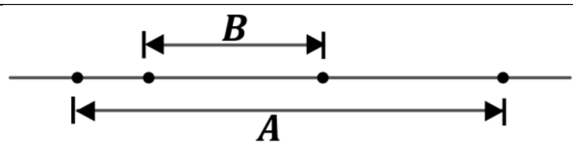
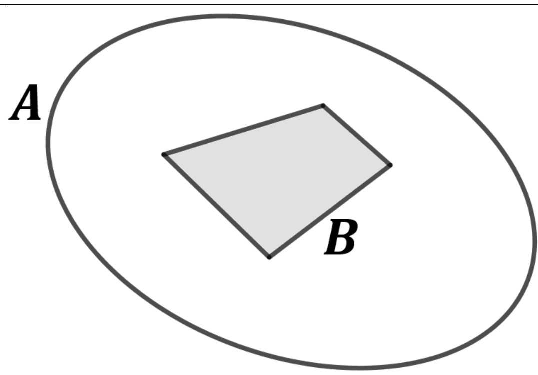
Chapter 6 Geometric Probability

Theorem 11. Problems with Equally Likely Regions

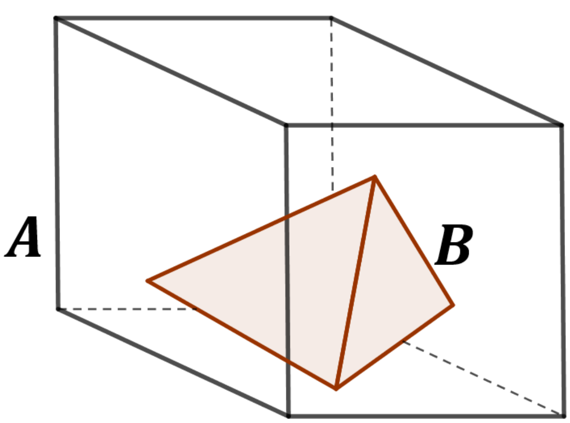
If a point is chosen at random from a region A (*i.e.* all points in region A are equally likely to be picked) and if region B is a subset of region A, then



$$P(\text{point picked is in region } B) = \frac{\text{Length}(B)}{\text{Length}(A)}$$



$$P(\text{point picked is in region } B) = \frac{\text{Area}(B)}{\text{Area}(A)}$$



$$P(\text{point picked is in region } B) = \frac{\text{Volume}(B)}{\text{Volume}(A)}.$$

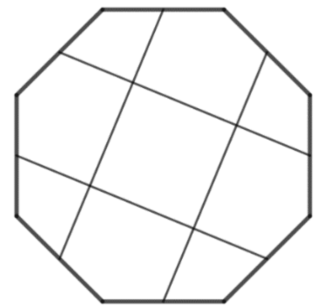
Theorem 11 is the continuous analogue of Theorem 5 - Problems with *Equally Likely* <u>*Outcomes*</u>.

To find probability with continuous sample spaces we would, *in general*, need to modify our definition of the probability of an event from sums over all outcomes to integrals over all outcomes – which of course, is a problem of calculus.

However, in the special cases where the continuum of outcomes in a region are assumed to be equally likely, the calculus problem of finding length, area or volume simplifies to a geometry problem of finding length, area or volume.

Exercises for Chapter 6

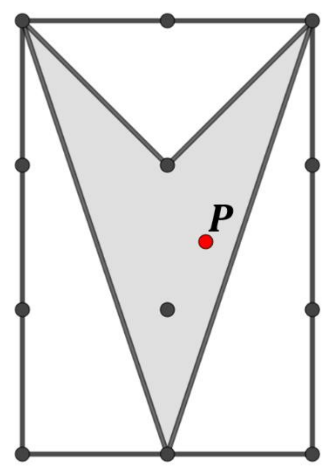
1. (5T174) Four segments are drawn from the midpoints of the sides of a regular octagon, creating a square, four congruent pentagons, and four congruent kites, as shown in the figure. If a point is chosen at random inside the octagon, determine exactly the probability that the point lies inside the square.



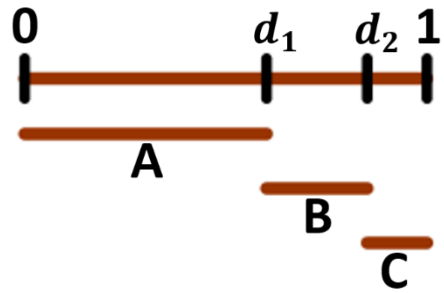
Still need to insert problems (5T115), (5D104), (5T104), (MB072), (5D024), (5C011), (5T993) here.

2. (5D914) The westbound Main Street bus on which I ride is scheduled to arrive at State Street at 8:10, but it actually arrives randomly within 4 minutes on either side of 8:10. The north bound State Street bus that I hope to catch is scheduled to arrive at Main Street at 8:14, but again its arrival is randomly distributed within 4 minutes either side of 8:14. What is the probability of my catching the 8:14 bus?

3. A rectangle and an arrowhead are drawn on a regularly spaced grid of lattice points. If a point P is chosen at random in the rectangle, what is the probability that P will be in the shaded arrowhead, as shown in the figure on the right? (Note: The point P can be *anywhere* in or on the rectangle and is not limited to the lattice points.)
(Source: December 2012 Calendar Problem #20, *The Mathematics Teacher*)

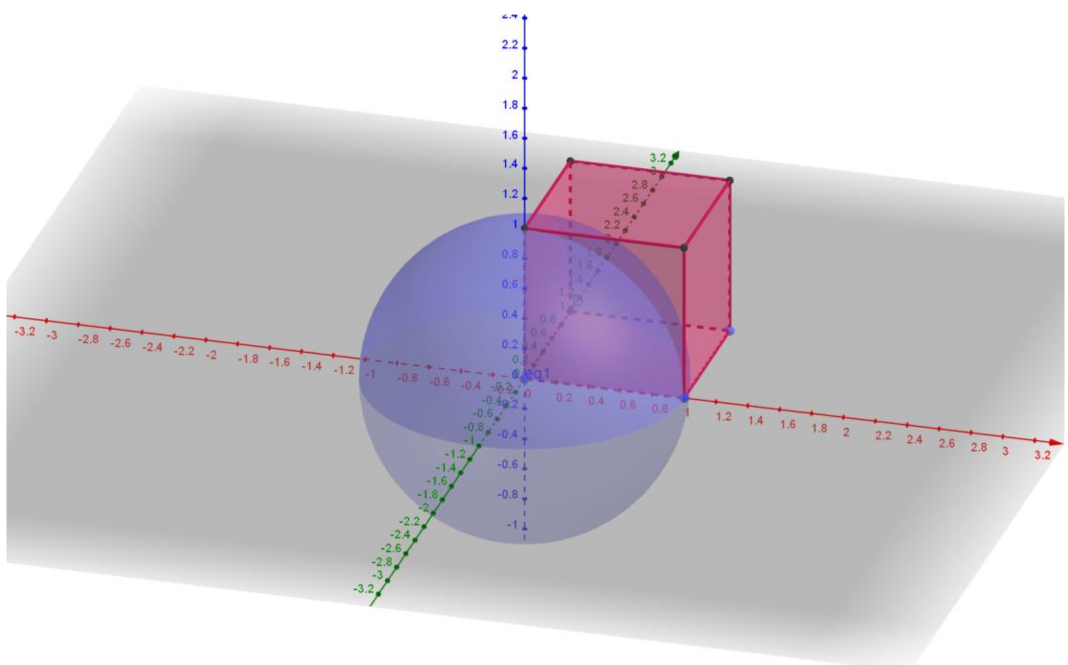


4. Suppose we select two points on a stick of unit length such that the two points are picked independently and such that all points along the stick are equally likely to be picked in both cases. Then suppose we break the stick at these two points and get three pieces of stick. What is the probability that the three pieces can be made into a triangle?



5. Mai and her mom are hoping to see each over lunch. Both Mai and her mom only take 15 minutes for lunch. Mai has to finish some reading before class and her mom has to finish some lecture prep before class. Both Mai and her mom are so busy that all they can promise is that they will get to the cafeteria, independently, at some random point between noon and 1 pm. What is the probability that they will actually be able to see each other over lunch?

6. If a point (x_0, y_0, z_0) is picked at random from the unit cube where $0 \leq x \leq 1, 0 \leq y \leq 1$, $0 \leq z \leq 1$, what is the probability that $x_0^2 + y_0^2 + z_0^2 \leq 1$?



7. The sum of two positive quantities is known. If all pairs of possible values are equally likely, prove that the probability that their product will not be less than five-ninths of the maximum possible product is $2/3$. (Source: *Probability*, James R. Gray, 1967, Page 31, Problem 13)

   Note: For two positive numbers x and y with $x + y = c$ for c fixed, the maximum possible value of xy occurs for $x = y = c/2$. Is that intuitive? To formally justify this, note that

$$xy = x(c - x) = cx - x^2 = -\left(x - \frac{c}{2}\right)^2 + \left(\frac{c}{2}\right)^2.$$

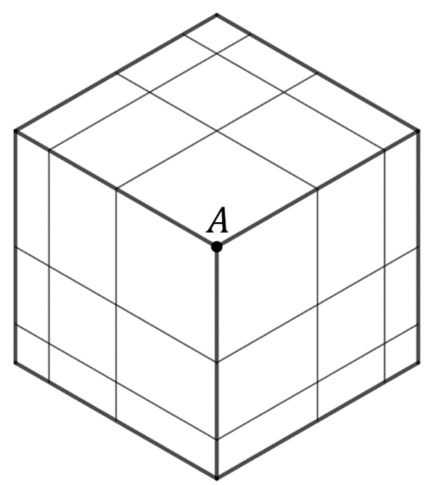
   So $xy \leq (c/2)^2$ and this only happens when $x = y = c/2$.

8. (a) If at a certain conference one of the delegates is equally likely to arrive at any time during an hour, find the probability that the greater of the times he was present or absent during that hour is at least n times the smaller.

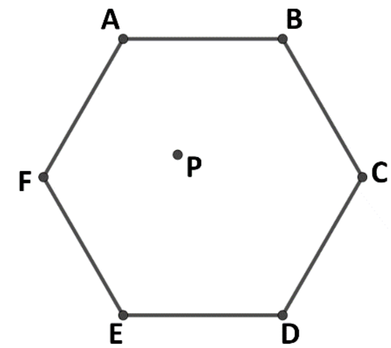
   (b) If a second delegate is equally likely to arrive, independently, at any time during the same hour, what is the probability that the arrivals are separated by at least forty minutes? (Source: *Probability*, James R. Gray, 1967, Page 32, Problem 14)

9. A straight line is divided at random into three parts. What is the probability that an acute-angled triangle can be formed by those three parts? (Source: *Probability*, James R. Gray, 1967, Page 33, Problem 20)

10. Each edge of a cube measures 6 cm. The three edges passing through vertex A are divided into segments of 3 cm, 2 cm, and 1 cm, starting from point A. Then the cube is cut along the planes parallel to its faces and passing through the points of division (see figure). The pieces are then put into a bag and shaken before a single piece is randomly selected and drawn out. What is the probability that this piece will have dimensions 3 cm $\times$ 2 cm $\times$ 1 cm? (Source: Combinatorics-polynomials-probability, *Quantum: The Student Magazine of Math and Science*, Nikolay Vasilyev and Victor Gutenmacher, March/April 1993, pp 18-22, 62.)

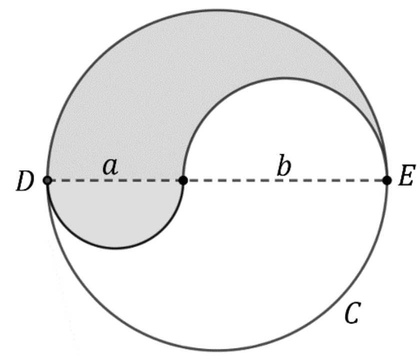


11. Let P be a randomly chosen interior point of the regular hexagon $ABCDEF$ as shown.



Find the probability that there exists a perpendicular line segment from P to each of the six sides $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DE}$, $\overline{EF}$, and $\overline{FA}$ of this hexagon. (Source: *Mathematics Teacher*, April 1990 Calendar, Problem 17.)

12. Let $\overline{DE}$ be a diameter of circle C. The shaded (yin) portion of the yin-yang type symbol shown is constructed by adding on a semicircle of diameter a and removing a semicircle of diameter b from the upper half of circle C. Assume the centers of both semicircles are on diameter $\overline{DE}$.



What is the probability that a randomly selected point from the interior of circle C is in the shaded (yin) portion of this yin-yang type symbol?

Project 6 How Likely is it for Two Numbers to be Relatively Prime?

What is the probability that two numbers picked independently and at random from the set of all positive integers will have no common divisor?

The answer is $6/\pi^2$ which is just a bit larger than 60%. So how does π make its way into a problem that does not involve circles?

There is documentation that this problem with its fascinating answer was posed in lectures given by Chebyshev (founding father of Russian mathematics) in the 19th century. This problem is sometimes referred to as **Chebyshev's Problem** in his honor. However, there is some scholarship that sources the problem to lectures given by the German mathematician Dirichlet nearly a half century prior to Chebyshev. What can be said with certainty is that this problem has caught the attention of many famous mathematicians.

Let a and b be the two numbers picked from $\mathbb{Z}^+$ (the set of all positive integers). What is the probability that a is divisible by 3? Clearly every third number in $\mathbb{Z}^+$ is a multiple of three, so $P(a \text{ is divisible by } 3) = 1/3$.

By the same logic $P(a \text{ is divisible by } n) = 1/n$ for every $n \in \mathbb{Z}^+$. And as a and b are picked *independently* from $\mathbb{Z}^+$, $P(a \text{ and } b \text{ are both divisible by } n) = 1/n^2$.

Therefore,

$$P\big(a \text{ and } b \text{ are } \underline{\text{not}} \text{ both divisible by } n\big) = 1 - \frac{1}{n^2}.$$

The positive integers a and b will have no common divisor if and only if they are relatively prime. That is, provided a and b are not both divisible by the same prime number. (Why? Because every potential common divisor can be written as the product of prime numbers so to be divisible by any number is to be divisible by all primes in the prime factorization of that number).

Let E_k be the event that a and b are not divisible by the k^{th} prime number. To be precise, let $q_1, q_2, q_3, q_4, q_5, q_6, q_7, \ldots$ be the sequence of prime numbers $2,3,5,7,11,13,17, \ldots$. So, for example, E_6 is the event that a and b are not both divisible by 13.

Then

$$P(a \text{ and } b \text{ have no common divisor})$$

$$= P(E_1 \cap E_2 \cap E_3 \cap E_4 \cap \cdots).$$

Finding this probability would be *much* easier if the events $E_1, E_2, E_3, E_4, \ldots$ were *independent* because if these events were independent then

$$P(E_1 \cap E_2 \cap E_3 \cap E_4 \cap \cdots) = P(E_1)P(E_2)P(E_3)P(E_4)\cdots.$$

Roughly put, we would like to hope that knowing that a and b are not both divisible by $q_3 = 5$, for one example, does not give us any hint as to whether they are both divisible by $q_5 = 11$.

This doesn't seem *unreasonable* but that is no substitute for a proof! However, we are just going to have to accept this result without proof for this course.

Having made that assumption, we have

$$P(a \text{ and } b \text{ have no common divisor})$$
$$= P(E_1 \cap E_2 \cap E_3 \cap E_4 \cap E_5 \cap E_6 \cdots)$$
$$= P(E_1)P(E_2)P(E_3)P(E_4)P(E_5)P(E_6)\cdots$$
$$= \left(1 - \frac{1}{q_1^2}\right)\left(1 - \frac{1}{q_2^2}\right)\left(1 - \frac{1}{q_3^2}\right)\left(1 - \frac{1}{q_4^2}\right)\left(1 - \frac{1}{q_5^2}\right)\left(1 - \frac{1}{q_6^2}\right)\cdots.$$

Now *imagine* the process of multiplying out this infinite product. The process, *in theory*, would be to go through *all possible ways* of picking either the term 1 or $-\left(\frac{1}{2^2}\right)$ from the first factor $\left(1 - \frac{1}{2^2}\right)$ and then multiplying that by one of the two terms in the second factor $\left(1 - \frac{1}{3^2}\right)$ and then multiplying that by one of the two terms in the third factor $\left(1 - \frac{1}{5^2}\right)$, etc.

It would obviously be a big mess! But fortunately, there is trick that will actually make the process doable.

Instead of working to find $P(a \text{ and } b \text{ have no common divisor})$ let's consider finding

$$(\star) \qquad \frac{1}{P(a \text{ and } b \text{ have no common divisor})}.$$

If that turns out to be doable (and it is) then we could flip $(\star)$ back over when we're done to recapture $P(a \text{ and } b \text{ have no common divisor})$.

Consider

$$\frac{1}{P(a \text{ and } b \text{ have no common divisor})}$$

$$= \frac{1}{\left(1-\frac{1}{q_1^2}\right)\left(1-\frac{1}{q_2^2}\right)\left(1-\frac{1}{q_3^2}\right)\left(1-\frac{1}{q_4^2}\right)\left(1-\frac{1}{q_5^2}\right)\left(1-\frac{1}{q_6^2}\right)\cdots}$$

$$= \left(\frac{1}{1-\frac{1}{q_1^2}}\right)\left(\frac{1}{1-\frac{1}{q_2^2}}\right)\left(\frac{1}{1-\frac{1}{q_3^2}}\right)\left(\frac{1}{1-\frac{1}{q_4^2}}\right)\left(\frac{1}{1-\frac{1}{q_5^2}}\right)\left(\frac{1}{1-\frac{1}{q_6^2}}\right)\cdots$$

$$= \prod_{i=1}^{\infty}\left(\frac{1}{1-\frac{1}{q_i^2}}\right)$$

where again $q_1, q_2, q_3, q_4, q_5, q_6, q_7, \ldots$ is the sequence of prime numbers $2,3,5,7,11,13,17, \ldots$

We still have an infinite product at this point and it doesn't look to be any easier to work with. But now comes the trick! Consider the geometric sequence

$$\left(\frac{1}{q_i^2}\right)^0 + \left(\frac{1}{q_i^2}\right)^1 + \left(\frac{1}{q_i^2}\right)^2 + \cdots .$$

From the general formula for geometric sequences, we know that

$$\left(\frac{1}{q_i^2}\right)^0 + \left(\frac{1}{q_i^2}\right)^1 + \left(\frac{1}{q_i^2}\right)^2 + \cdots = \left(\frac{1}{1-\frac{1}{q_i^2}}\right).$$

So,

$$\frac{1}{P(a \text{ and } b \text{ have no common divisor})}$$

$$= \prod_{i=1}^{\infty}\left(\frac{1}{1-\frac{1}{q_i^2}}\right)$$

$$= \prod_{i=1}^{\infty}\left(\left(\frac{1}{q_i^2}\right)^0 + \left(\frac{1}{q_i^2}\right)^1 + \left(\frac{1}{q_i^2}\right)^2 + \cdots\right).$$

Now *imagine* the process of multiplying this infinite product *all* out. Is that possible? It is if you look at "just right"!

