Random Partition of a Set   2/28/01

A partition of the set $\mathcal{V}$ is a collection of disjoint nonempty sets whose union equals $\mathcal{V}$.

e.g. The 5 partitions of the set $\mathcal{V} = \{1,2,3\}$ are

$\{1,2,3\}$	$\{1,2\},\{3\}$	$\{1,3\},\{2\}$
$\{2,3\},\{1\}$	$\{1\},\{2\},\{3\}$	

We refer to each nonempty subset making up a partition of the set $\mathcal{V}$ as a ***block***. For any partition of the set $\mathcal{V}$ we refer to the number of blocks that have cardinality j as the ***multiplicity*** of j in that partition.

It is well known that β_n, the n^{th} Bell Number, counts the total number of partitions of a set of n elements and that $S(n,k)$, the Stirling Number of the Second Kind, counts the total number of partition of a set of n elements into exactly k blocks. The article *A Review of the Stirling Numbers, Their Generalizations and Statistical Applications*, Charalambides, Ch. A.; Singh, Jagbir; Communications in Statistics, Theory and Methods, Vol. 17, No. 8, 1988, pages 2533--2595, is an excellent resource on both Bell and Stirling Numbers.

Let $\mathcal{V}_n$ be any set of n (distinct) elements. If a partition of the set $\mathcal{V}_n$ is selected uniformly at random from the set of all β_n partitions of $\mathcal{V}_n$, we will refer to this as a ***random partition of*** $\boldsymbol{\mathcal{V}_n}$. If a partition of $\mathcal{V}_n$ is selected uniformly at random from the set of all $S(n,k)$ partitions of $\mathcal{V}_n$ with k blocks, we will refer to this as a ***random partition of*** $\boldsymbol{\mathcal{V}_n}$ ***with k blocks***.

We will need the following definitions.

$\mathbb{S}^\infty$: the infinite product space $\{0,1,\dots\} \times \{0,1,\dots\} \times \cdots$

$\mathbb{S}^\infty_n$: the set of all vectors $(s_1,s_2,\dots)$ in $\mathbb{S}^\infty$ such that $1s_1 + 2s_2 + \dots = n$

$\mathbb{S}^\infty_{n,t}$: the set of all vectors $(s_1,s_2,\dots)$ in $\mathbb{S}^\infty$ such that $\begin{matrix} 1s_1 + 2s_2 + \dots = n \\ s_1 + s_2 + \dots = t \end{matrix}$

For any $\mathcal{A} \subseteq \mathbb{S}^\infty$ define $\mathcal{A}_n = \mathcal{A} \cap \mathbb{S}^\infty_n$ and $\mathcal{A}_{n,t} = \mathcal{A} \cap \mathbb{S}^\infty_{n,t}$

We note that the condition that $1s_1 + 2s_2 + \ldots = n$ implies that $s_j = 0$ for all $j > n$. Hence all vectors in $\mathcal{A}_n$ and $\mathcal{A}_{n,t}$ are of the form $(a_1,\ldots,a_n,0,0,\ldots)$.

For all $\mathcal{A} \neq (0,0,\ldots)$, let $\mathbb{A}_n$ be the collection of n-dimensional vectors formed by taking each infinite-dimensional vector in $\mathcal{A}_n$ and truncating after a_n. So for example,

$$(a_1,\ldots,a_n,0,0,\ldots) \rightarrow (a_1,\ldots,a_n)$$

Define $\mathbb{A}_{n,t}$ similarly. For notational consistency it is necessary to separate out the case $\mathcal{A} = (0,0,\ldots)$.

Let X_j equal the multiplicity of j in a random partition of a set of n elements into t blocks.

We can construct the set of all partitions of a set of n elements into t blocks such that $X_1 = x_1, \ldots, X_n = x_n$ in the following manner.

Take any one of the $n!$ permutations of the n elements and use the first x_1 elements of that permutation to fill the first x_1 blocks, use the next $2x_2$ elements of that permutation to fill the next x_2 blocks, and so on. In total we would use the $1x_1 + 2x_2 + \ldots + nx_n = n$ elements to fill the $x_1 + x_2 + \ldots + x_n = t$ blocks.

This yields $n!$ set partitions but not all of these set partitions are distinct. In particular this count would assume that rearranging the x_j blocks of cardinality j amongst themselves leads to distinct set partitions - which they do not. Furthermore this count would assume that rearranging elements within a block leads to distinct set partitions - which they do not. Therefore it is necessary to divide the count of $n!$ by the number of ways to arrange the x_j blocks of cardinality j amongst themselves $(j = 1, \ldots, n)$ and by the number of ways to arrange the elements in each block.

It follows that there are

$$\frac{n!}{x_1!\cdots x_n!}\left(\frac{1}{1!}\right)^{x_1}\cdots\left(\frac{1}{n!}\right)^{x_n}$$

partitions of a set of n elements into t blocks such that $X_1 = x_1, \ldots, X_n = x_n$ provided $1x_1 + \ldots + nx_n = n$, $x_1 + \ldots + x_n = t$, and $x_j \in \{0, \ldots, n\}\ \forall j$

and

$$P((X_1, \ldots, X_n) = (x_1, \ldots, x_n)) = \begin{cases} \dfrac{\frac{n!}{x_1!\cdots x_n!}\left(\frac{1}{1!}\right)^{x_1}\cdots\left(\frac{1}{n!}\right)^{x_n}}{S(n,t)} & \substack{1x_1+\ldots+nx_n=n \\ x_1+\ldots+x_n=t \\ x_j\in\{0,1,\ldots,n\}\forall j} \\ 0 & \text{otherwise} \end{cases}$$

If we let W_j equal the multiplicity of j in a random partition of a set of n elements then it follows similarly that

$$P((W_1, \ldots, W_n) = (w_1, \ldots, w_n)) = \begin{cases} \frac{\frac{n!}{w_1! \cdots w_n!}\left(\frac{1}{1!}\right)^{w_1} \cdots \left(\frac{1}{n!}\right)^{w_n}}{\beta_n} & {\scriptstyle 1w_1 + \ldots + nw_n = n \atop w_j \in \{0,1,\ldots,n\} \forall j} \\ 0 & \text{otherwise} \end{cases}$$

Theorem 1.

$$\mathrm{E}(g^*(X_1, \ldots, X_n)) = \left(\frac{1}{t!S(n,t)}\right)\left(\frac{d^n}{d\lambda^n}\frac{d^t}{d\theta^t}\left(e^{\theta(e^\lambda - 1)}\mathrm{E}(g(Y_1, Y_2, \ldots))\right)\Big|_{\substack{\lambda=0\\ \theta=0}}\right)$$

where $g(a_1, a_2, \ldots)$ is any function and $g^*(a_1, \ldots, a_n) = g(a_1, \ldots, a_n, 0, 0, \ldots)$ and for $\mathcal{A} \neq (0, 0, \ldots)$

$$P((X_1, \ldots, X_n) \in \mathbb{A}_{n,t}) = \left(\frac{1}{t!S(n,t)}\right)\left(\frac{d^n}{d\lambda^n}\frac{d^t}{d\theta^t}\left(e^{\theta(e^\lambda - 1)}P((Y_1, Y_2, \ldots) \in \mathcal{A})\right)\Big|_{\substack{\lambda=0\\ \theta=0}}\right)$$

where $Y_1, Y_2, \ldots$ is an infinite sequence of independent Poisson random variables such that

$$P(Y_j = y) = \frac{e^{\left(\frac{-\theta\lambda^j}{j!}\right)}\left(\frac{\theta\lambda^j}{j!}\right)^y}{y!} \qquad y = 0, 1, 2, \ldots \text{ and } j = 1, 2, \ldots \qquad \blacksquare$$

Theorem 2.

$$\mathrm{E}(g^*(W_1, \ldots, W_n)) = \left(\frac{1}{\beta_n}\right)\left(\frac{d^n}{d\lambda^n}\left(e^{(e^\lambda - 1)}\mathrm{E}(g(Y_1, Y_2, \ldots))\right)\Big|_{\lambda=0}\right)$$

where $g(a_1, a_2, \ldots)$ is any function and $g^*(a_1, \ldots, a_n) = g(a_1, \ldots, a_n, 0, 0, \ldots)$ and for $\mathcal{A} \neq (0, 0, \ldots)$

$$P((W_1, \ldots, W_n) \in \mathbb{A}_n) = \left(\frac{1}{\beta_n}\right)\left(\frac{d^n}{d\lambda^n}\left(e^{(e^\lambda - 1)}P((Y_1, Y_2, \ldots) \in \mathcal{A})\right)\Big|_{\lambda=0}\right)$$

where $Y_1, Y_2, \ldots$ is an infinite sequence of independent Poisson random variables such that

$$P(Y_j = y) = \frac{e^{\left(\frac{-\lambda^j}{j!}\right)} \left(\frac{\lambda^j}{j!}\right)^y}{y!} \qquad y = 0, 1, 2, \ldots \text{ and } j = 1, 2, \ldots \qquad \blacksquare$$

Now suppose we randomly distribute n distinct objects into t distinct urns such that objects are distributed independently and are equally likely to go into any of the t urns. If one or more urns are still empty after this distribution we empty all of the urns and start the distribution process over. The distribution process stops as soon as we obtain a distribution of the n objects where each of the t urns has at least one object.

