Random Partitions of an Integer.

Define $\mathbb{S}^\infty$ to be the **infinite** product space $\{0,1,\dots\} \times \{0,1,\dots\} \times \cdots$ and let $\mathbb{S}_n^\infty$ be the set of all vectors $(s_1,s_2,\dots)$ in $\mathbb{S}^\infty$ such that $1s_1 + 2s_2 + 3s_3 + \dots = n$.

Let $\mathcal{A} \subseteq \mathbb{S}^\infty$ and define $\mathcal{A}_n = \mathcal{A} \cap \mathbb{S}_n^\infty$.

We note that the condition that $1s_1 + 2s_2 + 3s_3 + \dots = n$ implies that $s_j = 0$ for all $j > n$. Hence all vectors in $\mathcal{A}_n$ are of the form $(a_1,a_2,\dots,a_n, 0, 0, \dots)$.

For all $\mathcal{A} \neq (0, 0, \dots)$, let $\mathbb{A}_n$ be the collection of n-dimensional vectors formed by taking each infinite-dimensional vector in $\mathcal{A}_n$ and truncating after a_n. So for example,

$$(a_1,a_2,\dots,a_n, 0, 0, \dots) \to (a_1,a_2,\dots,a_n)$$

Define the random vector $(X_1, \dots, X_n)$ such that

$$\begin{aligned} &P((X_1, \dots, X_n) = (k_1, \dots, k_n)) \\ &= \begin{cases} \frac{1}{\psi(n)} & 1k_1 + \dots + nk_n = n, k_j \geq 0 \ \forall j \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

where $\psi(n)$ equals the number of partitions of the integer n. That is, the random vector $(X_1, \dots, X_n)$ is a randomly chosen partition of the integer n.

Define $Y_1, Y_2, \dots$ to be an infinite sequence of independent **Geometric** random variables such that

$$P(Y_j = k) = \left(1 - \lambda^j\right)\left(\lambda^j\right)^k \qquad k = 0, 1, 2, \dots \text{ and } j = 0, 1, 2, \dots$$

Theorem

For $\mathcal{A} \neq (0, 0, \dots)$

$$P((X_1, \dots, X_n) \in \mathbb{A}_n) = \left(\frac{1}{\psi(n)n!}\right) \frac{d^n}{d\lambda^n} \left(\left(\frac{1}{\prod_{j=1}^{\infty}(1 - \lambda^j)} \right) P((Y_1, Y_2, \dots) \in \mathcal{A}) \right) \Bigg|_{\lambda=0}$$

$$= \left(\frac{1}{\psi(n)n!}\right) \frac{d^n}{d\lambda^n} \left(\sum_{i=0}^{\infty} \psi(i)\lambda^i P((Y_1, Y_2, \dots) \in \mathcal{A}) \right) \Bigg|_{\lambda=0}$$

Proof

Let k_j be a nonnegative integer for $j = 0, 1, \cdots$. Then

$$P(Y_1 = k_1, Y_2 = k_2, \dots) = \prod_{j=1}^{\infty} \left(1 - \lambda^j\right) \left(\lambda^j\right)^{k_j}$$
$$= \left(\prod_{j=1}^{\infty} \left(1 - \lambda^j\right) \right) \lambda^{\left(\sum_{j=1}^{\infty} j k_j\right)}$$

and

$$P\left(\sum_{j=1}^{\infty} j Y_j = n \right) = \underset{\substack{(k_1, k_2, \dots) \ni \\ \sum_{j=1}^{\infty} j k_j = n}}{\sum \cdots \sum} P(Y_1 = k_1, Y_2 = k_2, \dots)$$
$$= \left(\prod_{j=1}^{\infty} \left(1 - \lambda^j\right) \right) \left(\underset{\substack{(k_1, k_2, \dots) \ni \\ \sum_{j=1}^{\infty} j k_j = n}}{\sum \cdots \sum} \left(\lambda^{\left(\sum_{j=1}^{\infty} j k_j\right)} \right) \right)$$
$$= \left(\prod_{j=1}^{\infty} \left(1 - \lambda^j\right) \right) \lambda^n \left(\underset{\substack{(k_1, k_2, \dots, k_n) \ni \\ \sum_{j=1}^{n} j k_j = n}}{\sum \cdots \sum} 1 \right)$$
$$= \left(\prod_{j=1}^{\infty} \left(1 - \lambda^j\right) \right) \lambda^n \psi(n)$$

where $\psi(n)$ equals the number of partitions of the the integer n.

