Random Labeled Trees 7/19/01

Theorem 1. 1-Shifted Poisson Randomization Theorem

Define $\mathbb{S}^n$ to be the product space $\{1,2,\ldots\} \times \cdots \times \{1,2,\ldots\}$ and let $\mathbb{S}^n_t$ be the set of all vectors $(s_1,\ldots,s_n)$ in $\mathbb{S}^n$ such that $s_1 + \ldots + s_n = t$.

Suppose that $(X_1,\ldots,X_n)$ is a random vector such that for all $(s_1,\ldots,s_n) \in \mathbb{S}^n_t$,

$$P(X_1 = s_1,\ldots,X_n = s_n) = \frac{(t-n)!}{(s_1-1)!\cdots(s_n-1)!}\,(p_1)^{s_1-1}\cdots(p_n)^{s_n-1}$$

where $p_1 + \ldots + p_n = 1$ and $p_j \geq 0$ for $j = 1,\ldots,n$.

Suppose that $(Y_1,\ldots,Y_n)$ is a vector of independent **1-shifted Poisson** random variables and that Y_j has parameter λp_j, $j = 1,\ldots,n$. That is, suppose that for all $s \in \{1,2,\ldots\}$,

$$P(Y_j = s) = \frac{e^{-\lambda p_j}(\lambda p_j)^{s_{j-1}}}{(s_j-1)!}$$

Then for $t \geq n$,

$$\mathrm{E}(g(X_1,\ldots,X_n)) = \frac{(t-n)!}{t!}\frac{\mathrm{d}^t}{\mathrm{d}\lambda^t}\left(e^{\lambda}\lambda^n \mathrm{E}(g(Y_1,\ldots,Y_n))\right)\Big|_{\lambda=0}$$

Let $\mathcal{A} \subseteq \mathbb{S}^n$ and define $\mathcal{A}_t = \mathcal{A} \cap \mathbb{S}^n_t$. Then for $t \geq n$,

$$P((X_1,\ldots,X_n) \in \mathcal{A}_t) = \frac{(t-n)!}{t!}\frac{\mathrm{d}^t}{\mathrm{d}\lambda^t}\left(e^{\lambda}\lambda^n P((Y_1,\ldots,Y_n) \in \mathcal{A}\,)\right)\Big|_{\lambda=0}$$

If we take $p_j = \frac{1}{n}$ for $j = 1,\ldots,n$ and $t = 2n-2$, then the random vector $(X_1,\ldots,X_n)$ models random labeled trees with n vertices where X_j equals the degree of the j^{th} vertex.

Theorem 2.

We will need the following definitions.

$\mathbb{S}^{\infty}$: the infinite product space $\{0,1,\ldots\} \times \{0,1,\ldots\} \times \cdots$

$\mathbb{S}^{\infty}_{n}$: the set of all vectors $(s_1,s_2,\ldots)$ in $\mathbb{S}^{\infty}$ such that $0s_1 + 1s_2 + \ \ldots = n-2$

$\mathbb{S}^{\infty}_{n,t}$: the set of all vectors $(s_1,s_2,\ldots)$ in $\mathbb{S}^{\infty}$ such that $\begin{matrix} 0s_1 + 1s_2 + \ \ldots = n-2 \\ s_1 + s_2 + \ \ldots = t \end{matrix}$

For any $\mathcal{A} \subseteq \mathbb{S}^{\infty}$ define $\mathcal{A}_n = \mathcal{A} \cap \mathbb{S}^{\infty}_{n}$ and $\mathcal{A}_{n,t} = \mathcal{A} \cap \mathbb{S}^{\infty}_{n,t}$

We note that the condition that $0s_1 + 1s_2 + \ \ldots = n-2$ implies that $s_j = 0$ for all $j > n-1$. Hence all vectors in $\mathcal{A}_n$ and $\mathcal{A}_{n,t}$ are of the form $(a_1,\ldots,a_{n-1}, 0, 0, \ldots)$.