Let $U_j, (j = 1, \ldots, t)$ equal the number of objects in the j^{th} urn after the distribution process has stopped. Then,

$$P((U_1, \ldots, U_t) = (u_1, \ldots, u_t)) = \begin{cases} \frac{n!}{u_1! \cdots u_t! t! S(n,t)} & \substack{u_1 + \ldots + u_t = n \\ u_j \in \{1,2,\ldots\} \ \forall j} \\ 0 & \text{else} \end{cases}$$

Let $V_j, (j = 1, \ldots, n)$ equal the number of (labeled) urns containing j objects after the distribution process has stopped. Then,

$$P((V_1, \ldots, V_n) = (v_1, \ldots, v_n)) = \begin{cases} \frac{\frac{n!}{v_1! \cdots v_n!} \left(\frac{1}{1!}\right)^{v_1} \cdots \left(\frac{1}{n!}\right)^{v_n}}{S(n,t)} & \substack{1v_1 + \ldots + nv_n = n \\ v_1 + \ldots + v_n = t \\ v_j \in \{0,1,\ldots,n\} \ \forall j} \\ 0 & \text{otherwise} \end{cases}$$

We note that the above probability distribution is based on a model where the t urns are distinguishable (labeled) but clearly the latter probability would be the same if the t urns were not labeled. However distributing n distinguishable objects into t like (unlabeled) urns is equivalent to partitioning a set of n objects into t blocks. That is

$$P((V_1, \ldots, V_n) \in \mathbb{A}_{n,t}) = P((X_1, \ldots, X_n) \in \mathbb{A}_{n,t})$$

where as before the $X_j, (j = 1, \ldots, t)$ equal the number of blocks with j elements in a random partition of a set of n elements into t blocks.

The distinction between the two probabilities is that in the case of labeled urns there are $tS(n,t)$ possible distributions and in the case of unlabeled urns there are only $S(n,t)$ possible distributions.

We will need the following definitions.

$\mathbb{U}^t$: the t-dimensional product space $\{1,2,\dots\} \times \cdots \times \{1,2,\dots\}$

$\mathbb{U}_n^t$: the set of all vectors $(u_1,\dots,u_t)$ in $\mathbb{U}^t$ such that $u_1 + \ \dots + u_t = n$

Define $\mathcal{U}_n^{\mathbb{A}} \subseteq \mathbb{U}^t$ as that set such that $(V_1, \dots, V_n) \in \mathbb{A}_{n,t} \Leftrightarrow (U_1, \dots, U_t) \in \mathcal{U}_n^{\mathbb{A}}$.

Theorem 3.

$$P((X_1, \dots, X_n) \in \mathbb{A}_{n,t}) = \frac{1}{t!S(n,t)} \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} \Big(\big(e^\lambda - 1\big)^t P\big((Y_1,\dots,Y_t) \in \mathcal{U}^{\mathbb{A}}\big) \Big) \Big|_{\lambda=0}$$

where $Y_1, \dots, Y_t$ are *iid* zero-truncated Poisson random variables with parameter λ. That is,

$$P(Y = y) = \begin{cases} \frac{e^{-\lambda}\lambda^y}{y!(1-e^{-\lambda})} & y \in \{1, 2, \dots\} \\ 0 & \text{else} \end{cases} \qquad \blacksquare$$

Applications

<u>Problem 1.</u>

(a) Show that the number of partitions of a set of n elements into t blocks containing exactly k blocks of cardinality v equals

$$\sum_{j=k}^{\min\left(t,\left\lfloor \frac{n-t}{v-1}\right\rfloor\right)} (-1)^{j-k}\left(\frac{n!}{k!\;(j-k)!\;(v!)^{j}(n-vj)!}\right)S(n-vj,t-j)$$

(b) Show that the number of partitions of a set of n elements containing exactly k blocks of cardinality v equals

$$\sum_{j=k}^{\left\lfloor \frac{n}{v}\right\rfloor}(-1)^{j-k}\left(\frac{n!}{k!\;(j-k)!\;(v!)^{j}(n-vj)!}\right)\beta_{n-vj}$$

provided $vk \leq n$.

(c) Show that the number of partitions of a set of n elements containing at least k blocks of cardinality v equals

$$\sum_{j=k}^{\left\lfloor \frac{n}{v}\right\rfloor}(-1)^{j-k}\left(\frac{n!}{j\;(k-1)!(j-k)!(v!)^{j}(n-vj)!}\right)\beta_{n-vj}$$

provided $vk \leq n$.

We note that the special case of (c) where $k=1$ is in agreement with Haigh, J., *Random Equivalence Relations*, <u>Journal of Combinatorial Theory</u>, <u>Series A</u>, Vol. 13, 1972, pages 287-295.

<u>Problem 2.</u>

(a) Show that the k^{th} factorial moment of the number of blocks containing v elements in a random partition of a set of n elements into t blocks equals

$$\frac{n!}{(n-vk)!(v!)^k}\frac{S(n-vk,t-k)}{S(n,t)} \quad \text{provided } v \le \left\lfloor \frac{n-t+k}{k} \right\rfloor \text{ and } t \ge k$$

We note that the special case where $k = 1$ is in agreement with Proposition 2.4 in Recski, A., *On Random Permutations*, Discrete Mathematics, 16, 1976, 173-177.

(b) As a check on the result in (a) verify that $\sum_{v=1}^{n} \binom{n}{v} \frac{S(n-v,t-1)}{S(n,t)} = t.$

(c) Show that the k^{th} factorial moment of the number of blocks containing v elements in a random partition of a set of n elements equals

$$\frac{n!}{(n-vk)!(v!)^k}\frac{\beta_{n-vk}}{\beta_n} \quad \text{provided } v \le \left\lfloor \frac{n}{k} \right\rfloor$$

We note that the special case where $k = 1$ is in agreement with Proposition 1.1 in Recski, A., *On Random Permutations*, Discrete Mathematics, 16, 1976, 173-177.

Problem 3.

(a) Show that the expected number of multiplicities in a random partition of a set of n elements into t blocks that equal k is

$$\sum_{v=1}^{n} \sum_{j=k}^{\min\left(t,\left\lfloor \frac{n-t}{v-1} \right\rfloor\right)} (-1)^{j-k} \left(\frac{n!}{k!\,(j-k)!\,(v!)^j (n-vj)!} \right) \frac{S(n-vj,t-j)}{S(n,t)}$$

(b) Show that the expected number of multiplicities in a random partition of a set of n elements that equal k is

$$\sum_{t=1}^{n} \sum_{v=1}^{n} \sum_{j=k}^{\min\left(t,\left\lfloor \frac{n-t}{v-1} \right\rfloor\right)} (-1)^{j-k} \left(\frac{n!}{k!\,(j-k)!\,(v!)^j (n-vj)!} \right) \frac{S(n-vj,t-j)}{\beta_n}$$

Problem 4.

Show that the expected number of blocks in a random partition of a set with n elements equals

$$\frac{\beta_{n+1} - \beta_n}{\beta_n}$$

in agreement with L. H. Harper, *Stirling Behavior is Asymptotically Normal*, Annals of Mathematical Statistics, Vol 38, 1967, pages 410-414.