It is necessary to separate out the case $\mathcal{A} = (0, 0, \dots)$ because in this case

$$n = 1s_1 + 2s_2 + 3s_3 + \dots = (1 \cdot 0) + (2 \cdot 0) + (3 \cdot 0) + \dots = 0$$

and it does not make notational sense to use n as an index for $\mathbb{A}_n$.

It follows that for $\mathcal{A} \neq (0, 0, \dots)$

$$P((Y_1,Y_2,\dots)\in\mathcal{A})=\sum_{n=0}^{\infty}P\left((Y_1,Y_2,\dots)\in\mathcal{A}\;\Big|\;\sum_{j=1}^{\infty}jY_j=n\right)P\left(\sum_{j=1}^{\infty}jY_j=n\right)$$
$$=\sum_{n=0}^{\infty}\sum_{\mathcal{A}_n}\left(\frac{\lambda^n\prod_{j=1}^{\infty}(1-\lambda^j)}{\lambda^n\psi(n)\prod_{j=1}^{\infty}(1-\lambda^j)}\right)\left(\lambda^n\psi(n)\prod_{j=1}^{\infty}(1-\lambda^j)\right)$$
$$=\sum_{n=1}^{\infty}\sum_{\mathcal{A}_n}\left(\frac{1}{\psi(n)}\right)\left(\psi(n)\prod_{j=1}^{\infty}(1-\lambda^j)\right)\lambda^n$$
$$=\sum_{n=1}^{\infty}P((X_1,\dots,X_n)\in\mathbb{A}_n)\left(\psi(n)\prod_{j=1}^{\infty}(1-\lambda^j)\right)\lambda^n$$

Therefore

$$\left(\frac{1}{\prod_{j=1}^{\infty}(1-\lambda^j)}\right)P((Y_1,Y_2,\dots)\in\mathcal{A})=\sum_{n=1}^{\infty}P((X_1,\dots,X_n)\in\mathbb{A}_n)(\psi(n))\lambda^n$$

and

$$\frac{d^r}{d\lambda^r}\left(\left(\frac{1}{\prod_{j=1}^{\infty}(1-\lambda^j)}\right)P((Y_1,Y_2,\dots)\in\mathcal{A})\right)\Bigg|_{\lambda=0}$$
$$=\sum_{n=1}^{\infty}P((X_1,\dots,X_n)\in\mathbb{A}_n)(\psi(n))r!\mathrm{I}_{\{r\}}(n)$$
$$=P((X_1,\dots,X_r)\in\mathbb{A}_r)(\psi(r))r!$$

Thus, for $\mathcal{A}\neq(0,0,\dots)$

$$P((X_1,\dots,X_n)\in\mathbb{A}_n)=\left(\frac{1}{\psi(n)n!}\right)\frac{d^n}{d\lambda^n}\left(\left(\frac{1}{\prod_{j=1}^{\infty}(1-\lambda^j)}\right)P((Y_1,Y_2,\dots)\in\mathcal{A})\right)\Bigg|_{\lambda=0}$$

Finally, we note that

$$\frac{1}{\prod_{j=1}^{\infty}(1-\lambda^j)} = \big(1+\lambda+\lambda^2+\dots\big)\big(1+\lambda^2+\lambda^4+\dots\big)\big(1+\lambda^3+\lambda^6+\dots\big)\cdots$$
$$= \sum_{i=0}^{\infty}\psi(i)\lambda^i$$

where $\psi(n)$ equals the number of partitions of the integer n. Hence we can write

$$P((X_1,\dots,X_n)\in\mathbb{A}_n) = \left(\frac{1}{\psi(n)n!}\right)\frac{d^n}{d\lambda^n}\left(\sum_{i=0}^{\infty}\psi(i)\lambda^i P((Y_1,Y_2,\dots)\in\mathcal{A})\right)\Bigg|_{\lambda=0}$$

Random Partitions of an Integer n into t parts.