For all $\mathcal{A} \neq (0, 0, \ldots)$, let $\mathbb{A}_n$ be the collection of $(n-1)$-dimensional vectors formed by taking each infinite-dimensional vector in $\mathcal{A}_n$ and truncating after a_n. So for example,

$$(a_1,\ldots,a_{n-1}, 0, 0, \ldots) \rightarrow (a_1,\ldots,a_{n-1})$$

Define $\mathbb{A}_{n,t}$ similarly. For notational consistency it is necessary to separate out the case $\mathcal{A} = (0, 0, \ldots)$

Define the random vector

$$P((X_1, \ldots, X_{n-1}) = (x_1, \ldots, x_{n-1})) = \begin{cases} \dfrac{\frac{t!\,(n-2)!}{x_1!\cdots x_{n-1}!(0!)^{x_1}\cdots((n-2)!)^{x_{n-1}}}}{t^{n-2}} & \begin{matrix} 0x_1+\ldots+(n-2)x_{n-1}=n-2 \\ x_1+\ldots+x_{n-1}=t \\ x_j \in \{0,1,\ldots,\} \forall j \end{matrix} \\ 0 & \text{otherwise} \end{cases}$$

where

$$\sum \cdots \sum_{\substack{x_1+\ldots+x_{n-1}=t \\ 0x_1+\ldots+(n-2)x_{n-1}=n-2 \\ x_i \in \{0,1\ldots\}}} \frac{t!(n-2)!}{x_1!\cdots x_{n-1}!(0!)^{x_1}\cdots((n-2)!)^{x_{n-1}}} = t^{n-2}$$

Consider an infinite sequence $Y_1, Y_2, \ldots$ of independent Poisson random variables where

$$P(Y_j = y) = \frac{e^{\left(\frac{-\theta\lambda^{j-1}}{(j-1)!}\right)}\left(\frac{\theta\lambda^{j-1}}{(j-1)!}\right)^y}{y!} \qquad y = 0, 1, 2, \ldots \text{ and } j = 1, 2, \ldots$$

Then

$$\mathrm{E}(g(X_1, \ldots, X_{n-1}, 0, 0, \ldots)) = \left(\frac{1}{n^{t-2}}\right)\left(\frac{d^{n-2}}{d\lambda^{n-2}}\frac{d^t}{d\theta^t}\left(e^{\theta e^\lambda}\mathrm{E}(g(Y_1, Y_2, \ldots))\right)\Big|_{\substack{\lambda=0\\\theta=0}}\right)$$

and for $\mathcal{A} \neq (0, 0, \ldots)$

$$P((X_1, \ldots, X_{n-1}) \in \mathbb{A}_{n,t}) = \left(\frac{1}{n^{t-2}}\right)\left(\frac{d^{n-2}}{d\lambda^{n-2}}\frac{d^t}{d\theta^t}\left(e^{\theta e^\lambda}P((Y_1, Y_2, \ldots) \in \mathcal{A})\right)\Big|_{\substack{\lambda=0\\\theta=0}}\right)$$

If we take $t = n$, then the random vector $(X_1, \ldots, X_{n-1})$ models random labeled trees with n vertices where X_j equals the number of vertices with degree j.

Problem 1.

The number of labeled trees with n vertices such that the first vertex has degree *k* equals

$$\binom{n-2}{k-1}(n-1)^{n-k-1}$$

Reference

This result agrees with "On Cayley's Formula for Counting Trees", Clarke, L. E., *Journal of the London Mathematical Society*, Vol. 33, 1958, 471 - 474.

Problem 2.

The number of labeled trees with n vertices such that exactly k vertices have degree 1 equals

$$\frac{n!}{k!}\, S(n-2, n-k)$$

Reference

This problem appears in Combinatorial Problems and Exercises, 2^{nd} Edition, Laszlo Lovasz, North-Holland, 1993, page 35, Problem 8.

Rényi, Alfréd, "Some Remarks on the Theory of Trees", *Magyar Tud. Akad. Math. Kutato Int. Kozl.*, 4 , 1959, 73 - 85.

