Random Compositions of an Integer.

A ***composition*** of n is a partition of n where the order of the parts is taken into account.

$e.g.$ The 8 compositions of $n = 4$ are

$1+1+1+1$	$1+2+1$	$2+2$	$3+1$
$1+1+2$	$2+1+1$	$1+3$	4

In contrast there are only 5 partitions of $n = 4$, namely

$1+1+1+1$	$2+2$	4
$1+1+2$	$1+3$	

We refer to each integer of a composition on n as a ***part***. We refer to the number of times a given integer occurs as a part in a composition as the ***multiplicity*** of that integer in that composition.

It is straightforward to show that there are $\binom{n-1}{t-1}$ compositions of n with exactly t parts and that there are $\sum_{t=1}^{n}\binom{n-1}{t-1} = 2^{n-1}$ compositions of n in total.

If a composition of n is picked uniformly at random from the set of all $\binom{n-1}{t-1}$ compositions of n with exactly t parts, we will refer to this as a ***random composition with t parts***.

If a composition of n is picked uniformly at random from the set of all 2^{n-1} compositions of n, we will refer to this as a ***random composition***.

We will need the following definitions.

$\mathbb{S}^\infty$: the **infinite** product space $\{0,1,\dots\} \times \{0,1,\dots\} \times \cdots$

$\mathbb{S}^\infty_n$: the set of all vectors $(s_1,s_2,\dots)$ in $\mathbb{S}^\infty$ such that $1s_1 + 2s_2 + \dots = n$

$\mathbb{S}^\infty_{n,t}$: the set of all vectors $(s_1,s_2,\dots)$ in $\mathbb{S}^\infty$ such that $\begin{array}{c} 1s_1 + 2s_2 + \dots = n \\ s_1 + s_2 + \dots = t \end{array}$

For any $\mathcal{A} \subseteq \mathbb{S}^\infty$ define $\mathcal{A}_n = \mathcal{A} \cap \mathbb{S}^\infty_n$ and $\mathcal{A}_{n,t} = \mathcal{A} \cap \mathbb{S}^\infty_{n,t}$.

We note that the condition that $1s_1 + 2s_2 + \dots = n$ implies that $s_j = 0$ for all $j > n$. Hence all vectors in $\mathcal{A}_n$ and $\mathcal{A}_{n,t}$ are of the form $(a_1,a_2,\dots,a_n,0,0,\dots)$.

For all $\mathcal{A} \neq (0, 0, \dots)$, let $\mathbb{A}_n$ be the collection of n-dimensional vectors formed by taking each infinite-dimensional vector in $\mathcal{A}_n$ and truncating after a_n. So for example,

$$(a_1, a_2, \dots, a_n, 0, 0, \dots) \rightarrow (a_1, a_2, \dots, a_n)$$

Define $\mathbb{A}_{n,t}$ in the same way by truncating after a_n in $\mathcal{A}_{n,t}$.

It is necessary to separate out the case $\mathcal{A} = (0, 0, \dots)$ because in this case

$$n = 1s_1 + 2s_2 + \dots = (1 \cdot 0) + (2 \cdot 0) + \dots = 0$$

and it does not make notational sense to use n as an index for $\mathbb{A}_n$.

Let X_j equal the multiplicity of j in a random composition of n with t parts. It follows directly that

$$P(X_1 = x_1, \dots, X_n = x_n) = \begin{cases} \frac{t!}{x_1! \cdots x_n!} \frac{1}{\binom{n-1}{t-1}} & \substack{1x_1 + 2x_2 + \dots + nx_n = n \\ x_1 + x_2 + \dots + x_n = t \\ x_j \in \{0,1,\dots,n\}\ \forall j} \\ 0 & \text{else} \end{cases}$$

and if we let W_j equal the multiplicity of j in a random composition of n, that

$$P(W_1 = w_1, \dots, W_n = w_n) = \begin{cases} \frac{(w_1 + \dots + w_n)!}{w_1! \cdots w_n!} \left(\frac{1}{2}\right)^{n-1} & \substack{1w_1 + 2w_2 + \dots + nw_n = n \\ w_j \in \{0,1,\dots,n\}\ \forall j} \\ 0 & \text{else} \end{cases}$$

Let $Y_1, Y_2, \dots$ be an infinite sequence of **independent** Poisson random variables where

$$P(Y_j = y) = \frac{\exp(-\lambda^j \theta)(\lambda^j \theta)^y}{y!} \qquad y = 0, 1, 2, \dots \text{ and } j = 0, 1, 2, \dots$$

Theorem 1 and its corollary which follow demonstrate how problems involving the dependent X_j's and the dependent W_j's can be reformulated in terms of the independent Y_j's.

Theorem 1.

$$E(g^*(X_1,...,X_n)) = \frac{1}{n!\binom{n-1}{t-1}}\left(\frac{d^n}{d\lambda^n}\frac{d^t}{d\theta^t}\left(e^{\left(\frac{\theta\lambda}{1-\lambda}\right)}E(g(Y_1,Y_2,\dots))\right)\bigg|_{\substack{\lambda=0\\ \theta=0}}\right)$$

Explain the relationship between $g^*()$ and $g()$.

For $\mathcal{A} \neq (0,0,\dots)$

$$P((X_1,...,X_n)\in \mathbb{A}_{n,t}) = \frac{1}{n!\binom{n-1}{t-1}}\left(\frac{d^n}{d\lambda^n}\frac{d^t}{d\theta^t}\left(e^{\left(\frac{\theta\lambda}{1-\lambda}\right)}P((Y_1,Y_2,\dots)\in\mathcal{A})\right)\bigg|_{\substack{\lambda=0\\ \theta=0}}\right)$$

Corollary 1.

$$P((W_1,...,W_n)\in\mathbb{A}_n) = \sum_{t=1}^{n} P((X_1,...,X_n)\in\mathbb{A}_{n,t})\frac{\binom{n-1}{t-1}}{2^{n-1}}$$

Proof of Theorem 1b. no no no no no!!!!!

Let y_j be a nonnegative integer for $j = 0, 1, \cdots$. Then

$$P(Y_1 = y_1, Y_2 = y_2,\dots) = \frac{e^{\left(-\sum_{j=1}^{\infty}\lambda^j\right)}\lambda^{\left(\sum_{j=1}^{\infty} jy_j\right)}}{\prod_{j=1}^{\infty}(y_j)!} = \frac{e^{\left(\frac{-\lambda}{1-\lambda}\right)}\lambda^{\left(\sum_{j=1}^{\infty} jy_j\right)}}{\prod_{j=1}^{\infty}(y_j)!}$$

and

$$
\begin{aligned}
P\left(\sum_{j=1}^{\infty} jY_j = n\right) &= \sum_{\substack{(y_1,y_2,\dots)\ni \\ \sum_{j=1}^{\infty} jy_j=n}} \cdots \sum P(Y_1 = y_1, Y_2 = y_2, \dots) \\
&= e^{\left(\frac{-\lambda}{1-\lambda}\right)} \lambda^n \left(\sum_{\substack{(y_1,y_2,\dots)\ni \\ \sum_{j=1}^{\infty} jy_j=n}} \cdots \sum \frac{1}{\prod_{j=1}^{\infty} (y_j)!} \right) \\
&= e^{\left(\frac{-\lambda}{1-\lambda}\right)} \lambda^n \left(\sum_{\substack{(y_1,y_2,\dots)\ni \\ \sum_{j=1}^{n} jy_j=n}} \cdots \sum \frac{1}{\prod_{j=1}^{n} (y_j)!} \right) \\
&= e^{\left(\frac{-\lambda}{1-\lambda}\right)} \lambda^n \, L(n)
\end{aligned}
$$

where $L(n) = \sum_{j=1}^{n} \frac{1}{j!} \binom{n-1}{j-1}$. It follows that for $\mathcal{A} \neq (0, 0, \dots)$