What would it look like when you finished? You would end up with the sum of *all possible* terms of the form

$$\left(\frac{1}{q_1^2}\right)^{m_1}\left(\frac{1}{q_2^2}\right)^{m_2}\left(\frac{1}{q_3^2}\right)^{m_3}\left(\frac{1}{q_4^2}\right)^{m_4}\cdots$$

$$=\left(\frac{1}{q_1^{m_1}q_2^{m_2}q_3^{m_3}q_4^{m_4}\cdots}\right)^2$$

where each $m_i \in \{0,1,2,3,\ldots\}$.

Do you recognize this yet? It will help if we replace the q_i will the numbers they represent.

$$\left(\frac{1}{2^{m_1}3^{m_2}5^{m_3}7^{m_4}11^{m_4}13^{m_5}17^{m_6}\cdots}\right)^2.$$

Do you now see that the denominator $2^{m_1}3^{m_2}5^{m_3}7^{m_4}11^{m_4}13^{m_5}17^{m_6}\cdots$ with each $m_i \in \{0,1,2,3,\ldots\}$ is the prime factorization of some integer?

And so, the sum over *all possible* terms of the form

$$\left(\frac{1}{q_1^{m_1}q_2^{m_2}q_3^{m_3}q_4^{m_4}\cdots}\right)^2$$

where each $m_i \in \{0,1,2,3,\ldots\}$ is the sum of $\left(\frac{1}{n}\right)^2$ over all possible positive integers n.

That is,

$$\frac{1}{P(a \text{ and } b \text{ have no common divisor})}$$

$$=\prod_{i=1}^{\infty}\left(\left(\frac{1}{q_i^2}\right)^0+\left(\frac{1}{q_i^2}\right)^1+\left(\frac{1}{q_i^2}\right)^2+\cdots\right)$$

$$=\sum_{n=1}^{\infty}\frac{1}{n^2}.$$

And fortunately, the great mathematician Leonard Euler of Basel, Switzerland, showed in 1735 that this sum equals $\pi^2/6$. It was a celebrated result at the time because many other famous mathematicians of his day had tried but failed to solve this problem. To this day, this result is called the **Basel Problem** in honor of Euler's hometown.

So there we have it,

$$\frac{1}{P(a \text{ and } b \text{ have no common divisor})} = \frac{\pi^2}{6}$$

which means

$$P(a \text{ and } b \text{ have no common divisor}) = \frac{6}{\pi^2}.$$

■

Project 7 Frog Jumping Problem (AMC 12B, 2010, Problem 18)

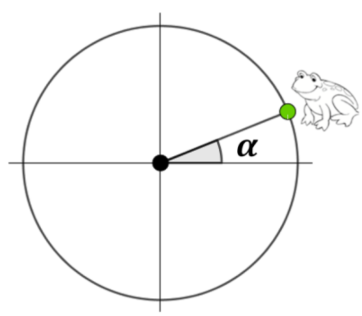
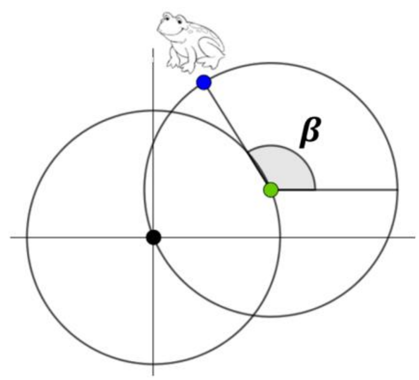
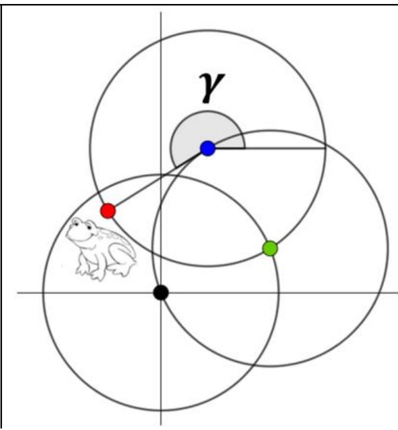
(A generalization of MSHSML Problem SD104)

A frog makes 3 jumps, each exactly 1 meter long. The directions of the jumps are chosen independently at random. What is the probability that the frog's final position is no more than 1 meter from its starting position?

(The above is the exact wording from the AMC exam but let me add some clarifying information. We can assume the frog starts at the origin of a fixed coordinate system and at each jump picks an angle at random from $[0,2\pi]$ radians independent of all previous jumps.)

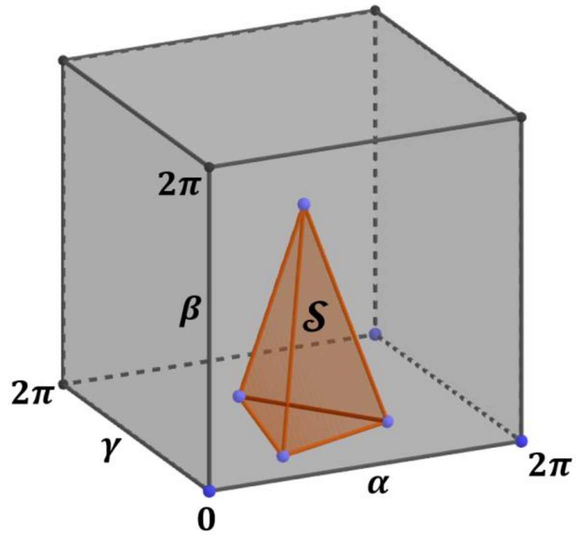
<u>Solution</u>

Let α, β and γ be the three angles the frog chooses. Then …

		
after the first jump the frog’s coordinates will be $(\cos(\alpha), \sin(\alpha))$,	after the second jump the frog’s coordinates will be $(\cos(\alpha)+\cos(\beta), \sin(\alpha)+\sin(\beta))$,	and after the third jump the frog’s coordinates will be $(\cos(\alpha)+\cos(\beta)+\cos(\gamma), \sin(\alpha)+\sin(\beta)+\sin(\gamma))$.

The problem is asking for the probability that the frog hops back inside the unit circle centered at the origin on the third jump.

We are given the information that all angles (α, β, γ) in the $2\pi \times 2\pi \times 2\pi$ cube shown below are equally likely to be selected by the frog.



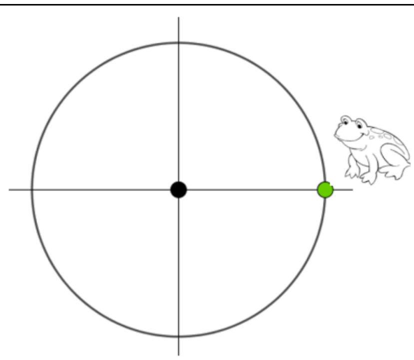
Let $\mathcal{S}$ (*hypothetically* the brown pyramid shown above) represent the subset of angles (α, β, γ) in this cube where the frog will end up jumping back into the unit circle centered at $(0,0)$.

In this case it would follow from the method of geometric probability that

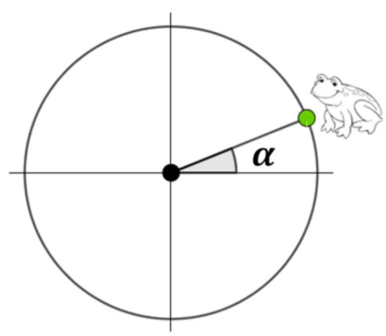
$$P(\text{frog hops back into the unit circle centered at } (0,0) \text{ after the third jump}) = \frac{\text{Vol}(\mathcal{S})}{\text{Vol}(\text{Cube})}.$$

A difficulty in this problem is that it is three dimensional which makes describing $\mathcal{S}$ and finding its volume more difficult than in previous geometric probability problems we have encountered.

Fortunately, we can use symmetry to reduce this to a two-dimensional problem. To illustrate this idea, consider the two cases shown below, when $\alpha = 0$ and when $\alpha = 0.384$

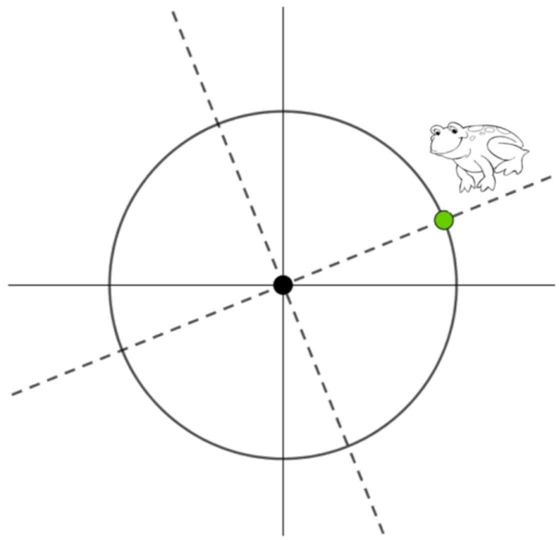


$\alpha = 0$ radians



$\alpha = 0.384$ radians

By imaging a new coordinate system (shown as the dotted lines in the figure below)

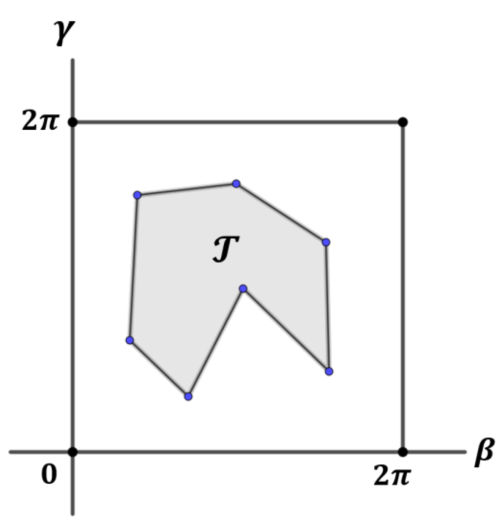


it is apparent from the symmetry of the circle that the probability that the frog will finally end up back inside the unit circle centered at the origin is the same whether $\alpha = 0$ or $\alpha = 0.384$ (or any other radian value in $[0,2\pi]$).

That is, the event that the frog ends up in the unit circle centered at the origin is ***independent*** of the value of α.

It follows that we can just **fix $\boldsymbol{\alpha = 0}$** without changing the probability that the frog ends up in the unit circle centered at the origin.

By fixing $\alpha = 0$ the problem becomes one of describing and finding the area of that two-dimensional region $\mathcal{T}$, inside the $2\pi \times 2\pi$ square of all possible values of the angles (β, γ), where the frog will end up jumping back into the unit circle centered at $(0,0)$.

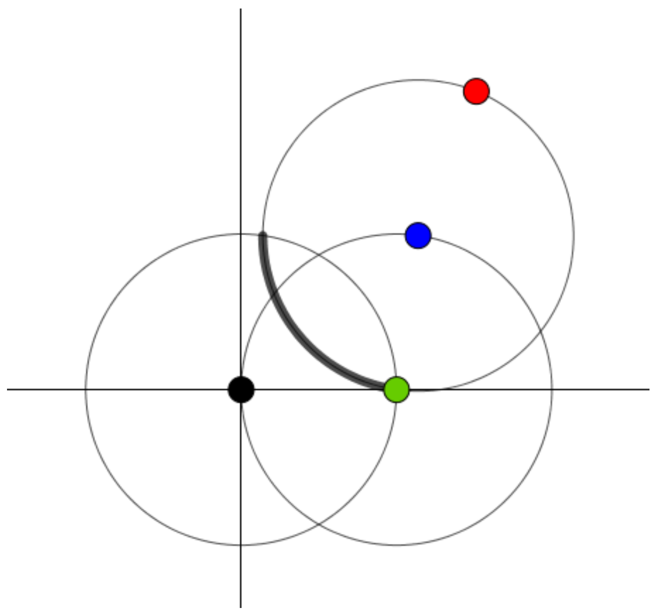


Once we find $\mathcal{T}$ then it will follow that

$$P(\text{frog hops back into the unit circle centered at } (0,0) \text{ after the third jump}) = \frac{\text{Area}(\mathcal{T})}{\text{Area}(\text{Square})}.$$

Our goal now is to find what the region $\mathcal{T}$ defined above will look like for any given (fixed) value of β, $0 \leq \beta \leq 2\pi$.

For any fixed β (*i.e.* keeping the blue dot fixed), the frog will be inside the unit circle centered at the origin when the red dot (the frog's position after the third jump) is somewhere on the darkened arc shown below.



This is, on the arc between the two points of intersection of the circle centered at $(0,0)$ and the circle centered at

$$(\cos(0) + \cos(\beta), \sin(0) + \sin(\beta)) = (1 + \cos(\beta), \sin(\beta)),$$

The equation of these two circles are:

$$(x - 0)^2 + (y - 0)^2 = 1$$

and

$$\left(x - (1 + \cos(\beta))\right)^2 + (y - \sin(\beta))^2 = 1.$$

For any fixed β we can solve for the points of intersection of these two circles by expanding the second equation and simplifying.

$$\begin{aligned} 1 &= \big(x - (1 + \cos(\beta))\big)^2 + (y - \sin(\beta))^2 \\ &= (x^2 + y^2) + (\sin^2(\beta) + \cos^2(\beta)) - 2\cos(\beta)\,x - 2\sin(\beta)\,y - 2x + 2\cos(\beta) + 1 \\ &= 1 + 1 - 2\cos(\beta)\,x - 2\sin(\beta)\,y - 2x + 2\cos(\beta) + 1. \end{aligned}$$

Note that we could make the substitution $x^2 + y^2 = 1$ because the points of intersection are on the circle where $x^2 + y^2 = 1$. We could make the substitution $\sin^2(\beta) + \cos^2(\beta) = 1$ because this is a Pythagorean identity, true for all angles. Bringing all terms to the same side, the second equation has simplified to $-\cos(\beta)\,x - \sin(\beta)\,y - x + \cos(\beta) + 1 = 0$. However,

$$-\cos(\beta)\,x - \sin(\beta)\,y - x + \cos(\beta) + 1 = 0$$

$\Leftrightarrow$

$$y = \left(\frac{1 + \cos(\beta)}{\sin(\beta)}\right)(1 - x).$$

Now substituting this formula for y in terms of x into the equation $x^2 + y^2 = 1$, we have the quadratic equation

$$x^2 + \left(\frac{1 + \cos(\beta)}{\sin(\beta)}\right)^2 (1 - x)^2 = 1.$$

After simplification this becomes

$$x^2 - (1 + \cos(\beta))x + \cos(\beta) = 0.$$

Using the quadratic formula we find

$$\begin{aligned} x &= \frac{(1 + \cos(\beta)) \pm \sqrt{(1 + \cos(\beta))^2 - 4\cos(\beta)}}{2} \\ &= \frac{(1 + \cos(\beta)) \pm \sqrt{(1 - \cos(\beta))^2}}{2} \\ &= \frac{(1 + \cos(\beta)) \pm (1 - \cos(\beta))}{2}. \end{aligned}$$

So, the x coordinate of the two points of intersection are $x = 1$ and $x = \cos(\beta)$. It follows that the (x, y) coordinates of the two points of intersection are

$$(x, y) = \left(1, \left(\frac{1 + \cos(\beta)}{\sin(\beta)}\right)(1 - 1)\right) = (1,0)$$

and

$$(x, y) = \left(\cos(\beta), \left(\frac{1 + \cos(\beta)}{\sin(\beta)}\right)(1 - \cos(\beta))\right)$$

$$= \left(\cos(\beta), \frac{1 - \cos^2(\beta)}{\sin(\beta)}\right)$$

$$= (\cos(\beta), \sin(\beta)).$$

Recall that the frog's position after the third jump (*i.e* the red dot in our diagrams) is

$$(1 + \cos(\beta) + \cos(\gamma), \sin(\beta) + \sin(\gamma)).$$

We want to set this position equal to the above two points of intersection and solve for γ as a function of β.

$$(1 + \cos(\beta) + \cos(\gamma), \sin(\beta) + \sin(\gamma)) = (1,0)$$

$$\Leftrightarrow \begin{cases} \cos(\beta) + \cos(\gamma) = 0 \\ \sin(\beta) + \sin(\gamma) = 0 \end{cases}$$

$$\Leftrightarrow \gamma = \begin{cases} \beta + \pi & 0 \leq \beta \leq \pi \\ \beta - \pi & \pi \leq \beta \leq 2\pi \end{cases}$$

and

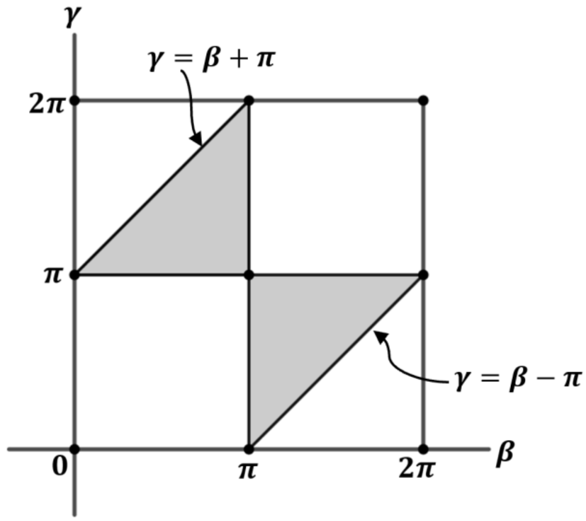
$$(1 + \cos(\beta) + \cos(\gamma), \sin(\beta) + \sin(\gamma)) = (\cos(\beta), \sin(\beta))$$

$$\Leftrightarrow \begin{cases} 1 + \cos(\gamma) = 0 \\ \sin(\gamma) = 0 \end{cases}$$

$$\Leftrightarrow \gamma = \pi.$$

This shows that for fixed $\beta \in [0, \pi]$ the range of γ where the frog will be inside the unit circle centered at $(0,0)$ is $\gamma \in [\pi, \beta + \pi]$ and for fixed $\beta \in [\pi, 2\pi]$ the range of γ is $\gamma \in [\beta - \pi, \pi]$.

We illustrate this as the shaded regions in the diagram below.



Because all points (β, γ) in the square region $0 \leq \beta \leq 2\pi$, $0 \leq \gamma \leq 2\pi$ are equally likely,

$$P(\text{frog hops back into the unit circle centered at } (0,0) \text{ after the third jump})$$

$$= \frac{\text{Area of the two shaded triangles}}{\text{Area in the square}}$$

$$= \frac{\frac{1}{2}\pi^2 + \frac{1}{2}\pi^2}{(2\pi)(2\pi)} = \frac{\pi^2}{4\pi^2} = \frac{1}{4}.$$

■

Chapter 7 Miscellaneous Topics

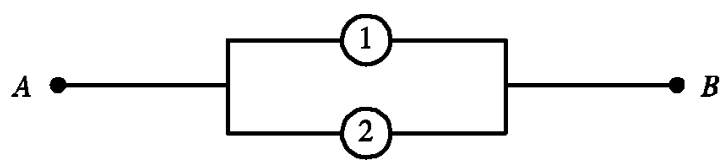
7.1 System Reliability

Electrical circuit problems offer a are nice application of working with using “or’s” and “and’s”.

