Random Permutations

A permutation of the objects $(1, \ldots, n)$ defines a mapping. For example, the permutation $\pi = (3, 1, 2, 4)$ of the objects $(1, 2, 3, 4)$ defines the mapping

$$1 \mapsto 3,\ 2 \mapsto 1,\ 3 \mapsto 2,\ 4 \mapsto 4$$

This same mapping could also be represented in the form

$$(1 \mapsto 3, 3 \mapsto 2, 2 \mapsto 1) \ \text{ and } \ (4 \mapsto 4)$$

or more succinctly as

$$(1, 3, 2), (4)$$

The separate parts are referred to as ***cycles*** of the permutation. As argued in Riordan, An Introduction to Combinatorial Analysis, Chapter 4, *The Cycles of Permutations*, it is easy to see that every permutation can be uniquely represented by its cycles provided we adopt the convention that expressions such as $(1, 3, 2)$, $(3, 2, 1)$, and $(2, 1, 3)$, which represent the same cycle, are indistinguishable. We note that in the literature of cycles of permutations it is standard notation to write a cycle with its smallest element in the first position.

Clearly $n!$ equals the total number of permutations of $(1, \ldots, n)$. It is well known that $|s(n, t)|$, the signless Stirling Number of the First Kind, counts the total number of permutations of $(1, \ldots, n)$ with exactly t cycles. The article *A Review of the Stirling Numbers, Their Generalizations and Statistical Applications*, Charalambides, Ch. A.; Singh, Jagbir; Communications in Statistics, Theory and Methods, Vol. 17, No. 8, 1988, pages 2533--2595, is an excellent resource on Stirling Numbers.

If a permutation of $(1, \ldots, n)$ is selected uniformly at random from the set of all $n!$ permutations of $(1, \ldots, n)$, we will refer to this as a ***random permutation***.

If a permutation of $(1, \ldots, n)$ is selected uniformly at random from the set of all $|s(n, t)|$ permutations of $(1, \ldots, n)$ with t cycles, we will refer to this as a ***random permutation with t cycles***.

We will need the following definitions.

$\mathbb{S}^\infty$: the infinite product space $\{0,1,\ldots\} \times \{0,1,\ldots\} \times \cdots$

$\mathbb{S}^\infty_n$: the set of all vectors $(s_1,s_2,\ldots)$ in $\mathbb{S}^\infty$ such that $1s_1 + 2s_2 + \ \ldots \ = n$

$\mathbb{S}^\infty_{n,t}$: the set of all vectors $(s_1,s_2,\ldots)$ in $\mathbb{S}^\infty$ such that $\begin{matrix} 1s_1 + 2s_2 + \ \ldots \ = n \\ s_1 + s_2 + \ \ldots \ = t \end{matrix}$

For any $\mathcal{A} \subseteq \mathbb{S}^\infty$ define $\mathcal{A}_n = \mathcal{A} \cap \mathbb{S}^\infty_n$ and $\mathcal{A}_{n,t} = \mathcal{A} \cap \mathbb{S}^\infty_{n,t}$

We note that the condition that $1s_1 + 2s_2 + \ \ldots \ = n$ implies that $s_j = 0$ for all $j > n$. Hence all vectors in $\mathcal{A}_n$ and $\mathcal{A}_{n,t}$ are of the form $(a_1,\ldots,a_n, 0, 0, \ldots)$.

For all $\mathcal{A} \neq (0, 0, \ldots)$, let $\mathbb{A}_n$ be the collection of n-dimensional vectors formed by taking each infinite-dimensional vector in $\mathcal{A}_n$ and truncating after a_n. So for example,

$$(a_1,\ldots,a_n, 0, 0, \ldots) \rightarrow (a_1,\ldots,a_n)$$

Define $\mathbb{A}_{n,t}$ similarly. For notational consistency it is necessary to separate out the case $\mathcal{A} = (0, 0, \ldots)$.

We will refer to a cycle with r elements as an ***r - cycle***. A permutation of n elements with k_1 1 - *cycles*, $\ldots$, k_n n - *cycles* is said to be of ***cycle class*** $(\boldsymbol{k_1}, \ldots, \boldsymbol{k_n})$.

Let X_j equal the number of j - *cycles* in a random permutation of $(1, \ldots, n)$ with t cycles.

We can construct the set of all permutations of a set of n elements into t cycles such that $X_1 = x_1, \ldots, X_n = x_n$ in the following manner.