Problem 3.

(a) The number of labeled trees with n vertices such that exactly c_i vertices have degree i, $i = 1, \ldots, n-1$, equals

$$\frac{n!(n-2)!}{c_1!\cdots c_{n-1}!(0!)^{c_1}(1!)^{c_2}\cdots((n-2)!)^{c_{n-1}}}$$

where $c_1 + \ldots + c_{n-1} = n$, $1c_1 + \ldots + (n-1)c_{n-1} = 2(n-1)$, and $c_i \in \{0, 1 \ldots\}$.

(b)

$$\sum_{\substack{c_1+\ldots+c_{n-1}=n \\ 0c_1+\ldots+(n-2)c_{n-1}=n-2 \\ c_i\in\{0,1\ldots\}}} \cdots \sum \frac{(2n-2)!}{c_1!\cdots c_{n-1}!(0!)^{c_1}(1!)^{c_2}\cdots((n-2)!)^{c_{n-1}}} = \binom{2n-2}{n} n^{n-2}$$

(c)

$$\sum_{\substack{c_1+\ldots+c_{n-1}=t \\ 0c_1+\ldots+(n-2)c_{n-1}=n-2 \\ c_i\in\{0,1\ldots\}}} \cdots \sum \frac{(n+t-2)!}{c_1!\cdots c_n!(0!)^{c_1}(1!)^{c_2}\cdots((n-1)!)^{c_n}} = \binom{n+t-2}{t} t^{n-2}$$

<u>Proof</u> (Theorem 1)

For all $(s_1,\ldots,s_n) \in \mathbb{S}^n$

$$P(Y_1 = s_1, \ldots, Y_n = s_n) = \prod_{i=1}^{n} P(Y_i = s_i)$$

$$= \prod_{i=1}^{n} \frac{e^{-\lambda p_j}(\lambda p_j)^{s_{j-1}}}{(s_j - 1)!}$$

$$= \frac{e^{-\lambda}\lambda^{s_1+\ldots+s_n-n}}{(t-n)!} \frac{(t-n)!}{(s_1-1)!\cdots(s_n-1)!} (p_1)^{s_1-1}\cdots(p_n)^{s_n-1}$$

$$P(Y_1 + \ldots + Y_n = t) = \sum_{\mathbb{S}_t^n} P(Y_1 = s_1,\ldots,Y_n = s_n)$$

$$= \sum_{\mathbb{S}_t^n} \frac{e^{-\lambda}\lambda^{s_1+\ldots+s_n-n}}{(t-n)!} \frac{(t-n)!}{(s_1-1)!\cdots(s_n-1)!} (p_1)^{s_1-1}\cdots(p_n)^{s_n-1}$$

$$= \frac{e^{-\lambda}\lambda^{t-n}}{(t-n)!} \sum_{\mathbb{S}_t^n} \frac{(t-n)!}{(s_1-1)!\cdots(s_n-1)!} (p_1)^{s_1-1}\cdots(p_n)^{s_n-1}$$

$$= \frac{e^{-\lambda}\lambda^{t-n}}{(t-n)!}$$

Thus,

$$P((Y_1,\ldots,Y_n)) \in \mathcal{A} | Y_1 + \ldots + Y_n = t)$$

$$= \frac{P((Y_1,\ldots,Y_n) \in \mathcal{A} \text{ and } Y_1 + \ldots + Y_n = t)}{P(Y_1+\ldots+Y_n = t)}$$

$$= \frac{P((Y_1,\ldots,Y_n) \in \mathcal{A}_t)}{P(Y_1+\ldots+Y_n = t)}$$

$$= \frac{\sum_{\mathcal{A}_t} \frac{e^{-\lambda}\lambda^{s_1+\ldots+s_n-n}}{(t-n)!} \frac{(t-n)!}{(s_1-1)!\cdots(s_n-1)!} (p_1)^{s_1-1}\cdots(p_n)^{s_n-1}}{\frac{e^{-\lambda}\lambda^{t-n}}{(t-n)!}}$$