$$
\begin{aligned}
P((Y_1, Y_2, \dots) \in \mathcal{A}) &= \sum_{n=0}^{\infty} P\left((Y_1, Y_2, \dots) \in \mathcal{A} \mid \sum_{j=1}^{\infty} jY_j = n\right) P\left(\sum_{j=1}^{\infty} jY_j = n\right) \\
&= \sum_{n=1}^{\infty} \sum_{\mathcal{A}_n} \left(\frac{\frac{e^{\left(\frac{-\lambda}{1-\lambda}\right)} \lambda^n}{\prod_{j=1}^{\infty} (y_j)!}}{e^{\left(\frac{-\lambda}{1-\lambda}\right)} \lambda^n L(n)} \right) \left(e^{\left(\frac{-\lambda}{1-\lambda}\right)} \lambda^n L(n)\right) \\
&= \sum_{n=1}^{\infty} \sum_{\mathbb{A}_n} \left(\frac{\frac{1}{\prod_{j=1}^{n} (y_j)!}}{L(n)} \right) \left(e^{\left(\frac{-\lambda}{1-\lambda}\right)} \lambda^n L(n)\right) \\
&= \sum_{n=1}^{\infty} P((X_1, \dots, X_n) \in \mathbb{A}_n) \left(e^{\left(\frac{-\lambda}{1-\lambda}\right)} \lambda^n L(n)\right)
\end{aligned}
$$

where the vector $(X_1, \dots, X_n)$ is a random composition of n.

Therefore,

$$
e^{\left(\frac{\lambda}{1-\lambda}\right)} P((Y_1, Y_2, \dots) \in \mathcal{A}) = \sum_{n=1}^{\infty} P((X_1, \dots, X_n) \in \mathbb{A}_n)(L(n))\lambda^n
$$

and

$$\frac{d^r}{d\lambda^r}\Big(e^{(\frac{\lambda}{1-\lambda})}P((Y_1,Y_2,\dots)\in\mathcal{A})\Big)\Big|_{\lambda=0}$$

$$=\sum_{n=1}^{\infty}P((X_1,...,X_n)\in\mathbb{A}_n)(L(n))r!\mathrm{I}_{\{r\}}(n)$$

$$=P((X_1,...,X_r)\in\mathbb{A}_r)(L(r))r!$$

Hence, for $\mathcal{A}\neq(0,0,\dots)$

$$P((X_1,...,X_n)\in\mathbb{A}_n)=\frac{1}{n!L(n)}\left(\frac{d^n}{d\lambda^n}\Big(e^{(\frac{\lambda}{1-\lambda})}P((Y_1,Y_2,\dots)\in\mathcal{A})\Big)\Big|_{\lambda=0}\right)$$

where the vector $(X_1,...,X_n)$ is a random composition of n. □

Proof of Theorem 1.

Let y_j be a nonnegative integer for $j=0,1,\cdots$. Then

$$P(Y_1=y_1,Y_2=y_2,\dots)=\frac{e^{\left(-\theta\sum_{j=1}^{\infty}\lambda^j\right)}\lambda^{\left(\sum_{j=1}^{\infty}jy_j\right)}\theta^{\left(\sum_{j=1}^{\infty}y_j\right)}}{\prod_{j=1}^{\infty}(y_j)!}$$

$$=\frac{e^{(\frac{-\theta\lambda}{1-\lambda})}\lambda^{\left(\sum_{j=1}^{\infty}jy_j\right)}\theta^{\left(\sum_{j=1}^{\infty}y_j\right)}}{\prod_{j=1}^{\infty}(y_j)!}$$

and

$$P\left(\sum_{j=1}^{\infty} jY_j = n \text{ and } \sum_{j=1}^{\infty} Y_j = t\right) = \sum_{\substack{(y_1,y_2,\dots)\ni \\ \sum_{j=1}^{\infty} jy_j=n \\ \sum_{j=1}^{\infty} y_j=t}} \cdots \sum P(Y_1 = y_1, Y_2 = y_2, \dots)$$

$$= \frac{e^{\left(\frac{-\theta\lambda}{1-\lambda}\right)}\lambda^n\theta^t}{t!}\left(\sum_{\substack{(y_1,y_2,\dots)\ni \\ \sum_{j=1}^{\infty} jy_j=n \\ \sum_{j=1}^{\infty} y_j=t}} \cdots \sum \frac{t!}{\prod_{j=1}^{\infty}(y_j)!}\right)$$

$$= \frac{e^{\left(\frac{-\theta\lambda}{1-\lambda}\right)}\lambda^n\theta^t}{t!}\left(\sum_{\substack{(y_1,y_2,\dots)\ni \\ \sum_{j=1}^{n} jy_j=n \\ \sum_{j=1}^{n} y_j=t}} \cdots \sum \frac{t!}{\prod_{j=1}^{n}(y_j)!}\right)$$

$$= \frac{e^{\left(\frac{-\theta\lambda}{1-\lambda}\right)}\lambda^n\theta^t}{t!}\binom{n-1}{t-1}$$

It follows that for $\mathcal{A} \neq (0, 0, \dots)$

$$P((Y_1, Y_2, \dots) \in \mathcal{A}) = \sum_{n=0}^{\infty}\sum_{t=0}^{\infty} P\left((Y_1, Y_2, \dots) \in \mathcal{A} \mid \sum_{j=1}^{\infty} jY_j = n, \sum_{j=1}^{\infty} Y_j = t\right) P\left(\sum_{j=1}^{\infty} jY_j = n, \sum_{j=1}^{\infty} Y_j = t\right)$$

$$= \sum_{n=1}^{\infty}\sum_{t=1}^{\infty}\sum_{\mathcal{A}_{n,t}}\left(\frac{\frac{e^{\left(\frac{-\theta\lambda}{1-\lambda}\right)}\lambda^n\theta^t}{\prod_{j=1}^{\infty}(y_j)!}}{\frac{e^{\left(\frac{-\theta\lambda}{1-\lambda}\right)}\lambda^n\theta^t}{t!}\binom{n-1}{t-1}}\right)\left(\frac{e^{\left(\frac{-\theta\lambda}{1-\lambda}\right)}\lambda^n\theta^t}{t!}\binom{n-1}{t-1}\right)$$

$$= \sum_{n=1}^{\infty}\sum_{t=1}^{\infty}\sum_{\mathbb{A}_{n,t}}\left(\frac{\frac{t!}{\prod_{j=1}^{n}(y_j)!}}{\binom{n-1}{t-1}}\right)\left(\frac{e^{\left(\frac{-\theta\lambda}{1-\lambda}\right)}\lambda^n\theta^t}{t!}\binom{n-1}{t-1}\right)$$

$$= \sum_{n=1}^{\infty}\sum_{t=1}^{\infty} P((X_1, \dots, X_n) \in \mathbb{A}_{n,t})\left(\frac{e^{\left(\frac{-\theta\lambda}{1-\lambda}\right)}\lambda^n\theta^t}{t!}\binom{n-1}{t-1}\right)$$

where the vector $(X_1, \dots, X_n)$ is a random composition of n with t parts.

Therefore,

$$e^{\left(\frac{\theta\lambda}{1-\lambda}\right)} P((Y_1, Y_2, \dots) \in \mathcal{A}) = \sum_{n=1}^{\infty} \sum_{t=1}^{\infty} P((X_1, \dots, X_n) \in \mathbb{A}_{n,t}) \left(\frac{1}{t!} \binom{n-1}{t-1} \right) \lambda^n \theta^t$$

and

$$\begin{aligned} &\frac{d^r}{d\lambda^r} \frac{d^s}{d\theta^s} \left(e^{\left(\frac{\theta\lambda}{1-\lambda}\right)} P((Y_1, Y_2, \dots) \in \mathcal{A}) \right) \Bigg|_{\substack{\lambda=0 \\ \theta=0}} \\ &\quad = \sum_{n=1}^{\infty} \sum_{t=1}^{\infty} P((X_1, \dots, X_n) \in \mathbb{A}_{n,t}) \left(\frac{1}{t!} \binom{n-1}{t-1} \right) r! \mathrm{I}_{\{r\}}(n) \, s! \mathrm{I}_{\{s\}}(t) \\ &\quad = P((X_1, \dots, X_r) \in \mathbb{A}_{r,s}) \left(\frac{1}{s!} \binom{r-1}{s-1} \right) r! \, s! \end{aligned}$$