Take any one of the $n!$ permutations of the n elements and use the first x_1 elements of that permutation to fill the x_1 1 - cycles, use the next $2x_2$ elements of that permutation to fill the x_2 2 - cycles, and so on. In total we would use the $1x_1 + 2x_2 + \ldots + nx_n = n$ elements to fill the $x_1 + x_2 + \ldots + x_n$ cycles.

This assignment yields $n!$ permutations but not all of these permutations are distinct. In particular this count would assume that rearranging the x_j j-cycles amongst themselves leads to distinct permutations - which they do not. Furthermore this count would assume that <u>all</u> rearrangements of the elements within a cycle leads to distinct permutations - which they do not.

Therefore it is necessary to divide the count of $n!$ by the number of ways to arrange the x_j

j-cycles amongst themselves $(j = 1, \ldots, n)$ and by the number of ways to arrange the elements in each cycle and not change the cycle.

It follows that there are

$$\frac{n!}{x_1! \cdots x_n!} \left(\frac{1}{1}\right)^{x_1} \cdots \left(\frac{1}{n}\right)^{x_n}$$

permutations of a set of $(1, \ldots, n)$ with cycle class $(x_1, \ldots, x_n)$ provided $1x_1 + \ldots + nx_n = n$ and $x_j \in \{0, \ldots, n\}\ \forall j$

and

$$P((X_1, \ldots, X_n) = (x_1, \ldots, x_n)) = \begin{cases} \dfrac{\frac{n!}{x_1! \cdots x_n!} \left(\frac{1}{1}\right)^{x_1} \cdots \left(\frac{1}{n}\right)^{x_n}}{|s(n,t)|} & \begin{matrix} 1x_1 + \ldots + nx_n = n \\ x_1 + \ldots + x_n = t \\ x_j \in \{0, 1, \ldots, n\} \forall j \end{matrix} \\ 0 & \text{otherwise} \end{cases}$$

If we define W_j as the number of *j-cycles* in a random permutation of $(1, \ldots, n)$ then it follows similarly that

$$P((W_1, \ldots, W_n) = (w_1, \ldots, w_n)) = \begin{cases} \frac{1}{w_1! \cdots w_n!} \left(\frac{1}{1}\right)^{w_1} \cdots \left(\frac{1}{n}\right)^{w_n} & \begin{matrix} 1w_1 + \ldots + nw_n = n \\ w_j \in \{0, 1, \ldots, n\} \forall j \end{matrix} \\ 0 & \text{otherwise} \end{cases}$$

We note that it follows from the law of total probability that

$$\sum_{\substack{1w_1 + \ldots + nw_n = n \\ w_j \in \{0,1,\ldots,n\} \forall j}} \frac{1}{w_1! \cdots w_n!} \left(\frac{1}{1}\right)^{w_1} \cdots \left(\frac{1}{n}\right)^{w_n} = 1$$

which is Cauchy's identity.

Theorem 1.

$$E(g^*(W_1, \ldots, W_n)) = \left(\frac{1}{n!}\right)\frac{d^n}{d\lambda^n}\left(\left(\frac{1}{1-\lambda}\right)E(g(Y_1, Y_2, \ldots))\right)\Bigg|_{\lambda=0}$$

where $g(a_1, a_2, \dots)$ is any function and $g^*(a_1, \dots, a_n) = g(a_1, \dots, a_n, 0, 0, \dots)$ and for $\mathcal{A} \neq (0, 0, \dots)$

$$P((W_1, \ldots, W_n) \in \mathbb{A}_n) = \left(\frac{1}{n!}\right)\frac{d^n}{d\lambda^n}\left(\left(\frac{1}{1-\lambda}\right)P((Y_1, Y_2, \dots) \in \mathcal{A})\right)\Bigg|_{\lambda=0}$$

where $Y_1, Y_2, \dots$ is an infinite sequence of independent Poisson random variables such that

$$P(Y_j = y) = \frac{\exp^{\left(\frac{-\lambda^j}{j}\right)}\left(\frac{\lambda^j}{j}\right)^y}{y!} \qquad y = 0, 1, 2, \dots \text{ and } j = 1, 2, \dots \quad \blacksquare$$

Theorem 2.