We will need the following definitions.

$\mathbb{S}^\infty$: the **infinite** product space $\{0,1,\dots\} \times \{0,1,\dots\} \times \cdots$

$\mathbb{S}^\infty_{n,t}$: the set of all vectors $(s_1,s_2,\dots)$ in $\mathbb{S}^\infty$ such that $\begin{array}{c} 1s_1 + 2s_2 + \dots = n \\ s_1 + s_2 + \dots = t \end{array}$

For any $\mathcal{A} \subseteq \mathbb{S}^\infty$ define $\mathcal{A}_{n,t} = \mathcal{A} \cap \mathbb{S}^\infty_{n,t}$.

We note that the condition that $1s_1 + 2s_2 + \dots = n$ implies that $s_j = 0$ for all $j > n$. Hence all vectors in $\mathcal{A}_{n,t}$ are of the form $(a_1,a_2,\dots,a_n,0,0,\dots)$.

For all $\mathcal{A} \neq (0,0,\dots)$, let $\mathbb{A}_{n,t}$ be the collection of n-dimensional vectors formed by taking each infinite-dimensional vector in $\mathcal{A}_{n,t}$ and truncating after a_n. So for example,

$$(a_1,a_2,\dots,a_n,0,0,\dots) \rightarrow (a_1,a_2,\dots,a_n)$$

It is necessary to separate out the case $\mathcal{A} = (0,0,\dots)$ because in this case

$$n = 1s_1 + 2s_2 + \dots = (1 \cdot 0) + (2 \cdot 0) + \dots = 0$$

and it does not make notational sense to use n as an index for $\mathbb{A}_{n,t}$.

Let X_j equal the multiplicity of j in a random partition of n with t parts. It follows from definition that

$$P(X_1 = x_1, \dots, X_n = x_n) = \begin{cases} \frac{1}{\psi(n,t)} & \begin{array}{c} 1x_1+2x_2+\dots+nx_n=n \\ x_1+x_2+\dots+x_n=t \\ x_j\in\{0,1,\dots,n\}\ \forall j \end{array} \\ 0 & \text{else} \end{cases}$$

where $\psi(n,t)$ equals the number of partitions of the integer n with t parts. That is, the random vector $(X_1, \dots, X_n)$ is a randomly chosen partition of the integer n with t parts.

Define $Y_1, Y_2, \dots$ to be an infinite sequence of independent **Geometric** random variables such that

$$P(Y_j = k) = \left(1 - \theta\lambda^j\right)\left(\theta\lambda^j\right)^k \qquad k = 0,1,2,\dots \text{ and } j = 0,1,2,\dots$$

Theorem

For $\mathcal{A} \neq (0,0,\dots)$

$$P((X_1,\dots,X_n)\in\mathbb{A}_{n,t}) = \left(\frac{1}{\psi(n,t)n!t!}\right)\frac{d^n}{d\lambda^n}\frac{d^t}{d\theta^t}\left(\left(\frac{1}{\prod_{j=1}^{\infty}(1-\theta\lambda^j)}\right)P((Y_1,Y_2,\dots)\in\mathcal{A})\right)\Bigg|_{\substack{\lambda=0\\ \theta=0}}$$

$$= \left(\frac{1}{\psi(n,t)n!t!}\right)\frac{d^n}{d\lambda^n}\frac{d^t}{d\theta^t}\left(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\psi(i,j)\lambda^i\theta^j P((Y_1,Y_2,\dots)\in\mathcal{A})\right)\Bigg|_{\substack{\lambda=0\\ \theta=0}}$$