Consider the following portion of an electric circuit with three relays. Current will flow from point A to point B if there is at least one closed path when the relays are activated. The relays may malfunction and not close when activated. Suppose that the relays act independently of one another and close properly when activated, with a probability of 0.9.

Compare the probability of current flowing from A to B in the series system shown below

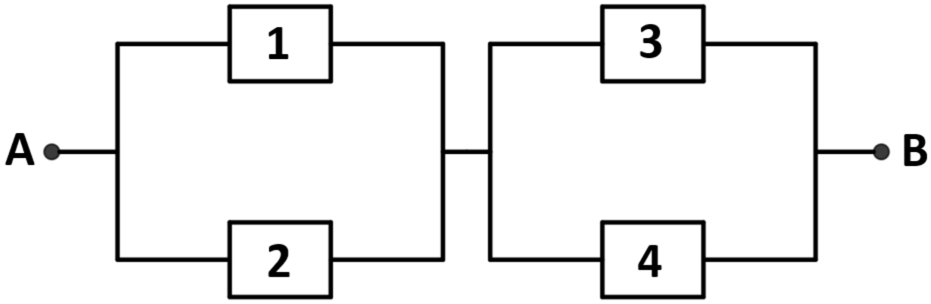
with the probability of flow in the parallel system shown below.



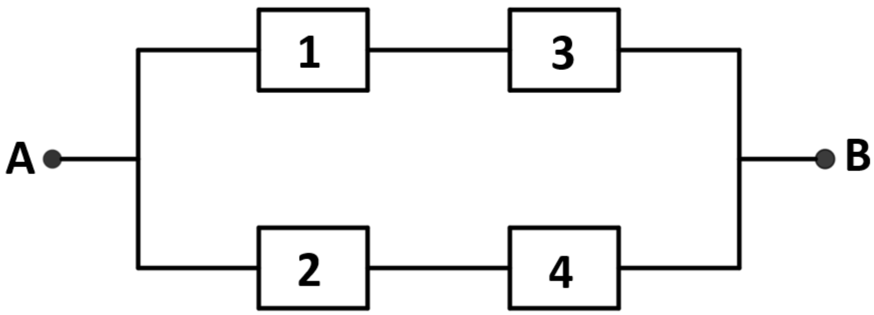
Example

Suppose that Switches 1 through 4 used in the construction of electric circuits will shut properly when activated with probability p. Assuming that these switches operate independently, will Design 1 or Design 2 have the higher probability that current will flow from point A to point B when the switches are activated?

Design 1



Design 2



<u>Solution</u>

Let C_j be the event that the j^{th} switch shuts properly when activated, $j = 1,2,3,4$. Then

$$P(\text{Design 1 works}) = P\big((C_1 \text{or } C_2) \text{ and } (C_3 \text{ or } C_4)\big)$$

$$= P\big((C_1 \cup C_2) \cap (C_3 \cup C_4)\big)$$

$$= P(C_1 \cup C_2)P(C_2 \cup C_4)$$

$$= \big(P(C_1) + P(C_2) - P(C_1 \cap C_2)\big)\big(P(C_1) + P(C_2) - P(C_1 \cap C_2)\big)$$

$$= (p + p - p^2)^2$$

$$= 4p^2 - 4p^3 + p^4$$

$$= p^2(2 - p)^2$$

$$
\begin{aligned}
P(\text{Design 2 works}) &= P\big((C_1 \text{and} C_3) \text{ or } (C_2 \text{and } C_4)\big) \\
&= P\big((C_1 \cap C_3) \cup (C_2 \cap C_4)\big) \\
&= P(C_1 \cap C_3) + P(C_2 \cap C_4) - P\big((C_1 \cap C_3) \cap (C_2 \cap C_4)\big) \\
&= p^2 + p^2 - p^4 \\
&= 2p^2 - p^4 \\
&= p^2(2 - p^2)
\end{aligned}
$$

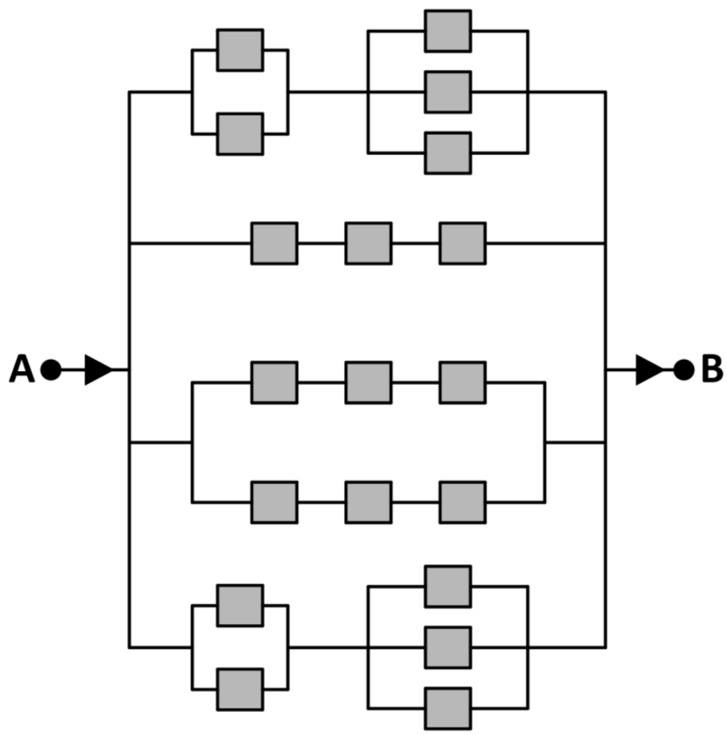
$$
(2-p)^2 - (2-p^2) = (4 - 4p + p^2) - 2 + p^2
$$

$$
\begin{aligned}
&= 2p^2 - 4p + 2 \\
&= 2(p^2 - 2p + 1) \\
&= 2(p-1)(p-1)
\end{aligned}
$$

■

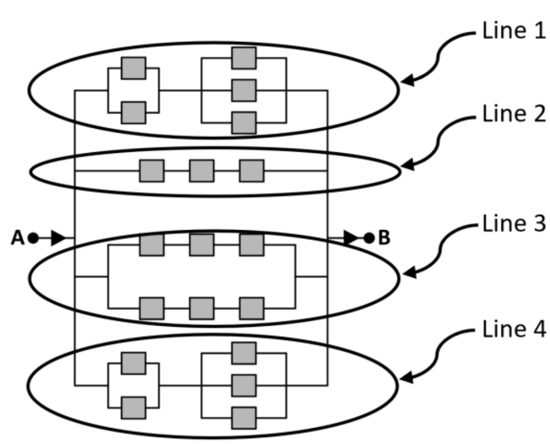
Example

Electricity can flow from point **A** to point **B** in the diagram shown below provided there is *at least one path* where every switch (the shaded small rectangles) on that path is working. Assume that each switch operates or fails independently of all other switches and assume that each switch will operate properly with probability $p = 0.90$. Under these assumptions, what is the probability that a current can flow from point **A** to point **B**.



<u>Solution</u>

The key to a messy looking (it's really not as messy as it seems) problem of this type is to break it down into many smaller problems and solve the smaller problems separately. We start by identifying four separate "lines" which are connected in ***parallel***.



Lines are connected in **parallel** if the System will work if and only if **at least one** Line works.

Let E_i be the event that Line i works. Then

$$P(\text{System works}) = P(E_1 \text{ or } E_2 \text{ or } E_3 \text{ or } E_4) = P(E_1 \cup E_2 \cup E_3 \cup E_4).$$

An important point of why this problem is not as hard as it looks is that all the switches act independently. And because each line is composed of separate switches (*i.e.* no two lines share a switch), the lines act independently.

That is, the events E_1, E_2, E_3 and E_4 are independent. Remember that this implies that the complements of these events E_1^c, E_2^c, E_3^c and E_4^c are also independent events.

First note that by the rule of complements

$$P(E_1 \cup E_2 \cup E_3 \cup E_4) = 1 - P((E_1 \cup E_2 \cup E_3 \cup E_4)^c)$$

and then by DeMorgan's Rule

$$P((E_1 \cup E_2 \cup E_3 \cup E_4)^c) = P(E_1^c \cap E_2^c \cap E_3^c \cap E_4^c).$$

That is,

$$P(E_1 \cup E_2 \cup E_3 \cup E_4) = 1 - P(E_1^c \cap E_2^c \cap E_3^c \cap E_4^c).$$

(It is actually easier if you just say what this means in words. The probability that at least one of the lines works equals 1 minus the probability that all of the lines fail.)

But as we noted above, E_1^c, E_2^c, E_3^c and E_4^c are independent events. So

$$\begin{aligned} P(E_1 \cup E_2 \cup E_3 \cup E_4) &= 1 - P(E_1^c \cap E_2^c \cap E_3^c \cap E_4^c) \\ &= 1 - P(E_1^c)P(E_2^c)P(E_3^c)P(E_4^c) \\ &= 1 - \big(1 - P(E_1)\big)\big(1 - P(E_2)\big)\big(1 - P(E_3)\big)\big(1 - P(E_4)\big). \end{aligned}$$

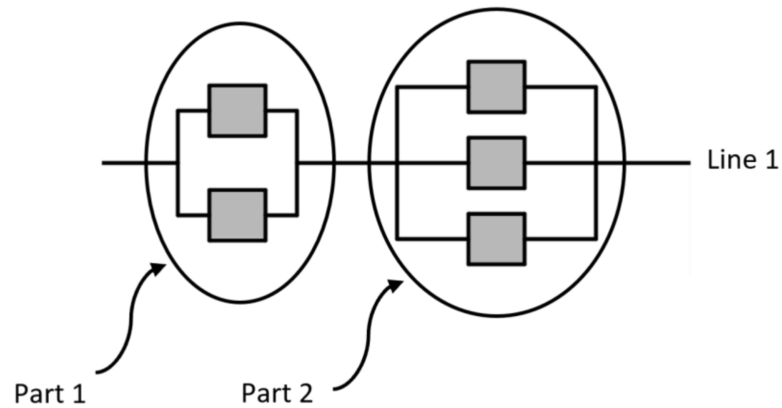
Notice what this is telling us in general.

> If a system (or a part of a system) is composed of n <u>independent</u> lines connected in <u>parallel</u>, then
>
> $$P(\text{System Works}) = 1 - \prod_{i=1}^{n}\big(1 - P(i^{th} \text{ Line Works})\big).$$

So we can always break up a system (or a part of a system) composed on n independent lines connected in parallel into n separate (and much easier) subproblems!

Line 1

Now let's focus on finding $P(\text{Line 1 Works})$. Notice that Line 1 is composed of two "parts".



Notice that both Part 1 and Part 2 have to work if order for Line 1 to work. When separate "parts" all have to work in order for a line to work we say the parts are connected in ***series***.

Parts of the same line are connected in **series** if the line will work if and only if **all parts** on that line work.

Let K_i be the event that Part i on a Line 1 works.

$$P(\text{Line 1 works}) = P(K_1 \text{ and } K_2) = P(K_1 \cap K_2).$$

Now the independence of the switches comes to our rescue again. Notice that each part is composed of separate switches (*i.e.* no switch is in both parts). Therefore, the parts act independently.

That is, the events K_1 and K_2 are independent.

Hence,

$$P(\text{Line 1 works}) = P(K_1 \cap K_2) = P(K_1)P(K_2).$$

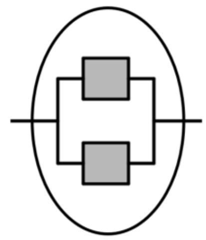
Notice what this is telling us in general.

If a line is composed of n independent parts connected in series, then

$$P(\text{Line Works}) = \prod_{i=1}^{n} P(i^{th} \text{ Part on that Line Works}).$$

So, similar to what we saw for lines connected in parallel, we can always break up a line composed on n independent parts connected in **series** into n separate (and much easier) subproblems!

Let's continue to simplify $P(\text{Line 1 works}) = P(K_1)P(K_2)$. Part 1 of Line 1 consists of two switches connected in parallel.



Hence, the probability that this part works is just

$$P(K_1) = 1 - \prod_{i=1}^{2}\Big(1 - P(i^{th} \text{ Switch of Part 1 Works})\Big)$$

$$P(K_1) = 1 - \big(1 - P(1^{st} \text{ Switch of Part 1 Works})\big)\Big(1 - P(2^{nd} \text{ Switch of Part 1 Works})\Big)$$

Now recall that we are assuming that *every* switch in the entire system works with probability p. Therefore

$$P(K_1) = 1 - (1-p)(1-p) = 1 - (1-p)^2.$$

By the same reasoning,

$$P(K_2) = 1 - (1-p)(1-p)(1-p) = 1 - (1-p)^3.$$

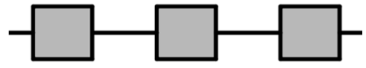
Therefore,

$$P(\text{Line 1 works}) = P(K_1 \cap K_2) = P(K_1)P(K_2)$$
$$= (1-(1-p)^2)(1-(1-p)^3).$$

That's Line 1 finished. Three more to go.

Line 2

The second line

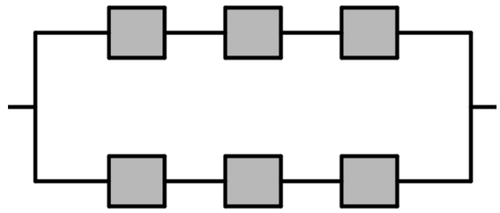


is easy because it is just three (independent) switches connected in series. So

$$P(\text{Line 2 Works}) = \prod_{i=1}^{3} P(i^{th} \text{ switch on that line works}) = p \cdot p \cdot p = p^3.$$

Line 3

The third line



is composed of two (sub)lines connected in **parallel**. These two (sub)lines are independent because they share no switches. Hence, from the general formula for independent lines connected in parallel,

$$P(\text{Line 3 Works}) = 1 - \prod_{i=1}^{2}\left(1 - P(i^{th}\text{ (sub)line of Line 3 Works})\right).$$

We can immediately notice that each subline of Line 3 is exactly what we encountered in Line 2. That is,

$$P(1^{st}\text{ (sub)line of Line 3 Works}) = P(2^{st}\text{ (sub)line of Line 3 Works}) = P(\text{Line 2 Works})$$

which we just determined to equal p^3. Therefore,

$$P(\text{Line 3 Works}) = 1 - \prod_{i=1}^{2}\left(1 - P(i^{th}\text{ (sub)line of Line 3 Works})\right)$$

$$= 1 - \left(1 - P(1^{st}\text{ (sub)line of Line 3 Works})\right)\left(1 - P(2^{nd}\text{ (sub)line of Line 3 Works})\right)$$

$$= 1 - (1 - p^3)(1 - p^3) = 1 - (1 - p^3)^2.$$

Line 4

We see that Line 4 is a repeat of Line 1. Hence

$$P(\text{Line 4 Works}) = P(\text{Line 1 Works}) = (1 - (1 - p)^2)(1 - (1 - p)^3).$$

System

Bringing this all together

$$P(\text{System Works}) = 1 - \prod_{i=1}^{4}\left(1 - P(i^{th}\text{ Line Works})\right)$$

where

$$\boldsymbol{P}(\textbf{Line 1 Works}) = \boldsymbol{P}(\textbf{Line 4 Works})$$
$$= (1-(1-p)^2)(1-(1-p)^3) = (1-(1-0.9)^2)(1-(1-.9)^3)$$
$$= (1-0.1^2)(1-0.1^3) = (0.99)(0.999) = \mathbf{0.98901}$$

$$\boldsymbol{P}(\textbf{Line 2 Works}) = p^3 = (0.9)^3 = \mathbf{0.729}$$

$$\boldsymbol{P}(\textbf{Line 3 Works}) = 1-(1-p^3)^2 = 1-(1-0.729)^2 = \mathbf{0.926559}.$$

Therefore,

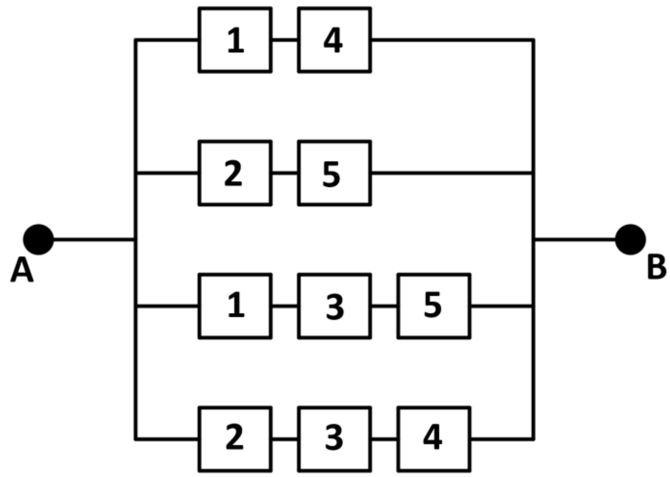
$$P(\text{System Works}) = 1 - \prod_{i=1}^{4}\left(1 - P(i^{th}\text{ Line Works})\right)$$
$$= 1-(1-0.98901)(1-0.729)(1-0.926559)(1-0.98901)$$
$$= 0.9999975961727$$

So, this system is almost sure to work!

■

Example ?? Again

Looking back at Example ?? we might note that we also had four lines connected in parallel in that problem because the system works if and only if *at least one* of the four lines shown in the diagram below works. That is, if and only if all switches work properly along at least one of these four lines.



Now the approach we took in Example ??? showed how we could take advantage of lines connected in parallel to make the problem much easier to handle.

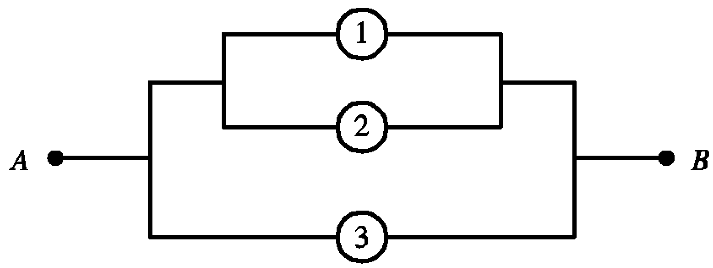
Could we use the approach of Example ??? in Example ??? ? Why or why not?

Exercises for Chapter 7, Section 1

Consider the following portion of an electric circuit with three relays. Current will flow from point A to point B if there is at least one closed path when the relays are activated. The relays may malfunction and not close when activated. Suppose that the relays act independently of one another and close properly when activated, with a probability of 0.9.

a What is the probability that current will flow when the relays are activated?

b Given that current flowed when the relays were activated, what is the probability that relay 1 functioned?