$$E(g^*(X_1, \ldots, X_n)) = \frac{d^n}{d\lambda^n}\frac{d^t}{d\theta^t}\left(\left(\frac{1}{1-\lambda}\right)^\theta\left(\frac{1}{t!|s(n,t)|}\right)E(g(Y_1, Y_2, \ldots))\right)\Bigg|_{\substack{\lambda=0\\ \theta=0}}$$

where $g(a_1, a_2, \dots)$ is any function and $g^*(a_1, \dots, a_n) = g(a_1, \dots, a_n, 0, 0, \dots)$ and for $\mathcal{A} \neq (0, 0, \dots)$

$$P((X_1, \ldots, X_n) \in \mathbb{A}_{n,t}) = \frac{d^n}{d\lambda^n}\frac{d^t}{d\theta^t}\left(\left(\frac{1}{1-\lambda}\right)^\theta\left(\frac{1}{t!|s(n,t)|}\right)P((Y_1, Y_2, \dots) \in \mathcal{A})\right)\Bigg|_{\substack{\lambda=0\\ \theta=0}}$$

where $Y_1, Y_2, \dots$ is an infinite sequence of independent Poisson random variables such that

$$P(Y_j = y) = \frac{\exp^{\left(\frac{-\theta\lambda^j}{j}\right)}\left(\frac{\theta\lambda^j}{j}\right)^y}{y!} \qquad y = 0, 1, 2, \dots \text{ and } j = 1, 2, \dots \quad \blacksquare$$

Now consider the set of all possible ways to distribute n distinguishable keys onto t distinct key rings such that no key ring is left empty and where the position of keys on a key ring matters, but only up to circular shifts. Suppose we pick a distribution from this set uniformly at random.

Let U_j, $(j = 1, \ldots, t)$ equal the number of keys on the j^{th} key ring. Then,

$$P((U_1, \ldots, U_t) = (u_1, \ldots, u_t)) = \begin{cases} \dfrac{\frac{n!}{u_1 \cdots u_t}}{t!\,|s(n,t)|} & \substack{u_1+\ldots+u_t=n \\ u_j\in\{1,2,\ldots\}\ \forall j} \\ 0 & \text{else} \end{cases}$$

Let V_j, $(j = 1, \ldots, n)$ equal the number of (distinguishable) key rings containing j keys. Then,

$$P((V_1, \ldots, V_n) = (v_1, \ldots, v_n)) = \begin{cases} \dfrac{\frac{n!}{v_1! \cdots v_n!}\left(\frac{1}{1}\right)^{v_1} \cdots \left(\frac{1}{n}\right)^{v_n}}{|s(n,t)|} & \substack{1v_1+\ldots+nv_n=n \\ v_1+\ldots+v_n=t \\ v_j\in\{0,1,\ldots,n\}\ \forall j} \\ 0 & \text{otherwise} \end{cases}$$

We note that the above probability distribution is based on a model where the t key rings are distinguishable (e.g. different colors) but clearly the latter probability would be the same if the t key rings were the same color or not. However distributing n distinguishable keys onto t like key rings is equivalent to forming a permutation of $(1, \ldots, n)$ with t cycles. That is, for all $\mathbb{A}_{n,t} \subset \mathbb{S}_{n,t}^{\infty}$

$$P((V_1, \ldots, V_n) \in \mathbb{A}_{n,t}) = P((X_1, \ldots, X_n) \in \mathbb{A}_{n,t})$$

where as before the X_j, $(j = 1, \ldots, t)$ equal the number of cycles with j elements in a random permutation of $(1, \ldots, n)$ with t cycles.

We will need the following definitions.

$\mathbb{U}^t$: the t-dimensional product space $\{1,2,\ldots\} \times \cdots \times \{1,2,\ldots\}$

$\mathbb{U}_n^t$: the set of all vectors $(u_1,\ldots, u_t)$ in $\mathbb{U}^t$ such that $u_1 + \ldots + u_t = n$

Define $\mathcal{U}_n^{\mathbb{A}} \subseteq \mathbb{U}^t$ as that set such that $(V_1, \ldots, V_n) \in \mathbb{A}_{n,t} \Leftrightarrow (U_1, \ldots, U_t) \in \mathcal{U}_n^{\mathbb{A}}$.