Problem 5.

(a) Suppose that a partition is picked uniformly at random from the set of $S(n,t)$ partitions of a set of n elements into exactly t blocks and from that partition a block is then picked uniformly at random. Show that the expected size of this block is

$$\frac{n}{t}$$

(b) Suppose that a partition is picked uniformly at random from the set of β_n partitions of a set of n elements and from that partition a block is then picked uniformly at random. Show that the expected size of this block is

$$\frac{n}{\beta_n}\sum_{t=1}^{n}\frac{S(n,t)}{t}$$

(c) Suppose that a block is picked uniformly at random from the set of all $\sum_{t=1}^{n} tS(n,t)$ blocks in the β_n partitions of a set of n elements. Show that the expected size of this block is

$$\frac{n\beta_n}{\beta_{n+1} - \beta_n}$$

(d) Suppose that a partition is picked uniformly at random from the set of $S(n,t)$ partitions of a set of n elements into exactly t blocks and from that partition an element is selected uniformly at random. Show that the expected size of the block containing the selected element is

$$\frac{(n-1)S(n-1,t)}{S(n,t)}+1$$

(e) Suppose that a partition is picked uniformly at random from the set of β_n partitions of a set of n elements and from that partition an element is selected uniformly at random. Show that the expected size of the block containing the selected element is

$$\frac{(n-1)\beta_{n-1}}{\beta_n}+1$$

Problem 6.

Show that the number of partitions of a set of n elements into t blocks where exactly r of the t blocks have an even number of elements is

$$\frac{1}{t!}\binom{t}{r}\left(\frac{1}{2^t}\right)\sum_{j=0}^{2r}\sum_{k=0}^{t-r}(-1)^{j+k}\binom{2r}{j}\binom{t-r}{k}(t-j-2k)^n$$

The special case $r=t$ (all blocks have an even number of elements) simplifies to

$$\frac{1}{t!}\left(\frac{1}{2^t}\right)\sum_{j=0}^{2t}(-1)^j\binom{2t}{j}(t-j)^n$$

and the special case $r=0$ (all blocks have an odd number of elements) to

$$\frac{1}{t!}\left(\frac{1}{2^t}\right)\sum_{k=0}^{t}(-1)^k\binom{t}{k}(t-2k)^n$$

in agreement with (2.24 and 2.25) of *Set Partitions*, L. Carlitz, Fibonacci Quarterly, Nov. 1976, pages 327 – 342, apart from an obvious typo in equation 2.24.

Problem 7.

Suppose that a partition is picked uniformly at random from the set of β_n partitions of a set of n elements and from that partition a block size is picked uniformly at random from the set of all block sizes in that partition. Show that the expected size of the selected block size is

Proof (Theorem 1)

Consider an infinite sequence $Y_1, Y_2, \ldots$ of independent Poisson random variables where

$$P(Y_j = y) = \frac{e^{\left(\frac{-\lambda^j}{j!}\right)}\left(\frac{\lambda^j}{j!}\right)^y}{y!} \qquad y = 0, 1, 2, \ldots \text{ and } j = 1, 2, \ldots$$

Let y_j be a nonnegative integer for $j = 1, 2, \cdots$. Then

$$\begin{aligned} P(Y_1 = y_1, Y_2 = y_2, \ldots) &= \frac{e^{\left(-\sum\limits_{j=1}^{\infty}\frac{\lambda^j}{j!}\right)}\lambda^{\left(\sum\limits_{j=1}^{\infty} jy_j\right)}\prod\limits_{j=1}^{\infty}\left(\frac{1}{j!}\right)^{y_j}}{\prod\limits_{j=1}^{\infty}(y_j)!} \\ &= \frac{e^{(-e^{\lambda}+1)}\lambda^{\left(\sum\limits_{j=1}^{\infty} jy_j\right)}\prod\limits_{j=1}^{\infty}\left(\frac{1}{j!}\right)^{y_j}}{\prod\limits_{j=1}^{\infty}(y_j)!} \end{aligned}$$

and

$$\begin{aligned} P\left(\sum_{j=1}^{\infty} jY_j = n\right) &= \sum_{\substack{(y_1, y_2, \ldots)\ni \\ \sum_{j=1}^{\infty} jy_j = n}} \cdots \sum P(Y_1 = y_1, Y_2 = y_2, \ldots) \\ &= \frac{e^{(-e^{\lambda}+1)}\lambda^n}{n!}\left(\sum_{\substack{(y_1, y_2, \ldots)\ni \\ \sum_{j=1}^{\infty} jy_j = n}} \cdots \sum n!\left(\frac{\prod\limits_{j=1}^{\infty}\left(\frac{1}{j!}\right)^{y_j}}{\prod\limits_{j=1}^{\infty}(y_j)!}\right)\right) \\ &= \frac{e^{(-e^{\lambda}+1)}\lambda^n}{n!}\beta_n \end{aligned}$$

It follows that for $\mathcal{A} \neq (0, 0, \ldots)$

$$P((Y_1,Y_2,\dots)\in\mathcal{A})=\sum_{n=0}^{\infty}P\left((Y_1,Y_2,\dots)\in\mathcal{A}\;\middle|\;\sum_{j=1}^{\infty}jY_j=n\right)P\left(\sum_{j=1}^{\infty}jY_j=n\right)$$

$$=\sum_{n=0}^{\infty}\sum_{\mathcal{A}_n}\left(\frac{\dfrac{e^{\left(-e^{\lambda}+1\right)}\lambda^n\prod_{j=1}^{\infty}\left(\frac{1}{j!}\right)^{y_j}}{\prod_{j=1}^{\infty}(y_j)!}}{\dfrac{e^{\left(-e^{\lambda}+1\right)}\lambda^n}{n!}\beta_n}\right)\left(\frac{e^{\left(-e^{\lambda}+1\right)}\lambda^n}{n!}\beta_n\right)$$

$$=\sum_{n=1}^{\infty}\sum_{\mathbb{A}_n}\left(\frac{\frac{n!}{y_1!y_2!\cdots y_n!}\left(\frac{1}{1!}\right)^{y_1}\left(\frac{1}{2!}\right)^{y_2}\cdots\left(\frac{1}{n!}\right)^{y_n}}{\beta_n}\right)\left(\frac{e^{\left(-e^{\lambda}+1\right)}\lambda^n}{n!}\beta_n\right)$$

$$=\sum_{n=1}^{\infty}P((W_1,\ldots,W_n)\in\mathbb{A}_n)\left(\frac{e^{\left(-e^{\lambda}+1\right)}\lambda^n}{n!}\beta_n\right)$$

Therefore

$$e^{\left(e^{\lambda}-1\right)}P((Y_1,Y_2,\dots)\in\mathcal{A})=\sum_{n=1}^{\infty}P((W_1,\ldots,W_n)\in\mathbb{A}_n)\left(\frac{\beta_n}{n!}\right)\lambda^n$$

and

$$\frac{d^r}{d\lambda^r}\left(e^{\left(e^{\lambda}-1\right)}P((Y_1,Y_2,\dots)\in\mathcal{A})\right)\Bigg|_{\lambda=0}$$
$$=\sum_{n=1}^{\infty}P((W_1,\ldots,W_n)\in\mathbb{A}_n)\left(\frac{\beta_n}{n!}\right)r!\mathrm{I}_{\{r\}}(n)$$

Therefore, for $\mathcal{A}\neq(0,0,\dots)$

$$P((W_1,\ldots,W_n)\in\mathbb{A}_n)=\left(\frac{1}{\beta_n}\right)\left(\frac{d^n}{d\lambda^n}\left(e^{\left(e^{\lambda}-1\right)}P((Y_1,Y_2,\dots)\in\mathcal{A})\right)\Bigg|_{\lambda=0}\right)\qquad\square$$

Proof (Theorem 2)

Consider an infinite sequence $Y_1, Y_2, \ldots$ of independent Poisson random variables where

$$P(Y_j = y) = \frac{e^{\left(\frac{-\theta\lambda^j}{j!}\right)}\left(\frac{\theta\lambda^j}{j!}\right)^y}{y!} \qquad y = 0, 1, 2, \ldots \text{ and } j = 1, 2, \ldots$$