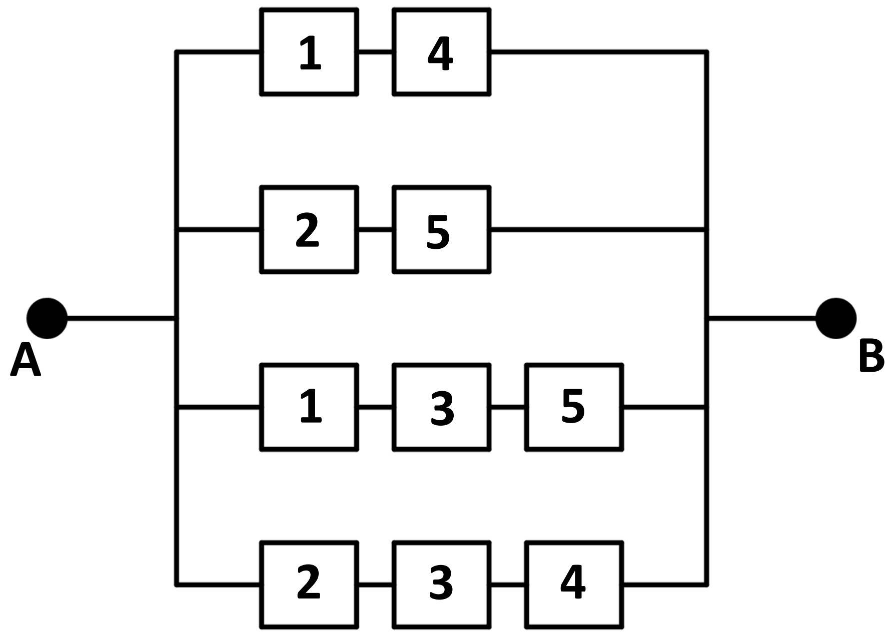
Solution

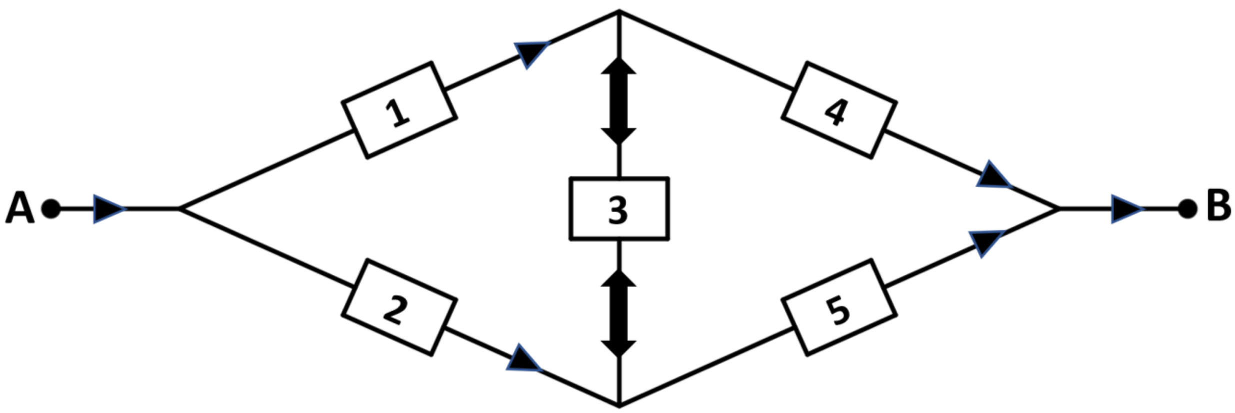
(a) Let C_i be the event that Relay i works properly. Then

$$P(\text{current flows from } A \text{ to } B) = P((C_1 \text{or} C_2) \text{or} C_3)$$
$$= P(C_1 \cup C_2 \cup C_3)$$
$$= 1 - P((C_1 \cup C_2 \cup C_3)')$$
$$= 1 - P((C_1)' \cap (C_2)' \cap (C_3)')$$
$$= 1 - P((C_1)')P((C_2)')P((C_3)')$$
$$= 1 - (0.1)(0.1)(0.1) = 1 - 0.001 = 0.999$$

(b)

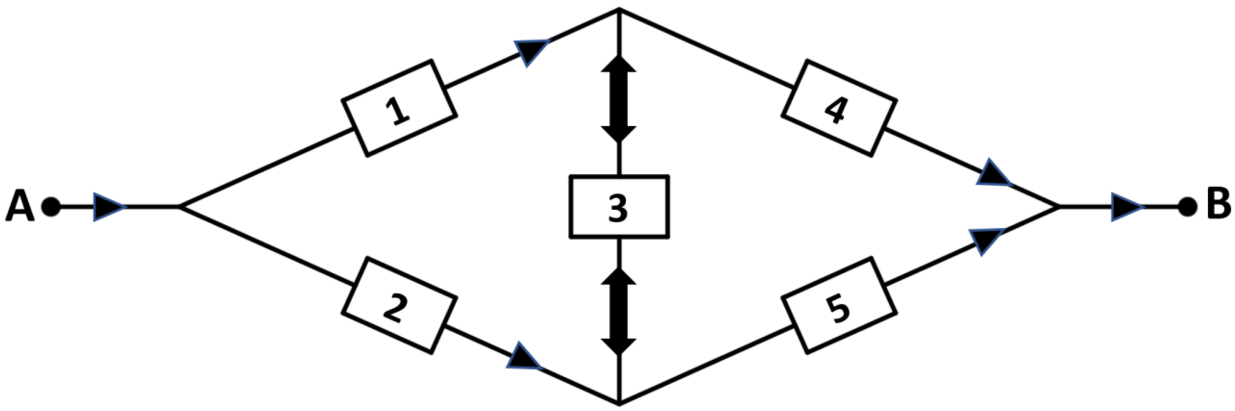
$$P\big(C_1 \big| (C_1 \cup C_2 \cup C_3)\big) = \frac{P\big(C_1 \cap (C_1 \cup C_2 \cup C_3)\big)}{P(C_1 \cup C_2 \cup C_3)}$$
$$= \frac{P(C_1)}{P(C_1 \cup C_2 \cup C_3)} = \frac{0.9}{0.999} = \frac{1000}{1111} = 0.\overline{900}$$


1
4
2
5
A
B
1
3
5
2
3
4


1
4
A
3
B
2
5

Project 7 The Bridge Design

The circuit shown below is called a *bridge*. Let C_j be the event that the j^{th} switch shuts properly when activated, $j = 1,2,3,4,5$. Assume the switches operate independently of each other and let $p = P(C_j)$ for all $j = 1,2,3,4,5$ for some $0 < p < 1$. Find the probability that the current will flow from point A to point B when the switches are activated.



Solution

$$P(\text{Current flows}) = P\big((C_1 \text{ and } C_4) \text{ or } (C_2 \text{ and } C_5) \text{ or } (C_1 \text{ and } C_3 \text{ and } C_5) \text{ or } (C_2 \text{ and } C_3 \text{ and } C_4)\big)$$

$$= P\big((C_1 \cap C_4) \cup (C_2 \cap C_5) \cup (C_1 \cap C_3 \cap C_5) \cup (C_2 \cap C_3 \cap C_4)\big).$$

Can we apply Property (vi)

(vi) If events $E_1, E_2, \dots, E_n$ are mutually exclusive then $P(E_1 \cup E_2 \cup \dots \cup E_n) = P(E_1) + P(E_2) + \dots + P(E_n)$.

and simplify this last expression to

$$P(C_1 \cap C_4) + P(C_2 \cap C_4) + P(C_1 \cap C_3 \cap C_5) + P(C_2 \cap C_3 \cap C_4)\ ?$$

No! The events $(C_1 \cap C_4)$, $(C_2 \cap C_4)$, $(C_1 \cap C_3 \cap C_5)$ and $(C_2 \cap C_3 \cap C_4)$ are **not mutually exclusive**. As just one example, we note that

$$P\big((C_1 \cap C_4) \cap (C_2 \cap C_4)\big) = P(C_1 \cap C_2 \cap C_4 \cap C_4)$$

$$= P\big(C_1 \cap C_2 \cap (C_4 \cap C_4)\big)$$

$$= P(C_1 \cap C_2 \cap C_4) = P(C_1)P(C_2)P(C_4) = p^3 \neq 0.$$

Hence, $(C_1 \cap C_4) \cap (C_2 \cap C_4) \neq \emptyset$ by the contrapositive of Property (ii), $P(\emptyset) = 0$.

Therefore, we have to apply the principle of inclusion-exclusion, the generalized version of Property (vi), which is valid even when the events are *not* mutually exclusive.

$$P(\text{Current flows})$$
$$= P\big((C_1 \cap C_4) \cup (C_2 \cap C_5) \cup (C_1 \cap C_3 \cap C_5) \cup (C_2 \cap C_3 \cap C_4)\big)$$
$$= P(C_1 \cap C_4) + P(C_2 \cap C_5) + P(C_1 \cap C_3 \cap C_5) + P(C_2 \cap C_3 \cap C_4)$$
$$-P\big((C_1 \cap C_4) \cap (C_2 \cap C_5)\big) - P\big((C_1 \cap C_4) \cap (C_1 \cap C_3 \cap C_5)\big)$$
$$-P\big((C_1 \cap C_4) \cap (C_2 \cap C_3 \cap C_4)\big) - P\big((C_2 \cap C_5) \cap (C_1 \cap C_3 \cap C_5)\big)$$
$$-P\big((C_2 \cap C_5) \cap (C_2 \cap C_3 \cap C_4)\big) - P\big((C_1 \cap C_3 \cap C_5) \cap (C_2 \cap C_3 \cap C_4)\big)$$
$$+P\big((C_1 \cap C_4) \cap (C_2 \cap C_5) \cap (C_1 \cap C_3 \cap C_5)\big)$$
$$+P\big((C_1 \cap C_4) \cap (C_2 \cap C_5) \cap (C_2 \cap C_3 \cap C_4)\big)$$
$$+P\big((C_1 \cap C_4) \cap (C_1 \cap C_3 \cap C_5) \cap (C_2 \cap C_3 \cap C_4)\big)$$
$$+P\big((C_2 \cap C_5) \cap (C_1 \cap C_3 \cap C_5) \cap (C_2 \cap C_3 \cap C_4)\big)$$
$$-P\big((C_1 \cap C_4) \cap (C_2 \cap C_5) \cap (C_1 \cap C_3 \cap C_5) \cap (C_2 \cap C_3 \cap C_4)\big)$$

$$= P(C_1 \cap C_4) + P(C_2 \cap C_5) + P(C_1 \cap C_3 \cap C_5) + P(C_2 \cap C_3 \cap C_4)$$
$$-P(C_1 \cap C_2 \cap C_4 \cap C_5) - P(C_1 \cap C_3 \cap C_4 \cap C_5)$$
$$-P(C_1 \cap C_2 \cap C_3 \cap C_4) - P(C_1 \cap C_2 \cap C_3 \cap C_5)$$
$$-P(C_2 \cap C_3 \cap C_4 \cap C_5) - P(C_1 \cap C_2 \cap C_3 \cap C_4 \cap C_5)$$
$$+P(C_1 \cap C_2 \cap C_3 \cap C_4 \cap C_5)$$
$$+P(C_1 \cap C_2 \cap C_3 \cap C_4 \cap C_5)$$
$$+P(C_1 \cap C_2 \cap C_3 \cap C_4 \cap C_5)$$
$$+P(C_1 \cap C_2 \cap C_3 \cap C_4 \cap C_5)$$
$$-P(C_1 \cap C_2 \cap C_3 \cap C_4 \cap C_5)$$

$$= P(C_1)P(C_4) + P(C_2)P(C_5) + P(C_1)P(C_3)P(C_5) + P(C_2)P(C_3)P(C_4)$$

$$-P(C_1)P(C_2)P(C_4)P(C_5) - P(C_1)P(C_3)P(C_4)P(C_5)$$

$$-P(C_1)P(C_2)P(C_3)P(C_4) - P(C_1)P(C_2)P(C_3)P(C_5)$$

$$-P(C_2)P(C_3)P(C_4)P(C_5) - P(C_1)P(C_2)P(C_3)P(C_4)P(C_5)$$

$$+P(C_1)P(C_2)P(C_3)P(C_4)P(C_5)$$

$$+P(C_1)P(C_2)P(C_3)P(C_4)P(C_5)$$

$$+P(C_1)P(C_2)P(C_3)P(C_4)P(C_5)$$

$$+P(C_1)P(C_2)P(C_3)P(C_4)P(C_5)$$

$$-P(C_1)P(C_2)P(C_3)P(C_4)P(C_5)$$

$$= 2p^2 + 2p^3 - 5p^4 + 2p^5.$$

■

Note: Make sure not to gloss over how we came to the result

$$P\big((C_1 \cap C_4) \cap (C_2 \cap C_5) \cap (C_1 \cap C_3 \cap C_5) \cap (C_2 \cap C_3 \cap C_4)\big) = p^5,$$

for example. The details of the simplification show that

$$P\big((C_1 \cap C_4) \cap (C_2 \cap C_5) \cap (C_1 \cap C_3 \cap C_5) \cap (C_2 \cap C_3 \cap C_4)\big)$$

$$= P\big((C_1 \cap C_1) \cap (C_2 \cap C_2) \cap (C_3 \cap C_3) \cap (C_4 \cap C_4) \cap (C_5 \cap C_5)\big)$$

$$= P(C_1 \cap C_2 \cap C_3 \cap C_4 \cap C_5)$$

$$= P(C_1)P(C_2)P(C_3)P(C_4)P(C_5) = p^5.$$

7.2. Geometric Series in Probability

Geometric Series

$$p^0 + p^1 + p^2 + \cdots = \frac{1}{1-p} \text{ provided } |p| < 1.$$

Remember that for $|p| < 1$,

$$\sum_{x=a}^{b} p^x = \frac{p^a - p^{b+1}}{1-p} \quad \text{and} \quad \sum_{x=a}^{\infty} p^x = \frac{p^a}{1-p}.$$

But we can simplify this sum. It is just a geometric series. Remember that for $|q| < 1$,

$$\sum_{x=a}^{b} q^x = \frac{q^a - q^{b+1}}{1-q} \quad \text{and} \quad \sum_{x=a}^{\infty} q^x = \frac{q^a}{1-q}.$$

Why? Let $S = q^a + q^{a+1} + q^{a+2} + \cdots + q^b$. Then multiplying both sides by q gives

$$S \cdot q = q(q^a + q^{a+1} + \cdots + q^b) = q^{a+1} + q^{a+2} + \cdots + q^{b+1} .$$

Subtracting Sq from S we get

$$\begin{aligned} S - S \cdot q &= (q^a + q^{a+1} + \cdots + q^b) - (q^{a+1} + q^{a+2} + \cdots + q^{b+1}) \\ &= q^a - q^{b+1}. \end{aligned}$$

So,

$$q^a - q^{b+1} = S - Sq = S(1-q) \Longrightarrow S = \frac{q^q - q^{b+1}}{1-q}.$$

Note the following general result about the geometric random variable (*i.e.* negative binomial with $r = 1$). If $X \sim$geometric(p), then $P(X > a) = (1-p)^a$. Why?

$$P(X > a) = \sum_{x=a+1}^{\infty} \binom{x-1}{1-1} p^1 (1-p)^{x-1} = p \sum_{x=a}^{\infty} (1-p)^x$$

$$= p\left(\frac{(1-p)^a - (1-p)^\infty}{p}\right) = (1-p)^a.$$

Exercises for Chapter 7, Section 2

1. (TI1711) Allie and Bubs play a game. Allie starts and flips a coin twice. If she gets two heads she wins. If not, then Bubs flips a coin twice. If he gets two heads, he wins. If Bubs fails to get two heads, then it's Allie's turn again. The game continues in this fashion; the first person to get two heads on their turn wins. What is the probability Allie wins?

2. Suppose you know from experience that 1% of the parts coming off an assembly line at a local manufacturing plant are defective.

   (i) What is the probability that a lot of 500 will have less than 3 defectives in it?

   (ii) Suppose you cannot tell by just looking whether a part is defective but rather have to subject the part to a test in order to tell. What is the probability that you will have to test more than 150 parts before you find a defective one?

3. Adam rolls a well-balanced die until he gets a 6. Amy rolls the same die until she rolls an odd number. What is the probability that Adam rolls the die more times than Amy does?

4. A and B are involved in a duel. The rules of the duel are that they get to pick up their guns and shoot at each other simultaneously. If one or both are hit, then the duel is over. If both shots miss, then they repeat the process. Suppose that the results of the shots are independent and that each shot of A will hit B with probability p_A, and each shot of B will hit A with probability p_B. What is

   (a) the probability that A is not hit?

   (b) the probability that both duelists are hit?

   (c) the probability that the duel ends after the n^{th} round of shots?

   (d) the conditional probability that the duel ends after the n^{th} round of shots given that A is not hit?

   (e) the conditional probability that the duel ends after the n^{th} round of shots given that both duelists are hit?

5. Amelia has a coin that lands heads with probability $1/3$, and Blaine has a coin that lands on heads with probability $2/5$. Amelia and Blaine alternately toss their coins until someone gets a head; the first one to get a head wins. All coin tosses are independent. Amelia goes first. The probability that Amelia wins is p/q, where p and q are relatively prime positive integers. What is $q - p$? (Source: AMC 10a Problem 18 2017)

6. A fair die is rolled until a 6 appears.

   (a) What is the probability that it will take exactly 6 tries?

(b) What is the probability that it will take at least 6 tries?

(c) Is this experiment more likely to end on an even numbered trial or an odd numbered trial?

(d) If the first roll is not a six, what is the probability it will take at least 3 tries to get a 6?

Project 8 First Player Advantage

3. **First player advantage.** Suppose two players alternate tossing a coin. The player who gets the first head wins. Obviously, the player who gets to toss first has an advantage in this game. How much of an advantage does this player have? In other words, what is the probability that the player who tosses first wins the game, assuming the coin is a fair coin that comes up heads and tails with probability 1/2 each?

Solution: *The first player wins if and only if the sequence starts with H, TTH, TTTTH, etc. The probability for this is*

$$\frac{1}{2}+\left(\frac{1}{2}\right)^3+\left(\frac{1}{2}\right)^5+\cdots=\frac{1/2}{1-1/4}=\boxed{2/3}.$$

Let p equal the probability that the first player wins

$$p=\frac{1}{2}+\frac{1}{4}p$$

$$p\left(1-\frac{1}{4}\right)=\frac{1}{2}$$

$$p=\frac{\frac{1}{2}}{\frac{3}{4}}=\frac{2}{3}$$

Three players, A, B, and C, take turns throwing a single die, A leads. As soon as a player tosses a one, that player drops out of the game and the remaining players continue rolling the die until everyone has rolled a one. What is the probability that A tosses the first one, B tosses the second one, and C tosses the third one?

Let p equal the probability that A wins.

$$p=\frac{1}{6}+\left(\frac{5}{6}\right)^3 p$$

$\frac{216}{1001}$.

$$p\left(1-\left(\frac{5}{6}\right)^3\right)=\frac{1}{6}$$

$$p=\frac{\frac{1}{6}}{\left(1-\left(\frac{5}{6}\right)^3\right)}=\frac{36}{91}$$

Now let p equal the probability that B is the next winner

$$p=\frac{1}{6}+\left(\frac{5}{6}\right)^2 p$$

$$p\left(1-\left(\frac{5}{6}\right)^2\right)=\frac{1}{6}$$

$$p=\frac{\frac{1}{6}}{\left(1-\left(\frac{5}{6}\right)^2\right)}=\frac{6}{11}$$

$$\left(\frac{36}{91}\right)\left(\frac{6}{11}\right)=\frac{216}{1001}$$

Geometric Series and a Probability Problem
Author(s): Curtis Cooper
Source: *The American Mathematical Monthly*, Vol. 93, No. 2 (Feb., 1986), pp. 126-127

19. Adam rolls a well-balanced die until he gets a 6. Amy rolls the same die until she rolls an odd number. What is the probability that Adam rolls the die more times than Amy does?