Theorem 3.

$$P((X_1,\ldots,X_n)\in\mathbb{A}_{n,t}) = \frac{1}{t!|s(n,t)|}\frac{\mathrm{d}^n}{\mathrm{d}\theta^n}\left((-\ln(1-\theta))^t P\left((Y_1,\ldots,Y_t)\in\mathcal{U}^{\mathbb{A}}\right)\right)\Big|_{\theta=0}$$

where $Y_1,\ldots,Y_t$ are *iid* logarithmic series random variables with parameter θ. That is,

$$P(Y=y) = \begin{cases} \frac{\theta^y}{y(-\ln(1-\theta))} & y\in\{1,2,\ldots\} \\ 0 & \text{else} \end{cases}$$ ■

Problem 1.

(a)

The probability that a random permutation of $(1,2,\ldots,n)$ contains exactly k cycles of length j equals

$$\frac{1}{n!}\sum_{i=k}^{\left\lfloor \frac{n}{j}\right\rfloor}(-1)^{i-k}\binom{i}{k}\left(\frac{n!}{i!\,j^i}\right)$$

References

This result was derived by Goncharov, V.L. (1944), *Some Factors from Combinatorics*, Izv. Akad. Nauk SSSR, Ser. Mat. **8,** 3–48. Translated from Russian in Goncharov, V.L. (1962), *On the Field of Combinatory Analysis*, Translations from American Mathematical Society, **19,** 1–46.

(b) The probability that a random permutation of $(1,2,\ldots,n)$ into t cycles will contain exactly k cycles of length j is

$$\frac{1}{|s(n,t)|}\sum_{i=k}^{t}(-1)^{i-k}\binom{i}{k}\left(\frac{n!}{i!\,j^i}\right)\frac{|s(n-ij,t-i)|}{(n-ij)!}$$

(c) The probability that a random permutation of $(1,2,\ldots,n)$ into t cycles contains at

least k cycles of length j is

$$\frac{1}{|s(n,t)|}\sum_{i=k}^{t}(-1)^{i-k}\binom{i-1}{k-1}\left(\frac{n!}{i!\,j^i}\right)\frac{|s(n-ij,t-i)|}{(n-ij)!}$$

and the probability that a random permutation of $(1,2,\ldots,n)$ contains at least k cycles of length j is

$$\frac{1}{n!}\sum_{i=k}^{\left\lfloor \frac{n}{j}\right\rfloor}(-1)^{i-k}\binom{i-1}{k-1}\left(\frac{n!}{i!\,j^i}\right)$$

(d) The r^{th} descending factorial moment of the number of cycles of length j in a random permutation of $(1,2,\ldots,n)$, is

$$\frac{1}{j^r} \qquad 0\le r\le \left\lfloor \frac{n}{j}\right\rfloor$$

and the r^{th} descending factorial moment of the number of cycles of length j in a random permutation of $(1,2,\ldots,n)$ with t cycles is

$$\frac{n!}{(n-jr)!j^r}\,\frac{|s(n-jr,t-r)|}{|s(n,t)|} \qquad r+1\le t\le n-r(j-1)$$

References

The first of these two results can be found in Riordan, J. An Introduction to Combinatorial Analysis, page 84, Problem 12.

(e) The probability that a random permutation of $(1,2,\ldots,n)$ contains exactly k_1 cycles of length j_1,exactly k_2 cycles of length j_2, …, and exactly k_r cycles of length j_r is

$$\sum_{\substack{(i_1,\ldots,i_r)\ni\\ i_1k_1+\ldots+i_rk_r\le n\\ i_1\ge k_1,\ldots,\, i_r\ge k_r}}\cdots\sum(-1)^{(i_1+\ldots+i_r)-(k_1+\ldots+k_r)}\binom{i_1}{k_1}\cdots\binom{i_r}{k_r}\left(\frac{1}{i_1!\cdots i_r!\,j_1^{i_1}\cdots j_r^{i_r}}\right)$$

Problem 2.

The r^{th} descending factorial moment of the number of cycles in a random permutation of $(1, 2, \ldots, n)$ is

$$\frac{r!}{n!}|s(n+1, r+1)|$$

References

This result can be found in Riordan, J. An Introduction to Combinatorial Analysis,
page 71, equation (12).

Problem 3.

The number of permutations of $(1, 2, \ldots, n)$ which have k cycles, none of which is
an r cycle is

$$\sum_{j=0}^{\lfloor \frac{n}{r} \rfloor}(-1)^j\left(\frac{n!}{j!(n-rj)!r^j}\right)|s(n-rj, k-j)|$$

References

Riordan, J. An Introduction to Combinatorial Analysis, page 73, equation (18) is the special case $r = 1$. In the case $r = 1$ these numbers are referred to as the *Associated Stirling Numbers of the First Kind.*

Riordan gives a recurrence relation for general r in Problem 16(a), page 85.

