentlessard.com/bookproofs/sticks-in-the-woods/

10. Each edge of a cube measures 6 cm. The three edges passing through vertex A are divided into segments of 3 cm, 2 cm, and 1 cm, starting from point A. Then the cube is cut along the planes parallel to its faces and passing through the points of division (see figure). The pieces are then put into a bag and shaken before a single piece is randomly selected and drawn out. What is the probability that this piece will have dimensions 3 cm $\times$ 2 cm $\times$ 1 cm? (Source: Combinatorics-polynomials-probability, *Quantum: The Student Magazine of Math and Science*, Nikolay Vasilyev and Victor Gutenmacher, March/April 1993, pp 18-22, 62.)



Solution



We can uniquely identify each of the 27 pieces this cube will be cut into by an (x, y, z) "coordinate" system where the value of x is the width of a piece along the x axis, the value of y is the width of a piece along the y axis and the value of z is the width of a piece along the z axis.

Each of the 27 "coordinates" (x, y, z), $x \in \{1,2,3\}$, $y \in \{1,2,3\}$ and $z \in \{1,2,3\}$ corresponds uniquely to one of the 27 pieces.

With this system $(3,3,3)$ uniquely identifies the piece containing the corner point A.

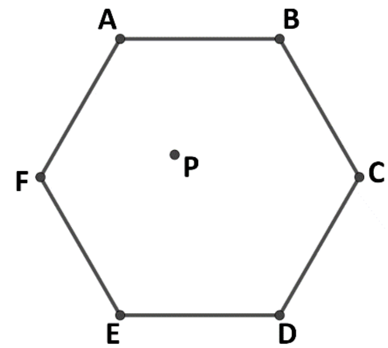
There are $3! = 6$ ways to arrange the three distinct numbers $(1,2,3)$ and each corresponds uniquely to a piece of this cube with dimensions 3 cm $\times$ 2 cm $\times$ 1 cm.

Therefore, the probability of randomly selecting a piece with these dimensions is the count ratio $6/27 = 2/9$.

■

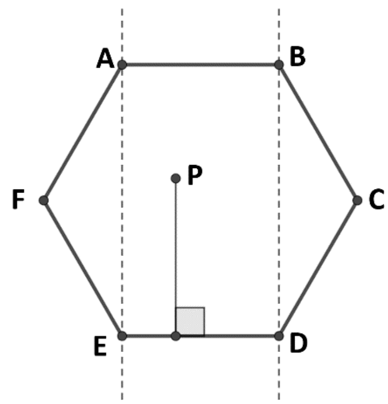
11. Let P be a randomly chosen interior point of the regular hexagon $ABCDEF$ as shown.

    Find the probability that there exists a perpendicular line segment from P to each of the six sides $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DE}$, $\overline{EF}$, and $\overline{FA}$ of this hexagon. (Source: *Mathematics Teacher*, April 1990 Calendar, Problem 17.)

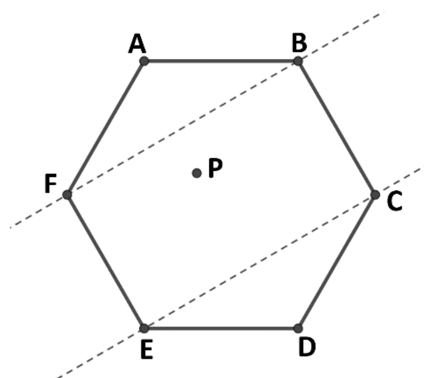


Solution

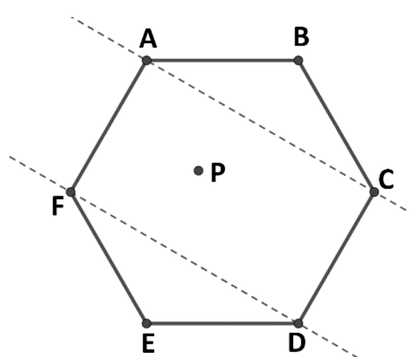
A perpendicular line segment from the interior point P to side $\overline{ED}$ only exists if P is inside or on the rectangle $ABDE$. Because $\overline{AB}$ and $\overline{ED}$ are necessarily parallel, it also follows P has to be inside or on rectangle $ABDE$ in order for a perpendicular from P to side $\overline{AB}$ to exist.