Let y_j be a nonnegative integer for $j = 1, 2, \cdots$. Then

$$P(Y_1 = y_1, Y_2 = y_2, \ldots) = \frac{e^{\left(-\sum_{j=1}^{\infty}\frac{\theta\lambda^j}{j!}\right)}\theta^{\left(\sum_{j=1}^{\infty}y_j\right)}\lambda^{\left(\sum_{j=1}^{\infty}jy_j\right)}\prod_{j=1}^{\infty}\left(\frac{1}{j!}\right)^{y_j}}{\prod_{j=1}^{\infty}(y_j)!}$$

$$= \frac{e^{\theta(-e^{\lambda}+1)}\theta^{\left(\sum_{j=1}^{\infty}y_j\right)}\lambda^{\left(\sum_{j=1}^{\infty}jy_j\right)}\prod_{j=1}^{\infty}\left(\frac{1}{j!}\right)^{y_j}}{\prod_{j=1}^{\infty}(y_j)!}$$

and

$$P\left(\sum_{j=1}^{\infty}jY_j = n \text{ and } \sum_{j=1}^{\infty}Y_j = t\right) = \sum_{\substack{(y_1,y_2,\ldots)\ni \\ \sum_{j=1}^{\infty}jy_j=n \\ \sum_{j=1}^{\infty}y_j=t}}\cdots\sum P(Y_1 = y_1, Y_2 = y_2, \ldots)$$

$$= \frac{e^{\theta(-e^{\lambda}+1)}\theta^t\lambda^n}{n!}\left(\sum_{\substack{(y_1,y_2,\ldots)\ni \\ \sum_{j=1}^{\infty}jy_j=n \\ \sum_{j=1}^{\infty}y_j=t}}\cdots\sum n!\left(\frac{\prod_{j=1}^{\infty}\left(\frac{1}{j!}\right)^{y_j}}{\prod_{j=1}^{\infty}(y_j)!}\right)\right)$$

$$= \frac{e^{\theta(-e^{\lambda}+1)}\theta^t\lambda^n}{n!}S(n,t)$$

It follows that

$$P((Y_1,Y_2,\dots)\in\mathcal{A}) = \sum_{n=0}^{\infty}\sum_{t=0}^{\infty} P\left((Y_1,Y_2,\dots)\in\mathcal{A} \mid \sum_{j=1}^{\infty} jY_j = n, \sum_{j=1}^{\infty} Y_j = t\right) P\left(\sum_{j=1}^{\infty} jY_j = n, \sum_{j=1}^{\infty} Y_j = t\right)$$

$$= \sum_{n=0}^{\infty}\sum_{t=0}^{\infty}\sum_{\mathcal{A}_{n,t}} \left(\frac{\frac{e^{\theta\left(-e^{\lambda}+1\right)}\theta^t\lambda^n \prod_{j=1}^{\infty}\left(\frac{1}{j!}\right)^{y_j}}{\prod_{j=1}^{\infty}(y_j)!}}{\frac{e^{\theta\left(-e^{\lambda}+1\right)}\theta^t\lambda^n}{n!}S(n,t)} \right) \left(\frac{e^{\theta\left(-e^{\lambda}+1\right)}\theta^t\lambda^n}{n!} S(n,t) \right)$$

$$= \sum_{n=1}^{\infty}\sum_{t=1}^{\infty}\sum_{\mathbb{A}_{n,t}} \left(\frac{\frac{n!}{y_1!y_2!\cdots y_n!}\left(\frac{1}{1!}\right)^{y_1}\left(\frac{1}{2!}\right)^{y_2}\cdots\left(\frac{1}{n!}\right)^{y_n}}{S(n,t)} \right) \left(\frac{e^{\theta\left(-e^{\lambda}+1\right)}\theta^t\lambda^n}{n!} S(n,t) \right)$$

$$= \sum_{n=1}^{\infty}\sum_{t=1}^{\infty} P((X_1,\dots,X_n)\in\mathbb{A}_{n,t}) \left(\frac{\mathrm{e}^{\theta\left(-e^{\lambda}+1\right)}\theta^t\lambda^n}{n!} S(n,t) \right)$$

Therefore

$$e^{\theta\left(e^{\lambda}-1\right)} P((Y_1,Y_2,\dots)\in\mathcal{A}) = \sum_{n=1}^{\infty}\sum_{t=1}^{\infty} P((X_1,\dots,X_n)\in\mathbb{A}_{n,t}) \left(\frac{S(n,t)}{n!}\right)\theta^t\lambda^n$$

and

$$\frac{d^r}{d\lambda^r}\frac{d^s}{d\theta^s}\left(e^{\theta\left(e^{\lambda}-1\right)} P((Y_1,Y_2,\dots)\in\mathcal{A})\right)\Big|_{\substack{\lambda=0\\ \theta=0}}$$
$$= \sum_{n=1}^{\infty}\sum_{t=1}^{\infty} P((X_1,\dots,X_n)\in\mathbb{A}_{n,t}) \left(\frac{S(n,t)}{n!}\right) r!s!\mathrm{I}_{\{r\}}(n)\mathrm{I}_{\{s\}}(t)$$

Therefore, for $\mathcal{A} \neq (0,0,\dots)$

$$P((X_1,\dots,X_n)\in\mathbb{A}_{n,t}) = \left(\frac{1}{t!S(n,t)}\right)\left(\frac{d^n}{d\lambda^n}\frac{d^t}{d\theta^t}\left(e^{\theta\left(e^{\lambda}-1\right)} P((Y_1,Y_2,\dots)\in\mathcal{A})\right)\Big|_{\substack{\lambda=0\\ \theta=0}}\right) \quad \square$$

Proof (Theorem 3)

$$P((X_1,\ldots,X_n)\in\mathbb{A}_{n,t})$$

$$= P((V_1,\ldots,V_n)\in\mathbb{A}_{n,t})$$

$$= P((U_1,\ldots,U_t)\in\mathcal{U}_n^{\mathbb{A}})$$

But by the Zero-Truncated Poisson Randomization Theorem,

$$P\big((U_1,\ldots,U_t)\in\mathcal{U}_n^{\mathbb{A}}\big)=\frac{1}{t!S(n,t)}\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\Big(\big(e^\lambda-1\big)^tP\big((Y_1,\ldots,Y_t)\in\mathcal{U}^{\mathbb{A}}\big)\Big)\Big|_{\lambda=0}$$

where $Y_1,\ldots,Y_t$ are *iid* zero-truncated Poisson random variables with parameter λ. That is,

$$P(Y=y)=\begin{cases}\frac{e^{-\lambda}\lambda^y}{y!(1-e^{-\lambda})} & y\in\{1,2,\ldots\}\\ 0 & \text{else}\end{cases}\qquad\square$$

Solutions

Problem 1(a)

This problem can be handled by a direct application of the inclusion-exclusion principle but we will apply Theorem 1 to illustrate its use.

Clearly the solution equals $S(n,t)P(X_v = k)$. Now define

$$\begin{aligned} \mathbb{A}_{n,t} &= \{(a_1,a_2,\ldots,a_n) | 1a_1 + \ldots + na_n = n, a_1 + \ldots + a_n = t \text{ and } a_v = k\} \\ \mathcal{A}_{n,t} &= \{a_1,a_2,\ldots,a_n, 0, 0, \ldots | 1a_1 + \ldots + na_n = n, a_1 + \ldots + a_n = t \text{ and } a_v = k\} \\ \mathcal{A} &= \{a_1,a_2,\ldots,a_n, a_{n+1}, a_{n+2}, \ldots | \, a_v = k\} \end{aligned}$$

Then by Theorem 1,

$$\begin{aligned} P(X_v = k) &= P((X_1, \ldots, X_n) \in \mathbb{A}_{n,t}) \\ &= \left(\frac{1}{t!S(n,t)}\right)\left(\frac{d^n}{d\lambda^n}\frac{d^t}{d\theta^t}\left(e^{\theta(e^\lambda - 1)} P((Y_1, Y_2, \ldots) \in \mathcal{A})\right)\Big|_{\substack{\lambda=0\\ \theta=0}}\right) \end{aligned}$$

where

$$\begin{aligned} P((Y_1, Y_2, \ldots) \in \mathcal{A}) &= \left(\prod_{i \neq v} P(Y_i \in \{0,1,\ldots\})\right) P(Y_v = k) \\ &= P(Y_v = k) = \frac{e^{\left(\frac{-\theta\lambda^v}{v!}\right)}\left(\frac{\theta\lambda^v}{v!}\right)^k}{k!} \end{aligned}$$