Proof

Let y_j be a nonnegative integer for $j = 0, 1, \cdots$. Then

$$P(Y_1 = y_1, Y_2 = y_2, \dots) = \prod_{j=1}^{\infty} \left(1 - \theta\lambda^j\right) \left(\theta\lambda^j\right)^{y_j}$$
$$= \left(\prod_{j=1}^{\infty} \left(1 - \theta\lambda^j\right)\right) \theta^{\left(\sum_{j=1}^{\infty} y_j\right)} \lambda^{\left(\sum_{j=1}^{\infty} j y_j\right)}$$

and

$$P\left(\sum_{j=1}^{\infty} jY_j = n \text{ and } \sum_{j=1}^{\infty} Y_j = t\right) = \sum_{\substack{(y_1, y_2, \dots) \ni \\ \sum_{j=1}^{\infty} j y_j = n \\ \sum_{j=1}^{\infty} y_j = t}} \cdots \sum P(Y_1 = y_1, Y_2 = y_2, \dots)$$
$$= \left(\prod_{j=1}^{\infty} \left(1 - \theta\lambda^j\right)\right) \left(\sum_{\substack{(y_1, y_2, \dots) \ni \\ \sum_{j=1}^{\infty} j y_j = n \\ \sum_{j=1}^{\infty} y_j = t}} \cdots \sum \left(\theta^{\left(\sum_{j=1}^{\infty} y_j\right)} \lambda^{\left(\sum_{j=1}^{\infty} j y_j\right)} \right) \right)$$
$$= \left(\prod_{j=1}^{\infty} \left(1 - \theta\lambda^j\right)\right) \theta^t \lambda^n \left(\sum_{\substack{(y_1, y_2, \dots) \ni \\ \sum_{j=1}^{\infty} j y_j = n \\ \sum_{j=1}^{\infty} y_j = t}} \cdots \sum 1 \right)$$
$$= \left(\prod_{j=1}^{\infty} \left(1 - \theta\lambda^j\right)\right) \theta^t \lambda^n \psi(n, t)$$

where $\psi(n, t)$ equals the number of partitions of the the integer n with t parts.

It is necessary to separate out the case $\mathcal{A} = (0, 0, \dots)$ because in this case

$$n = 1s_1 + 2s_2 + 3s_3 + \ \dots = (1 \cdot 0) + (2 \cdot 0) + (3 \cdot 0) + \ \dots = 0$$

and it does not make notational sense to use n as an index for $\mathbb{A}_{n,t}$.

It follows that for $\mathcal{A} \neq (0,0,\dots)$

$$
\begin{aligned}
P((Y_1,Y_2,\dots)\in\mathcal{A}) &= \sum_{n=0}^{\infty}\sum_{t=0}^{\infty}P\left((Y_1,Y_2,\dots)\in\mathcal{A} \mid \sum_{j=1}^{\infty}jY_j=n,\sum_{j=1}^{\infty}Y_j=t\right)P\left(\sum_{j=1}^{\infty}jY_j=n,\sum_{j=1}^{\infty}Y_j=t\right) \\
&= \sum_{n=1}^{\infty}\sum_{t=1}^{\infty}\sum_{\mathcal{A}_{n,t}}\left(\frac{\theta^t\lambda^n\prod_{j=1}^{\infty}(1-\theta\lambda^j)}{\theta^t\lambda^n\psi(n,t)\prod_{j=1}^{\infty}(1-\theta\lambda^j)}\right)\left(\prod_{j=1}^{\infty}(1-\theta\lambda^j)\psi(n,t)\right)\theta^t\lambda^n \\
&= \sum_{n=1}^{\infty}\sum_{t=1}^{\infty}\sum_{\mathcal{A}_{n,t}}\left(\frac{1}{\psi(n,t)}\right)\left(\psi(n,t)\prod_{j=1}^{\infty}(1-\theta\lambda^j)\right)\theta^t\lambda^n \\
&= \sum_{n=1}^{\infty}\sum_{t=1}^{\infty}\sum_{\mathbb{A}_{n,t}}\left(\frac{1}{\psi(n,t)}\right)\left(\psi(n,t)\prod_{j=1}^{\infty}(1-\theta\lambda^j)\right)\theta^t\lambda^n \\
&= \sum_{n=1}^{\infty}\sum_{t=1}^{\infty}P((X_1,\dots,X_n)\in\mathbb{A}_{n,t})\left(\psi(n,t)\prod_{j=1}^{\infty}(1-\theta\lambda^j)\right)\theta^t\lambda^n
\end{aligned}
$$