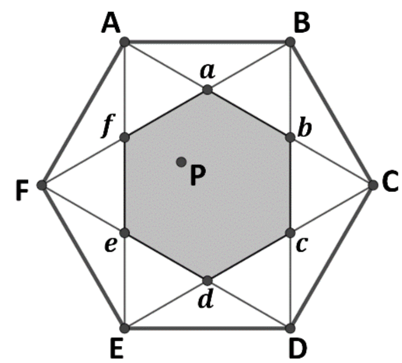
By the same reasoning, the point P must be inside or on rectangle $BCEF$ in order for a perpendicular from P to either $\overline{BC}$ or $\overline{EF}$ to exist.



And finally, the point P must be inside or on rectangle $ACDF$ in order for a perpendicular from P to either $\overline{CD}$ or $\overline{AF}$ to exist.



It follows that point P must be inside or on the *intersection* of rectangles $ABDE, BCEF$ and $ACDF$ in order that there exists a perpendicular line segment from P to each of the six sides of the regular hexagon $ABCDEF$.



That is, there exists a perpendicular line segment from P to each of the six sides of the regular hexagon $ABCDEF$ if and only if the point P is inside or on the shaded six-sided polygon $abcdef$. Furthermore, we can reason from symmetry that polygon $abcdef$ is itself a regular hexagon.

Because point P was a randomly chosen interior point of the regular hexagon $ABCDEF$, it follows from the principle of geometric probability that

$$P(\text{perpendicular line segments exist from } P \text{ to each of the six sides})$$

$$= \frac{\text{Area}(abcdef)}{\text{Area}(ABCDEF)}.$$

Two regular hexagons are necessarily similar and hence the ratio of areas equals the ratio of the squares of corresponding sides. That is,

$$\frac{\text{Area}(abcdef)}{\text{Area}(ABCDEF)} = \frac{(ab)^2}{(AB)^2}.$$

With some angle chasing we can verify that ΔaBb is equilateral and that $m\angle aAb = 30°$ and $m\angle AaB = 120°$. Applying the Law of Sines to ΔAaB we can establish that

$$\frac{\sin\,(120°)}{AB} = \frac{\sin\,(\angle 30°)}{aB}$$

which implies

$$AB = \sqrt{3}aB = \sqrt{3}ab.$$

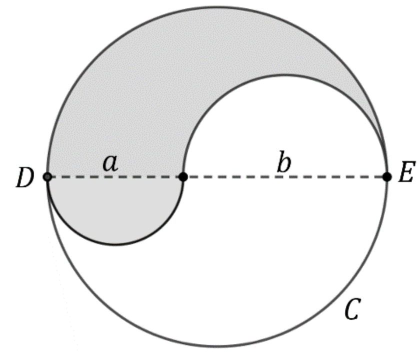
Hence,

$$P(\text{perpendicular line segments exist from } P \text{ to each of the six sides})$$

$$= \frac{\text{Area}(abcdef)}{\text{Area}(ABCDEF)} = \frac{(ab)^2}{(AB)^2} = \frac{(ab)^2}{\left(\sqrt{3}ab\right)^2} = \frac{1}{3}.$$

■

12. Let $\overline{DE}$ be a diameter of circle C. The shaded (yin) portion of the yin-yang type symbol shown is constructed by adding on a semicircle of diameter a and removing a semicircle of diameter b from the upper half of circle C. Assume the centers of both semicircles are on diameter $\overline{DE}$.



What is the probability that a randomly selected point from the interior of circle C is in the shaded (yin) portion of this yin-yang type symbol?

Solution

The area of circle C equals $\pi\left(\frac{a+b}{2}\right)^2$. The area of the added semicircle with diameter a equals $\frac{\pi}{2}\left(\frac{a}{2}\right)^2$ and the area of the subtracted semicircle with diameter b equals $\frac{\pi}{2}\left(\frac{b}{2}\right)^2$.

Because the point was randomly selected from the interior of C, the probability that the selected point falls in the shaded portion equals the ratio of the area of the shaded portion of this symbol to the area of circle C.

$$P(\text{random point belongs to the shaded portion})$$

$$= \frac{\left(\frac{\pi}{2}\right)\left(\frac{a+b}{2}\right)^2 + \left(\frac{\pi}{2}\right)\left(\frac{a}{2}\right)^2 - \left(\frac{\pi}{2}\right)\left(\frac{b}{2}\right)^2}{\pi\left(\frac{a+b}{2}\right)^2} = \frac{\frac{\pi}{4}a(a+b)}{\frac{\pi}{4}(a+b)^2} = \frac{a}{a+b}.$$

■