Therefore,

$$P(X_v = k) = \left(\tfrac{1}{t!S(n,t)}\right)\left(\tfrac{d^n}{d\lambda^n}\tfrac{d^t}{d\theta^t}\left(e^{\theta(e^\lambda - 1)}\,\frac{e^{\left(\frac{-\theta\lambda^v}{v!}\right)}\left(\frac{\theta\lambda^v}{v!}\right)^k}{k!}\right)\Big|_{\substack{\lambda=0\\ \theta=0}}\right)$$

$$= \left(\tfrac{1}{t!S(n,t)}\right)\frac{\left(\frac{1}{v!}\right)^k}{k!}\left(\tfrac{d^n}{d\lambda^n}\tfrac{d^t}{d\theta^t}\left(e^{\theta\left(e^\lambda - 1 - \frac{\lambda^v}{v!}\right)}\theta^k\lambda^{vk}\right)\Big|_{\substack{\lambda=0\\ \theta=0}}\right)$$

$$= \left(\tfrac{1}{S(n,t)}\right)\frac{\left(\frac{1}{v!}\right)^k}{k!(t-k)!}\left(\tfrac{d^n}{d\lambda^n}\left(\left(e^\lambda - 1 - \tfrac{\lambda^v}{v!}\right)^{t-k}\lambda^{vk}\right)\Big|_{\lambda=0}\right)$$

$$= \left(\frac{1}{S(n,t)}\right)\frac{\left(\frac{1}{v!}\right)^k}{k!(t-k)!}\left(\frac{d^n}{d\lambda^n}\left(\sum_{j=0}^{t-k}\binom{t-k}{j}(e^\lambda-1)^{t-k-j}\left(-\frac{\lambda^v}{v!}\right)^j\lambda^{vk}\right)\Bigg|_{\lambda=0}\right)$$

$$= \left(\frac{1}{S(n,t)}\right)\frac{\left(\frac{1}{v!}\right)^k}{k!(t-k)!}\left(\frac{d^n}{d\lambda^n}\left(\sum_{j=0}^{t-k}\frac{(t-k)!}{j!}\sum_{i=t-k-j}^{\infty}\frac{S(i,t-k-j)}{i!}\lambda^i\left(-\frac{\lambda^v}{v!}\right)^j\lambda^{vk}\right)\Bigg|_{\lambda=0}\right)$$

$$= \left(\frac{1}{S(n,t)}\right)\frac{\left(\frac{1}{v!}\right)^k}{k!(t-k)!}\left(\frac{d^n}{d\lambda^n}\left(\sum_{j=0}^{t-k}\sum_{i=t-k-j}^{\infty}(-1)^j\left(\frac{1}{v!}\right)^j\ \frac{(t-k)!}{j!}\ \frac{S(i,t-k-j)}{i!}\lambda^i\lambda^{vj}\lambda^{vk}\right)\Bigg|_{\lambda=0}\right)$$

$$= \left(\frac{1}{S(n,t)}\right)\frac{\left(\frac{1}{v!}\right)^k}{k!(t-k)!}\left(\frac{d^n}{d\lambda^n}\left(\sum_{j=0}^{t-k}\sum_{i=t-k-j}^{\infty}(-1)^j\left(\frac{1}{v!}\right)^j\ \frac{(t-k)!}{j!}\ \frac{S(i,t-k-j)}{i!}\lambda^{i+v(j+k)}\right)\Bigg|_{\lambda=0}\right)$$

$$= \left(\frac{1}{S(n,t)}\right)\frac{\left(\frac{1}{v!}\right)^k}{k!(t-k)!}\left(\sum_{j=0}^{t-k}\sum_{i=t-k-j}^{\infty}(-1)^j\left(\frac{1}{v!}\right)^j\ \frac{(t-k)!}{j!}\ \frac{S(i,t-k-j)}{i!}\ n!\ \mathrm{I}_{\{i+v(j+k)\}}(n)\right)$$

$$= \left(\frac{1}{S(n,t)}\right)\frac{\left(\frac{1}{v!}\right)^k}{k!(t-k)!}\left(\sum_{j=0}^{t-k}\sum_{i=t-k-j}^{\infty}(-1)^j\left(\frac{1}{v!}\right)^j\ \frac{(t-k)!}{j!}\ \frac{S(i,t-k-j)}{i!}\ n!\ \mathrm{I}_{\{n-v(j+k)\}}(i)\right)$$

$$= \left(\frac{1}{S(n,t)}\right)\frac{\left(\frac{1}{v!}\right)^k}{k!(t-k)!}\left(\sum_{j=0}^{t-k}(-1)^j\left(\frac{1}{v!}\right)^j\ \frac{(t-k)!}{j!}\ \frac{S(n-v(j+k),t-k-j)}{(n-v(j+k))!}\ n!\ \mathrm{I}_{\{t-k-j,\ldots\}}(n-v(j+k))\right)$$

$$= \left(\frac{1}{S(n,t)}\right)\frac{\left(\frac{1}{v!}\right)^k}{k!(t-k)!}\left(\sum_{j=0}^{\min\left(t-k,\left\lfloor\frac{n-(v-1)k-t}{v-1}\right\rfloor\right)}(-1)^j\left(\frac{1}{v!}\right)^j\ \frac{(t-k)!}{j!}\ \frac{S(n-v(j+k),t-k-j)}{(n-v(j+k))!}\ n!\right)$$

Therefore the number of partitions of the set of n elements into t blocks which contain exactly k blocks of cardinality v is

$$\sum_{j=k}^{\min\left(t,\left\lfloor\frac{n-t}{v-1}\right\rfloor\right)}(-1)^{j-k}\left(\frac{n!}{k!\ (j-k)!\ (v!)^j(n-vj)!}\right)S(n-vj,t-j) \qquad \square$$

Problem 1(b)

$$P((W_1, \ldots, W_n) \in \mathbb{A}_n) = \left(\frac{1}{\beta_n}\right)\left(\frac{d^n}{d\lambda^n}\left(e^{\left(e^{\lambda}-1\right)} P((Y_1, Y_2, \ldots) \in \mathcal{A})\right)\Big|_{\lambda=0}\right)$$

Clearly the solution equals $\beta_n \cdot P(W_v = k)$. Now define

$$\begin{aligned}
\mathbb{A}_n &= \{(a_1, a_2, \ldots, a_n) | 1a_1 + \ \ldots + na_n = n \text{ and } a_v = k\} \\
\mathcal{A}_n &= \{a_1, a_2, \ldots, a_n, 0, 0, \ldots | 1a_1 + \ \ldots + na_n = n \text{ and } a_v = k\} \\
\mathcal{A} &= \{a_1, a_2, \ldots, a_n, a_{n+1}, a_{n+2}, \ldots \,|\, a_v = k\}
\end{aligned}$$

Then by Theorem 2,

$$\begin{aligned}
P(W_v = k) &= P((W_1, \ldots, W_n) \in \mathbb{A}_n) \\
&= \left(\frac{1}{\beta_n}\right)\left(\frac{d^n}{d\lambda^n}\left(e^{\left(e^{\lambda}-1\right)} P((Y_1, Y_2, \ldots) \in \mathcal{A})\right)\Big|_{\lambda=0}\right)
\end{aligned}$$

where

$$\begin{aligned}
P((Y_1, Y_2, \ldots) \in \mathcal{A}) &= \left(\prod_{i \neq v} P(Y_i \in \{0,1,\ldots\})\right) P(Y_v = k) \\
&= P(Y_v = k) = \frac{e^{\left(\frac{-\lambda^v}{v!}\right)} \left(\frac{\lambda^v}{v!}\right)^k}{k!}
\end{aligned}$$