3. **First player advantage.** Suppose two players alternate tossing a coin. The player who gets the first head wins. Obviously, the player who gets to toss first has an advantage in this game. How much of an advantage does this player have? In other words, what is the probability that the player who tosses first wins the game, assuming the coin is a fair coin that comes up heads and tails with probability $1/2$ each?

AMC 10a Problem 18 2017

Amelia has a coin that lands heads with probability $1/3$, and Blaine has a coin that lands on heads with probability $2/5$. Amelia and Blaine alternately toss their coins until someone gets a head; the first one to get a head wins. All coin tosses are independent. Amelia goes first. The probability that Amelia wins is p/q, where p and q are relatively prime positive integers. What is $q - p$?

Whiskey, Marbles, and Potholes, Author(s): J. Chris Fisher and Denis Hanson

Problem A. *Two players take turns drawing a single marble from an urn containing one red marble and four that are drab. They continue to remove marbles until one player wins by drawing the red marble. For the game to be fair, how much should player I (who picks first) pay when player II puts up* \$2? (Answer: \$3)

Problem B. *Player II makes an offer to his opponent: For* 50¢ *he will increase the probability of an immediate victory by removing one of the drab marbles before the game begins (leaving one red and three drab). Would player I be well advised to accept the deal*? (Answer: *No*; a *yes*-response shows aptitude for Hollywood script writing.)

Main Problem. *An urn contains r red and d drab marbles. Two players take turns drawing a marble from the urn; what is the probability that player I (the player drawing first) is the first to draw a red marble*?

Another way to obtain a formula for $P_1(d, r)$ makes use of the observation that the probability of choosing the first red marble on the kth round has a negative *hypergeometric distribution*; that is

Alternative Problem. *One spring day the crews are at work repairing potholes in all odd-numbered avenues on the first block* (between Albert and 1st Street). *A stranger to town wants to drive from the corner of Victoria and Albert to the corner of dth avenue and rth street* (i.e. making d moves down the map, and r moves to the right). *What is the probability that, by chance alone, he chooses a route that avoids delays caused by the work crew*?

consequently, the probability of arriving at (d, r) without having encountered any road work is

$$P(d, r) = \frac{s+u}{t+v}.$$

This so-called *mediant of* s/t *and* u/v, long used by students to drive up the blood pressure of mathematics teachers around the world, is also a fundamental operation in the study of Farey series [**2**, Chapter 3].

The alternative problem above is, of course, a disguised version of the main problem. An easy way to see the equivalence is to note that each path to (d, r) corresponds to a sequence of d Ds and r Rs. Of the $\binom{r+d}{r}$ paths to (d, r), those that avoid road work correspond to sequences whose first R appears after an even number (possibly zero) of Ds. Evidently, choosing a D-R sequence at random is equivalent to continuing to play the game described in the main problem (after the winner has been decided) until the urn is empty.

Equalizing a Two-Person Alternation Game

Two players A and B alternately roll a die, with the stipulation that the first player to roll a "4" (or any fixed number) loses. The game is patently disadvantageous to the first player, since the sequence $ABABAB\ldots$ gives the probability $.\overline{54}$ for A to lose. The sequence $ABBAABBAA\ldots$ still yields probability $.508\ldots$ for A to lose.

Is there a sequence for which both players are equally likely to win?

If infinite sequences are permitted, the answer is "yes." To wit, consider the infinite series

$$\frac{1}{6} + \left(\frac{5}{6}\right)\left(\frac{1}{6}\right) + \left(\frac{5}{6}\right)^2\left(\frac{1}{6}\right) + \left(\frac{5}{6}\right)^3\left(\frac{1}{6}\right) + \cdots + \left(\frac{5}{6}\right)^{n-1}\left(\frac{1}{6}\right) + \cdots \qquad (1)$$

that converges to 1. The nth term is the probability of the game ending precisely on the nth turn.

Paper by Thorpe

A before B before C … problem and solution

137 Three players A, B and C agree to play a series of games with a biased coin which has a probability p $(0<p<1)$ for a showing a head. In each game, each of the three players tosses the coin once, a success being recorded if the toss leads to a head. The series is won by the first player to obtain a success in a game in which no other player obtains a success. If just two players obtain successes in the same game they continue to play without the other player who is deemed to have lost the series. If all three players obtain successes in the same game they all continue to play.

Determine the probabilities of the possible outcomes of the first game played, from the point of view of a particular player (say A). Hence deduce that u_n, the probability that A wins the series on the nth game, satisfies the difference equation

$$u_n=(1-3pq)u_{n-1}+2p^3q^2(1-2pq)^{n-2}, \quad \text{for } n\geqslant 2, \quad q=1-p.$$

Finally, if $\phi(z)=\sum_{r=1}^{\infty} u_r z^r$ is the generating function for u_n, verify that

$$\phi(z)=\frac{pq^2z[1-(1-2p)z]}{[1-(1-2pq)z][1-(1-3pq)z]},$$

so that $\phi(1)=\frac{1}{3}$. Explain very briefly the significance of this value of $\phi(1)$.

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Example 22. Russian Roulette

Russian roulette is played with a revolver equipped with a rotatable magazine of six shots. The revolver is loaded with one shot. The first duellist, A, rotates the magazine at random, points the revolver at his head and presses the trigger. If, afterwards, he is still alive, he hands the revolver to the other duellist, B, who acts in the same way as A. The players shoot alternately in this manner, until a shot goes off. Determine the probability that A is killed.

Let H_i be the event that "a shot goes off at the ith trial". The events H_i are mutually exclusive. The event H_i occurs if there are $i-1$ "failures" and then one "success". Hence we get (see Example 17)

$$P(H_i)=\left(\frac{5}{6}\right)^{i-1}\frac{1}{6}.$$

The probability we want is given by

$$P = P(H_1 \cup H_3 \cup H_5 \cup \cdots) = \frac{1}{6}\left[1 + \left(\frac{5}{6}\right)^2 + \left(\frac{5}{6}\right)^4 + \cdots\right]$$

$$= \frac{1}{6} \cdot \frac{1}{1 - (\frac{5}{6})^2} = \frac{6}{11}.$$

The problem can also be solved in another surprisingly simple way. Let T be the event that "A is killed on the first trial". We have

$$P(A \text{ is killed}) = P(T) + P(T^*)P(A \text{ is killed}|T^*).$$

But if A survives the first trial, the roles of A and B are interchanged and so $P(A \text{ is killed}|T^*) = P(B \text{ is killed}) = 1 - P(A \text{ is killed})$. Inserting this above, we find

$$P(A \text{ is killed}) = \frac{1}{6} + \frac{5}{6}[1 - P(A \text{ is killed})].$$

Solving this equation we find $P(A \text{ is killed}) = 6/11$. □

Box K Finishes Last

Suppose we repeatedly distribute identical balls into m distinguishable boxes. Let p_j equal the probability that a ball is distributed into Box j on any given trial. Assume all trials are independent.

(I) Find $P(\text{Each of boxes } 1,2,\dots,K-1 \text{ receives a ball before Box } K \text{ does})$.

Note that in this problem we are not excluding the possibility that some or all of Boxes $K + 1, \dots, m$ receive a ball before Box K does.

Solution (I)

Let $A_{i,K}$ be the event that Box i receives a ball before Box K does and let B_j be the event that the first ball distributed lands in Box j. Then,

$$P(\text{Each of Boxes } 1,2,\dots,K-1 \text{ receives a ball before Box } K \text{ does})$$

$$= P\left(A_{1,K} \cap \cdots \cap A_{K-1,K}\right) = \sum_{j=1}^{m} P\left(A_{1,K} \cap \cdots \cap A_{K-1,K} \cap B_j \middle| B_j\right) p_j$$

$$= \sum_{j=1}^{K-1} P\left(A_{1,K} \cap \cdots \cap A_{K-1,K} \cap B_j \middle| B_j\right) p_j + P\left(A_{1,K} \cap \cdots \cap A_{K-1,K} \cap B_K \middle| B_K\right) p_K + \sum_{j=K+1}^{m} P\left(A_{1,K} \cap \cdots \cap A_{K-1,K} \cap B_j \middle| B_j\right) p_j.$$

However,

$$P\left(\left(\bigcap_{i=1}^{K-1} A_{i,K}\right) \cap B_j \middle| B_j\right) = \begin{cases} P\left(\bigcap_{\substack{i=1 \\ i \neq j}}^{K-1} A_{i,K}\right) & j \leq K-1 \\ 0 & j = K \\ P\left(\bigcap_{i=1}^{K-1} A_{i,K}\right) & j \geq K+1 \end{cases}$$

where we define $P\left(\bigcap_{\substack{i=1 \\ i \neq 1}}^{K-1} A_{i,2}\right) = 1$. Thus,

$$P\left(A_{1,K} \cap \cdots \cap A_{K-1,K}\right)$$

$$= \sum_{j=1}^{K-1} P\left(\bigcap_{\substack{i=1 \\ i \neq j}}^{K-1} A_{i,K}\right) p_j + 0 + P\left(\bigcap_{i=1}^{K-1} A_{i,K}\right) \cdot p^\star$$

where $p^\star = \sum_{j=K+1}^{m} p_j$.

Solving for $P\left(A_{1,K} \cap \cdots \cap A_{K-1,K}\right)$, we see we have the recurrence relation

$$P\left(\bigcap_{i=1}^{K-1} A_{i,K}\right) = \frac{1}{(p_1 + p_2 + \cdots + p_K)}\left[\sum_{j=1}^{K-1} P\left(\bigcap_{\substack{i=1 \\ i\neq j}}^{K-1} A_{i,K}\right) p_j\right].$$

The simplest way to "solve" this recurrence is to start with the simplest cases and "guess" the general solution and then verify this guess by induction.

Case $K = 2$

$$P(\text{Box 1 receives a ball before Box 2})$$

$$= P\left(A_{1,2}\right) = \frac{1}{(p_1 + p_2)} P\left(\bigcap_{\substack{i=1 \\ i\neq 1}}^{2-1} A_{i,2}\right) p_1 = \frac{p_1}{p_1 + p_2}.$$

Case $K = 3$

$$P(\text{Box 1 and Box 2 receive a ball before Box 3})$$

$$= P\left(A_{1,3} \cap A_{2,3}\right) = \frac{1}{(p_1 + p_2 + p_3)}\left[\sum_{j=1}^{3-1} P\left(\bigcap_{\substack{i=1 \\ i\neq j}}^{3-1} A_{i,3}\right) p_j\right]$$

$$= \frac{1}{(p_1 + p_2 + p_3)}\left(P\left(A_{2,3}\right)p_1 + P\left(A_{1,3}\right)p_2\right)$$

$$= \frac{1}{(p_1 + p_2 + p_3)}\left(\frac{p_2}{p_2 + p_3} p_1 + \frac{p_1}{p_1 + p_3} p_2\right)$$

$$= \left(\frac{p_1 p_2}{p_1 + p_2 + p_3}\right)\left(\frac{1}{p_2 + p_3} + \frac{1}{p_1 + p_3}\right).$$

It is still difficult to see the general pattern so we continue on the case $K = 4$.

$\boxed{\text{Case } K = 4}$

$$P(\text{Box 1 and Box 2 and Box 3 receive a ball before Box 4})$$

$$= P\left(A_{1,4} \cap A_{2,4} \cap A_{3,4}\right) = \frac{1}{(p_1 + p_2 + p_3 + p_4)}\left[\sum_{j=1}^{4-1} P\left(\bigcap_{\substack{i=1 \\ i \neq j}}^{4-1} A_{i,4}\right) p_j\right]$$

$$= \left(\frac{p_1}{p_1 + p_2 + p_3 + p_4}\right) P\left(A_{2,4} \cap A_{3,4}\right)$$

$$+ \left(\frac{p_2}{p_1 + p_2 + p_3 + p_4}\right) P\left(A_{1,4} \cap A_{3,4}\right)$$

$$+ \left(\frac{p_3}{p_1 + p_2 + p_3 + p_4}\right) P\left(A_{1,4} \cap A_{2,4}\right)$$

$$= \left(\frac{p_1}{p_1 + p_2 + p_3 + p_4}\right)\left[\left(\frac{p_2 p_3}{p_2 + p_3 + p_4}\right)\left(\frac{1}{p_3 + p_4} + \frac{1}{p_2 + p_4}\right)\right]$$

$$+ \left(\frac{p_2}{p_1 + p_2 + p_3 + p_4}\right)\left[\left(\frac{p_1 p_3}{p_1 + p_3 + p_4}\right)\left(\frac{1}{p_3 + p_4} + \frac{1}{p_1 + p_4}\right)\right]$$

$$+ \left(\frac{p_3}{p_1 + p_2 + p_3 + p_4}\right)\left[\left(\frac{p_1 p_2}{p_1 + p_2 + p_4}\right)\left(\frac{1}{p_2 + p_4} + \frac{1}{p_1 + p_4}\right)\right]$$

$$= \sum_{(j_1, j_2, j_3) \in \mathbb{P}_3} \frac{p_1 p_2 p_3}{\left(p_{j_1} + p_4\right)\left(p_{j_1} + p_{j_2} + p_4\right)\left(p_{j_1} + p_{j_2} + p_{j_3} + p_4\right)}$$

where $\mathbb{P}_3$ is the set of all permutations of all the numbers $\{1,2,3\}$. In the general case, (which we leave to the reader to verify by induction), we have

$$P(\text{Each of Boxes } 1,2,\dots,K-1 \text{ receives a ball before Box } K \text{ does})$$

$$= \sum_{(j_1, \dots, j_{K-1}) \in \mathbb{P}_{K-1}} \frac{p_1 p_2 p_3 \cdots p_{K-1}}{\left(p_{j_1} + p_K\right)\left(p_{j_1} + p_{j_2} + p_K\right) \cdots \left(p_{j_1} + p_{j_2} + \cdots + p_{j_{K-1}} + p_K\right)}$$

where $\mathbb{P}_{K-1}$ is the set of all permutations of all the numbers $\{1,2,\dots,K-1\}$.

A solution equivalent to that in (I) is derived in "Repeated Independent Trials and a Class of Dice Problems", September 1964, American Mathematical Monthly, Edward Thorp, 778 - 781. Thorp's uses the General Probability Theorem in his derivation. In that article, Thorp gives a conjectured upper and lower bound for

$$P(\text{Each of Boxes } 1,2,\dots,K-1 \text{ receives a ball before Box } K \text{ does}).$$

These conjectures were proven correct by C. W. Burrill, September 1966, American Mathematical Monthly, 738 - 741.

89.91 A generalised coin-tossing problem

The following problem was considered in [1]: 'A fair coin is tossed successively until either two heads occur in a row or three tails occur in a row. What is the probability that the sequence ends with two heads?' We are interested in the generalisation of this to an arbitrary number of both heads and tails in the above. Specifically, we consider tossing a fair coin until either n heads or m tails appear in a row for $n, m \in \mathbb{N}$. Then $p(n, m)$ is the probability that the sequence ends with n heads. Firstly, note that $p(n, m) = 1 - p(m, n)$ for $n, m \in \mathbb{N}$ and so also $p(n, n) = \frac{1}{2}$. Thus, the original problem considered in [1] is to calculate $p(2,3)$. The value $p(2,3) = \frac{7}{10}$ was calculated in [1] and we shall use the methods developed there in the more general problem. Perhaps surprisingly, $p(n, m)$ has the following rather simple form:

Theorem 1: Let $n, m \in \mathbb{N}$. Then

$$p(n, m) = \frac{2^m - 1}{2^m + 2^n - 2}.$$

19. Adam rolls a well-balanced die until he gets a 6. Amy rolls the same die until she rolls an odd number. What is the probability that Adam rolls the die more times than Amy does?

Solution

Let X be the number of rolls it takes Adam to get a 6. Let Y be the number of rolls it takes Amy to get an odd number.

Then $X \sim$Negative Binomial$(r = 1, p = 1/6)$ and $Y \sim$Negative Binomial$(r = 1, p = 1/2)$

Also, we know that X and Y are independent random variables.