Hence, for $\mathcal{A} \neq (0, 0, \dots)$

$$P((X_1, \dots, X_n) \in \mathbb{A}_{n,t}) = \frac{1}{n! \binom{n-1}{t-1}} \left(\frac{d^n}{d\lambda^n} \frac{d^t}{d\theta^t} \left(e^{\left(\frac{\theta\lambda}{1-\lambda}\right)} P((Y_1, Y_2, \dots) \in \mathcal{A}) \right) \Bigg|_{\substack{\lambda=0 \\ \theta=0}} \right)$$

where the vector $(X_1, \dots, X_n)$ is a random composition of n with t parts. □

Proof of Corollary 1.

$$\begin{aligned} P((W_1, \dots, W_n) \in \mathbb{A}_n) &= \sum_{t=1}^{n} P((W_1, \dots, W_n) \in \mathbb{A}_n | T = t) \; P(T = t) \\ &= \sum_{t=1}^{n} P((W_1, \dots, W_n) \in \mathbb{A}_n | T = t) \; \frac{\binom{n-1}{t-1}}{2^{n-1}} \\ &= \sum_{t=1}^{n} P((X_1, \dots, X_n) \in \mathbb{A}_{n,t}) \; \frac{\binom{n-1}{t-1}}{2^{n-1}} \end{aligned}$$ □

Theorem 2.

Let $\mathbb{S}^n$ be the product space $\{1,2,\dots\} \times \cdots \times \{1,2,\dots\}$ and let $\mathbb{S}_n^t$ be the set of all vectors $(s_1,\dots,s_t)$ in $\mathbb{S}^t$ such that $s_1 + \dots + s_t = n$.

Define $(X_{1,n}, \ldots, X_{t,n})$ to be that random vector which is equally likely to be any value in $\mathbb{S}_n^t$ and define $Y_1, \ldots, Y_t$ to be iid geometric random variables on $y \in \{1,2,\ldots\}$ with parameter p,

i.e.

$$P(Y = y) \ = \ p(1-p)^{y-1} \qquad y \in \{1,2,\ldots\} \text{ and } 0 \le p \le 1.$$

Then for $n \ge t$,

$$E(g^*(X_{1,n}, \ldots, X_{t,n})) = \frac{(-1)^n}{\binom{n-1}{t-1} n!} \cdot \frac{\mathrm{d}^n}{\mathrm{d}p^n} \left(\left(\frac{1-p}{p} \right)^t E(g(Y_1, \ldots, Y_t)) \right) \Bigg|_{p=1}$$

Let $\mathcal{A} \subset \mathbb{S}^t$ and define $\mathcal{A}_n = \mathcal{A} \cap \mathbb{S}_n^t$, then

$$P((X_{1,n}, \ldots, X_{t,n}) \in \mathcal{A}_n) = \frac{(-1)^n}{\binom{n-1}{t-1} n!} \cdot \frac{\mathrm{d}^n}{\mathrm{d}p^n} \left(\left(\frac{1-p}{p} \right)^t P((Y_1, \ldots, Y_t) \in \mathcal{A}) \right) \Bigg|_{p=1}$$

$$= \frac{(-1)^n}{\binom{n-1}{t-1} n!} \cdot \sum_{j=0}^{\infty} \binom{t+j-1}{j} \frac{\mathrm{d}^n}{\mathrm{d}p^n} \left((1-p)^{t+j} P((Y_1, \ldots, Y_t) \in \mathcal{A}) \right) \Big|_{p=1}$$

Applications

Problem 1.

How many parts would we expect to equal i in a random composition of n with t parts? That is, find $E(X_i)$. As a check on your answer make sure that $\sum_{i=1}^{n} E(X_i) = t$.

Answer

$$E(X_i) = \begin{cases} \frac{t}{\binom{n-1}{t-1}} \binom{n-i-1}{t-2} & {i \in \{1,\ldots,n-1\} \atop t \in \{1,\ldots,n-i+1\}} \\ 1 & i = n,\ t = 1 \\ 0 & \text{else} \end{cases}$$

Proof

In the case $i \in \{1, \ldots, n-1\}$ and $t \in \{1, \ldots, n-i+1\}$ we have

$$\begin{aligned}
E(X_i) &= \frac{1}{n!\binom{n-1}{t-1}} \left(\frac{d^n}{d\lambda^n} \frac{d^t}{d\theta^t} \left(e^{\left(\frac{\theta\lambda}{1-\lambda}\right)} E(Y_i) \right) \Big|_{\substack{\lambda=0 \\ \theta=0}} \right) \\
&= \frac{1}{n!\binom{n-1}{t-1}} \left(\frac{d^n}{d\lambda^n} \frac{d^t}{d\theta^t} \left(e^{\left(\frac{\theta\lambda}{1-\lambda}\right)} (\theta\lambda^i) \right) \Big|_{\substack{\lambda=0 \\ \theta=0}} \right) \\
&= \frac{1}{n!\binom{n-1}{t-1}} \left(\frac{d^n}{d\lambda^n} \left(\lambda^i \frac{d^t}{d\theta^t} \left(\theta e^{\theta\left(\frac{\lambda}{1-\lambda}\right)} \right) \Big|_{\theta=0} \right) \Big|_{\lambda=0} \right) \\
&= \frac{1}{n!\binom{n-1}{t-1}} \left(\frac{d^n}{d\lambda^n} \left(\lambda^i t \left(\frac{\lambda}{1-\lambda} \right)^{t-1} \right) \Big|_{\lambda=0} \right) \\
&= \frac{t}{n!\binom{n-1}{t-1}} \left(\sum_{j=0}^{\infty} \binom{(t-1)+j-1}{j} \frac{d^n}{d\lambda^n} \left(\lambda^{i+t-1+j} \right) \Big|_{\lambda=0} \right) \\
&= \frac{t}{\binom{n-1}{t-1}} \left(\sum_{j=0}^{\infty} \binom{(t-1)+j-1}{j} \mathrm{I}_{\{n-i-t+1\}}(j) \right) \\
&= \frac{t}{\binom{n-1}{t-1}} \binom{n-i-1}{t-2}
\end{aligned}$$

As a check on our answer, we note that for $t \in \{2, \ldots, n\}$

$$\sum_{i=1}^{n} E(X_i) = \sum_{i=1}^{n-t+1} \frac{t}{\binom{n-1}{t-1}} \binom{n-i-1}{t-2} + \sum_{i=n-t+2}^{n} 0$$
$$= \frac{t}{\binom{n-1}{t-1}} \sum_{i=1}^{n-t+1} \binom{n-i-1}{t-2}$$
$$= \frac{t}{\binom{n-1}{t-1}} \sum_{i=0}^{n-t} \binom{(t-2)+i}{t-2}$$
$$= t$$

In this last step we used Identity 1.48 of Gould's *Combinatorial Identities*. Namely,

$$\sum_{k=0}^{n} \binom{k+x}{r} = \binom{n+x+1}{r+1} - \binom{x}{r+1}.$$

In the case $t = 1$ we have trivially that

$$\sum_{i=1}^{n} E(X_i) = \sum_{i=1}^{n-1} 0 + \sum_{i=n}^{n} 1 = 1 = t$$

Problem 2.

How many compositions of n with t parts are there such that every part takes on a value between l and u inclusive?

Answer

$$\sum_{i=0}^{t} (-1)^i \binom{t}{i} \binom{n - t(l-1) - i(u-l+1) - 1}{t-1}$$

Notes:

The above solution simplifies to $\binom{n-t(l-1)-1}{t-1}$ in the special case $u = n$ and to $\binom{t}{n-t}$ in the special case $l = 1$ and $u = 2$.

That is,

$$\sum_{i=0}^{t} (-1)^i \binom{t}{i} \binom{n - t(l-1) - i(n-l+1) - 1}{t-1} = \binom{n - t(l-1) - 1}{t-1}$$

and

$$\sum_{i=0}^{t}(-1)^i\binom{t}{i}\binom{n-2i-1}{t-1}=\binom{t}{n-t}$$

Proof

This problem can be easily handled via generating functions but we will use Theorem 1 to illustrate its application.

