Random Rooted Labeled Forests of Height ≤ 1
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The number of rooted labeled forests with n vertices and t trees each of height ≤ 1 equals $\binom{n}{t} t^{n-t}$.

Let M_n equals the number of rooted labeld forests with n vertices where each tree has height ≤ 1.

It follows that $M_n = \sum_{t=1}^{n} \binom{n}{t} t^{n-t}$.

Let C_j equal the number of trees with j vertices in a rooted labeled forest with n vertices and a total of t trees, each of height ≤ 1.

Then,

$$\frac{n!}{C_1! \cdots C_n! (0!)^{C_1} \cdots ((n-1)!)^{C_n}}$$

equals the number of rooted labeled forests with n vertices where each of the trees has height ≤ 1 and there are exactly C_j trees with degree j, $1C_1 + 2C_2 + \ldots + nC_n = n$.

Furthermore,

$$\sum_{\substack{c_1+\ldots+c_n=t \\ 1c_1+\ldots+nc_n=n \\ c_i \in \{0,1\ldots\}}} \cdots \sum \frac{n!}{c_1! \cdots c_n! (0!)^{c_1} \cdots ((n-1)!)^{c_n}} = \binom{n}{t} t^{n-t}$$

Let $X_j =$ number of trees with j vertices in a random rooted labeled forest with n vertices where all trees have ≤ 1.

Then,

$$P(X_1 = x_1, \ldots, X_n = x_n) = \frac{n!}{x_1! \cdots x_n! (0!)^{x_1} \cdots ((n-1)!)^{x_n} M_n}$$

for all $(x_1, \ldots, x_n)$ such that $1x_1 + \ldots + nx_n = n$ and $x_i \in \{0, 1 \ldots\}$.

Now consider an infinite sequence $Y_1, Y_2, \ldots$ of independent Poisson random variables where

$$P(Y_j = y) = \frac{e^{\left(\frac{-\lambda^j}{(j-1)!}\right)}\left(\frac{\lambda^j}{(j-1)!}\right)^y}{y!} \qquad y = 0, 1, 2, \ldots \text{ and } j = 1, 2, \ldots$$

Let y_j be a nonnegative integer for $j = 1, 2, \cdots$. Then

$$P(Y_1 = y_1, Y_2 = y_2, \ldots) = \frac{e^{\left(-\sum_{j=1}^{\infty} \frac{\lambda^j}{(j-1)!}\right)} \lambda^{\left(\sum_{j=1}^{\infty} j y_j\right)} \prod_{j=1}^{\infty} \left(\frac{1}{(j-1)!}\right)^{y_j}}{\prod_{j=1}^{\infty} (y_j)!}$$

$$= \frac{e^{(-\lambda e^{\lambda})} \lambda^{\left(\sum_{j=1}^{\infty} j y_j\right)}}{\prod_{j=1}^{\infty} (y_j)!((j-1)!)^{y_j}}$$

and

$$P\left(\sum_{j=1}^{\infty} jY_j = n\right) = \underset{\substack{(y_1,y_2,\dots)\ni \\ \sum_{j=1}^{\infty} jy_j=n}}{\sum\cdots\sum} P(Y_1 = y_1, Y_2 = y_2, \dots)$$

$$= \frac{e^{\left(-\lambda e^{\lambda}\right)}\lambda^n}{n!}\left(\underset{\substack{(y_1,y_2,\dots)\ni \\ \sum_{j=1}^{\infty} jy_j=n}}{\sum\cdots\sum}\left(\frac{n!}{\prod_{j=1}^{\infty}(y_j)!((j-1)!)^{y_j}}\right)\right)$$

$$= \frac{e^{\left(-\lambda e^{\lambda}\right)}\lambda^n}{n!}\left(\underset{\substack{(y_1,y_2,\dots)\ni \\ \sum_{j=1}^{n} jy_j=n}}{\sum\cdots\sum}\left(\frac{n!}{\prod_{j=1}^{n}(y_j)!((j-1)!)^{y_j}}\right)\right)$$

$$= \frac{e^{\left(-\lambda e^{\lambda}\right)}\lambda^n}{n!}M_n$$

It follows that for $\mathcal{A} \neq (0, 0, \dots)$

$$P((Y_1, Y_2, \dots) \in \mathcal{A})$$

$$= \sum_{n=1}^{\infty} P\left((Y_1, Y_2, \dots) \in \mathcal{A} \mid \sum_{j=1}^{\infty} jY_j = n\right) P\left(\sum_{j=1}^{\infty} jY_j = n\right)$$

$$= \sum_{n=1}^{\infty} \sum_{\mathbb{A}_n} \left(\frac{\frac{e^{\left(-\lambda e^{\lambda}\right)}\lambda^n}{\prod_{j=1}^{n}\left(y_j\right)!((j-1)!)^{y_j}}}{\frac{e^{\left(-\lambda e^{\lambda}\right)}\lambda^n}{n!}M_n}\right)\left(\frac{e^{\left(-\lambda e^{\lambda}\right)}}{n!}M_n\right)\lambda^n$$

$$= \sum_{n=1}^{\infty} \sum_{\mathbb{A}_n} \left(\frac{n!}{\prod_{j=1}^{n}(y_j)!((j-1)!)^{y_j}M_n}\right)\left(\frac{e^{\left(-\lambda e^{\lambda}\right)}}{n!}M_n\right)\lambda^n$$

$$= \sum_{n=1}^{\infty} P((X_1, ..., X_n) \in \mathbb{A}_n) \left(\frac{e^{\left(-\lambda e^{\lambda}\right)}}{n!} M_n \right) \lambda^n$$

Therefore

$$e^{\lambda e^{\lambda}} P((Y_1, Y_2, \dots) \in \mathcal{A}) = \sum_{n=1}^{\infty} P((X_1, ..., X_n) \in \mathbb{A}_n) \left(\frac{M_n}{n!} \right) \lambda^n$$

and

$$\frac{d^t}{d\lambda^t} \left(e^{\lambda e^{\lambda}} P((Y_1, Y_2, \dots) \in \mathcal{A}) \right) \Big|_{\lambda=0} \\ = \sum_{n=1}^{\infty} P((X_1, ..., X_n) \in \mathbb{A}_n) \left(\frac{M_n}{n!} \right) t! \mathrm{I}_{\{t\}}(n)$$

Therefore, for $\mathcal{A} \neq (0, 0, \dots)$

$$P((X_1, ..., X_n) \in \mathbb{A}_n) = \left(\frac{1}{M_n} \right) \left(\frac{d^n}{d\lambda^n} \left(e^{\lambda e^{\lambda}} P((Y_1, Y_2, \dots) \in \mathcal{A}) \right) \Big|_{\lambda=0} \right)$$

Note:

$$1 + \sum_{n=1}^{\infty} \frac{M_n}{n!} \lambda^n = e^{\lambda e^{\lambda}}$$