Therefore

$$
\left(\frac{1}{\prod_{j=1}^{\infty}(1-\theta\lambda^j)}\right)P((Y_1,Y_2,\dots)\in\mathcal{A}) = \sum_{n=1}^{\infty}\sum_{t=1}^{\infty}P((X_1,\dots,X_n)\in\mathbb{A}_{n,t})(\psi(n,t))\,\theta^t\lambda^n
$$

and

$$
\begin{aligned}
&\frac{d^r}{d\lambda^r}\frac{d^s}{d\theta^s}\left(\left(\frac{1}{\prod_{j=1}^{\infty}(1-\theta\lambda^j)}\right)P((Y_1,Y_2,\dots)\in\mathcal{A})\right)\Bigg|_{\substack{\lambda=0\\ \theta=0}} \\
&= \sum_{n=1}^{\infty}\sum_{t=1}^{\infty}P((X_1,\dots,X_n)\in\mathbb{A}_{n,t})(\psi(n,t))r!s!\mathrm{I}_{\{r\}}(n)\mathrm{I}_{\{s\}}(t) \\
&= P((X_1,\dots,X_r)\in\mathbb{A}_{r,s})(\psi(r,s))r!s!
\end{aligned}
$$

Thus, for $\mathcal{A} \neq (0,0,\dots)$

$$P((X_1,\dots,X_n) \in \mathbb{A}_{n,t}) = \left(\frac{1}{\psi(n,t)n!t!}\right)\frac{d^n}{d\lambda^n}\frac{d^t}{d\theta^t}\left(\left(\frac{1}{\prod_{j=1}^{\infty}(1-\theta\lambda^j)}\right)P((Y_1,Y_2,\dots) \in \mathcal{A})\right)\Bigg|_{\substack{\lambda=0\\ \theta=0}}$$

Finally, we note that

$$\begin{aligned}\frac{1}{\prod_{j=1}^{\infty}(1-\theta\lambda^j)} &= \left(1+(\theta\lambda)^1+(\theta\lambda)^2+\dots\right)\left(1+\left(\theta\lambda^2\right)^1+\left(\theta\lambda^2\right)^2+\dots\right)\cdots \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\psi(i,j)\theta^j\lambda^i\end{aligned}$$

where $\psi(n,t)$ equals the number of partitions of the integer n into t parts. Hence we can write

$$P((X_1,\dots,X_n) \in \mathbb{A}_{n,t}) = \left(\frac{1}{\psi(n,t)n!t!}\right)\frac{d^n}{d\lambda^n}\frac{d^t}{d\theta^t}\left(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\psi(i,j)\lambda^i\theta^j P((Y_1,Y_2,\dots) \in \mathcal{A})\right)\Bigg|_{\substack{\lambda=0\\ \theta=0}}$$

Applications: Random Partitions of an Integer n.

(1) How many partitions of the positive integer n contain exactly k copies of j ?

$Answer:\ \psi(n-jk)-\psi(n-j(k+1))$

Clearly the solution equals $\psi(n)\cdot P(X_j=k)$. Now define

$\mathbb{A}_n=\{(a_1,a_2,\ldots,a_n)|1a_1+2a_2+\ \ldots\ +na_n=n \text{ and } a_j=k\}$
$\mathcal{A}_n=\{a_1,a_2,\ldots,a_n,0,0,\ldots|1a_1+2a_2+\ \ldots\ +na_n=n \text{ and } a_j=k\}$
$\mathcal{A}=\{a_1,a_2,\ldots,a_n,a_{n+1},a_{n+2},\ldots\,|\,a_j=k\}$

Then,

$$\begin{aligned}P(X_j=k)&=P((X_1,\ldots,X_n)\in\mathbb{A}_n)\\&=\left(\frac{1}{\psi(n)n!}\right)\frac{d^n}{d\lambda^n}\left(\sum_{i=0}^{\infty}\psi(i)\lambda^iP((Y_1,Y_2,\ldots)\in\mathcal{A})\right)\Bigg|_{\lambda=0}\end{aligned}$$

where

$$\begin{aligned}P((Y_1,Y_2,\ldots)\in\mathcal{A})&=\left(\prod_{i\neq j}P(Y_i\in\{0,1,\ldots\})\right)P(Y_j=k)\\&=P(Y_j=k)=\left(1-\lambda^j\right)\left(\lambda^j\right)^k\end{aligned}$$