Problem 4.

(a) The probability that exactly k cycle lengths are multiples of m in a random permutation of $(1, 2, \ldots, n)$ with t cycles is

$$\frac{n!}{|s(n,t)|}\sum_{j=k}^{t}\sum_{i=j}^{\lfloor\frac{n-t+j}{m}\rfloor}(-1)^{j-k}\binom{j}{k}\left(\frac{1}{m^j i!(n-mi)!}\right)|s(i,j)|\,|s(n-mi,t-j)|$$

(b) The probability that every cycle length in a random permutation of $(1,2,\ldots,n)$ with t cycles is a multiple of m (taking $k=t$ in above problem) simplifies to

$$\frac{\left|s\left(\frac{n}{m},t\right)\right|}{|s(n,t)|}\frac{n!}{\left(\frac{n}{m}\right)!}\frac{1}{m^t}$$

References

L. Carlitz, *Set Partitions*, Fibonacci Quarterly, Nov. 1976, pages 327-342 gives the formula in 4(b) for the case $m=2$.

Riordan, An Introduction to Combinatorial Analysis, Problem 18, pages 86-87, gives a table of the values of the above for $m=2$ and $n\leq 8$.

(c) The number of permutations of $(1,\ldots,n)$ where every cycle length belongs to the set $\{s,s+m,s+2m,\ldots\}$ for some integers s and m, $0\leq s<m$ is

$$n!\,\frac{\mathrm{d}^n}{\mathrm{d}\theta^n}\left(\prod_{j=0}^{m-1}\left(\frac{1}{1-\zeta^j\theta}\right)^{\frac{\zeta^{-sj}}{m}}\right)\Bigg|_{\theta=0}$$

where $\zeta=e^{2\pi i/m}$.

(d) The number of permutations of $(1,\ldots,n)$ where every cycle length is a multiple of

m (special case of 4(c) with $s=0$) equals

$$n!\binom{\frac{n}{m}+\frac{1}{m}-1}{\frac{n}{m}}$$

References

This result can be found in Sachkov, Probabilistic Methods in Combinatorial Analysis, Chapter 5, “Random Permutations”, page 151.

Notes:

Goulden and Jackson, Combinatorial Enumeration, Wiley-Interscience Series in Discrete Mathematics, 1983, Problem 3.3.12(a), page 188 give the answer in the form

$$\frac{n!}{\left(\frac{n}{m}\right)!\,m^{\left(\frac{n}{m}\right)}}\prod_{j=1}^{\frac{n}{m}-1}(jm+1)$$

Bolker and Gleason, *Counting Permutations*, Journal of Combinatorial Theory, Series A, Vol. 29, 1980, pages 236-242 give the answer in the form

$$\prod_{j=1}^{n}(n-j+\theta_m(j)) \qquad \text{where } \theta_m(j)=\begin{cases}1 & m \text{ divides } j\\ 0 & \text{else}\end{cases}$$

(e) The number of permutations of $(1,\ldots,n)$ where every cycle length belongs to the set $\left\{\frac{m}{2},\frac{m}{2}+m,\frac{m}{2}+2m,\ldots\right\}$ for even m (special case of 4(c) with $s=\frac{m}{2}$ and even m) equals

$$n!\sum_{j=0}^{\frac{2n}{m}}\binom{\frac{1}{m}}{\frac{2n}{m}-j}\binom{\frac{1}{m}+j-1}{j}$$

provided $\frac{m}{2}$ divides n.

References

This result can be found in Sachkov, Probabilistic Methods in Combinatorial Analysis, Chapter 5, "Random Permutations", page 151. However there is a misprint where the above sum starts at $j=1$ instead of $j=0$.

(f) The number of permutations of $(1,\ldots,n)$ where no cycle is a multiple of m is

$$n!\sum_{i=0}^{\lfloor\frac{n}{m}\rfloor}(-1)^i\binom{\frac{1}{m}}{i}$$

References

This result can be found in Goulden and Jackson, Combinatorial Enumeration, Wiley-Interscience Series in Discrete Mathematics, 1983, Problem 3.3.12(b), page 188.