$$= \sum_{\mathcal{A}_t} \frac{(t-n)!}{(s_1-1)!\cdots(s_n-1)!} (p_1)^{s_1-1}\cdots(p_n)^{s_n-1}$$

$$= P((X_1,\ldots,X_n) \in \mathcal{A}_t)$$

Therefore,

$$P((Y_1,\ldots,Y_n) \in \mathcal{A})$$

$$= \sum_{t=n}^{\infty} P((Y_1,\ldots,Y_n) \in \mathcal{A} \,|\textstyle\sum Y_i = t)P(\textstyle\sum \mathrm{Y}_i = t\,)$$

$$= \sum_{t=n}^{\infty} P((X_1,\ldots,X_n) \in \mathcal{A}_t)\tfrac{e^{-\lambda}\lambda^{t-n}}{(t-n)!}$$

and

$$e^{\lambda}\lambda^{n} P((Y_1,\ldots,Y_n) \in \mathcal{A}) = \sum_{t=n}^{\infty} P((X_1,\ldots,X_n) \in \mathcal{A}_t)\frac{1}{(t-n)!}\lambda^t$$

It follows that

$$\tfrac{\mathrm{d}^r}{\mathrm{d}\lambda^r}(e^{\lambda}\lambda^{n}P((Y_1,\ldots,Y_n) \in \mathcal{A}\,))\big|_{\lambda=0}$$

$$= \tfrac{\mathrm{d}^r}{\mathrm{d}\lambda^r}\left(\sum_{t=n}^{\infty} P((X_1,\ldots,X_n) \in \mathcal{A}_t)\tfrac{1}{(t-n)!}\lambda^t\right)\Bigg|_{\lambda=0}$$

$$= \sum_{t=n}^{\infty} P((X_1,\ldots,X_n) \in \mathcal{A}_t)\tfrac{1}{(t-n)!}\left(\tfrac{\mathrm{d}^r}{\mathrm{d}\lambda^r}\lambda^t\big|_{\lambda=0}\right)$$

$$= \sum_{t=n}^{\infty} P((X_1,\ldots,X_n) \in \mathcal{A}_t)\tfrac{t!}{(t-n)!(t-r)!}\mathrm{I}_{\{r\}}(t)$$

$$= P((X_1,\ldots,X_n) \in \mathcal{A}_r)\tfrac{r!}{(r-n)!}$$

Thus for $r \geq n$,

$$P((X_1,\ldots,X_n) \in \mathcal{A}_r) = \frac{(r-n)!}{r!} \frac{\mathrm{d}^r}{\mathrm{d}\lambda^r} \left(e^{\lambda} \lambda^n P((Y_1,Y_2,\ldots,Y_n) \in \mathcal{A}\ )\right)\Big|_{\lambda=0}$$

$\square$

<u>Proof</u> (Theorem 2)

Let y_j be a nonnegative integer for $j = 1, 2, \cdots$. Then

$$P(Y_1 = y_1, Y_2 = y_2, \ldots) = \frac{e^{\left(-\sum\limits_{j=1}^{\infty} \frac{\theta\lambda^{j-1}}{(j-1)!}\right)} \lambda^{\left(\sum\limits_{j=1}^{\infty}(j-1)y_j\right)} \theta^{\left(\sum\limits_{j=1}^{\infty} y_j\right)} \prod\limits_{j=1}^{\infty} \left(\frac{1}{(j-1)!}\right)^{y_j}}{\prod\limits_{j=1}^{\infty}(y_j)!}$$

$$= \frac{e^{\left(-\theta e^{\lambda}\right)} \lambda^{\left(\sum\limits_{j=1}^{\infty}(j-1)y_j\right)} \theta^{\left(\sum\limits_{j=1}^{\infty} y_j\right)}}{\prod\limits_{j=1}^{\infty}(y_j)!((j-1)!)^{y_j}}$$