Therefore,

$$P(W_v = k) = \left(\frac{1}{\beta_n}\right)\left(\frac{d^n}{d\lambda^n}\left(e^{\left(e^{\lambda}-1\right)} \frac{e^{\left(\frac{-\lambda^v}{v!}\right)} \left(\frac{\lambda^v}{v!}\right)^k}{k!}\right)\Bigg|_{\lambda=0}\right)$$

$$= \left(\frac{1}{\beta_n}\right)\frac{1}{(v!)^k k!}\left(\frac{d^n}{d\lambda^n}\left(e^{\left(e^{\lambda}-1\right)} e^{\left(-\frac{\lambda^v}{v!}\right)} \lambda^{vk}\right)\Big|_{\lambda=0}\right)$$

$$= \left(\frac{1}{\beta_n}\right)\frac{1}{(v!)^k k!}\left(\frac{d^n}{d\lambda^n}\left(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^j \frac{\beta_i}{i!j!(v!)^j} \lambda^{i+v(j+k)}\right)\Bigg|_{\lambda=0}\right)$$

$$= \left(\frac{1}{\beta_n}\right)\frac{1}{(v!)^k k!}\left(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^j \frac{\beta_i}{i!j!(v!)^j} n!\ \mathrm{I}_{\{i+v(j+k)\}}(n)\right)$$

$$= \left(\frac{1}{\beta_n}\right) \frac{1}{(v!)^k k!} \left(\sum_{j=0}^{\infty} \sum_{i=0}^{\infty} (-1)^j \frac{\beta_i}{i! j! (v!)^j} n!\ \mathrm{I}_{\{n-v(j+k)\}}(i) \right)$$

$$= \left(\frac{1}{\beta_n}\right) \frac{1}{(v!)^k k!} \left(\sum_{j=0}^{\infty} (-1)^j \frac{\beta_{n-v(j+k)}}{(n-v(j+k))! j! (v!)^j} n!\ \mathrm{I}_{\{0,1,\ldots\}}(n - v(j+k)) \right)$$

$$= \left(\frac{1}{\beta_n}\right) \frac{1}{k!} \left(\sum_{j=0}^{\lfloor \frac{n-vk}{v} \rfloor} (-1)^j \frac{\beta_{n-v(j+k)}}{(n-v(j+k))! j! (v!)^{j+k}} n! \right) \mathrm{I}_{\{0,1,\ldots\}}(n - vk)$$

$$= \left(\frac{1}{\beta_n}\right) \sum_{j=0}^{\lfloor \frac{n-vk}{v} \rfloor} (-1)^j \frac{n!}{k!\ j!\ (v!)^{j+k} (n-v(j+k))!}\ \beta_{n-v(j+k)} \quad \text{provided } n - vk \geq 0$$

Therefore the number of partitions of the set of n elements which contain exactly k blocks of cardinality v is

$$\sum_{j=k}^{\lfloor \frac{n}{v} \rfloor} (-1)^{j-k} \left(\frac{n!}{k!\ (j-k)!\ (v!)^j (n-vj)!} \right) \beta_{n-vj}$$

provided $n - vk \geq 0$ □

Problem 1(c)

By Theorem 3, the number of partitions of a set of n elements <u>into t blocks</u> containing at least k blocks of cardinality v equals

$$\frac{1}{t!} \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} \Big(\big(e^\lambda - 1\big)^t P\big((Y_1,\ldots,Y_t) \in \mathcal{U}^{\mathbb{A}}\big) \Big) \Big|_{\lambda=0}$$

where

$$\mathcal{U}^{\mathbb{A}} = \{(u_1, \ldots, u_t) \in \mathbb{U}^t \mid \text{at least } k \text{ of the } t \text{ values } (u_1, \ldots, u_t) \text{ equal } v\}$$

and $Y_1, \ldots, Y_n$ are *iid* zero-truncated Poisson random variables with parameter λ. That is,

$$P(Y=y)=\begin{cases}\frac{e^{-\lambda}\lambda^y}{y!(1-e^{-\lambda})} & y\in\{1,2,\dots\}\\ \\ 0 & \text{else}\end{cases}$$

Summing this result over all t yields the total number of partitions of a set of n elements containing at least k blocks of cardinality v.

However as an application of the Zero-Truncated Poisson Randomization Theorem we showed that the number of ways that n distinguishable balls can be distributed among t distinguishable urns such that at least k urns will contain v balls and no urn is left empty equals

$$\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\left((e^{\lambda}-1)^t P((Y_1,\dots,Y_t)\in\mathcal{U}^{\mathbb{A}})\right)\Big|_{\lambda=0}$$

$$=\sum_{j=k}^{\min\left(t,\lfloor\frac{n-t}{v-1}\rfloor\right)}(-1)^{j-k}\left(\frac{n!\,t!}{j\,(k-1)!(j-k)!(v!)^j(n-vj)!}\right)S(n-vj,t-j)$$

Dividing this result by $t!$ and summing this result over all possible values of t gives

$$\sum_{t=k}^{n-vk+k}\sum_{j=k}^{\min\left(t,\lfloor\frac{n-t}{v-1}\rfloor\right)}(-1)^{j-k}\left(\frac{n!}{j\,(k-1)!(j-k)!(v!)^j(n-vj)!}\right)S(n-vj,t-j)$$

$$=\sum_{j=k}^{\lfloor\frac{n}{v}\rfloor}\sum_{t=j}^{n-vk+j}(-1)^{j-k}\left(\frac{n!}{j\,(k-1)!(j-k)!(v!)^j(n-vj)!}\right)S(n-vj,t-j)$$

$$=\sum_{j=k}^{\lfloor\frac{n}{v}\rfloor}(-1)^{j-k}\left(\frac{n!}{j\,(k-1)!(j-k)!(v!)^j(n-vj)!}\right)\left(\sum_{t=j}^{n-vk+j}S(n-vj,t-j)\right)$$

$$=\sum_{j=k}^{\lfloor\frac{n}{v}\rfloor}(-1)^{j-k}\left(\frac{n!}{j\,(k-1)!(j-k)!(v!)^j(n-vj)!}\right)\beta_{n-vj}\qquad\square$$

Problem 2(a)

It is well known and easy to verify that the k^{th} factorial moment of a Poisson random variable with parameter τ is just τ^k. Therefore it follows from Theorem 1 that if we define $Y_v \sim$ Poisson$\left(\frac{\theta\lambda^v}{v!}\right)$ then

$$\mathrm{E}(X_v(X_v-1)\cdots(X_v-k+1))$$

$$= \left(\tfrac{1}{t!S(n,t)}\right)\left(\tfrac{d^n}{d\lambda^n}\tfrac{d^t}{d\theta^t}\left(e^{\theta\left(e^\lambda-1\right)}\mathrm{E}(Y_v(Y_v-1)\cdots(Y_v-k+1))\right)\Big|_{\substack{\lambda=0\\ \theta=0}}\right)$$

$$= \left(\tfrac{1}{t!S(n,t)}\right)\left(\tfrac{d^n}{d\lambda^n}\tfrac{d^t}{d\theta^t}\left(e^{\theta\left(e^\lambda-1\right)}\left(\tfrac{\theta\lambda^v}{v!}\right)^k\right)\Big|_{\substack{\lambda=0\\ \theta=0}}\right)$$

$$= \left(\tfrac{1}{t!S(n,t)}\right)\left(\tfrac{t!}{(t-k)!}\right)\left(\tfrac{d^n}{d\lambda^n}\left(\left(e^\lambda-1\right)^{t-k}\left(\tfrac{\lambda^v}{v!}\right)^k\right)\Big|_{\lambda=0}\right)\mathrm{I}_{\{k,k+1,\ldots\}}(t)$$

$$= \left(\tfrac{1}{S(n,t)}\right)\left(\tfrac{d^n}{d\lambda^n}\left(\tfrac{1}{(t-k)!}\left(e^\lambda-1\right)^{t-k}\tfrac{\lambda^{vk}}{(v!)^k}\right)\Big|_{\lambda=0}\right)\mathrm{I}_{\{k,k+1,\ldots\}}(t)$$

$$= \left(\tfrac{1}{(v!)^k S(n,t)}\right)\left(\tfrac{d^n}{d\lambda^n}\left(\sum_{j=t-k}^{\infty}\tfrac{1}{j!}S(j,t-k)\lambda^{j+vk}\right)\Bigg|_{\lambda=0}\right)\mathrm{I}_{\{k,k+1,\ldots\}}(t)$$

$$= \left(\tfrac{1}{(v!)^k S(n,t)}\right)\left(\sum_{j=t-k}^{\infty}\tfrac{1}{j!}S(j,t-k)\; n!\;\mathrm{I}_{\{n-vk\}}(j)\right)\mathrm{I}_{\{k,k+1,\ldots\}}(t)$$

$$= \tfrac{n!}{(n-vk)!(v!)^k}\tfrac{S(n-vk,t-k)}{S(n,t)}\;\mathrm{I}_{\{t-k,t-k+1,\ldots\}}(n-vk)\;\mathrm{I}_{\{k,k+1,\ldots\}}(t)$$

$$= \tfrac{n!}{(n-vk)!(v!)^k}\tfrac{S(n-vk,t-k)}{S(n,t)}\quad\text{provided}\;\; v \le \left\lfloor\tfrac{n-t+k}{k}\right\rfloor \text{ and } t \ge k.$$