Let $\mathcal{B} = \{\{1,\ldots,l-1\}\cup\{u+1,\ldots,n\}\}$.

Clearly the solution equals $\binom{n-1}{t-1}\cdot P(X_j=0, j\in\mathcal{B})$.

Now define

$$\mathbb{A}_{n,t}=\{(a_1,a_2,\ldots,a_n)|1a_1+\ \ldots\ +na_n=n, a_1+\ \ldots\ +a_n=t \text{ and } a_j=0, j\in\mathcal{B}\}$$
$$\mathcal{A}_{n,t}=\{a_1,a_2,\ldots,a_n,0,0,\ldots|1a_1+\ \ldots\ +na_n=n, a_1+\ \ldots\ +a_n=t \text{ and } a_j=0, j\in\mathcal{B}\}$$
$$\mathcal{A}=\{a_1,a_2,\ldots,a_n,a_{n+1},a_{n+2},\ldots|\ a_j=0, j\in\mathcal{B}\}$$

Then,

$$P(X_j=0,j\in\mathcal{B})=P((X_1,\ldots,X_n)\in\mathbb{A}_{n,t})$$
$$=\frac{1}{n!\binom{n-1}{t-1}}\left(\frac{d^n}{d\lambda^n}\frac{d^t}{d\theta^t}\left(e^{\left(\frac{\theta\lambda}{1-\lambda}\right)}P((Y_1,Y_2,\ldots)\in\mathcal{A})\right)\Big|_{\substack{\lambda=0\\ \theta=0}}\right)$$

where

$$P((Y_1,Y_2,\ldots)\in\mathcal{A})=\left(\prod_{i\notin\mathcal{B}}P(Y_i\in\{0,1,\ldots\})\right)P(Y_j=0,j\in\mathcal{B})$$
$$=\prod_{j\in\mathcal{B}}P(Y_j=0)=\prod_{j\in\mathcal{B}}\left(\frac{e^{(-\lambda^j\theta)}(\lambda^j\theta)^0}{0!}\right)$$

Therefore,

$$
\begin{aligned}
P(X_j = 0, j \in \mathcal{B}) &= \frac{1}{n!\binom{n-1}{t-1}} \left(\frac{d^n}{d\lambda^n} \frac{d^t}{d\theta^t} \left(e^{\left(\frac{\theta\lambda}{1-\lambda}\right)} \prod_{j\in\mathcal{B}} e^{(-\lambda^j \theta)} \right) \Bigg|_{\substack{\lambda=0\\ \theta=0}} \right) \\
&= \frac{1}{n!\binom{n-1}{t-1}} \left(\frac{d^n}{d\lambda^n} \frac{d^t}{d\theta^t} \left(e^{\theta\left(\frac{\lambda}{1-\lambda} - (\lambda^1 + \ldots + \lambda^{l-1}) - (\lambda^{u+1} + \ldots + \lambda^n)\right)} \right) \Bigg|_{\substack{\lambda=0\\ \theta=0}} \right) \\
&= \frac{1}{n!\binom{n-1}{t-1}} \left(\frac{d^n}{d\lambda^n} \frac{d^t}{d\theta^t} \left(e^{\theta\left(\frac{\lambda}{1-\lambda} - \frac{\lambda}{1-\lambda}(1-\lambda^{l-1}) - \frac{\lambda^{u+1}}{1-\lambda}(1-\lambda^{n-u})\right)} \right) \Bigg|_{\substack{\lambda=0\\ \theta=0}} \right) \\
&= \frac{1}{n!\binom{n-1}{t-1}} \left(\frac{d^n}{d\lambda^n} \frac{d^t}{d\theta^t} \left(e^{\theta\left(\frac{\lambda^l - \lambda^{u+1} + \lambda^{n+1}}{1-\lambda}\right)} \right) \Bigg|_{\substack{\lambda=0\\ \theta=0}} \right) \\
&= \frac{1}{n!\binom{n-1}{t-1}} \frac{d^n}{d\lambda^n} \left(\left(\frac{\lambda^l - \lambda^{u+1} + \lambda^{n+1}}{1-\lambda} \right)^t \right) \Bigg|
\end{aligned}
$$

At this point we can avoid some unnecessary algebra by noticing that the term $\frac{\lambda^{n+1}}{1-\lambda}$ can be dropped in as much its expansion will only lead to terms λ^c, $c > n$ and the n^{th} derivative of these terms evaluated at $\lambda = 0$ will all equal 0.

$$
\begin{aligned}
P(X_j = 0, j \in \mathcal{B}) &= \frac{1}{n!\binom{n-1}{t-1}} \frac{d^n}{d\lambda^n} \left(\left(\frac{\lambda^l - \lambda^{u+1} + \lambda^{n+1}}{1-\lambda} \right)^t \right) \Bigg|_{\lambda=0} \\
&= \frac{1}{n!\binom{n-1}{t-1}} \frac{d^n}{d\lambda^n} \left(\left(\frac{\lambda^l - \lambda^{u+1}}{1-\lambda} \right)^t \right) \Bigg|_{\lambda=0} \\
&= \frac{1}{n!\binom{n-1}{t-1}} \sum_{i=0}^{t} (-1)^i \binom{t}{i} \frac{d^n}{d\lambda^n} \left(\frac{\lambda^{lt+(u+1-l)i}}{(1-\lambda)^t} \right) \Bigg|_{\lambda=0} \\
&= \frac{1}{n!\binom{n-1}{t-1}} \sum_{i=0}^{t} \sum_{j=0}^{\infty} (-1)^i \binom{t}{i} \binom{t+j-1}{j} \frac{d^n}{d\lambda^n} \left(\lambda^{lt+(u+1-l)i+j} \right) \Bigg|_{\lambda=0} \\
&= \frac{1}{n!\binom{n-1}{t-1}} \sum_{i=0}^{t} \sum_{j=0}^{\infty} (-1)^i \binom{t}{i} \binom{t+j-1}{j} n! \left(\mathrm{I}_{\{n-lt-(u+1-l)i\}}(j) \right) \\
&= \frac{1}{\binom{n-1}{t-1}} \sum_{i=0}^{t} (-1)^i \binom{t}{i} \binom{t + (n-lt-(u+1-l)i) - 1}{n - lt - (u+1-l)i} \\
&= \frac{1}{\binom{n-1}{t-1}} \sum_{i=0}^{t} (-1)^i \binom{t}{i} \binom{n - t(l-1) - i(u-l+1) - 1}{t-1} \qquad \square
\end{aligned}
$$

Problem 3.

How many compositions of n with t parts are there for which the multiplicity of the integer j equals k ?

Answer

$$\sum_{i=0}^{t-k}(-1)^i\binom{t}{k}\binom{t-k}{i}\binom{n-j(i+k)-1}{n-j(i+k)-(t-k-i)}$$

Proof

Clearly the solution equals $\binom{n-1}{t-1}\cdot P(X_j=k)$.