and

$$P\left(\sum_{j=1}^{\infty}(j-1)Y_j = n-2, \sum_{j=1}^{\infty}Y_j = t\right) = \sum_{\substack{(y_1,y_2,\dots)\ni \\ \sum_{j=1}^{\infty}(j-1)y_j=n-2 \\ \sum_{j=1}^{\infty}y_j=t}} \cdots \sum P(Y_1 = y_1, Y_2 = y_2, \dots)$$

$$= \frac{e^{\left(-\theta e^{\lambda}\right)}\lambda^{n-2}\theta^t}{(n-2)!\,t!}\left(\sum_{\substack{(y_1,y_2,\dots)\ni \\ \sum_{j=1}^{\infty}(j-1)y_j=n-2 \\ \sum_{j=1}^{\infty}y_j=t}} \cdots \sum \left(\frac{(n-2)!\,t!}{\prod_{j=1}^{\infty}(y_j)!((j-1)!)^{y_j}}\right)\right)$$

$$= \frac{e^{\left(-\theta e^{\lambda}\right)}\lambda^{n-2}\theta^t}{(n-2)!\,t!}\left(\sum_{\substack{(y_1,y_2,\dots)\ni \\ \sum_{j=1}^{n-1}(j-1)y_j=n-2 \\ \sum_{j=1}^{n-1}y_j=t}} \cdots \sum \left(\frac{(n-2)!\,t!}{\prod_{j=1}^{n-1}(y_j)!((j-1)!)^{y_j}}\right)\right)$$

$$= \frac{e^{\left(-\theta e^{\lambda}\right)}\lambda^{n-2}\theta^t}{(n-2)!\,t!}t^{n-2}$$

It follows that for $\mathcal{A} \neq (0,0,\dots)$

$P((Y_1, Y_2, \dots) \in \mathcal{A})$

$$= \sum_{n=2}^{\infty}\sum_{t=0}^{\infty} P\left((Y_1, Y_2, \dots) \in \mathcal{A} \mid \sum_{j=1}^{\infty}(j-1)Y_j = n-2, \sum_{j=1}^{\infty}Y_j = t\right) P\left(\sum_{j=1}^{\infty}(j-1)Y_j = n-2, \sum_{j=1}^{\infty}Y_j = t\right)$$

$$= \sum_{n=2}^{\infty}\sum_{t=0}^{\infty}\sum_{\mathcal{A}_{n,t}}\left(\frac{\frac{e^{\left(-\theta e^{\lambda}\right)}\lambda^{n-2}\theta^t}{\prod_{j=1}^{\infty}\left(y_j\right)!((j-1)!)^{y_j}}}{\frac{e^{\left(-\theta e^{\lambda}\right)}\lambda^{n-2}\theta^t}{(n-2)!\,t!}t^{n-2}}\right)\left(\frac{e^{\left(-\theta e^{\lambda}\right)}\lambda^{n-2}\theta^t}{(n-2)!\,t!}t^{n-2}\right)$$

$$= \sum_{n=3}^{\infty} \sum_{t=1}^{\infty} \sum_{\mathbb{A}_{n,t}} \left(\frac{\frac{t!\,(n-2)!}{y_1! \cdots y_{n-1}! (0!)^{y_1} \cdots ((n-2)!)^{y_{n-1}}}}{t^{n-2}} \right) \left(\frac{e^{\left(-\theta e^{\lambda}\right)} \lambda^{n-2} \theta^t}{(n-2)!\, t!} t^{n-2} \right)$$

$$= \sum_{n=3}^{\infty} \sum_{t=1}^{\infty} P((X_1, ..., X_{n-1}) \in \mathbb{A}_{n,t}) \left(\frac{e^{\left(-\theta e^{\lambda}\right)} \lambda^{n-2} \theta^t}{(n-2)!\, t!} t^{n-2} \right)$$