Therefore,

$$\begin{aligned}P(X_j=k)&=\left(\frac{1}{\psi(n)n!}\right)\frac{d^n}{d\lambda^n}\left(\sum_{i=0}^{\infty}\psi(i)\lambda^i\left(1-\lambda^j\right)\left(\lambda^j\right)^k\right)\Bigg|_{\lambda=0}\\&=\left(\frac{1}{\psi(n)n!}\right)\left(\sum_{i=0}^{\infty}\psi(i)\,\frac{d^n}{d\lambda^n}\left(\lambda^{i+jk}-\lambda^{i+j(k+1)}\right)\right)\Bigg|_{\lambda=0}\\&=\left(\frac{1}{\psi(n)n!}\right)\sum_{i=0}^{\infty}\psi(i)\,n!\left(\mathrm{I}_{\{i+jk\}}(n)-\mathrm{I}_{\{i+j(k+1)\}}(n)\right)\\&=\left(\frac{1}{\psi(n)}\right)\sum_{i=0}^{\infty}\psi(i)\left(\mathrm{I}_{\{n-jk\}}(i)-\mathrm{I}_{\{n-j(k+1)\}}(i)\right)\\&=\frac{\psi(n-jk)-\psi(n-j(k+1))}{\psi(n)}\end{aligned}$$

It follows that $\psi(n - jk) - \psi(n - j(k+1))$ is the number of partitions of n which contain exactly k copies of j.

(2) How many partitions of the positive integer n into t parts contain exactly k copies of v?

$Answer: \ \psi(n-vk,t-k)-\psi(n-v(k+1),t-k-1)$

Clearly the solution equals $\psi(n,t)\cdot P(X_v=k)$. Now define

$$\begin{aligned}\mathbb{A}_{n,t}&=\{(a_1,a_2,\ldots,a_n)|1a_1+\ \ldots\ +na_n=n,a_1+\ \ldots\ +a_n=t \text{ and } a_v=k\}\\ \mathcal{A}_{n,t}&=\{a_1,a_2,\ldots,a_n,0,0,\ldots|1a_1+\ \ldots\ +na_n=n,a_1+\ \ldots\ +a_n=t \text{ and } a_v=k\}\\ \mathcal{A}&=\{a_1,a_2,\ldots,a_n,a_{n+1},a_{n+2},\ldots|\ a_v=k\}\end{aligned}$$

Then,

$$\begin{aligned}P(X_v=k)&=P((X_1,\ldots,X_n)\in\mathbb{A}_{n,t})\\ &=\left(\frac{1}{\psi(n,t)n!t!}\right)\frac{d^n}{d\lambda^n}\frac{d^t}{d\theta^t}\left(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\psi(i,j)\lambda^i\theta^jP((Y_1,Y_2,\ldots)\in\mathcal{A})\right)\Bigg|_{\substack{\lambda=0\\ \theta=0}}\end{aligned}$$

where

$$\begin{aligned}P((Y_1,Y_2,\ldots)\in\mathcal{A})&=\left(\prod_{i\neq v}P(Y_i\in\{0,1,\ldots\})\right)P(Y_v=k)\\ &=P(Y_v=k)=(1-\theta\lambda^v)(\theta\lambda^v)^k\end{aligned}$$