Now define

$$\begin{aligned}
\mathbb{A}_{n,t} &= \{(a_1,a_2,\ldots,a_n)|1a_1+\ \ldots+na_n=n, a_1+\ \ldots+a_n=t \text{ and } a_j=k\}\\
\mathcal{A}_{n,t} &= \{a_1,a_2,\ldots,a_n,0,0,\ldots|1a_1+\ \ldots+na_n=n, a_1+\ \ldots+a_n=t \text{ and } a_j=k\}\\
\mathcal{A} &= \{a_1,a_2,\ldots,a_n,a_{n+1},a_{n+2},\ldots|\ a_j=k\}
\end{aligned}$$

Then,

$$\begin{aligned}
P(X_j=k) &= P((X_1,\ldots,X_n)\in\mathbb{A}_{n,t})\\
&= \frac{1}{n!\binom{n-1}{t-1}}\left(\frac{d^n}{d\lambda^n}\frac{d^t}{d\theta^t}\left(e^{\left(\frac{\theta\lambda}{1-\lambda}\right)}P((Y_1,Y_2,\ldots)\in\mathcal{A})\right)\Bigg|_{\substack{\lambda=0\\\theta=0}}\right)
\end{aligned}$$

where

$$\begin{aligned}
P((Y_1,Y_2,\ldots)\in\mathcal{A}) &= \left(\prod_{i\neq j}P(Y_i\in\{0,1,\ldots\})\right)P(Y_j=k)\\
&= P(Y_j=k)=\frac{e^{(-\lambda^j\theta)}(\lambda^j\theta)^k}{k!}
\end{aligned}$$

Therefore,

$$
\begin{aligned}
P(X_j = k) &= \frac{1}{n!\binom{n-1}{t-1}} \left(\frac{d^n}{d\lambda^n} \frac{d^t}{d\theta^t} \left(e^{\left(\frac{\theta\lambda}{1-\lambda}\right)} \frac{e^{(-\lambda^j\theta)}(\lambda^j\theta)^k}{k!} \right) \Bigg|_{\substack{\lambda=0\\ \theta=0}} \right) \\
&= \frac{1}{n!k!\binom{n-1}{t-1}} \left(\frac{d^n}{d\lambda^n} \frac{d^t}{d\theta^t} \left(e^{\theta\left(\frac{\lambda}{1-\lambda}-\lambda^j\right)} \theta^k \lambda^{jk} \right) \Big|_{\substack{\lambda=0\\ \theta=0}} \right) \\
&= \frac{\binom{t}{k}}{n!\binom{n-1}{t-1}} \left(\frac{d^n}{d\lambda^n} \left(\left(\frac{\lambda}{1-\lambda} - \lambda^j \right)^{t-k} \lambda^{jk} \right) \Bigg|_{\lambda=0} \right) \\
&= \frac{\binom{t}{k}}{n!\binom{n-1}{t-1}} \left(\sum_{i=0}^{t-k} (-1)^i \binom{t-k}{i} \frac{d^n}{d\lambda^n} \left(\left(\frac{\lambda}{1-\lambda} \right)^{t-k-i} \left(\lambda^j\right)^i \lambda^{jk} \right) \Bigg|_{\lambda=0} \right) \\
&= \frac{\binom{t}{k}}{n!\binom{n-1}{t-1}} \left(\sum_{i=0}^{t-k} \sum_{l=0}^{\infty} (-1)^i \binom{t-k}{i} \binom{t-k-i+l-1}{l} \frac{d^n}{d\lambda^n} \left(\lambda^{t-k-i} \lambda^l \lambda^{ji} \lambda^{jk} \right) \Bigg|_{\lambda=0} \right) \\
&= \frac{\binom{t}{k}}{n!\binom{n-1}{t-1}} \left(\sum_{i=0}^{t-k} \sum_{l=0}^{\infty} (-1)^i \binom{t-k}{i} \binom{t-k-i+l-1}{l} \frac{d^n}{d\lambda^n} \left(\lambda^{t-k-i+l+j(i+k)} \right) \Bigg|_{\lambda=0} \right) \\
&= \frac{\binom{t}{k}}{n!\binom{n-1}{t-1}} \left(\sum_{i=0}^{t-k} \sum_{l=0}^{\infty} (-1)^i \binom{t-k}{i} \binom{t-k-i+l-1}{l} n!\, \mathrm{I}_{\{n-t+k+i-j(i+k)\}}(l) \right) \\
&= \frac{\binom{t}{k}}{\binom{n-1}{t-1}} \sum_{i=0}^{t-k} (-1)^i \binom{t-k}{i} \binom{t-k-i+(n-t+k+i-j(i+k))-1}{n-t+k+i-j(i+k)} \\
&= \frac{1}{\binom{n-1}{t-1}} \sum_{i=0}^{t-k} (-1)^i \binom{t}{k} \binom{t-k}{i} \binom{n-j(i+k)-1}{n-j(i+k)-(t-k-i)}
\end{aligned}
$$

Hence there are

$$
\sum_{i=0}^{t-k} (-1)^i \binom{t}{k} \binom{t-k}{i} \binom{n-j(i+k)-1}{n-j(i+k)-(t-k-i)}
$$

compositions of n with t parts for which the multiplicity of the integer j equals k. □

Problem 4.

How many multiplicities in a random composition of n with t parts do we expect to equal k ?

Answer

$$\frac{1}{\binom{n-1}{t-1}} \sum_{j=1}^{n} \sum_{i=0}^{t-k} (-1)^i \binom{t}{k} \binom{t-k}{i} \binom{n-j(i+k)-1}{n-j(i+k)-(t-k-i)}$$

Proof

The number of multiplicities equaling k in a random composition of n with t parts is

$$\sum_{j=1}^{n} \mathrm{I}_{\{k\}}(X_j)$$

Therefore,

$$\begin{aligned} E\left(\sum_{j=1}^{n} \mathrm{I}_{\{k\}}(X_j)\right) &= \sum_{j=1}^{n} P(X_j = k) \\ &= \frac{1}{\binom{n-1}{t-1}} \sum_{j=1}^{n} \sum_{i=0}^{t-k} (-1)^i \binom{t}{k} \binom{t-k}{i} \binom{n-j(i+k)-1}{n-j(i+k)-(t-k-i)} \qquad \square \end{aligned}$$

Problem 5.

How many multiplicities in a random composition of n do we expect to equal k ?

Answer

$$\frac{1}{2^{n-1}} \sum_{t=1}^{n} \sum_{j=1}^{n} \sum_{i=0}^{t-k} (-1)^i \binom{t}{k} \binom{t-k}{i} \binom{n-j(i+k)-1}{n-j(i+k)-(t-k-i)}$$

Hitczenko and Savage consider Problem 5 in their paper "*On the Multiplicity of Parts in a Random Composition of a Large Integer*", Pawel Hitczenko, Carla Savage, October, 1999. A preprint of their paper is available at **http://www.csc.ncsu.edu/faculty/savage/**. While they do not find an exact result as we have, they do show that

$$
\begin{aligned}
E\left(\sum_{j=1}^{n} \mathrm{I}_{\{k\}}(W_j)\right) &\approx \binom{\lfloor n/2 \rfloor}{k} \sum_{j=1}^{\infty} \left(2^{-j}\right)^k \left(1-2^{-j}\right)^{\lfloor n/2 \rfloor - k} \\
&\approx 1/(k \ln(2)) \qquad \text{for large } n.
\end{aligned}
$$

Proof

The number of multiplicities equaling k in a random composition of n is

$$
\sum_{j=1}^{n} \mathrm{I}_{\{k\}}(W_j).
$$

Let T represent the number of parts in a random composition of n. Then,

$$
\begin{aligned}
E\left(\sum_{j=1}^{n} \mathrm{I}_{\{k\}}(W_j)\right) &= E\left(E\left(\sum_{j=1}^{n} \mathrm{I}_{\{k\}}(W_j)\right)\middle| T\right) \\
&= \sum_{t=1}^{n} E\left(\sum_{j=1}^{n} \mathrm{I}_{\{k\}}(W_j)\middle| T=t\right) P(T=t) \\
&= \sum_{t=1}^{n} E\left(\sum_{j=1}^{n} \mathrm{I}_{\{k\}}(X_j)\right) P(T=t) \\
&= \sum_{t=1}^{n} E\left(\sum_{j=1}^{n} \mathrm{I}_{\{k\}}(X_j)\right) \frac{\binom{n-1}{t-1}}{2^{n-1}} \\
&= \sum_{t=1}^{n} \left(\frac{1}{\binom{n-1}{t-1}} \sum_{j=1}^{n} \sum_{i=0}^{t-k} (-1)^i \binom{t}{k} \binom{t-k}{i} \binom{n-j(i+k)-1}{n-j(i+k)-(t-k-i)} \right) \frac{\binom{n-1}{t-1}}{2^{n-1}} \\
&= \frac{1}{2^{n-1}} \sum_{t=1}^{n} \sum_{j=1}^{n} \sum_{i=0}^{t-k} (-1)^i \binom{t}{k} \binom{t-k}{i} \binom{n-j(i+k)-1}{n-j(i+k)-(t-k-i)} \qquad \square
\end{aligned}
$$

Problem 6.