Therefore

$$e^{\theta e^{\lambda}} P((Y_1, Y_2, \dots) \in \mathcal{A}) = \sum_{n=3}^{\infty} \sum_{t=1}^{\infty} P((X_1, ..., X_{n-1}) \in \mathbb{A}_{n,t}) \left(\frac{t^{n-2}}{(n-2)!\, t!} \right) \lambda^{n-2} \theta^t$$

and

$$\frac{d^{r-2}}{d\lambda^{r-2}} \frac{d^s}{d\theta^s} \left(e^{\theta e^{\lambda}} P((Y_1, Y_2, \dots) \in \mathcal{A}) \right) \Big|_{\substack{\lambda=0 \\ \theta=0}}$$
$$= \sum_{n=3}^{\infty} \sum_{t=1}^{\infty} P((X_1, ..., X_{n-1}) \in \mathbb{A}_{n,t}) \left(\frac{t^{n-2}}{(n-2)!\, t!} \right) r! s! \mathrm{I}_{\{r\}}(n) \mathrm{I}_{\{s\}}(t)$$

Therefore, for $\mathcal{A} \neq (0, 0, \dots)$

$$P((X_1, ..., X_{n-1}) \in \mathbb{A}_{n,t}) = \left(\frac{1}{n^{t-2}} \right) \left(\frac{d^{n-2}}{d\lambda^{n-2}} \frac{d^t}{d\theta^t} \left(e^{\theta e^{\lambda}} P((Y_1, Y_2, \dots) \in \mathcal{A}) \right) \Big|_{\substack{\lambda=0 \\ \theta=0}} \right)$$

and finally,

$$P((X_1, ..., X_{n-1}) \in \mathbb{A}_{n,n}) = \left(\frac{1}{n^{n-2}} \right) \left(\frac{d^{n-2}}{d\lambda^{n-2}} \frac{d^n}{d\theta^n} \left(e^{\theta e^{\lambda}} P((Y_1, Y_2, \dots) \in \mathcal{A}) \right) \Big|_{\substack{\lambda=0 \\ \theta=0}} \right)$$

Solutions

Problem 1

$$P((X_1,\ldots,X_n) \in \mathcal{A}_{2n-2}) = \frac{(n-2)!}{(2n-2)!}\frac{\mathrm{d}^{2n-2}}{\mathrm{d}\lambda^{2n-2}}\left(e^{\lambda}\lambda^{n}P((Y_1,\ldots,Y_n) \in \mathcal{A}\ )\right)\Big|_{\lambda=0}$$

$P((Y_1,\ldots,Y_n) \in \mathcal{A}\ )$

$$= P(Y_1 = k \text{ and } Y_j \in \{1,2,\ldots\},\ j = 2,3,\ldots)$$

$$= \frac{e^{-\frac{\lambda}{n}}\left(\frac{\lambda}{n}\right)^{k-1}}{(k-1)!} \times 1$$

$P((X_1,\ldots,X_n) \in \mathcal{A}_{2n-2})$

$$= \frac{(n-2)!}{(2n-2)!}\frac{\mathrm{d}^{2n-2}}{\mathrm{d}\lambda^{2n-2}}(e^{\lambda}\lambda^{n}P((Y_1,\ldots,Y_n) \in \mathcal{A}\ ))|_{\lambda=0}$$

$$= \frac{(n-2)!}{(2n-2)!}\frac{\mathrm{d}^{2n-2}}{\mathrm{d}\lambda^{2n-2}}\left(e^{\lambda}\lambda^{n}\frac{e^{-\frac{\lambda}{n}}\left(\frac{\lambda}{n}\right)^{k-1}}{(k-1)!}\right)\Bigg|_{\lambda=0}$$

$$= \frac{(n-2)!}{(2n-2)!}\left(\frac{1}{n^{k-1}(k-1)!}\right)\frac{\mathrm{d}^{2n-2}}{\mathrm{d}\lambda^{2n-2}}\left(e^{\lambda\left(\frac{n-1}{n}\right)}\lambda^{n+k-1}\right)\Big|_{\lambda=0}$$