(g) The probability that a random permutation of $(1, 2, \ldots, n)$ contains an even number of cycles all of which have odd length equals

$$\binom{n}{n/2}\left(\frac{1}{2}\right)^n$$

References

This result can be found in Wilf, generatingfunctionology, 2nd edition, page 84.

Problem 5.

Let c_n^c (c_n^o) equal the number of even (odd) permutations of $(1, 2, \ldots, n)$ and let $\xi_{n,k}^{(e)}$ $\left(\xi_{n,k}^{(o)}\right)$ equal the number of even (odd) permutations of $(1, 2, \ldots, n)$ with k cycles.

(a)

$$c_n^c = \begin{cases} 1 & n = 1 \\ \frac{n!}{2} & n \geq 2 \end{cases} \quad \text{and} \quad c_n^o = \begin{cases} 0 & n = 1 \\ \frac{n!}{2} & n \geq 2 \end{cases}$$

References

This result can be found in Riordan, John, An Introduction to Combinatorial Analysis, Problem 20, page 87-88.

Note:

The "standard" proof wherein one demonstrates a bijection by switching the position of elements $n-1$ and n in any permutation and noting that the parity changes is a much simplier proof but the present approach illustrates another aspect of Theorem 1.

(b)

$$\xi_{n,k}^{(e)} = \frac{|s(n,k)| + s(n,k)}{2} = \begin{cases} |s(n,k)| & n-k \text{ is even} \\ 0 & n-k \text{ is odd} \end{cases}$$

and

$$\xi_{n,k}^{(o)} = \frac{|s(n,k)| - s(n,k)}{2} = \begin{cases} 0 & n-k \text{ is even} \\ |s(n,k)| & n-k \text{ is odd} \end{cases}$$

References

This result can be found in Sachkov, Probabilistic Methods in Combinatorial Analysis, page 158.

Note:

Applying Theorem 2 is a useful demonstration but a simplier proof follows from the observation that for fixed n and t such that $1x_1 + \ldots + nx_n = n$ and $x_1 + \ldots + x_n = t$ with $x_j \in \{0, 1, \ldots, n\} \forall j$, then

$$x_2 + x_4 + x_6 + \ldots \text{ is even } \Leftrightarrow n - t \text{ is even.}$$

It follows from this observation that

$$\xi_{n,k}^{(e)} = \sum_{\substack{1x_1+\ldots+nx_n=n \\ x_1+\ldots+x_n=t \\ x_2+x_4+x_6+\ldots=\text{even} \\ x_j\in\{0,1,\ldots,n\}\forall j}} \frac{n!}{x_1!\cdots x_n!}\left(\frac{1}{1}\right)^{x_1}\cdots\left(\frac{1}{n}\right)^{x_n}$$

$$= \begin{cases} \displaystyle\sum_{\substack{1x_1+\ldots+nx_n=n \\ x_1+\ldots+x_n=t \\ x_j\in\{0,1,\ldots,n\}\forall j}} \frac{n!}{x_1!\cdots x_n!}\left(\frac{1}{1}\right)^{x_1}\cdots\left(\frac{1}{n}\right)^{x_n} & n-t = \text{even} \\ 0 & n-t = \text{odd} \end{cases}$$

$$= \begin{cases} |s(n,t)| & n-t = \text{even} \\ 0 & n-t = \text{odd} \end{cases}$$

Problem 6.

(a) How many permutations of $(1, 2, \ldots, n)$ are there for which the longest run of

1-cycles is less than or equal to k ?

(b) How many permutations of $(1, 2, \ldots, n)$ with t-*cycles* are there for which the longest run of 1-cycles is less than or equal to k ?

(c) How many permutations of $(1, 2, \ldots, n)$ are there which have exactly r runs of length k of 1-*cycles*?

(d) How many permutations of $(1, 2, \ldots, n)$ with t-*cycles* are there which have exactly
r runs of length k of 1-*cycles*?

Problem 7.

Define $W_j = \begin{cases} 1 & \text{if } X_j = v \\ 0 & \text{else} \end{cases}$

Define $T_j = \begin{cases} 1 & \text{if } X_j > 0 \\ 0 & \text{else} \end{cases}$

(a) How many permutations of $(1, 2, \ldots, n)$ are there for which exactly r of the values
in the cycle class $(k_1, \ldots, k_n)$ equal v? i.e. $N(W_1 + \ldots + W_n = r)$

(b) $\mathrm{E}\Big((W_1 + \ldots + W_n)_{(r)}\Big)$

(c) $P(T_1 + \ldots + T_n = r)$

(d) $\mathrm{E}\Big((T_1 + \ldots + T_n)_{(r)}\Big)$

Problem 8.