Suppose a part is picked uniformly at random from a random composition of n with t parts. Let V represent the multiplicity of this part. Find $P(V=v)$.

Answer

$$
P(V=v) = \frac{1}{\binom{n-1}{t-1}} \sum_{j=1}^{n} \sum_{i=0}^{t-v} (-1)^i \binom{t-1}{v-1} \binom{t-v}{i} \binom{n-j(i+v)-1}{n-j(i+v)-(t-v-i)}
$$

Note: Using the relationship $\sum_{v=0}^{n} P(V=v)=1$ we can establish the identity

$$\sum_{v=0}^{n}\sum_{j=1}^{n}\sum_{i=0}^{t-v}(-1)^{i}\binom{t-1}{v-1}\binom{t-v}{i}\binom{n-j(i+v)-1}{n-j(i+v)-(t-v-i)}=\binom{n-1}{t-1}$$

Proof

(Method 1.)

Define the random variables

$$R_j = \text{multiplicity of part } j,\ j \in \{1,\dots,t\}$$
$$S = \text{part selected in our above two phased random process}$$

Then,

$$\begin{aligned} P(V=v) &= \sum_{j=1}^{t} P(V=v|S=j)P(S=j) \\ &= \sum_{j=1}^{t} P(R_j=v)P(S=j) \end{aligned}$$

It is clear that $R_1, R_2, \dots, R_t$ are not independent random variables, but they are exchangeable random variables. Furthermore $P(S=j)=\frac{1}{t}$ for all j. Therefore

$$P(V=v)=P(R_1=v).$$

Now define the random variable W to be the value of part 1.

Then,

$$P(V=v)=P(R_1=v)=\sum_{j=1}^{n} P(R_1=v \text{ and } W=j)$$

However, we notice that

part 1 has multiplicity v and part 1 has the value j

$\Leftrightarrow$

value j has multiplicity v and part 1 has the value j

That is $P(R_1 = v \text{ and } W = j) = P(X_j = v \text{ and } W = j)$

But we also notice that there exists a natural one to one correspondence between the set of all compositions of n with t parts such that value j has multiplicity v and part 1 has the value j and the set of all compositions of $n-j$ with $t-1$ parts such that the value j has multiplicity $v-1$.

From Problem 3 we have that there are

$$\sum_{i=0}^{t-v} (-1)^i \binom{t-1}{v-1} \binom{t-v}{i} \binom{n-j(i+v)-1}{n-j(i+v)-(t-v-i)}$$

such compositions. Hence,

$$P(R_1 = v, W = j) = \frac{1}{\binom{n-1}{t-1}} \sum_{i=0}^{t-v} (-1)^i \binom{t-1}{v-1} \binom{t-v}{i} \binom{n-j(i+v)-1}{n-j(i+v)-(t-v-i)}$$

and

$$\begin{aligned} P(V = v) &= \sum_{j=1}^{n} P(R_1 = v \text{ and } W = j) \\ &= \frac{1}{\binom{n-1}{t-1}} \sum_{j=1}^{n} \sum_{i=0}^{t-v} (-1)^i \binom{t-1}{v-1} \binom{t-v}{i} \binom{n-j(i+v)-1}{n-j(i+v)-(t-v-i)} \end{aligned}$$

(Method 2.)

Let S represent the number of multiplicities equaling v in a random composition of n with t parts. That is

$$S = \sum_{j=1}^{n} \mathrm{I}_{\{v\}}(X_j)$$

Then,

$$P(V = v) = \sum_{s=0}^{t} P(V = v | \, S = s) P(S = s)$$

$$= \sum_{s=0}^{t} \left(\tfrac{vs}{t}\right) P(S = s)$$

$$= \tfrac{v}{t}\left(\sum_{s=0}^{t} sP(S = s)\right)$$

$$= \tfrac{v}{t}E(S)$$

$$= \tfrac{v}{t}E\left(\sum_{j=1}^{n} \mathrm{I}_{\{v\}}(X_j)\right)$$

$$= \tfrac{v}{t}\sum_{j=1}^{n} P(X_j = v)$$

$$= \tfrac{v}{t}\sum_{j=1}^{n} \left(\tfrac{1}{\binom{n-1}{t-1}} \sum_{i=0}^{t-v} (-1)^i \tbinom{t}{v}\tbinom{t-v}{i}\tbinom{n-j(i+v)-1}{n-j(i+v)-(t-v-i)}\right)$$

$$= \tfrac{\boldsymbol{v}}{t\binom{n-1}{t-1}} \sum_{j=1}^{n} \sum_{i=0}^{t-v} (-1)^i \tbinom{t}{v}\tbinom{t-v}{i}\tbinom{n-j(i+v)-1}{n-j(i+v)-(t-v-i)}$$

$$= \tfrac{\mathbf{1}}{\binom{n-1}{t-1}} \sum_{j=1}^{n} \sum_{i=0}^{t-v} (-1)^i \tbinom{t-1}{v-1}\tbinom{t-v}{i}\tbinom{n-j(i+v)-1}{n-j(i+v)-(t-v-i)}$$

Problem 7.

Suppose a part is picked uniformly at random from a random composition of n. Let V represent the multiplicity of this part. Find $P(V = v)$.

Answer

$$P(V = v) = \frac{1}{2^{n-1}} \sum_{t=1}^{n}\sum_{j=1}^{n}\sum_{i=0}^{t-k} (-1)^i \binom{t-1}{v-1}\binom{t-v}{i}\binom{n-j(i+v)-1}{n-j(i+v)-(t-v-i)}$$

Proof

Let the random variable T represent the number of parts in the random composition of n selected.

$$
\begin{aligned}
P(V=v) &= \sum_{t=1}^{n} P(V=v|\,T=t)P(T=t) \\
&= \sum_{t=1}^{n} P(V=v|\,T=t)\frac{\binom{n-1}{t-1}}{2^{n-1}} \\
&= \sum_{t=1}^{n}\left(\frac{1}{\binom{n-1}{t-1}}\sum_{j=1}^{n}\sum_{i=0}^{t-v}(-1)^i\binom{t-1}{v-1}\binom{t-v}{i}\binom{n-j(i+v)-1}{n-j(i+v)-(t-v-i)}\right)\frac{\binom{n-1}{t-1}}{2^{n-1}} \\
&= \frac{1}{2^{n-1}}\sum_{t=1}^{n}\sum_{j=1}^{n}\sum_{i=0}^{t-k}(-1)^i\binom{t-1}{v-1}\binom{t-v}{i}\binom{n-j(i+v)-1}{n-j(i+v)-(t-v-i)} \qquad \square
\end{aligned}
$$

Problem 8.

Suppose a part is picked uniformly at random from a random composition of n with t parts. Let Z represent the value of this part. Find $E(Z)$.

Answer

$$E(Z) = \frac{n}{t}$$

Of course this answer follows immediately from the fact that the parts are exchangeable random variables but will we apply Theorem 2 as a well of illustrating its use.

Proof

Let Q_j represent the value of the j^{th} part.

Let $g^*(X_{1,n},\ldots,X_{t,n}) = Q_1$ and $g(Y_1,\ldots,Y_t) = Y_1$.