$$= \frac{(n-2)!}{(2n-2)!}\left(\frac{1}{n^{k-1}(k-1)!}\right)\left(\frac{n-1}{n}\right)^{((2n-2)-(n+k-1))}\frac{(2n-2)!}{((2n-2)-(n+k-1))!}$$

provided $2n-2 \geq n+k-1$

$$= \frac{(n-2)!}{(2n-2)!}(n-1)^{n-k-1}\left(\frac{1}{n}\right)^{(n-2)}\frac{(2n-2)!}{(n-k-1)!(k-1)!}$$

$$= \frac{\binom{n-2}{k-1}(n-1)^{n-k-1}}{n^{n-2}}$$

Therefore, the number of labeled trees on n vertices such that v_1 has degree k equals

$$\binom{n-2}{k-1}(n-1)^{n-k-1}$$

Problem 2

$$P((Y_1,\ldots,Y_n) \in \mathcal{A}\ )$$

$$= P(\text{exactly } k \text{ of } (Y_1,\ldots,Y_n) \text{ equal } 1)$$

$$= \tbinom{n}{k}(P(Y_1 = 1))^k(1 - P(Y_1 = 1))^{n-k}$$

$$= \tbinom{n}{k}\left(\tfrac{e^{-\frac{\lambda}{n}}\left(\frac{\lambda}{n}\right)^{1-1}}{(1-1)!}\right)^k\left(1 - \tfrac{e^{-\frac{\lambda}{n}}\left(\frac{\lambda}{n}\right)^{1-1}}{(1-1)!}\right)^{n-k}$$

$$= \tbinom{n}{k}\left(e^{-\frac{\lambda}{n}}\right)^k\left(1 - e^{-\frac{\lambda}{n}}\right)^{n-k}$$

$$= \tbinom{n}{k}\sum_{j=0}^{n-k}(-1)^j\binom{n-k}{j}\left(e^{-\frac{\lambda}{n}}\right)^{k+j}$$

Therefore,

$$P((X_1,\ldots,X_n) \in \mathcal{A}_{2n-2})$$

$$= \tfrac{(n-2)!}{(2n-2)!}\tfrac{\mathrm{d}^{2n-2}}{\mathrm{d}\lambda^{2n-2}}(e^{\lambda}\lambda^n P((Y_1,\ldots,Y_n) \in \mathcal{A}\ ))|_{\lambda=0}$$

$$= \tfrac{(n-2)!}{(2n-2)!}\tbinom{n}{k}\ \sum_{j=0}^{n-k}(-1)^j\binom{n-k}{j}\left(\tfrac{\mathrm{d}^{2n-2}}{\mathrm{d}\lambda^{2n-2}}\left(e^{\lambda\left(\frac{n-k-j}{n}\right)}\lambda^n\right)\Big|_{\lambda=0}\right)$$

$$= \tfrac{\binom{n}{k}}{n^{n-2}}\ \sum_{j=0}^{n-k}(-1)^j\binom{n-k}{j}(n-k-j)^{(n-2)}$$

$$= \tfrac{n!}{k!n^{n-2}}\ \left(\tfrac{1}{(n-k)!}\sum_{j=0}^{n-k}(-1)^j\binom{n-k}{j}(n-k-j)^{(n-2)}\right)$$

$$= \tfrac{n!}{k!n^{n-2}}\ S(n-2, n-k)$$

Hence, the number of labeled trees with n vertices such that exactly k vertices have degree 1 equals

$$\frac{n!}{k!}\, S(n-2, n-k)$$

Problem 3(a)

We begin by noting that

$$c_1 + \ldots + c_{n-1} = \text{total number of vertices } = n$$

and

$$\begin{aligned}
&0c_1 + 1c_2 + \ldots + (n-2)c_{n-1} \\
&\quad = (1c_1 + 2c_2 + \ldots + (n-1)c_{n-1}) - (c_1 + \ldots + c_{n-1}) \\
&\quad = \text{sum of all degrees} - \text{total number of vertices} \\
&\quad = 2(\text{number of edges}) - \text{total number of vertices} \\
&\quad = 2(n-1) - n \\
&\quad = n-2
\end{aligned}$$