Therefore,

$$\begin{aligned}P(X_j=k)&=\left(\frac{1}{\psi(n,t)n!t!}\right)\frac{d^n}{d\lambda^n}\frac{d^t}{d\theta^t}\left(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\psi(i,j)\lambda^i\theta^j(1-\theta\lambda^v)(\theta\lambda^v)^k\right)\Bigg|_{\substack{\lambda=0\\ \theta=0}}\\ &=\left(\frac{1}{\psi(n,t)n!t!}\right)\left(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\psi(i,j)\,\frac{d^n}{d\lambda^n}\frac{d^t}{d\theta^t}\left(\theta^{j+k}\lambda^{i+vk}-\theta^{j+k+1}\lambda^{i+v(k+1)}\right)\right)\Bigg|_{\substack{\lambda=0\\ \theta=0}}\\ &=\left(\frac{1}{\psi(n,t)n!t!}\right)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\psi(i,j)\,n!t!\left(\mathrm{I}_{\{j+k\}}(t)\mathrm{I}_{\{i+vk\}}(n)-\mathrm{I}_{\{j+k+1\}}(t)\mathrm{I}_{\{i+v(k+1)\}}(n)\right)\\ &=\left(\frac{1}{\psi(n,t)}\right)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\psi(i,j)\left(\mathrm{I}_{\{t-k\}}(j)\mathrm{I}_{\{n-vk\}}(i)-\mathrm{I}_{\{t-k-1\}}(j)\mathrm{I}_{\{n-v(k+1)\}}(i)\right)\\ &=\frac{\psi(n-vk,t-k)-\psi(n-v(k+1),t-k-1)}{\psi(n,t)}\end{aligned}$$

It follows that $\psi(n - vk, t - k) - \psi(n - v(k+1), t - k - 1)$ is the number of partitions of n with t parts which contain exactly k copies of j.

(2) Find $P((X_1, \ldots, X_n) \in \mathbb{A}_n \mid X_1 = 0)$ or more generally find $P((X_1, \ldots, X_n) \in \mathbb{A}_n \mid X_i = x_i, i \in J)$

(3) Find

$$\mathrm{E}\left(\left(\sum_{j=1}^{n} X_j\right)_r\right)$$

where $(m)_r = m(m-1)\cdots(m-r+1)$.

(4) How many partitions of n have k (cycles ?), no one of which is a unitary cycle?

(5) How many partitions of n have k (cycles ?) of length r?

Isn't this just asking for $P(X_r = k)$?

(6) r^{th} longest and r^{th} shortest (cycles ?) for fixed $(X_1, \ldots, X_n)$

(7) Partitions which avoid a given pattern.

(8) Record values.

Let $X_1^{(n)}, \ldots, X_n^{(n)}$ be the <u>order statistics</u> of $X_1, \ldots, X_n$. Find $P\left(X_i^{(n)} \in \mathcal{R}\right)$

Problem 4.

How many multiplicities in a random partition of n do we expect to equal k ?

Answer

$$\frac{1}{\psi(n)}\sum_{j=1}^{n}\left(\psi(n-jk)-\psi(n-j(k+1))\right)$$

This result is Theorem 2 in "*On the Multiplicity of Parts in a Random Partition*", Corteel, Pittel, Savage, Wilf, Random Structures and Algorithms, pages $185-197$.

Proof

The number of multiplicities equaling k in a random composition of n is

$$\sum_{j=1}^{n}\mathrm{I}_{\{k\}}(X_j)$$

Therefore,

$$\begin{aligned} E\left(\sum_{j=1}^{n}\mathrm{I}_{\{k\}}(X_j)\right) &= \sum_{j=1}^{n}P(X_j=k)\\ &= \frac{1}{\psi(n)}\sum_{j=1}^{n}\left(\psi(n-jk)-\psi(n-j(k+1))\right) \qquad \square \end{aligned}$$

Problem 10.

Let U represent the number of distinct integers that occur in a random partition of n.
Find $E(U)$.

Answer

$$\frac{\psi(0)+\psi(1)+\ldots+\psi(n-1)}{\psi(n)}$$

This result is (5) in "*On the Multiplicity of Parts in a Random Partition*", Corteel, Pittel, Savage, Wilf, Random Structures and Algorithms, pages $185-197$.

Proof

$$\begin{aligned}
E(U) &= E\left(\sum_{j=1}^{n}\left(1-\mathrm{I}_{\{0\}}(X_j)\right)\right) \\
&= n-\sum_{j=1}^{n} P(X_j=0) \\
&= n-\sum_{j=1}^{n}\left(\frac{\psi(n-j\cdot 0)-\psi(n-j(0+1))}{\psi(n)}\right) \\
&= n-\sum_{j=1}^{n}\left(\frac{\psi(n)-\psi(n-j)}{\psi(n)}\right) \\
&= \frac{\psi(0)+\psi(1)+\ldots+\psi(n-1)}{\psi(n)}
\end{aligned}$$