The argument made in Problem 1 could be made to show that $P(Z=z) = P(Q_1=z)$, therefore by Theorem 2,

$$
\begin{aligned}
E(Z) = E(Q_1) = &\ \frac{(-1)^n}{\binom{n-1}{t-1} n!} \frac{\mathrm{d}^n}{\mathrm{d}p^n} \left(\left(\frac{1-p}{p} \right)^t E(Y_1) \right) \Bigg|_{p=1} \\
= &\ \frac{(-1)^n}{\binom{n-1}{t-1} n!} \frac{\mathrm{d}^n}{\mathrm{d}p^n} \left(\left(\frac{1-p}{p} \right)^t \left(\frac{1}{p} \right) \right) \Bigg|_{p=1} \\
= &\ \frac{(-1)^n}{\binom{n-1}{t-1} n!} \frac{\mathrm{d}^n}{\mathrm{d}p^n} \left(\sum_{i=0}^{\infty} \binom{(t+1)+i-1}{i} (1-p)^{t+i} \right) \Bigg|_{p=1} \\
= &\ \frac{(-1)^n}{\binom{n-1}{t-1} n!} \sum_{i=0}^{\infty} \binom{(t+1)+i-1}{i} \left(\frac{\mathrm{d}^n}{\mathrm{d}p^n} (1-p)^{t+i} \Bigg|_{p=1} \right) \\
= &\ \frac{(-1)^n}{\binom{n-1}{t-1} n!} \sum_{i=0}^{\infty} \binom{(t+1)+i-1}{i} \left((-1)^n n! \, \mathrm{I}_{\{n-t\}}(i) \right) \\
= &\ \frac{1}{\binom{n-1}{t-1}} \binom{(t+1)+(n-t)-1}{n-t} \\
= &\ \frac{n}{t}
\end{aligned}
$$

Problem 9.

Suppose a part is picked uniformly at random from a random composition of n. Let Z represent the value of this part. Find $E(Z)$.

Answer

$$
E(Z) = 2 - \left(\frac{1}{2} \right)^{n-1}
$$

Proof

$$
\begin{aligned}
E(Z) &= E(Z|T) \\
&= \sum_{t=1}^{n} E(Z|T=t)P(T=t) \\
&= \sum_{t=1}^{n} \frac{n}{t} \frac{\binom{n-1}{t-1}}{2^{n-1}} \\
&= \frac{1}{2^{n-1}} \sum_{t=1}^{n} \binom{n}{t} \\
&= \frac{1}{2^{n-1}} \left(\sum_{t=0}^{n} \binom{n}{t} - 1 \right) \\
&= \frac{1}{2^{n-1}} (2^n - 1) \\
&= 2 - \left(\frac{1}{2}\right)^{n-1}
\end{aligned}
$$

Problem 10.

Let U represent the number of distinct integers that occur in a random composition of n with t parts. Find $E(U)$.

Answer

$$
E(U) = n - \frac{1}{\binom{n-1}{t-1}} \sum_{j=1}^{n} \sum_{i=0}^{t} (-1)^i \binom{t}{i} \binom{n-ij-1}{n-ij-(t-i)}
$$

Proof

$$
\begin{aligned}
E(U) &= E\left(\sum_{j=1}^{n} \left(1 - \mathrm{I}_{\{0\}}(X_j)\right) \right) \\
&= n - \sum_{j=1}^{n} P(X_j = 0) \\
&= n - \sum_{j=1}^{n} \left(\frac{1}{\binom{n-1}{t-1}} \sum_{i=0}^{t} (-1)^i \binom{t}{i} \binom{n-ij-1}{n-ij-(t-i)} \right) \\
&= n - \frac{1}{\binom{n-1}{t-1}} \sum_{j=1}^{n} \sum_{i=0}^{t} (-1)^i \binom{t}{i} \binom{n-ij-1}{n-ij-(t-i)}
\end{aligned}
$$

Problem 6 (another approach that really doesn't work!).

Suppose a part is picked uniformly at random from a random composition. Let V represent the multiplicity of this part. Find $P(V=v)$. As a check on your answer make sure that $\sum_{v=0}^{n} P(V=v)=1$.

Answer

Proof

Let T represent the number of parts in a random composition.

Let V_t represent the multiplicity of a part picked uniformly at random from a random composition with t parts.

Let S_t^v represent the number of multiplicities equaling v in a random composition of n with t parts. That is

$$S_t^v = \sum_{j=1}^{n} \mathrm{I}_{\{v\}}(X_j)$$

Then,

$$\begin{aligned}
P(V=v) &= \sum_{t=1}^{n} P(V=v \mid T=t) P(T=t) \\
&= \sum_{t=1}^{n} P(V=v \mid T=t) \frac{\binom{n-1}{t-1}}{2^{n-1}} \\
&= \sum_{t=1}^{n} P(V_t=v) \frac{\binom{n-1}{t-1}}{2^{n-1}} \\
&= \sum_{t=1}^{n} \left(\sum_{s=0}^{t} P(V_t=v \mid S_t^v=s) P(S_t^v=s) \right) \frac{\binom{n-1}{t-1}}{2^{n-1}} \\
&= \sum_{t=1}^{n} \sum_{s=0}^{t} \left(\frac{vs}{t} \right) P(S_t^v=s) \frac{\binom{n-1}{t-1}}{2^{n-1}} \\
&= \sum_{t=1}^{n} \sum_{s=0}^{t} \frac{vs}{t} \frac{\binom{n-1}{t-1}}{2^{n-1}} P\left(\sum_{j=1}^{n} \mathrm{I}_{\{v\}}(X_j) = s \right)
\end{aligned}$$

Let A_j represent the event that $X_j = v$. Then

$$P\left(\sum_{j=1}^{n} \mathrm{I}_{\{v\}}(X_j) = s\right) = P(\text{exactly } s \text{ of the events } A_1, \ldots, A_n \text{ occur})$$
$$= \sum_{r=0}^{n-s} (-1)^r \binom{r+s}{s} \mathbb{S}_{r+s}$$

where

$$\mathbb{S}_{r+s} = \begin{cases} \displaystyle\sum_{(j_1,\ldots,j_{r+s})\in\mathbb{C}_{r+s}} P(A_{j_1} \cap \cdots \cap A_{j_{r+s}}) & 1 \leq r+s \leq n \\ 0 & \text{else} \end{cases}$$

and $\mathbb{C}_{r+s}$ is the set of all samples of size $r+s$ drawn without replacement from $\{1,\ldots,n\}$, when the order of sampling is considered unimportant.

Finally, we can solve for $P(A_{j_1} \cap \cdots \cap A_{j_{r+s}})$ via Theorem 1.

$$P(A_{j_1}\cap\cdots\cap A_{j_{r+s}}) = P(X_{j_1}=v,\cdots,X_{j_{r+s}}=v)$$

$$= \frac{1}{n!\binom{n-1}{t-1}}\left(\frac{d^n}{d\lambda^n}\frac{d^t}{d\theta^t}\Big(e^{\left(\frac{\theta\lambda}{1-\lambda}\right)}P(Y_{j_1}=v,\cdots,Y_{j_{r+s}}=v)\Big)\Big|_{\substack{\lambda=0\\ \theta=0}}\right)$$

$$= \frac{1}{n!(v!)^{r+s}\binom{n-1}{t-1}}\left(\frac{d^n}{d\lambda^n}\frac{d^t}{d\theta^t}\Big(e^{\left(\frac{\theta\lambda}{1-\lambda}\right)}e^{(-\theta(\lambda^{j_1}+\ldots+\lambda^{j_{r+s}}))}\theta^{(r+s)v}\lambda^{v(j_1+\ldots+j_n)}\Big)\Big|_{\substack{\lambda=0\\ \theta=0}}\right)$$

$$= \frac{1}{n!(v!)^{r+s}\binom{n-1}{t-1}}\left(\frac{d^n}{d\lambda^n}\frac{d^t}{d\theta^t}\Big(e^{\left(-\theta\left(\frac{\lambda}{1-\lambda}-\lambda^{j_1}-\ldots-\lambda^{j_{r+s}}\right)\right)}\theta^{(r+s)v}\lambda^{v(j_1+\ldots+j_n)}\Big)\Big|_{\substack{\lambda=0\\ \theta=0}}\right)$$

$$= \frac{t!}{n!(v!)^{r+s}(t-(r+s)v)!\binom{n-1}{t-1}}\left(\frac{d^n}{d\lambda^n}\left(\left(\frac{\lambda}{1-\lambda}-\lambda^{j_1}-\ldots-\lambda^{j_{r+s}}\right)^{t-(r+s)v}\lambda^{v(j_1+\ldots+j_n)}\right)\Big|_{\lambda=0}\right)$$

$$= \frac{t!}{n!(v!)^{r+s}(t-(r+s)v)!\binom{n-1}{t-1}}\sum_{i=0}^{t-(r+s)k}\sum_{l=0}^{\infty}\sum_{\mathbb{C}_{r+s}}(-1)^{\boldsymbol{i}}\binom{t-(r+s)v}{i}$$