The problem is asking for $P(X > Y)$.

$$P(X > Y) = P\big((X, Y) \in \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3), \dots\}\big)$$

$$= \sum_{x=2}^{\infty} \sum_{y=1}^{x-1} P(X = x, Y = y)$$

$$= \sum_{x=2}^{\infty} \sum_{y=1}^{x-1} P(X = x) P(Y = y)$$

$$= \sum_{x=2}^{\infty} \sum_{y=1}^{x-1} \binom{x-1}{1-1} \left(\frac{1}{6}\right)^1 \left(\frac{5}{6}\right)^{x-1} \binom{y-1}{1-1} \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^{y-1}$$

$$= \left(\frac{1}{6}\right)^1 \left(\frac{1}{2}\right)^1 \sum_{x=2}^{\infty} \sum_{y=1}^{x-1} \left(\frac{5}{6}\right)^{x-1} \left(\frac{1}{2}\right)^{y-1}$$

$$= \left(\frac{1}{6}\right)^1 \left(\frac{1}{2}\right)^1 \sum_{x=2}^{\infty} \left(\frac{5}{6}\right)^{x-1} \left(\sum_{y=1}^{x-1} \left(\frac{1}{2}\right)^{y-1} \right).$$

Let $S_x = \sum_{y=1}^{x-1} \left(\frac{1}{2}\right)^{y-1}$

$$S_x = \left(\frac{1}{2}\right)^0 + \left(\frac{1}{2}\right)^1 + \left(\frac{1}{2}\right)^2 + \cdots + \left(\frac{1}{2}\right)^{(x-1)-1}$$

$$S_x \left(\frac{1}{2}\right)^1 = \left(\left(\frac{1}{2}\right)^0 + \left(\frac{1}{2}\right)^1 + \left(\frac{1}{2}\right)^2 + \cdots + \left(\frac{1}{2}\right)^{(x-1)-1}\right)\left(\frac{1}{2}\right)^1$$

$$= \left(\frac{1}{2}\right)^1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots + \left(\frac{1}{2}\right)^{x-1}$$

$$S_x - S_x \left(\frac{1}{2}\right)^1$$

$$= \left(\left(\frac{1}{2}\right)^0 + \left(\frac{1}{2}\right)^1 + \cdots + \left(\frac{1}{2}\right)^{x-2}\right) - \left(\left(\frac{1}{2}\right)^1 + \left(\frac{1}{2}\right)^2 + \cdots + \left(\frac{1}{2}\right)^{x-1}\right)$$

$$= \left(\frac{1}{2}\right)^0 - \left(\frac{1}{2}\right)^{x-1}$$

but

$$S_x - S_x \left(\frac{1}{2}\right)^1 = S_x \left(1 - \left(\frac{1}{2}\right)^1\right)$$

So

$$S_x - S_x \left(\frac{1}{2}\right)^1 = S_x \left(1 - \left(\frac{1}{2}\right)^1\right) = \left(\frac{1}{2}\right)^0 - \left(\frac{1}{2}\right)^{x-1}$$

$$S_x = \frac{\left(\frac{1}{2}\right)^0 - \left(\frac{1}{2}\right)^{x-1}}{1 - \left(\frac{1}{2}\right)^1} = 2\left(1 - \left(\frac{1}{2}\right)^{x-1}\right)$$

So,

$$P(X > Y) = \left(\frac{1}{6}\right)^1 \left(\frac{1}{2}\right)^1 \sum_{x=2}^{\infty} \left(\frac{5}{6}\right)^{x-1} \cdot S_x = \left(\frac{1}{6}\right)^1 \left(\frac{1}{2}\right)^1 \sum_{x=2}^{\infty} \left(\frac{5}{6}\right)^{x-1} \cdot \left(2\left(1 - \left(\frac{1}{2}\right)^{x-1}\right)\right)$$

$$= \left(\frac{1}{6}\right)^1 \left(\frac{1}{2}\right)^1 2\left(\sum_{x=2}^{\infty} \left(\frac{5}{6}\right)^{x-1} - \sum_{x=2}^{\infty} \left(\frac{5}{6} \cdot \frac{1}{2}\right)^{x-1}\right)$$

Let

$$S_n = p^1 + p^2 + p^3 + \cdots + p^{n-1}$$

Then,

$$S = \sum_{x=2}^{\infty} p^{x-1} = p^1 + p^2 + p^3 + \cdots = \lim_{n \to \infty} S_n$$

$$S_n = p^1 + p^2 + p^3 + \cdots + p^{n-1}$$

$$S_n \cdot p^1 = (p^1 + p^2 + p^3 + \cdots + p^{n-1}) \cdot p^1$$

$$= p^2 + p^3 + p^4 + \cdots + p^n$$

$$S_n - S_n \cdot p^1$$

$$= (p^1 + p^2 + p^3 + \cdots + p^{n-1}) - (p^2 + p^3 + p^4 + \cdots + p^n)$$

$$= 1 - p^n$$

but

$$S_n - S_n \cdot p^1 = S_n(1 - p^1)$$

So

$$S_n - S_n \cdot p^1 = S_n(1 - p^1) = p^1 - p^n$$

$$S_n = \frac{p^2 - p^n}{1 - p^1}$$

$$S = \sum_{x=2}^{\infty} p^{x-1} = p^1 + p^2 + p^3 + \cdots = \lim_{n\to\infty} S_n$$

$$= \lim_{n\to\infty} S_n$$

$$= \lim_{n\to\infty} \left(\frac{p^1 - p^n}{1 - p^1}\right)$$

$$= \frac{p^1 - p^{\infty}}{1 - p^1} = \frac{p^1 - 0}{1 - p^1} \quad \text{(because } 0 < p < 1\text{)}$$

$$= \frac{p^1}{1 - p}$$

So,

$$P(X > Y) = \left(\frac{1}{6}\right)^1 \left(\frac{1}{2}\right)^1 2\left(\sum_{x=2}^{\infty}\left(\frac{5}{6}\right)^{x-1} - \sum_{x=2}^{\infty}\left(\frac{5}{6}\cdot\frac{1}{2}\right)^{x-1}\right)$$

$$= \left(\frac{1}{6}\right)^1 \left(\frac{1}{2}\right)^1 2\left(\frac{\frac{5}{6}}{1-\left(\frac{5}{6}\right)} - \frac{\frac{5}{12}}{1-\left(\frac{5}{12}\right)}\right)$$

$$= \left(\frac{1}{6}\right)\left(\frac{\frac{5}{6}}{\frac{1}{6}} - \frac{\frac{5}{12}}{\frac{7}{12}}\right)$$

$$= \left(\frac{1}{6}\right)\left(5 - \frac{5}{7}\right)$$

$$= \left(\frac{5}{6}\right)\left(1 - \frac{1}{7}\right)$$

$$= \frac{5}{7}$$

■

(63) A and B are involved in a duel. The rules of the duel are that they get to pick up their guns and shoot at each other simultaneously. If one or both are hit, then the duel is over. If both shots miss, then they repeat the process. Suppose that the results of the shots are independent and that each shot of A will hit B with probability p_A, and each shot of B will hit A with probability p_B. What is

(a) the probability that A is not hit?

(b) the probability that both duelists are hit?

(c) the probability that the duel ends after the n^{th} round of shots?

(d) the conditional probability that the duel ends after the n^{th} round of shots given that A is not hit?

(e) the conditional probability that the duel ends after the n^{th} round of shots given that both duelists are hit?

Solution

Let $p_A = P(A \text{ hits } B \text{ on any given shot})$ and let $p_B = P(B \text{ hits } A \text{ on any given shot})$.

(a) $P(A \text{ not hit})$

$$= \sum_{j=1}^{\infty} P\Big(A \text{ and } B \text{ both miss on their first } j-1 \text{ shots}$$

$$\text{AND } A \text{ hits on } j^{th} \text{ shot but } B \text{ misses on } j^{th} \text{ shot}\Big)$$

$$= \sum_{j=1}^{\infty} P\Big(A \text{ and } B \text{ both miss on their first } j-1 \text{ shots}\Big)$$

$$\cdot \; P\Big(A \text{ hits on } j^{th} \text{ shot but } B \text{ misses on } j^{th} \text{ shot}\Big)$$

$$= \sum_{j=1}^{\infty} (1-p_A)^{j-1}(1-p_B)^{j-1} \cdot p_A \cdot (1-p_B)$$

$$= p_A \cdot (1 - p_B) \cdot \left(\sum_{j=1}^{\infty} (1 - p_A)^{j-1} (1 - p_B)^{j-1} \right)$$

$$= p_A \cdot (1 - p_B) \cdot \left(\sum_{j=1}^{\infty} \big((1 - p_A)(1 - p_B) \big)^{j-1} \right)$$

$$= p_A \cdot (1 - p_B) \cdot \left(\sum_{k=0}^{\infty} \big((1 - p_A)(1 - p_B) \big)^{k} \right)$$

(Recall our in class discussion about "change of variable".

Here I let $k = j - 1.$)

$$= p_A \cdot (1 - p_B) \left(\frac{1}{1 - (1 - p_A)(1 - p_B)} \right)$$

(Recall our in class discussion on geometric series.)

$$= \frac{p_A (1 - p_B)}{1 - (1 - p_A)(1 - p_B)}$$

(b) $P(A \text{ and } B \text{ both hit})$

$$= \sum_{j=1}^{\infty} P\Big(A \text{ and } B \text{ both miss on their first } j - 1 \text{ shots}$$

$$\text{AND } A \text{ hits on } j^{th} \text{ shot and } B \text{ hit on } j^{th} \text{ shot}\Bigg)$$

$$= \sum_{j=1}^{\infty} P\left(A \text{ and } B \text{ both miss on their first } j-1 \text{ shots}\right)$$

$$\cdot \quad P\left(A \text{ hits on } j^{th} \text{ shot and } B \text{ hit on } j^{th} \text{ shot}\right)$$

$$= \sum_{j=1}^{\infty} (1-p_A)^{j-1}(1-p_B)^{j-1} \cdot p_A \cdot p_B$$

$$= p_A \cdot p_B \cdot \left(\sum_{j=1}^{\infty} (1-p_A)^{j-1}(1-p_B)^{j-1}\right)$$

$$= p_A \cdot p_B \cdot \left(\sum_{j=1}^{\infty} \big((1-p_A)(1-p_B)\big)^{j-1}\right)$$

$$= p_A \cdot p_B \cdot \left(\sum_{k=0}^{\infty} \big((1-p_A)(1-p_B)\big)^{k}\right)$$

$$= p_A \cdot p_B \cdot \left(\frac{1}{1-(1-p_A)(1-p_B)}\right)$$

$$\frac{p_A \cdot p_B}{1-(1-p_A)(1-p_B)}$$

(c) $P(\text{duel ends after the } n^{th} \text{ round of shots})$

$$= P\Big(A \text{ and } B \text{ both miss on their first } n-1 \text{ shots}$$
$$\text{AND } A \text{ and } B \text{ don't both miss on their} n^{th} \text{ shot}\Big)$$

$$= P\Big(A \text{ and } B \text{ both miss on their first } n-1 \text{ shots}\Big)$$
$$\cdot\ P(A \text{ and } B \text{ don't both miss on their } n^{th} \text{ shot})$$

$$= \Big((1-p_A)^{n-1}(1-p_B)^{n-1}\Big)\Big(1-(1-p_A)(1-p_B)\Big)$$

(d) $P(\text{duel ends after the } n^{th} \text{ round of shots } / A \text{ not hit})$

$$= P\Big(A \text{ and } B \text{ both miss on their first } n-1 \text{ shots}$$
$$\text{AND } A \text{ hits but } B \text{ misses on} n^{th} \text{ shot}\Big) / P(A \text{ not hit})$$

$= P\left(A \text{ and } B \text{ both miss on their first } n-1 \text{ shots}\right)$

$\cdot \quad P(A \text{ hits but } B \text{ misses on } n^{th} \text{ shot}) \;/\; P(A \text{ not hit})$

$$= \frac{\left((1-p_A)^{n-1}(1-p_B)^{n-1}\right)\left(p_A(1-p_B)\right)}{\left(\frac{p_A(1-p_B)}{1-(1-p_A)(1-p_B)}\right)}$$

$$= \left((1-p_A)^{n-1}(1-p_B)^{n-1}\right)\left(1-(1-p_A)(1-p_B)\right)$$

(e) $P(\text{duel ends after the } n^{th} \text{ round of shots } / A \text{ and } B \text{ both hit})$

$= P\Big(A \text{ and } B \text{ both miss on their first } n-1 \text{ shots}$

$\text{AND } A \text{ hits but } B \text{ misses on} n^{th} \text{ shot}\Big) \;/\; P(A \text{ and } B \text{ both hit})$

$= P\left(A \text{ and } B \text{ both miss on their first } n-1 \text{ shots}\right)$

$\cdot\, P(A \text{ and } B \text{ both hit } n^{th} \text{ shot}) \;/\; P(A \text{ and } B \text{ both hit})$

$$= \frac{\left((1-p_A)^{n-1}(1-p_B)^{n-1}\right)\left(p_A p_B\right)}{\left(\frac{p_A \cdot p_B}{1-(1-p_A)(1-p_B)}\right)}$$

$$= \left((1-p_A)^{n-1}(1-p_B)^{n-1}\right)\left(1-(1-p_A)(1-p_B)\right)$$

Note that parts (c), (d) and (e) have the same answers!

■

7.3 Event A Occurs Before Event B

Consider an experiment consisting of independent and identical replications of a game with sample space Ω. Let A and B represent two disjoint events in Ω. Suppose this game is repeated until event A occurs or event B occurs for the first time. That is, until the outcome belongs to A or B for the first time.

For example, we can imagine that we continue to roll a pair of dice until a sum of either 5 (event A) or 7 (event B) appears for the first time.

On any given trial, let $p = P(A)$, $q = P(B)$, $r = P(N) = 1 - p - q$.

Show that under this set up

$$\boxed{P(\text{event } A \text{ occurs before event } B) = \frac{P(A)}{P(A) + P(B)} = \frac{p}{p+q}.}$$

<u>Solution 1</u> (Geometric Series Approach)

Let N represent the event "neither A nor B". That is, $N = A' \cap B'$. Let (N, N, A), for example, be our notation for the situation where "Neither" occurs on the first and second trials and event A occurs for the first time on the third trial. Using this notation, we can represent the event "A occurs before B" as

$$(A \text{ before } B) \equiv N \text{ or } (N, A) \text{ or } (N, N, A) \text{ or } (N, N, N, A) \text{ or } (N, N, N, N, A) \text{ or } \cdots.$$

We recognize that events $N, (N, A), (N, N, A), (N, N, N, A), \cdots$ are all disjoint. Therefore

$P(A \text{ before } B)$

$= P(A) + P(N, A) + P(N, N, A) + P(N, N, N, A) + P(N, N, N, N, A) + \cdots$

$= p + rp + r^2 p + r^3 p + r^4 p + \cdots$

$= p(r^0 + r^1 + r^2 + r^3 + r^4 + \cdots)$

$= p\left(\frac{1}{1-r}\right)$

$= p\left(\frac{1}{p+q}\right).$

■

Solution 2 (First Step Analysis Approach)

For any event E, suppose we let E_1 represent the event that event E occurs on the first trial. Then

$$\begin{aligned} P(A \text{ before } B) &= P(A \text{ before } B|A_1)P(A_1) + P(A \text{ before } B|B_1)P(B_1) \\ &\quad + P(A \text{ before } B|N_1)P(N_1) \\ &= (1 \cdot p) + (0 \cdot q) + P(A \text{ before } B|N_1) \cdot (1 - p - q) \\ &= p + P(A \text{ before } B|N_1) \cdot (1 - p - q). \end{aligned}$$

But *intuitively*,

$$P(A \text{ before } B) = P(A \text{ before } B|N_1).$$

That is, knowing that you got a nuisance ball on the first draw does not change the ultimate probability that you will get an A ball before a B ball.

After making this substitution in the previous expression for $P(A \text{ before } B)$, we end up with $P(A \text{ before } B)$ occurring on both sides of the equation. That is,

$$\begin{aligned} P(A \text{ before } B) &= p + P(A \text{ before } B|N_1) \cdot (1 - p - q) \\ &= p + P(A \text{ before } B) \cdot (1 - p - q). \end{aligned}$$

Algebraically solving for $P(A \text{ before } B)$ yields

$$P(A \text{ before } B) = \frac{p}{1 - (1 - p - q)} = \frac{p}{p + q} = \frac{P(A)}{P(A) + P(B)}.$$

■

Solution 3 (Conditional Probability by Rescaling Approach)

Suppose an urn contains n_1 balls labeled A, n_2 balls labeled B and n_3 balls labeled N such that

$$p = \frac{n_1}{n_1 + n_2 + n_3}, \; q = \frac{n_2}{n_1 + n_2 + n_3} \; \text{ and } \; r = \frac{n_3}{n_1 + n_2 + n_3}.$$

Then our experiment could be modeled as drawing balls from this urn *with replacement* until we get a ball labeled A or a ball labeled B for the first time. The presence of the n_3 balls labeled N are "nuisance" balls in as much as every time we draw one, we just sigh, throw it back in and try again.

The more nuisance N balls there are in the urn the more draws it will take, *on average*, to finish this experiment. But it is *intuitively* clear that the ultimate probability of drawing an A ball before drawing a B ball will not depend on how many nuisance N balls are in the urn.

In particular, $P(A \text{ before } B)$ would not change if we initially just removed *all* the nuisance N balls from the urn. This would be convenient for us because then on the very first draw we would be able to tell whether an A ball occurred before a B ball (or vice versa).

Removing the nuisance balls will rescale the probability of drawing an A ball or a B ball on any given draw but it will *not* change the relative probability of these two events. Specifically, if we let $\Omega^\star$ represent the new sample space once the n_3 nuisance N balls are removed from the urn, then we can see that the relative probability of drawing an A ball to that of drawing a B ball is the same for the original sample space Ω and the reduced sample space $\Omega^\star$.

$$\frac{P_\Omega(A)}{P_\Omega(B)} = \frac{\dfrac{n_1}{n_1+n_2+n_3}}{\dfrac{n_2}{n_1+n_2+n_3}} = \frac{n_1}{n_2}$$

and similarly

$$\frac{P_{\Omega^\star}(A)}{P_{\Omega^\star}(B)} = \frac{\dfrac{n_1}{n_1+n_2}}{\dfrac{n_2}{n_1+n_2}} = \frac{n_1}{n_2}.$$

This description is the exact idea behind *conditional probability* - some outcomes are removed from the (original) sample space but the relative probability of the remaining outcomes is not changed.

In general, the relative probabilities of outcomes in the reduced sample space will not change as long as all we *scale* (multiply) each remaining outcome by the *same* factor k where k is chosen to guarantee that $P_{\Omega^\star}(\Omega^\star) = 1$.

In particular, we can find k by solving

$$1 = P_{\Omega^\star}(A) + P_{\Omega^\star}(B) = kP_\Omega(A) + kP_\Omega(B)$$

for k. In this case we find

$$k = \frac{1}{P_\Omega(A) + P_\Omega(B)}.$$

Thus,

$$P_{\Omega^\star}(A) = k \cdot P_\Omega(A) = \frac{P_\Omega(A)}{P_\Omega(A) + P_\Omega(B)}$$

and

$$P_{\Omega^\star}(B) = k \cdot P_\Omega(B) = \frac{P_\Omega(B)}{P_\Omega(A) + P_\Omega(B)}.$$

It follows from this argument that

$$P_\Omega(A \text{ before } B) = P_{\Omega^\star}(A \text{ before } B) = P_{\Omega^\star}(A) = \frac{P_\Omega(A)}{P_\Omega(A) + P_\Omega(B)}.$$

Also notice that this formula for $P_{\Omega^\star}(A)$ is consistent with our clear understanding that once we remove the n_3 Nuisance balls they the resulting probability of drawing an A ball on any draw must equal the number of A balls (n_1) in the urn over the total number balls in the urn $(n_1 + n_2)$. Notice that we can work backwards to see that, in fact, this is the case.

$$P_{\Omega^\star}(A) = \frac{P_\Omega(A)}{P_\Omega(A) + P_\Omega(B)} = \frac{\frac{n_1}{n_1 + n_2 + n_3}}{\left(\frac{n_1}{n_1 + n_2 + n_3}\right) + \left(\frac{n_2}{n_1 + n_2 + n_3}\right)} = \frac{n_1}{n_1 + n_2}.$$

■

Example A pair of dice is rolled until a sum of either 5 or 7 appears. Find the probability that a sum of 5 occurs before a sum of 7.

Using the geometric series approach we find

$$\begin{aligned}
&P(5 \text{ before a } 7) \\
&= P(5) + P(N,5) + P(N,N,5) + P(N,N,N,5) + P(N,N,N,N,5) + \cdots \\
&= \left(\frac{4}{36}\right) + \left(\frac{26}{36}\right)\left(\frac{4}{36}\right) + \left(\frac{26}{36}\right)^2\left(\frac{4}{36}\right) + \left(\frac{26}{36}\right)^3\left(\frac{4}{36}\right) + \cdots \\
&= \left(\frac{4}{36}\right)\left(\frac{1}{1 - \left(\frac{26}{36}\right)}\right)
\end{aligned}$$

$$= \left(\frac{4}{36}\right)\left(\frac{36}{10}\right) = \frac{4}{10} = \frac{2}{5}$$

and using the derived general formula we again find that

$$P(5 \text{ before a } 7) = \frac{P(5)}{P(5) + P(7)} = \frac{4/36}{(4/36) + (6/36)} = \frac{4}{10} = \frac{2}{5}.$$

■

The general conclusion from this example is that the probability of a particular event (*e.g.* Amy wins two games before Charlie wins two games) in a game where ties are possible will equal the probability of that same event in a version of the game where ties are not possible provided we adjust the probability of any player winning a game to their conditional probability of winning a game given the information that the game did not end in a tie.