How many permutations of $(1, 2, \ldots, n)$ are there for which all cycle lengths are between l and u inclusive?

i.e. $(X_1 + \ldots + X_{l-1}) + (X_{u+1} + \ldots + X_n) = 0$

Problem 9.

(a) Suppose a permutation of $(1, 2, \ldots, n)$ is picked at random form the set of all $n!$ permutations of $(1, 2, \ldots, n)$ and from this permutation a cycle is picked at random. Let W represent the length of this cycle.

Find $P(W = w)$ and $\mathrm{E}\big(W_{(r)}\big)$.

(b) Suppose a permutation of $(1, 2, \ldots, n)$ is picked at random form the set of all permutations of $(1, 2, \ldots, n)$ with t cycles and from this permutation a cycle is picked at random. Let W represent the length of this cycle.

Find $P(W = w)$ and $\mathrm{E}\big(W_{(r)}\big)$

(c) Suppose a cycle is picked at random from the set of all cycles. (explain). Let W represent the length of this cycle.

Find $P(W = w)$ and $\mathrm{E}\big(W_{(r)}\big)$

(d) Suppose a permutation of $(1, 2, \ldots, n)$ is picked at random and an element is picked at random from that permutation. Let W represent the length of this cycle containing the randomly picked element.

Find $P(W = w)$ and $\mathrm{E}\big(W_{(r)}\big)$

Goulden & Jackson, page 190, Problem 3.3.19.

> Show that the number of permutations of $(1, 2, \ldots, n)$ in which the cycle containing n has length m, is $(n-1)!$, for any $m = 1, \ldots, n$.

Lovasz, page 29, problem 3. Shows that the probability that the cycle containing "1" has length k is $\frac{1}{n}$ for $k = 1, 2, \ldots, n$.

Hence expectation follows immediately.

Grusho, A.A. *Properties of random permutations with constraints on the maximum cycle length*, Probabilistic Methods in Discrete Mathematics, (Petrozavodsk, 1992), pages 60-63 considers this problem with the additional constraint that no cycle can have length greater than c.

Also look at other problems similar to this covered in section on Random Set Partitions

Problem 10.

$P(X_2 + X_4 + X_6 + \ldots = r)$

Problem 11.

Determine the number of permutations of $(1, 2, \ldots, n)$ for which the number of r-*cycles* equals the number of s-*cycles*.

References

This problem is discussed in Riordan, John, An Introduction to Combinatorial Analysis, Problem 15(b), page 84-85.

Problem 12.

Determine the number of permutations of $(1, 2, \ldots, n)$ which have no j-*cycles* for any $j > 2$.

References

This problem is discussed in Riordan, John, An Introduction to Combinatorial Analysis, Problem 17, page 85-86.

Problem 13.

Determine the number of even (odd) permutations of $(1, 2, \ldots, n)$ which have no 1-*cycles*.

References

This problem is discussed in Riordan, John, An Introduction to Combinatorial Analysis, Problem 21, page 88-89.

Problem 14.

Show that the number of permutations of $(1, 2, \ldots, n)$ which have k cycles, none of which is a 1 cycle, 2 cycle, $\ldots$, or r cycle is

Howard, F. T. refers to these numbers as the ***r*-associated Stirling Numbers of the First Kind** in Fibonacci Quarterly, *Associated Stirling Numbers*, Vol 18, no. 4, 1980, pages 303-315.

Problem 15.

Record values.

Goldie, Charles (1989). "Records, permutations and greatest convex minorants", *Mathematical Proceedings of the Cambridge Philosophical Society*, **106**, no. 1, pp. 169-177.

> "Let Π be a random element of S_n, the set of permutations of $\mathbb{N}_n$, all $n!$ elements of S_n being equally likely. Π may be written as a product of cycles. Let us say that $i \in \mathbb{N}_n$ is a *new-cycle index* if i does not belong to the cycles containing $1, \ldots, i-1$. The random set of new-cycle indices is denoted $\mathcal{C}$. It always contains 1. Stam, theorem 3) has shown that the events $i \in \mathcal{C}$ are independent, with respective probabilities $1/i$.
>
> [Stam, A.J. (1983). "Cycles of random permutations", *Ars Combinatoria*, **16**, pages 43-48.]