$$\times\binom{t-(r+s)v-i+l-1}{l}\frac{i!}{c_1!\cdots c_{r+s}!}$$

$$= \frac{t!}{n!(v!)^{r+s}(t-(r+s)v)!\binom{n-1}{t-1}}\sum_{i=0}^{t-(r+s)k}\sum_{l=0}^{\infty}\sum_{\mathbb{C}_{r+s}}(-1)^{\boldsymbol{i}}\binom{t-(r+s)v}{i}\binom{t-(r+s)v-i+l-1}{l}$$

$$\times\frac{i!}{c_1!\cdots c_{r+s}!}\left(\frac{d^n}{d\lambda^n}\Big(\lambda^{v(j_1+\ldots+j_n)+(t-(r+s)v-i+l)+(j_1c_1+\ldots j_{r+s}c_{r+s})}\Big)\Big|_{\lambda=0}\right)$$

$$= \frac{t!}{(v!)^{r+s}(t-(r+s)v)!\binom{n-1}{t-1}}\sum_{i=0}^{t-(r+s)k}\sum_{\mathbb{C}_{r+s}}(-1)^{\boldsymbol{i}}\binom{t-(r+s)v}{i}$$

$$\times\binom{n-v(j_1+\ldots+j_n)-(j_1c_1+\ldots j_{r+s}c_{r+s})-1}{n-v(j_1+\ldots+j_n)-(t-(r+s)v-i)-(j_1c_1+\ldots j_{r+s}c_{r+s})}\frac{i!}{c_1!\cdots c_{r+s}!}$$

Problem .

Let W represent the number of distinct part sizes in a random composition of n with t parts. Find $P(W = w)$.

Answer

Proof

Let S_t represent the number of multiplicities equaling 0 in a random composition of n with t parts. That is

$$S_t = \sum_{j=1}^{n} \mathrm{I}_{\{0\}}(X_j)$$

Let A_j represent the event that $X_j = 0$. Then

$$P(W = w) = P(S_t = n - w)$$

$$\begin{aligned} P\left(\sum_{j=1}^{n} \mathrm{I}_{\{0\}}(X_j) = n - w\right) &= P(\text{exactly } n - w \text{ of the events } A_1, \ldots, A_n \text{ occur}) \\ &= \sum_{r=0}^{n-(n-w)} (-1)^r \binom{r + (n - w)}{n - w} \mathbb{S}_{r+(n-w)} \\ &= \sum_{r=0}^{w} (-1)^r \binom{n - w + r}{n - w} \mathbb{S}_{r+n-w} \end{aligned}$$

where

$$\mathbb{S}_{r+n-w} = \begin{cases} \sum_{(j_1,\ldots,j_{r+n-w}) \in \mathbb{C}_{r+n-w}} P(A_{j_1} \cap \cdots \cap A_{j_{r+n-w}}) & 1 \leq r + n - w \leq n \\ 0 & \text{else} \end{cases}$$

and $\mathbb{C}_{r+n-w}$ is the set of all samples of size $r + n - w$ drawn without replacement from $\{1,\ldots,n\}$, when the order of sampling is considered unimportant.

Finally, we can solve for $P(A_{j_1} \cap \cdots \cap A_{j_{r+n-w}})$ via Theorem 1.

$$P(A_{j_1} \cap \cdots \cap A_{j_{r+n-w}}) = P(X_{j_1} = 0, \cdots, X_{j_{r+n-w}} = 0)$$

$$= \frac{1}{n!\binom{n-1}{t-1}} \left(\frac{d^n}{d\lambda^n} \frac{d^t}{d\theta^t} \left(e^{\left(\frac{\theta\lambda}{1-\lambda}\right)} P(Y_{j_1} = 0, \cdots, Y_{j_{r+n-w}} = 0) \right) \Big|_{\substack{\lambda=0 \\ \theta=0}} \right)$$

$$= \frac{1}{n!\binom{n-1}{t-1}} \left(\frac{d^n}{d\lambda^n} \frac{d^t}{d\theta^t} \left(e^{\left(\theta\left(\frac{\lambda}{1-\lambda} - \lambda^{j_1} - \ldots - \lambda^{j_{r+n-w}}\right)\right)} \right) \Big|_{\substack{\lambda=0 \\ \theta=0}} \right)$$

$$= \frac{1}{n!\binom{n-1}{t-1}} \left(\frac{d^n}{d\lambda^n} \left(\left(\frac{\lambda}{1-\lambda} - \lambda^{j_1} - \ldots - \lambda^{j_{r+n-w}} \right)^t \right) \Big|_{\lambda=0} \right)$$

$$= \frac{1}{n!\binom{n-1}{t-1}} \left(\frac{d^n}{d\lambda^n} \left(\left(\frac{\lambda - (1-\lambda)(\lambda^{j_1} + \ldots + \lambda^{j_{r+n-w}})}{1-\lambda} \right)^t \right) \Big|_{\lambda=0} \right)$$

$$= \frac{1}{n!\binom{n-1}{t-1}} \left(\frac{d^n}{d\lambda^n} \left(\sum_{i=0}^{\infty} \binom{t+i-1}{i} \lambda^i (\lambda - (1-\lambda)(\lambda^{j_1} + \ldots + \lambda^{j_{r+n-w}}))^t \right) \Big|_{\lambda=0} \right)$$

$$= \frac{1}{n!\binom{n-1}{t-1}} \left(\frac{d^n}{d\lambda^n} \left(\sum_{i=0}^{\infty} \sum_{l=0}^{t} \binom{t+i-1}{i} \binom{t}{l} \lambda^i \lambda^l (1-\lambda)^{t-l} (-1)^{t-l} (\lambda^{j_1} + \ldots + \lambda^{j_{r+n-w}})^{t-l} \right) \Big|_{\lambda=0} \right)$$

Problem 6.

Suppose a part is picked uniformly at random from a random composition of n with t parts. Let V represent the multiplicity of this part. Find $P(V = v)$.

Answer

$$P(V = v) = \frac{\mathbf{1}}{\binom{n-1}{t-1}} \sum_{j=1}^{n} \sum_{i=0}^{t-v} (-1)^i \binom{t-1}{v-1}\binom{t-v}{i}\binom{n-j(i+v)-1}{n-j(i+v)-(t-v-i)}$$

Proof

Let S represent the number of multiplicities equaling v in a random composition of n with t parts. That is

$$S = \sum_{j=1}^{n} \mathrm{I}_{\{v\}}(X_j)$$

Then,

$$P(V = v) = \sum_{s=0}^{t} P(V = v|\, S = s)P(S = s)$$

$$= \sum_{s=0}^{t} \left(\tfrac{vs}{t}\right) P(S = s)$$

$$= \tfrac{v}{t}\left(\sum_{s=0}^{t} sP(S = s)\right)$$

$$= \tfrac{v}{t} E(S)$$

$$= \tfrac{v}{t} E\left(\sum_{j=1}^{n} \mathrm{I}_{\{v\}}(X_j)\right)$$

$$= \tfrac{v}{t}\sum_{j=1}^{n} P(X_j = v)$$

$$= \tfrac{v}{t}\sum_{j=1}^{n}\left(\frac{1}{\binom{n-1}{t-1}} \sum_{i=0}^{t-v} (-1)^i \binom{t}{v}\binom{t-v}{i}\binom{n-j(i+v)-1}{n-j(i+v)-(t-v-i)}\right)$$

$$= \frac{\boldsymbol{v}}{t\binom{n-1}{t-1}} \sum_{j=1}^{n} \sum_{i=0}^{t-v} (-1)^i \binom{t}{v}\binom{t-v}{i}\binom{n-j(i+v)-1}{n-j(i+v)-(t-v-i)}$$

$$= \frac{1}{\binom{n-1}{t-1}} \sum_{j=1}^{n} \sum_{i=0}^{t-v} (-1)^i \binom{t-1}{v-1} \binom{t-v}{i} \binom{n-j(i+v)-1}{n-j(i+v)-(t-v-i)}$$