This is a simple trick that can save you a lot of time in a contest setting. Removing the possibility of ties makes the problem easier to model and easier to solve.

Exercises for Chapter 7, Section 3

1. (a) Two friends, Amy and Charlie like to play a game that can either end in a win for one of the players or in a tie. For any game let $P(\text{Amy wins}) = a$, $P(\text{Charlie wins}) = c$ and let $P(\text{tie}) = b = 1 - a - c$. Assume that the outcomes of successive games are independent.

   Amy and Charlie have decided ahead of time that if a game ends in a tie they will start a new game and continue to do this until one of them gets an outright win. Find the probability that Amy will earn a win before Charlie.

   (b) Continue with the same details as in part (a) except now find the probability that Amy will earn <u>two</u> wins before Charlie earns two wins.

   (c) How would the result in part (II) change if Amy and Charlie were playing a game where it was not possible for the game to end in a tie?

2. If A and B play a series of games in each of which the probability that A wins is p and that B wins is $q = 1 - p$, find

   (a) the probability that A wins two games before B wins three games.

   (b) the probability that A is the first player to win two successive games.

   (Source: *Probability*, James R. Gray, 1967, page 35, problem 30)

3. Three men, A, B, C have respective probabilities p, q, r of succeeding each time they attempt a certain task. They organize a competition with a prize for the first to succeed, each being allowed one attempt at a time in rotation in the order $A, B, C, A, B, C, A, \ldots$. If under these conditions their chances of winning the prize are equal, express q and r as functions of p and hence show that $0 \leq p \leq 1/3$.

   Assuming p, q, r to satisfy these relations examine whether there are any possible values of p for which it would benefit B to support the proposal to reverse the order of attempts after each round so that attempts are made in the order $A, B, C, C, B, A, A, B, C, C, \ldots$.

   (Source: *Probability*, James R. Gray, 1967, page 37, problem 29(

Project 8 Penney's Nontransitive Pennies

(TT092) Alec offers to play a game with Teddy. Alec will flip a coin repeatedly until either **HHH** (three consecutive heads) appears – in which case Alec wins – or until a three-flip sequence of Teddy's choosing appears. Teddy can't chose **HHH**, but he can choose any other three-flip sequence, as long as he declares it before the game begins. Presuming that Teddy makes the best possible choice, determine the probability that he will win.

Solution

Alec has at least a $1/8$ chance of winning (if **HHH** comes up immediately). Teddy wants to ensure that if a tail shows somewhere in the first three flips, that he will win before Alec does. The best strategy is to choose **THH**. This keeps Alec at a $1/8$ chance of winning, making Teddy's chance $7/8$.

■

Problem 21. Patterns in Repeated Trials

Consider a series of independent trials where on each trial there are S possible outcomes $\{O_1, \dots, O_S\}$ and suppose that on every trial

$$P\left(\{O_j\}\right) = p_j,\ \ j = 1, \dots, S.$$

Suppose the trials continue until a particular pattern of outcomes is observed. Let N_A equal the number of trials required to observe pattern A for the first time.

e.g. Consider rolling a die until you observe the pattern $A = \{1,2\}$. If we observed the sequence

$$2\ \ 1\ \ 4\ \ 6\ \ 3\ \ 3\ \ 1\ \ 6\ \ \underline{1\ \ 2}$$

Then N_A would be 10 for this particular sequence of outcomes.

The first problem we will consider is finding a formula for $E(N_A)$ for a general pattern A.

The second part of this problem involves finding the probability that pattern A will occur before pattern B.

The solution to both parts of this problem involves the critical points of one pattern with respect to another pattern. The definition of critical points goes back to "Patterns in Repeated Trials", Bizley, M.T.L., 1962, Journal of the Institute of Actuaries, 88, 360 - 366.

Definition

The k^{th} letter of pattern A is a *critical point* with respect to pattern B provided the last k letters of pattern B and first k letters of pattern A are identical (same letters in the same order).

Define

$$c_k(A|B) = \begin{cases} 1 & \text{if the } k^{th} \text{ letter of pattern } A \text{ is a critical point wrt to pattern } B \\ 0 & \text{else.} \end{cases}$$

Example Let $A = \{0,1,0,2\}$ and $B = \{0,2,1\}$. Then

$$\begin{array}{lllllllll} A & 0 & 1 & 0 & 2 & & & & \\ A & 0 & 1 & 0 & 2 & & & & c_4(A|A) = 1 \\ & & 0 & 1 & 0 & 2 & & & c_3(A|A) = 0 \\ & & & 0 & 1 & 0 & 2 & & c_2(A|A) = 0 \\ & & & & 0 & 1 & 0 & 2 & c_1(A|A) = 0 \end{array}$$

$$\begin{array}{lllllllll} B & 0 & 2 & 1 & & & & & \\ A & 0 & 1 & 0 & 2 & & & & c_3(A|B) = 0 \\ & & 0 & 1 & 0 & 2 & & & c_2(A|B) = 0 \\ & & & 0 & 1 & 0 & 2 & & c_1(A|B) = 0 \end{array}$$

$$\begin{array}{llllllll} A & 0 & 1 & 0 & 2 & & & \\ B & 0 & 2 & 1 & & & & c_4(B|A) = 0 \\ & & 0 & 2 & 1 & & & c_3(B|A) = 0 \\ & & & 0 & 2 & 1 & & c_2(B|A) = 1 \\ & & & & 0 & 2 & 1 & c_1(B|A) = 0 \end{array}$$

$$\begin{array}{lllllll} B & 0 & 2 & 1 & & & \\ B & 0 & 2 & 1 & & & c_3(B|B) = 1 \\ & & 0 & 2 & 1 & & c_2(B|B) = 0 \\ & & & 0 & 2 & 1 & c_1(B|B) = 0 \end{array}$$

We need a few more definitions before stating the main results.

Definition

Patterns A and B are reduced if neither pattern is just a piece of the other.

For example :

$$A = \{0, 1, 0, 2, 2\} \quad \text{and} \quad B = \{\, 0{,}2{,}1 \,\} \quad \Longrightarrow \quad A \text{ and } B \text{ are reduced patterns}$$
$$C = \{0, \underline{1, 0, 2}, 2\} \quad \text{and} \quad D = \{\, \underline{1{,}0{,}2} \,\} \quad \Longrightarrow \quad C \text{ and } D \text{ are } \underline{\text{not}} \text{ reduced patterns.}$$

Definition

Let $A = \{O_{a_1}, \dots, O_{a_r}\}$ and $B = \{O_{b_1}, \dots, O_{b_t}\}$. Define

$$(A * B) = \sum_{j=1}^{t} \left(\prod_{k=1}^{\min\{j,r\}} \frac{1}{p_{a_k}} \right) \cdot c_j(A|B).$$

Definition

Let $N_{A|B}$ equal the number of additional trials needed for pattern A to occur given that pattern B just occurred.

Theorem

(1) $E(N_A) = (A * A)$
(2) If patterns A and B are reduced, then $E(N_{A|B}) = (A * A) - (A * B)$
(3) If patterns A and B are reduced, then

$$P(A \text{ occurs before } B) = \frac{E(N_B) + E(N_{A|B}) - E(N_A)}{E(N_{B|A}) + E(N_{A|B})}$$

$$= \frac{(B * B) - (A * B)}{\big((B * B) - (A * B)\big) + \big((A * A) - (B * A)\big)}.$$

Applications

(4) Show that when flipping a fair coin,

$$P\big(\text{pattern } \{T,H,H\} \text{ appears before pattern } \{H,H,H\}\big) = \frac{7}{8}.$$

Note that when flipping a fair coin 3 times, all 8 possible outcomes

$$\big\{(H,H,H),(H,H,T),(H,T,H),(T,H,H),(H,T,T),(T,H,T),(T,T,H),(T,T,T)\big\}$$

are equally likely to occur. On this basis one might conclude that the answer to (4) should be $1/2$. What is wrong with this reasoning?

(5) Suppose we say that pattern A *beats* pattern B if

$$P\big(\text{pattern } A \text{ appears before pattern } B\big) > \frac{1}{2}.$$

Also suppose we use the notation $A \xrightarrow{b} B$ to mean pattern A beats pattern B. Show

$$T\ H\ H \xrightarrow{b} H\ H\ T \xrightarrow{b} H\ T\ T \xrightarrow{b} T\ T\ H \xrightarrow{b} T\ H\ H.$$

That is, show that the operation $\xrightarrow{b}$ is *nontransitive*.

(6) When flipping a fair coin show that

$$E(HTHH) < E(THTH)$$

but

$$T\ H\ T\ H \xrightarrow{b} H\ T\ H\ H.$$

That is, more likely than not, pattern $T\ H\ T\ H$ will occur before pattern $H\ T\ H\ H$ but on average pattern $T\ H\ T\ H$ will take longer to occur than pattern $H\ T\ H\ H$. Is this a logical contradiction?

(7) “A man was taken prisoner by pirates, who did not know what to do with him. Finally, the captain decided to write the letters L, I, V, E and K on a die, leaving one side blank. The die was to be thrown until one the words LIVE or KILL was formed by consecutive letters. The pirates, who liked to gamble, were enthusiastic about the idea. The captain asked if the prisoner had any last wish before the gambling started. ‘Yes’, he said, ‘I would be glad if you could replace the word KILL by DEAD’. The captain agreed and wrote the letters L, I, V, E, A and D on the die.”

Did the prisoner increase his odds of surviving by making this request?

Blom and Thorburn “How Many Random Digits are Required Until Given Sequences are Obtained?”, Journal of Applied Probability, 19, 518 - 531, 1982.

7.4 Dice Problems

Suppose that a fair m-sided die labeled from 1 to m is rolled r times. Show that the probability of getting a sum of n is

$$\frac{1}{m^r}\sum_{u=0}^{\left\lfloor\frac{n-r}{m}\right\rfloor}(-1)^u\binom{n-mu-1}{n-mu-r}\binom{r}{u}.$$

Solution

Let $\Psi(r,m,n)$ equal the number of solutions to

$$x_1+x_2+\cdots+x_r=n,\ \text{ with }\ x_j\in\{1,2,\dots,m\}, j=1,2,\dots,r.$$

If we let x_j equal the number showing on the j^{th} roll of the die then it is clear that the probability of getting a sum of n is

$$\frac{\Psi(r,m,n)}{m^r}.$$

Thus, the problem reduces to one of finding $\Psi(r,m,n)$. It is clear that $\Psi(r,m,n)$ equals the coefficient of y^n in the polynomial $h(y)$ given by

$$h(y)=(y^1+y^2+\cdots+y^m)^r.$$

Now,

$$h(y)=y^r(y^0+y^1+\cdots+y^{m-1})^r$$

$$=y^r(y^0+y^1+y^2\dots)^r(1-y^m)^r$$

$$=y^r\left(\sum_{j=0}^{\infty}\binom{r+j-1}{j}y^j\right)\left(\sum_{j=0}^{r}(-1)^j\binom{r}{j}y^{mj}\right)$$

$$=y^r\left(\sum_{j=0}^{\infty}a_jy^j\right)\left(\sum_{i=0}^{\infty}b_iy^i\right)$$

$$= y^r \left(\sum_{k=0}^{\infty} c_k y^k \right)$$

where

$$a_j = \binom{r+j-1}{j}, \quad j = 0,1,2,\dots$$

$$b_i = \begin{cases} (-1)^{i/m} \binom{r}{i/m} & i = 0, m, 2m, 3m, \dots, rm \\ 0 & \text{else} \end{cases}$$

and

$$c_k = \sum_{i=0}^{k} b_i a_{k-i}.$$

The coefficient of y^n in the polynomial

$$h(y) = y^r \sum_{k=0}^{\infty} c_k y^k$$

is c_{n-r}. Therefore,

$$\begin{aligned} \Psi(r,m,n) = c_{n-r} &= \sum_{i=0}^{n-r} b_i a_{n-r-i} \\ &= \sum_{i=0}^{n-r} b_i \binom{n-i-1}{n-i-r} \\ &= \sum_{u=0}^{\left\lfloor \frac{n-r}{m} \right\rfloor} b_{mu} \binom{n-mu-1}{n-mu-r} \end{aligned}$$

because $b_i = 0$ except for $i = 0, m, 2m, \dots, rm$. Therefore

$$\Psi(r,m,n) = \sum_{u=0}^{\left\lfloor \frac{n-r}{m} \right\rfloor} (-1)^u \binom{r}{u} \binom{n-mu-1}{n-mu-r}.$$

■

Theorem

When a fair m-sided die labeled from 1 to m is rolled r times, the probability of getting a sum of n, ($n = r, r + 1, \dots, mr$) equals the probability of getting a sum of $(m + 1)r - n$.

Proof

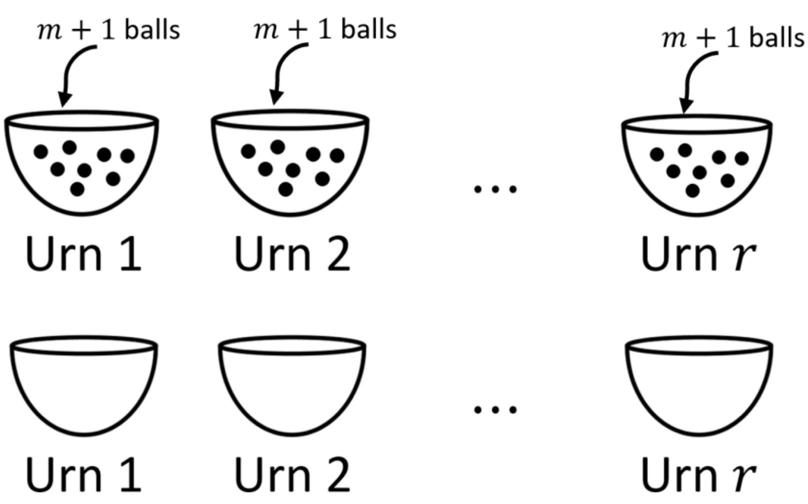
Let x_j equal the number showing on the j^{th} roll of the fair m-sided die labeled from 1 to m, $j = 1,2, \dots, r$.

As discussed in the previous result, the set of all ways for the sum of the rolls to equal n is the same as the set of solutions $(x_1, x_2, \dots, x_r)$ such that

$$x_1 + x_2 + \cdots + x_r = n, \text{ with } x_j \in \{1,2, \dots, m\}, j = 1,2, \dots, r.$$

But this is the same as the set of all ways to distribute n identical balls into r labeled urns with at least one ball per urn but no more than m balls per urn.

Now switch gears and imagine a set of r labeled urns that each start off filled with $m + 1$ identical balls.



Suppose we remove x_j balls from Urn j in the top row and put these x_j balls into Urn j in the bottom row according to the restrictions that $x_1 + x_2 + \cdots + x_r = n$ and $x_j \in \{1,2, \dots, m\}$.

We can see that this is just an alternative way of distributing n identical balls into r labeled urns (the bottom row) with at least one ball per urn but no more than m balls per urn.

If we let y_j equal the number of balls left in Urn j in the top row after removing x_j balls then it follows that $y_j = (m+1) - x_j$ and

$$y_1 + y_2 + \cdots + y_r = (m+1)r - (x_1 + \cdots + x_r) = (m+1)r - n.$$

Furthermore, $x_j \in \{1,2,\ldots,m\}$ implies that the range of possible values for y_j will be from $(m+1) - m = 1$ to $(m+1) - 1 = m$. That is, $y_j \in \{1,2,\ldots,m\}$, $j = 1,2,\ldots,r$.

This shows there is a one-to-one correspondence between the ways to distribute n identical balls into the bottom row of r distinct urns with at least one ball but no more than m balls in each urn and the ways to leave $(m+1)r - n$ identical balls in the top row of r distinct urns by removing at least one but no more than m balls from any one urn.

That is, there is a one-to-one correspondence between the set of solutions $(x_1, x_2, \ldots, x_r)$ such that

$$x_1 + x_2 + \cdots + x_r = n, \text{ with } x_j \in \{1,2,\ldots,m\}, j = 1,2,\ldots,r$$

and the set of solutions $(y_1, y_2, \ldots, y_r)$ such that

$$y_1 + y_2 + \cdots + y_r = (m+1)r - n, \text{ with } y_j \in \{1,2,\ldots,m\}, j = 1,2,\ldots,r.$$

But the variable name y_j is arbitrary and we could just as well refer to this as the set of solutions $(x_1, x_2, \ldots, x_r)$ such that

$$x_1 + x_2 + \cdots + x_r = (m+1)r - n, \text{ with } x_j \in \{1,2,\ldots,m\}, j = 1,2,\ldots,r.$$

It follows that when a fair m-sided die labeled from 1 to m is rolled r times, the probability of getting a sum of n, $(n = r, r+1, \ldots, mr)$ equals the probability of getting a sum of $(m+1)r - n$.

■

Example

Suppose we roll three fair 6-sided dice. What is the probability of obtaining a sum of 16 points?

Solution

We could compute this by applying the general result

$$P(\text{sum} = 16) = \frac{1}{6^3} \sum_{u=0}^{\left\lfloor \frac{16-3}{6} \right\rfloor} (-1)^u \binom{16 - 6u - 1}{16 - 6u - 3} \binom{3}{u}.$$

But it is easier to recognize that $P(\text{sum} = 16) = P(\text{sum} = (6+1) \cdot 3 - 16 = 5)$ and to find $P(\text{sum} = 5)$ by brute force. There are six ways for 3 dice to sum to 5. Namely,

$$\{1,1,3\}, \{1,3,1\}, \{3,1,1\}, \{1,2,2\}, \{2,1,2\}, \{2,2,1\}.$$

Hence,

$$P(\text{sum} = 16) = P(\text{sum} = 5) = \frac{6}{6^3} = \frac{1}{36}.$$

■

Exercises for Chapter 7, Section 4

1. (4C833) Four balls marked $1,2,3,4$ are placed in an urn. One ball is drawn, its number recorded, and the ball is returned to the urn. This process is repeated, and then repeated once more. Each ball is equally likely to be drawn on each occasion. If the sum of the numbers recorded is 9, what is the probability that the ball numbered 3 was drawn all three times?