Problem 2(b)

As a check on our work we note that $X_1 + \ldots + X_n = t$ and hence

$$t = \mathrm{E}(X_1 + \ldots + X_n) = \sum_{v=1}^{n} \mathrm{E}(X_v)$$

which can serve as a check on the above result for $\mathrm{E}(X_v)$. We note that

$$\sum_{v=1}^{n} \binom{n}{v} \frac{S(n-v,t-1)}{S(n,t)} = \frac{1}{S(n,t)} \sum_{v=1}^{n} \binom{n}{v} S(n-v, t-1)$$

$$= \frac{1}{S(n,t)} \sum_{v=0}^{n-1} \binom{n}{v} S(v, t-1)$$

$$= \frac{1}{S(n,t)} \sum_{v=t-1}^{n-1} \binom{n}{v} S(v, t-1)$$

$$= \frac{1}{S(n,t)} \left(\sum_{v=t-1}^{n} \binom{n}{v} S(v, t-1) \ - S(n, t-1) \right)$$

$$= \frac{1}{S(n,t)} (S(n+1, t) \ - S(n, t-1))$$

$$= \frac{1}{S(n,t)} (tS(n,t)) = t.$$

In this proof we used two well known recurrence relations for the Stirling Numbers of the Second Kind. Namely,

$$S(n+1, k+1) = \sum_{r=k}^{n} \binom{n}{r} S(r, k)$$

and

$$S(n+1, k) = S(n, k-1) + kS(n, k)$$

Problem 2(c)

By Theorem 2, if we define $Y_v \sim \text{Poisson}\left(\frac{\lambda^v}{v!}\right)$ then

$$\mathrm{E}(W_v(W_v - 1)\cdots(W_v - k + 1))$$

$$= \left(\frac{1}{\beta_n}\right) \left(\frac{d^n}{d\lambda^n} \left(e^{(e^\lambda - 1)} \mathrm{E}(Y_v(Y_v - 1)\cdots(Y_v - k + 1)) \right) \Big|_{\lambda=0} \right)$$

$$= \left(\frac{1}{\beta_n}\right)\left(\frac{d^n}{d\lambda^n}\left(e^{\left(e^\lambda-1\right)}\left(\frac{\lambda^v}{v!}\right)^k\right)\bigg|_{\lambda=0}\right)$$

$$= \left(\frac{1}{\beta_n}\right)\frac{1}{(v!)^k}\left(\frac{d^n}{d\lambda^n}\left(\sum_{j=0}^{\infty}\frac{1}{j!}\beta_j\lambda^{vk+j}\right)\Bigg|_{\lambda=0}\right)$$

$$= \left(\frac{1}{\beta_n}\right)\frac{1}{(v!)^k}\left(\sum_{j=0}^{\infty}\frac{\beta_j}{j!}\; n!\; \mathrm{I}_{\{n-vk\}}(j)\right)$$

$$= \frac{n!}{(n-vk)!(v!)^k}\;\frac{\beta_{n-vk}}{\beta_n} \qquad \text{provided } \; v \le \left\lfloor \frac{n}{k} \right\rfloor$$

Problem 3(a)

We have defined the random variable X_v as the multiplicity of v in a random partition of a set of n elements into t blocks. It follows that

$$\sum_{v=1}^{n} \mathrm{I}_{\{k\}}(X_v)$$

is the number of multiplicities that equal k in a random partition of a set of n elements into t blocks. The problem asks for the expected value of this sum. However

$$\mathrm{E}\left(\sum_{v=1}^{n} \mathrm{I}_{\{k\}}(X_v)\right) = \sum_{v=1}^{n} P(X_v = k)$$

and from Problem 1(a)

$$P(X_v = k) \;\; = \;\; \sum_{j=k}^{\min\left(t,\left\lfloor \frac{n-t}{v-1}\right\rfloor\right)} (-1)^{j-k}\left(\frac{n!}{k!\;(j-k)!\;(v!)^j(n-vj)!}\right)\frac{S(n-vj,t-j)}{S(n,t)}$$

Hence the expected number of multiplicities equaling k in a random partition of a set of n elements into t blocks is

$$\sum_{v=1}^{n}\;\sum_{j=k}^{\min\left(t,\left\lfloor \frac{n-t}{v-1}\right\rfloor\right)} (-1)^{j-k}\left(\frac{n!}{k!\;(j-k)!\;(v!)^j(n-vj)!}\right)\frac{S(n-vj,t-j)}{S(n,t)} \qquad \square$$

Problem 3(b)

We have defined the random variable W_v as the multiplicity of v in a random partition of a set of n elements. It follows that

$$\sum_{v=1}^{n} \mathrm{I}_{\{k\}}(W_v)$$

is the number of multiplicities that equal k in a random partition of a set of n elements. The problem asks for the expected value of this sum. However, if we let T equal the number of blocks in a random partition, then

$$\mathrm{E}\left(\sum_{v=1}^{n} \mathrm{I}_{\{k\}}(W_v)\right)$$

$$= \mathrm{E}\left(\mathrm{E}\left(\sum_{v=1}^{n} \mathrm{I}_{\{k\}}(W_v)\right)\middle| T\right)$$

$$= \sum_{t=1}^{n} \mathrm{E}\left(\sum_{v=1}^{n} \mathrm{I}_{\{k\}}(X_v)\right) P(T=t)$$

$$= \sum_{t=1}^{n} \sum_{v=1}^{n} \sum_{j=k}^{\min\left(t, \left\lfloor \frac{n-t}{v-1} \right\rfloor\right)} (-1)^{j-k} \left(\frac{n!}{k!\,(j-k)!\,(v!)^j (n-vj)!}\right) \frac{S(n-vj, t-j)}{\beta_n} \qquad \square$$

Problem 4

Let T equal the number of blocks in a random partition of a set with n elements.

$$\mathrm{E}(T) = \sum_{t=1}^{n} t P(T=t)$$

$$= \sum_{t=1}^{n} t \frac{S(n,t)}{\beta_n}$$

$$= \frac{1}{\beta_n} \sum_{t=1}^{n} (S(n+1,t) - S(n,t-1))$$

$$= \frac{1}{\beta_n}\left(\left(\sum_{t=1}^{n+1} S(n+1,t) - S(n+1,n+1)\right) - \left(\sum_{t=1}^{n+1} S(n,t-1) - S(n,n)\right)\right)$$

$$= \frac{1}{\beta_n}((\beta_{n+1} - 1) - (\beta_n - 1))$$

$$= \frac{\beta_{n+1} - \beta_n}{\beta_n} \qquad \square$$

Problem 5(a)