Let $\mathcal{F}$ be the set of record times. That is,

$$\mathcal{F} := \{i \in \mathbb{N}_n : X_i = \max(X_1, \ldots, X_i)\}$$

Let $\mathcal{R}$ be the set of record values. That is,

$$\mathcal{R} := \{X_i : i \in \mathcal{F}\}$$

Let $X_1^{(n)}, \ldots, X_n^{(n)}$ be the order statistics of $X_1, \ldots, X_n$

Theorem 3.1 The events $\{X_i^{(n)} \in \mathcal{R}\}$, $i = 1, \ldots, n$ are independent with probabilities

$$P\left(X_i^{(n)} \in \mathcal{R}\right) = \frac{1}{n+1-i}$$

Proof

$X_i^{(n)} \in \mathcal{R}$ if and only if $X_i^{(n)}$occurs *before* $X_{i+1}^{(n)}, \ldots, X_n^{(n)}$ in the finite sequence $X_1, \ldots, X_n$. Whatever order $X_{i+1}^{(n)}, \ldots, X_n^{(n)}$ occur in among themselves, there is probability $1/(n+1-i)$ that $X_i^{(n)}$ occurs earlier. Thus given information on which of $X_{i+1}^{(n)}, \ldots, X_n^{(n)}$ are record values, there is always probability $1/(n+1-i)$ that $X_i^{(n)}$ is a record value.

(?) I give this argument here because I wonder if this argument is (1) rigorous and (2) can be used to prove the hook length formula. Note that in case where $X_1, \ldots, X_n$ is a permutation of $1, \ldots, n$, then $X_j^{(n)} = j$.

Let

$$I_j = \begin{cases} 1 & X_j^{(n)} \in \mathcal{R} \\ 0 & \text{else} \end{cases}$$

Consider $P(I_j = 1 \mid I_{j+1} = 1, \ldots, I_n = 1)$

We are given that $X_{j+1}^{(n)}, \ldots, X_n^{(n)}$ are record values so we know that

$X_{j+1}^{(n)}$occurs *before* $X_{j+2}^{(n)}, \ldots, X_n^{(n)}$ in the finite sequence $X_1, \ldots, X_n$

and

$X_{j+2}^{(n)}$occurs *before* $X_{j+3}^{(n)}, \ldots, X_n^{(n)}$ in the finite sequence $X_1, \ldots, X_n$

and

$X_{n-1}^{(n)}$occurs *before* $X_n^{(n)}$in the finite sequence $X_1, \ldots, X_n$

but (as the argument goes) that tells us nothing about whether $X_j^{(n)}$ occurs *before* $X_{j+1}^{(n)}, \ldots, X_n^{(n)}$ in the finite sequence $X_1, \ldots, X_n$. The conditioning only tells us about the position of $X_{j+1}^{(n)}, \ldots, X_n^{(n)}$ relative to each other.

Therefore

$$P(I_j = 1 \mid I_{j+1} = 1, \ldots, I_n = 1) = P(I_j = 1)$$

and subsequently

$$\begin{aligned} &P(I_1 = 1, I_2 = 1, \ldots, I_n = 1) \\ &\quad = P(I_n = 1)P(I_{n-1} = 1 \mid I_n = 1)\cdots P(I_1 = 1 \mid I_2 = 1, \ldots, I_n = 1) \\ &\quad = P(I_n = 1)P(I_{n-1} = 1)\cdots P(I_1 = 1) \end{aligned}$$

which shows independence.

Karamata-Stirling laws.

The KS_n probability law is defined to be that of $Z_1 + \ldots + Z_n$ where

$Z_1, \ldots, Z_n$ are independent and $P(Z_i = 1) = 1/i,\ P(Z_i = 0) = 1 - 1/i.$

Explicitly,

$$P(Z_1 + \ldots + Z_n = k) = \frac{|s(n,k)|}{n!} \qquad (k = 0, 1, \ldots, n)$$

KS_n is the distribution of

(*i*) the number of cycles in a random permutation of *n* objects
(*ii*) the number of records in *n* exchangeable unequal r.v.s.
(*iii*) the number of sides in the gem of an *n*-step random walk with

exchangeable rationally independent increments”

Problem 16.