2. What is the probability of obtaining a sum of ten points in a throw of three symmetrical six-sided dice, each with faces numbered $1,2,3,4,5,6$? (Source: *Probability*, James R. Gray, 1967, page 30, problem 4)

3. An ordinary symmetrical six-sided die is thrown four times and the sum of the four numbers is 12. What is the probability that the sum of the numbers in the first two throws was 4? (Source: *Probability*, James R. Gray, 1967, page 38, problem 44)

4. When rolling 12 standard 6-sided dice, the probability that the sum of the numbers rolled on the 12 dice is 69 can be expressed in $N/6^{12}$. Find the sum of the digits of N. (2019 AMC 10C, Problem 8, a mock contest created by AoPS user fidgetboss_4000)

5. When n standard 6-sided dice are rolled, the probability of obtaining a sum of 1994 is greater than zero and is the same as the probability of obtaining a sum of *S.* What is the smallest possible value of S? (Source: Problem 30 of 1994 AHSME)

Project 9 Equally Likely Dice Sums are Impossible

1. *Equally likely dice sums.* A problem often given in a first probability course is to prove that it is not possible for two independent and identically distributed random variables X_1 and X_2 (which take on values $1, 2, 3, 4, 5, 6$) to be such that $P[X_1 + X_2 = k] = 1/11$ for $k = 2, \ldots, 12$, and hence to "conclude that it is impossible to weight a pair of dice so that the probability of occurrence of every sum from 2 to 12 will be the same." (E.g., Parzen [1, p. 327].) Students usually prove this result by solving the system $P[X_1 + X_2 = k] = 1/11 \quad (k = 2, \ldots, 12)$ for $P[X_1 = j] \equiv p_j (j = 1, \ldots, 6)$ and showing that $p_1 + \cdots + p_6 > 1$.

Equally Likely Dice Sums Do Not Exist
Author(s): Edward J. Dudewicz and Ronald E. Dann
Source: *The American Statistician*, Vol. 26, No. 4 (Oct., 1972), pp. 41-42

2. *The problem*. Let X_1 and X_2 be two independent discrete random variables (which take on values $1, \ldots, n$ where $n \geq 2$ is a fixed positive integer). Denote $P[X_1 = i] = p_i \geq 0$ $(1 \leq i \leq n)$, $\mathrm{P}[X_2 = j] = q_j \geq 0$ $(1 \leq j \leq n)$ where $p_1 + \cdots + p_n = q_1 + \cdots + q_n = 1$. Prove that there do not exist $p_1, \ldots, p_n$; $q_1, \ldots, q_n$ such that $P[X_1 + X_2 = k] = 1/(2n-1)$ for $k = 2, \ldots, 2n$. This fact may have application in Bayesian analysis, where one might take a prior distribution on such sums: the uniform distribution on sums is impossible.

REFERENCE

[1] Parzen, E. *Modern Probability Theory and Its Applications*, John Wiley & Sons, Inc., New York, 1960.

Project 10 Elchanan Mossel's Problem

A fair 6-sided die is rolled repeatedly until a 6 is obtained. Let X equal the number of rolls required. Let A be the event that none of the rolls prior to getting a 6 were odd numbers.

(a) Find $P(X = k)$

(b) Find $P(X = k|A)$.

Solution

(a) X follows a Negative Binomial distribution where we are waiting to get our first success (getting a 6) with $p = P(\text{success}) = P(\text{roll a 6}) = 1/6$.. Hence,

$$P(X = k) = (1 - p)^{k-1}p^1 = \left(\frac{5}{6}\right)^{k-1}\left(\frac{1}{6}\right), \quad k = 1,2,3, \dots \ .$$

(b) Imagine for a moment a fair three-sided die with the faces numbered as 2,4 and 6. Suppose this die is rolled repeatedly until a 6 is obtained. Let Y equal the number of rolls required.

Is $P(X = k|A) = P(Y = k)$? The two probabilities have certain elements in common but unfortunately they are not identical. (But the differences are subtle and easy to miss.)

It will help if we broaden the problem a bit. Suppose that we roll a fair six-sided die until we get any of the numbers $\{1,3,5 \text{ or } 6\}$ for the first time. Then we stop. Let T equal the total number of rolls performed before we stop. In this case T follows a Negative Binomial distribution where we are waiting for our first success (getting a $1,3,5$ or 6) with $p = P(\text{success}) = 4/6 = 2/3$. Hence,

$$P(T = k) = (1 - p)^{k-1}p^1 = \left(\frac{1}{3}\right)^{k-1}\left(\frac{2}{3}\right), \quad k = 1,2,3, \dots \ .$$

Let C_j be the event that the number $j \in \{1,3,5,6\}$ was the number we stopped on in the above experiment. By symmetry $P(C_1) = P(C_2) = P(C_3) = P(C_6) = 1/4$.

But it also follows by symmetry that $P(C_1|T = k) = P(C_3|T = k) = P(C_5|T = k) = P(C_6|T = k) = 1/4$.

Notice that $P(C_6|T = k) = P(C_6)$. This means that the events C_6 and $T = k$ are independent.

Therefore,

$$P(T = k|C_6) = P(T = k) = (1 - p)^{k-1}p^1 = \left(\frac{1}{3}\right)^{k-1}\left(\frac{2}{3}\right), \quad k = 1,2,3, \ldots \ .$$

But $P(X = k|A) = P(T = k|C_6)$.

Hence,

$$P(X = k|A) = P(T = k) = \left(\frac{1}{3}\right)^{k-1}\left(\frac{2}{3}\right), \quad k = 1,2,3, \ldots \ .$$

If they were identical then we would have a quick solution to our problem because Y follows a Negative Binomial distribution where we are waiting to get our first success (getting a 6) with $p = P(\text{success}) = P(\text{roll a } 6) = 1/3$. Hence,

$$P(Y = k) = (1 - p)^{k-1}p^1 = \left(\frac{2}{3}\right)^{k-1}\left(\frac{1}{3}\right), \quad k = 1,2,3, \ldots \ .$$

To sort out what is going on in a probability problem it is often helpful to go through a thought experiment of getting an approximate answer through a simulation.

We could simulate $X|A$ by rolling a fair (six-sided) die until you got a six or any odd number, whichever comes first.

If you get an odd number first, just toss out that case altogether.

If you get a six first, count how many rolls were required (including the roll where the 6 occurred). Keep a tally of how many rolls were required to get that six.

Now start the process over. Roll the die until you either (any) odd number or a six. If the six comes first, record in your tallies how many rolls were required to get the six.

Do this over and over until you have a lot of data collected. Then to approximate $P(X = k|A)$ you would calculate the percentage of times in your tallies that it took k rolls to get the first 6.

In theory, if you were patient enough to collect an infinite amount of data, your approximation would become the exact answer.

Now imagine that you kept separate tallies just as discussed above except that you swap out the 6 for a 1. That is, you roll a fair (six-sided) die until you got a 1 or any of the numbers $\{3,5,6\}$.

If you get any of $\{3,5$ or $6\}$ first, just toss out that case altogether. If you get a 1 first, count how many rolls were required (including the roll where the 1 occurred). Keep a tally of how many rolls were required to get that 1.

Do this over and over until you have a lot of data collected. Then to approximate $P(X = k|A)$ you would calculate the percentage of times in your tallies that it took k rolls to get the first 6.

It is real tempting to think of this as being equivalent to rolling a fair three-sided die with sides labeled 2,4 and 6. If this is true, then

$$P(X = k|A) = \left(\frac{2}{3}\right)^{k-1}\left(\frac{1}{3}\right),\ k = 1,2,3,\ldots\ .$$

Let's give some notation to the situation of a three

But is this correct? To see that it is not we need to describe an experiment that would allow us to simulate an answer to $P(X = k|A)$ and then compare that with the experiment of rolling a fair three-sided die with sides labeled 2,4 and 6 until we get a 6.

7.5 Recursion

Exercises for Chapter 7, Section 5

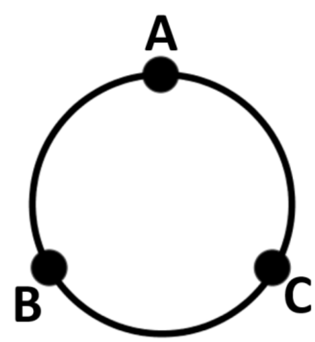
(5T175) A fair coin is flipped 7 times. Determine exactly the probability that two heads never happen on two successive tosses. Express the answer as a quotient of two relative prime integers.

Solution

Let n be the number of tosses and $f(n)$ the number of sequences without two consecutive heads. Then $f(1) = 2$, H or T; $f(2) = 3$, HT, TH, or TT; and $f(3) = 5, HTH, TTH, HTT, THT$, or TTT. To find $f(4)$, add TH to the sequences of length two or add T to the sequences of length 3 to obtain: $HTTH, THTH, TTTH, HTHT, TTHT, HTTT, THTT$, or $TTTT$. So $f(4) = 3 + 5 = 8$ and $f(5) = 5 + 8 = 13$, $f(6) = 8 + 13 = 21$, and $f(7) = 13 + 21 = 34$. There are $2^7 = 128$ possible sequences, so the probability of no consecutive Heads in seven flips of a coin is $34/128 = 17/64$.

■

1. Felix the cat has three favorite spots for napping. Whenever Felix gets bored he gets up from his current spot and randomly moves to one of his other two favorite spots.



If Felix starts in position A, what is the probability that Felix will end up back in position A after his twentieth move?

2. If u_n is the probability that in n tosses of a symmetrical coin three or more consecutive heads do not turn up, show that for $n \geq 4$,

$$u_{n-1} - u_n = \frac{1}{16} u_{n-4},$$

and hence evaluate u_7. (Source: *Probability*, James R. Gray, 1967, page 144, problem 7)

3. The respective probabilities of heads and tails when a biased coin is tossed are p and $1 - p$. If u_n denotes the probability that two heads in succession do not occur in n trials, show that

$$u_{n+2} = (1 - p)u_{n+1} + p(1 - p)u_n.$$

Hence find the value of u_n when $p = 2/3$. (Source: *Probability*, James R. Gray, 1967, page 144, problem 9)

4. A tetrahedron which has three green faces and one red face is placed with one face in contact with a table. It is then moved from its initial position by rotating it about one of the edges in contact with the table, all three edges being equally likely, until an adjacent face rests in contact with the table. A series of such moves is performed. If p_n denotes the probability that the red face is in contact with the table after n moves, shown that

$$p_{n+1} = \frac{1}{3}(1 - p_n).$$

Find p_n when initially (i) the red face and (ii) a green face was in contact with the table.

If the initial face in contact with the table was chosen at random from the four faces, what is the probability that the red face will be in contact with the table after n moves?

If the initial face in contact with the table was chosen at random then $p_0 = 1/4$ and hence

$$p_n = \left(\frac{1}{3}\right)^n \left(\frac{1}{4} - \frac{1}{4}\right) + \frac{1}{4} = \frac{1}{4}.$$

(Source: *Probability*, James R. Gray, 1967, Page 145, Problem 14)

5. Justine has a coin that will come up the same as the last flip $2/3$ of the time. She flips it and it comes up heads. She flips it 2010 more times. What is the probability that the last flip is heads? (Source: https://faculty.math.illinois.edu/~hildebr/ putnam/training19/ probability1.pdf)

Project 11 The Gambler’s Ruin

A gambler has a $7/13$ chance of winning a game and $6/13$ chance of losing. He bets $\$10$ each game. He starts with $\$100$. What is the probability that he reaches $\$150$ before going broke?

7.6 Method of Inclusion – Exclusion

Critical Point: In an Inclusion-Exclusion problem when, for example, you are finding $P(A \cap B)$ as part of $P(A \cup B \cup C \cup D \cup E)$, you are making no assumptions on whether events C, D and/or E occurred or not. In other words, when finding $P(A \cap B)$, we ignore events C, D and E altogether.

Exercises for Chapter 7, Section 6

1. (5T143) Four women each store a distinct hat in the same box. If all four women reach into the box randomly and independently, what is the probability that no woman picks her own hat?
2. A bag contains a proportion p_1 of white balls, p_2 of black balls and p_3 of red balls where $0 < p_1 + p_2 + p_3 \leq 1$. Balls are drawn at random, one at a time with replacement. Find the probability that n draws are required to select each color at least once. (Source: *Probability*, James Gray, Oliver & Boyd, 1967, page 43, Problem 60.)

7.7 Probability and Number Theory

Theorem

Let $\{\alpha_1, \dots, \alpha_k\}$ be the set of prime divisors of the positive integer n and let m be a number picked at random from $1,2, \dots, n$. Let A_i be the event that m is divisible by $\alpha_i, i = 1,2, \dots, k$. Then $A_1, A_2, \dots, A_k$ are mutually independent events and $P(A_i) = 1/\alpha_i$.

Schleicher, Lackmann (Eds), An Invitation to Mathematics, From Competitions to Research

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Problem 1. *A schoolteacher is in charge of some children. She wants to select two of them at random, and notices that it is exactly an even chance (50%) that they are of the same sex. What can be said about how many children of each sex there are?*

(find other problems like this in MSHSML problems)

Let us suppose that there are α boys and β girls. Then the number of (unordered) pairs of children is $\frac{1}{2}(\alpha+\beta)(\alpha+\beta-1)$, while the number of pairs of opposite sex is $\alpha\beta$. So

$$(\alpha+\beta)(\alpha+\beta-1) = 4\alpha\beta \tag{1}$$

which when rearranged gives

$$(\alpha-\beta)^2 = \alpha+\beta\,. \tag{2}$$

If we write $n = \alpha - \beta \in \mathbb{Z}$, then $\alpha = \frac{n(n+1)}{2}$ and $\beta = \frac{n(n-1)}{2}$. This means that α and β are consecutive triangular numbers, with α being the larger if $n > 0$ and β being the larger if $n < 0$.

Problem 2. *A man has a selection of socks of three different colours, which he keeps in a bag. Two socks of the same colour may be assumed to form a pair. He finds that if he pulls out two socks at random then there is exactly an even chance that they will form a pair. What can be said about how many socks of each colour he has?*

Exercises for Chapter 7, Section 7

1. A drawer contains a mixture of red socks and blue socks, at most 1991 in all. It so happens that when 2 socks are selected randomly without replacement, the probability is exactly one-half that both are red or both are blue. What is the largest possible number of red socks in the drawer? (Source: Mathematics Teacher, Calendar Problem #30, September 1991.)

7.8 Probability and Theory of Equations

Exercises for Chapter 7, Section 8

Răzvan Gelca, Titu Andreescu, *Putnam and Beyond*, Second Edition

Problem 1079, Section 6.5, Probability, page 337. Gelca and Andreescu site N. Negoescu, *Probleme cu … Probleme* (*Problems with … Problems*), Editura Facla, 1975, (Romanian) as the original source.

Three students take an exam. Assume they worked independently and that the exams were scored independently.

Let A_i be the event that exactly i of these students pass the exam for $i = 0,1,2,3$. Assume that $P(A_0) = 2/5$, $P(A_1) = 13/50$, $P(A_2) = 3/20$, $P(A_3) = 1/60$.

Let p_i equal the probability that the i^{th} person passes the exam, $i = 1,2,3$.

Find p_1, p_2 and p_3.

Solution

We can express the probability of each event A_0, A_1, A_2, A_3 in terms of p_1, p_2, p_3 as

$$\begin{aligned}
P(A_0) &= (1-p_1)(1-p_2)(1-p_3) \\
P(A_1) &= p_1(1-p_2)(1-p_3) + (1-p_1)p_2(1-p_3) + (1-p_1)(1-p_2)p_3 \\
P(A_2) &= p_1p_2(1-p_3) + p_1(1-p_2)p_3 + (1-p_1)p_2p_3 \\
P(A_3) &= p_1p_2p_3.
\end{aligned}$$

Define the polynomial $Q(x) = \big(p_1x + (1-p_1)\big)\big(p_2x + (1-p_2)\big)\big(p_3x + (1-p_3)\big)$.

Notice that the coefficient of x^i of $Q(x)$ is $P(A_i)$. That is,

$$\begin{aligned}
Q(x) &= P(A_3)x^3 + P(A_2)x^2 + P(A_1)x^1 + P(A_0) \\
&= \frac{1}{60}x^3 + \frac{3}{20}x^2 + \frac{13}{30}x + \frac{2}{5}.
\end{aligned}$$

Another way of saying this is that $Q(x)$ is the probability generating function for the number of successes (student passes the exam) in the $n = 3$ trials.

The zeros of $Q(x) = \big(p_1x + (1 - p_1)\big)\big(p_2x + (1 - p_2)\big)\big(p_3x + (1 - p_3)\big)$ are

$$r_i = \frac{-(1 - p_i)}{p_i} = 1 - \frac{1}{p_i}, \quad i = 1,2,3.$$

From the form

$$\begin{aligned} Q(x) &= \frac{1}{60}(x^3 + 9x^2 + 26x + 24) \\ &= \frac{1}{60}(x + 2)(x + 3)(x + 4) \end{aligned}$$

we can also determine that the roots of $Q(x)$ are $-2, -3$ and -4. Therefore,

$$\begin{aligned} -2 = r_1 &= 1 - \frac{1}{p_1} \Leftrightarrow p_1 = \frac{1}{3} \\ -3 = r_2 &= 1 - \frac{1}{p_2} \Leftrightarrow p_2 = \frac{1}{4} \\ -4 = r_3 &= 1 - \frac{1}{p_3} \Leftrightarrow p_3 = \frac{1}{5}. \end{aligned}$$

■

Is there a Theory of Equations "shortcut" for solving the general problem

Find p_1, p_2 and p_3 if

$$\begin{aligned} a_0 &= (1 - p_1)(1 - p_2)(1 - p_3) \\ a_1 &= p_1(1 - p_2)(1 - p_3) + (1 - p_1)p_2(1 - p_3) + (1 - p_1)(1 - p_2)p_3 \\ a_2 &= p_1p_2(1 - p_3) + p_1(1 - p_2)p_3 + (1 - p_1)p_2p_3 \\ a_3 &= p_1p_2p_3. \end{aligned}$$

Solution

Define the polynomial $Q(x) = \big(p_1x + (1 - p_1)\big)\big(p_2x + (1 - p_2)\big)\big(p_3x + (1 - p_3)\big)$.

Notice that the coefficient of x^i of $Q(x)$ is a_i. That is,

$$Q(x) = a_3x^3 + a_2x^2 + a_1x^1 + a_0.$$

But also notice that the roots r_1, r_2, r_3 of $Q(x)$ are $r_i = 1 - \frac{1}{p_i}$. So

$$p_i = \frac{1}{1 - r_i} \text{ where } r_1, r_2, r_3 \text{ are the roots of } Q(x).$$

7.9 Assortment of Problems

Exercises for Chapter 7, Section 9

1. Ten tickets are numbered $1,2,3, \ldots, 10$ respectively. Five tickets are selected at a time with replacement. What is the probability that the highest number appearing on a selected ticket is k? (Source: *Probability*, James R. Gray, 1967, page 15, Example 1.12)

(5T175) A fair coin is flipped 7 times. Determine exactly the probability that two heads never happen on two successive tosses. Express the answer as a quotient of two relative prime integers.

Solution

Let n be the number of tosses and $f(n)$ the number of sequences without two consecutive heads. Then $f(1) = 2$, H or T; $f(2) = 3$, HT, TH, or TT; and $f(3) = 5, HTH, TTH, HTT, THT$, or TTT. To find $f(4)$, add TH to the sequences of length two or add T to the sequences of length 3 to obtain: $HTTH, THTH, TTTH, HTHT, TTHT, HTTT, THTT$, or $TTTT$. So $f(4) = 3 + 5 = 8$ and $f(5) = 5 + 8 = 13$, $f(6) = 8 + 13 = 21$, and $f(7) = 13 + 21 = 34$. There are $2^7 = 128$ possible sequences, so the probability of no consecutive Heads in seven flips of a coin is $34/128 = 17/64$.

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