The given two step procedure for selecting a block is equivalent to selecting a block uniformly at random from the set of all $tS(n,t)$ blocks making up the $S(n,t)$ partitions of a set of n elements into exactly t blocks. As before, we let X_j equal the multiplicity of j in a random partition of a set of n elements into t blocks. Then,

$$P(L_t = v) = \frac{\text{number of blocks of size } v \text{ among the set of all } tS(n,t) \text{ blocks}}{tS(n,t)}$$

$$= \frac{\sum_{k=0}^{\min(\lfloor \frac{n}{v} \rfloor, t)} kP(X_v = k)S(n,t)}{tS(n,t)}$$

$$= \frac{\mathrm{E}(X_v)}{t} = \binom{n}{v} \frac{S(n-v,t-1)}{tS(n,t)}.$$

It follows that

$$\mathrm{E}(L_t) = \sum_{v=1}^{n} vP(L_t = v)$$

$$= \sum_{v=1}^{n} v \binom{n}{v} \frac{S(n-v,t-1)}{tS(n,t)}$$

$$= \frac{1}{tS(n,t)} \sum_{v=1}^{n} n \binom{n-1}{n-v} S(n-v, t-1)$$

$$= \frac{1}{tS(n,t)} \sum_{v'=0}^{n-1} n \binom{n-1}{v'} S(v', t-1)$$

$$= \frac{n}{tS(n,t)} S(n,t) = \frac{n}{t}. \qquad \square$$

Problem 5(b)

Let T represent the number of blocks in a randomly selected partition. Then

$$\mathrm{E}(L) = \mathrm{E}(\mathrm{E}(L|T))$$

$$= \sum_{t=1}^{n} \mathrm{E}(L_t) P(T = t)$$

$$= \sum_{t=1}^{n} \frac{n}{t} \frac{S(n,t)}{\beta_n}$$

$$= \frac{n}{\beta_n} \sum_{t=1}^{n} \frac{S(n,t)}{t} \qquad \square$$

Problem 6

From Theorem 3 the solution will equal

$$\frac{1}{t!} \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} \Big(\left(e^{\lambda} - 1\right)^t P\big((Y_1, \ldots, Y_t) \in \mathcal{U}^{\mathbb{A}}\big) \Big) \Big|_{\lambda=0}$$

where

$$\mathcal{U}^{\mathbb{A}} = \{(u_1, \ldots, u_t) \in \mathbb{U}^t \mid \text{exactly } r \text{ of the } t \text{ values } (u_1, \ldots, u_t) \text{ are even numbers}\}$$

and $Y_1, \ldots, Y_n$ are *iid* zero-truncated Poisson random variables with parameter λ. That is,

$$P(Y = y) = \begin{cases} \frac{e^{-\lambda} \lambda^y}{y!(1-e^{-\lambda})} & y \in \{1, 2, \ldots\} \\ 0 & \text{else} \end{cases}$$

The zero-truncated Poisson distribution is a member of the Power Series Family with $a_y = \frac{1}{y!}$, $b(\lambda) = e^{\lambda} - 1$, and $\tau = 1$. Hence from Theorem **PSMM**, taking $m = 2$, we find $\delta = 1$ and

$$P(Y_j \in \{2, 4, \ldots\}) = \frac{1}{2} \left(\frac{b(\lambda) + b(-\lambda)}{b(\lambda)} \right)$$

$$= \frac{1}{2} \left(\frac{(e^{\lambda} - 1) + (e^{-\lambda} - 1)}{(e^{\lambda} - 1)} \right)$$

$$= \frac{1}{e^{\lambda} - 1} \left(\frac{e^{\lambda} + e^{-\lambda}}{2} - 1 \right)$$

$$= \frac{1}{e^{\lambda}-1} \frac{\left(e^{\frac{\lambda}{2}} - e^{\frac{-\lambda}{2}}\right)^2}{2}$$

and

$$P(Y_j \in \{1, 3, \dots\}) = \frac{1}{2}\left(\frac{b(\lambda) - b(-\lambda)}{b(\lambda)}\right)$$

$$= \frac{1}{2}\left(\frac{(e^{\lambda}-1) - (e^{-\lambda}-1)}{(e^{\lambda}-1)}\right)$$

$$= \frac{1}{e^{\lambda}-1}\left(\frac{e^{\lambda} - e^{-\lambda}}{2}\right)$$

Therefore,

$$P\big((Y_1,\dots,Y_t) \in \mathcal{U}^{\mathbb{A}}\big) = \binom{t}{r}\left(\frac{1}{e^{\lambda}-1}\left(\frac{\left(e^{\frac{\lambda}{2}} - e^{\frac{-\lambda}{2}}\right)^2}{2}\right)\right)^r \left(\frac{1}{e^{\lambda}-1}\left(\frac{e^{\lambda} - e^{-\lambda}}{2}\right)\right)^{t-r}$$

and

$$\frac{1}{t!}\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\left((e^{\lambda}-1)^t P((Y_1,\dots,Y_t) \in \mathcal{U}^{\mathbb{A}})\right)\Big|_{\lambda=0}$$

$$= \frac{1}{t!}\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\left((e^{\lambda}-1)^t \binom{t}{r}\left(\frac{1}{e^{\lambda}-1}\left(\frac{\left(e^{\frac{\lambda}{2}} - e^{\frac{-\lambda}{2}}\right)^2}{2}\right)\right)^r \left(\frac{1}{e^{\lambda}-1}\left(\frac{e^{\lambda} - e^{-\lambda}}{2}\right)\right)^{t-r}\right)\Bigg|_{\lambda=0}$$

$$= \frac{1}{t!}\binom{t}{r}\left(\frac{1}{2^t}\right)\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\left(\left(e^{\frac{\lambda}{2}} - e^{\frac{-\lambda}{2}}\right)^{2r}(e^{\lambda} - e^{-\lambda})^{t-r}\right)\Big|_{\lambda=0}$$

$$= \frac{1}{t!}\binom{t}{r}\left(\frac{1}{2^t}\right)\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\left(\sum_{j=0}^{2r}\sum_{k=0}^{t-r}(-1)^{j+k}\binom{2r}{j}\binom{t-r}{k}e^{\lambda(t-j-2k)}\right)\Bigg|_{\lambda=0}$$

$$= \frac{1}{t!}\binom{t}{r}\left(\frac{1}{2^t}\right)\sum_{j=0}^{2r}\sum_{k=0}^{t-r}(-1)^{j+k}\binom{2r}{j}\binom{t-r}{k}\left(\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}e^{\lambda(t-j-2k)}\right)\Big|_{\lambda=0}$$

$$= \frac{1}{t!}\binom{t}{r}\left(\frac{1}{2^t}\right)\sum_{j=0}^{2r}\sum_{k=0}^{t-r}(-1)^{j+k}\binom{2r}{j}\binom{t-r}{k}(t-j-2k)^n \qquad \square$$

Problem 7

Notes:

For using *Derive*, define

$$S(a,b) = (1 - chi(0,a,\infty,0))(1 - chi(0,b,\infty,0)) + Stirling2(a,b)$$

because *Derive* mistakenly defines $Stirling2(0,0) = 0$.

$$\frac{\mathrm{d}^t}{\mathrm{d}x^t}\left(e^{ax}x^k\right)\Big|_{x=0} = \frac{t!}{(t-k)!}a^{t-k}\ \mathrm{I}_{\{k,k+1,\ldots\}}(t)$$

and

$$\frac{1}{k!}\left(e^{\lambda}-1\right)^k = \sum_{n=k}^{\infty}\frac{1}{n!}S(n,k)\lambda^n$$

and

$$e^{\left(e^{\lambda}-1\right)} = \sum_{n=0}^{\infty}\frac{1}{n!}\beta_n\lambda^n$$

Also read

Arratia, R. and Tavare, S. (1992). “Independent process approximations for random combinatorial structures.” *Advances in Mathematics*. ???

$\mu_{[k]}$: k^{th} Descending Factorial Moment of X $\quad$ $\mathrm{E}(X(X-1)\cdots(X-k+1))$

Suppose $X \sim$ Poisson(λ). Then

$$\mu_{[k]} = \sum_{j=0}^{\infty} j(j-1)\cdots(j-k+1)P(X=j)$$

$$= \sum_{j=k}^{\infty} j(j-1)\cdots(j-k+1)P(X=j)$$

$$= \sum_{j=k}^{\infty} j(j-1)\cdots(j-k+1)\frac{e^{-\lambda}\lambda^{j}}{j!}$$

$$= \lambda^{k} \sum_{j=k}^{\infty} \frac{e^{-\lambda}\lambda^{j-k}}{(j-k)!} = \lambda^{k}$$