Also,

$$\begin{aligned}
&P((Y_1,\ldots,Y_n) \in \mathcal{A}\ ) \\
&\quad = P\left(\bigcap_{i=1}^{n-1} (c_i \text{ of } (Y_1,\ldots,Y_n) \text{ equal } i)\right) \\
&\quad = \tfrac{n!}{c_1!\cdots c_{n-1}!} (P(Y_1 = 1))^{c_1} \cdots (P(Y_1 = n-1))^{c_{n-1}} \\
&\quad = \tfrac{n!}{c_1!\cdots c_{n-1}!} \left(\tfrac{e^{-\frac{\lambda}{n}} \left(\frac{\lambda}{n}\right)^{1-1}}{(1-1)!}\right)^{c_1} \cdots \left(\tfrac{e^{-\frac{\lambda}{n}} \left(\frac{\lambda}{n}\right)^{(n-1)-1}}{((n-1)-1)!}\right)^{c_{n-1}} \\
&\quad = \tfrac{n!}{c_1!\cdots c_{n-1}!(0!)^{c_1}(1!)^{c_2}\cdots((n-2)!)^{c_{n-1}} n^{0c_1+1c_2+\ldots+(n-2)c_{n-1}}} \left(\lambda^{0c_1+1c_2+\ldots+(n-2)c_{n-1}} e^{-\frac{\lambda}{n}(c_1+\ldots+c_{n-1})}\right)
\end{aligned}$$

$$= \frac{n!}{c_1!\cdots c_{n-1}!(0!)^{c_1}(1!)^{c_2}\cdots((n-2)!)^{c_{n-1}}n^{n-2}}(\lambda^{n-2}e^{-\lambda})$$

Therefore,

$$\begin{aligned}
&P((X_1,\ldots,X_n) \in \mathcal{A}_{2n-2}) \\
&= \frac{(n-2)!}{(2n-2)!}\frac{\mathrm{d}^{2n-2}}{\mathrm{d}\lambda^{2n-2}}(e^{\lambda}\lambda^n P((Y_1,\ldots,Y_n) \in \mathcal{A}\ ))|_{\lambda=0} \\
&= \frac{(n-2)!}{(2n-2)!}\frac{n!}{c_1!\cdots c_{n-1}!(0!)^{c_1}(1!)^{c_2}\cdots((n-2)!)^{c_{n-1}}n^{n-2}}\frac{\mathrm{d}^{2n-2}}{\mathrm{d}\lambda^{2n-2}}(e^{\lambda}\lambda^n\lambda^{n-2}e^{-\lambda})|_{\lambda=0} \\
&= \frac{(n-2)!}{(2n-2)!}\frac{n!}{c_1!\cdots c_{n-1}!(0!)^{c_1}(1!)^{c_2}\cdots((n-2)!)^{c_{n-1}}n^{n-2}}\frac{\mathrm{d}^{2n-2}}{\mathrm{d}\lambda^{2n-2}}(\lambda^{2n-2})|_{\lambda=0} \\
&= \frac{n!(n-2)!}{c_1!\cdots c_{n-1}!(0!)^{c_1}(1!)^{c_2}\cdots((n-2)!)^{c_{n-1}}n^{n-2}}
\end{aligned}$$

and the result follows by multiplying through by n^{n-2}, the total number of labeled trees on n vertices.

Problem 3(b)

This follows immediately from 3(a) by considering the identity

$$\sum_{\substack{c_1+\ldots+c_{n-1}=n \\ 1c_1+\ldots+(n-1)c_{n-1}=2(n-1) \\ c_i\in\{0,1\ldots\}}}\cdots\sum P((X_1,\ldots,X_n) \in \mathcal{A}_{2n-2}) = 1$$