# MSHSML Meet 1, Event A
# Study Guide

## 1A Pre-algebra Topics (no calculators)

Fractions to add and express as the quotient of two relatively prime integers
Complex fractions and continued fractions
Decimals, repeating decimals
Percentage, interest, and discount
Least common multiple, greatest common divisor
Number bases; change of base
(extra) Modular arithmetic, number theory

## 1 Contents

# 2 Percentage, Interest, and Discounts

## 2.1 Sales Tax

If the ticket price of an item equals $d$ and the sales tax equals $k\%$ then the total price $p$ equals

$$p = d + \left(\frac{k}{100}\right) \cdot d = d\left(1 + \frac{k}{100}\right).$$

## 2.2 Discounts (or Mark Downs)

If the ticket price of an item equals $d$ and you receive a $k\%$ discount then the total price $p$ equals

$$p = d - \left(\frac{k}{100}\right) \cdot d = d\left(1 - \frac{k}{100}\right).$$

## 2.3 Increases (or Mark Ups)

If the ticket price of an item was equal to $d$ and then the store applied a $k\%$ increase then the total price $p$ equals

$$p = d + \left(\frac{k}{100}\right) \cdot d = d\left(1 + \frac{k}{100}\right).$$

Notice this acts the same way as a sales tax.

## 2.4 Multiple Discounts (Additive and Chained)

**Additive discounts** of $k_1$ and $k_2$ percent applied to an original price of $p_0$ calculates to the new price $p_1$

$$p_1 = \left(1 - \frac{k_1 + k_2}{100}\right) p_0$$

while **chained** (or successive) **discounts** of $k_1$ and $k_2$ percent applied to an original price of $p_0$ calculates to the new price $p_1$

$$p_1 = \left(1 - \frac{k_1}{100}\right)\left(1 - \frac{k_2}{100}\right) p_0.$$

**Example 1.**

**(Version 1)**

You go shopping and the shoes you are interested in buying have just gone on sale. The store has advertised that the shoes you want are $20\%$ off the marked price. Additionally, you have a loyal customer card for this store that allows you can take 10% off the in-store price of everything in addition to any other sales going on. How much will you pay for this pair of shoes whose price in the store is $\$100$.

**(Version 2)**

You go shopping and the shoes you are interested in buying have just gone on sale. The store has advertised that the shoes you want are $20\%$ off the marked price. Additionally, you have a coupon for this store that says you can take another 10% off the reduced price after the $20\%$ mark down. How much will you pay for this pair of shoes whose price in the store is $\$100$.

**Solution**

Both versions of this problem represent successive discounts. While sounding very similar to each other there is a difference you have to take into account.

The store's $20\%$ off sale reduces the shoes from $\$100$ to $\$80$. In Version 1 of the story your loyal customer card allows you to take (an additional) $10\%$ off the in-store price of $\$100$. So you can use your loyalty card to save another $\$10$ and buy the shoes for $\$70$.

In Version 2, you get the same initial $20\%$ off to reduce the shows from $\$100$ to $\$80$. But then your coupon says you can take another $10\%$ off the reduced price of $\$80$. So with the coupon you get another $\$8$ off and are able to buy the shoes for $\$72$.

In Version 1 the calculations are

$$\$100 - \left(\frac{20}{100}\right)\$100 - \left(\frac{10}{100}\right)(\mathbf{\$100}) = \$100\left(1 - \frac{20}{100} - \frac{10}{100}\right)$$

That is, in this version you add the $20\%$ and $10\%$ together to get a $30\%$ discount.

In Version 2 the calculations are

$$\$100 - \left(\frac{20}{100}\right)\$100 - \left(\frac{10}{100}\right)\left(\mathbf{\$100} - \left(\frac{\mathbf{20}}{\mathbf{100}}\right)\mathbf{\$100}\right) = \$100\left(1 - \frac{20}{100}\right)\left(1 - \frac{10}{100}\right).$$

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Version 1 is an example of ***additive discounts*** where both discounts are applied to the original selling price. Version 2 is an example of ***chained discounts*** where the second discount coupon is applied to the discounted price obtained after using the first coupon. Chained discounts are also called ***successive discounts***.

**Example 2.**

A DVD player with a list price of $\$100$ is marked down $30\%$. If John gets an employee discount of $20\%$ off the sale price, how much does John pay for the DVD player?

**Solution**

First, we need to figure out whether these are additive or chained discounts. The story says that John gets a second discount *off the sale price*. So this is an example of a chained discount.

$$p_1 = \left(1 - \frac{30}{100}\right)\left(1 - \frac{20}{100}\right)(\$100) = \$56.$$

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**Example 3.**

Mrs. Dorgan's gross monthly income is $\$5{,}000$. If $15\%$ is withheld for income taxes, $7\%$ for social security, and $2\%$ for insurance, what is her net monthly income (after deducting these expenses)?

**Solution**

Are these additive or chained discounts? Each withholding is based on her gross monthly so these are additive discounts.

$$p_1 = \left(1 - \frac{15 + 7 + 2}{100}\right)(\$5000) = \$3800$$

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## 2.5 Multiple Increases

We also can have both additive and chained percent increases.

**<u>Additive increases</u>** of $k_1$ and $k_2$ percent applied to an original price of $p_0$ calculates to the new price $p_1$

$$p_1 = \left(1 + \frac{k_1 + k_2}{100}\right) p_0$$

while **<u>chained</u>** (or successive) **<u>increases</u>** of $k_1$ and $k_2$ percent applied to an original price of $p_0$ calculates to the new price $p_1$

$$p_1 = \left(1 + \frac{k_1}{100}\right)\left(1 + \frac{k_2}{100}\right) p_0.$$

**Example 4.**

You got a 3% salary raise in each of your first two years at your first job after college. Your starting salary was \$50,000. How much are you earning after your two raises?

**Solution**

Are these additive or chained percent increases? When you get a raise you get a percent increase on your current salary, not your starting salary. That is, pay raises are chained increases.

$$p_1 = \left(1 + \frac{3}{100}\right)\left(1 + \frac{3}{100}\right)(\$50{,}000) = \$53{,}045.00.$$

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**Example 5.**

Suppose you make $\$100{,}000$ a year. Additionally, the company you work for puts an extra $25\%$ of your salary into a separate company retirement plan Furthermore the company subsidizes your medical insurance in the amount of another $20\%$ of your salary. If you take into account the money this company pays in salary, retirement and medical benefits, how much are you really making each year?

**Solution**

Are these additive or chained percent increases? They are additive increases because the company is only putting the $25\%$ towards retirement on your salary of $\$100{,}00$, not $25\%$ on your salary after taking your medical benefits into account (*i.e.* $\$120{,}000$).

$$p_1 = \left(1 + \frac{25}{100} + \frac{20}{100}\right)(\$100{,}000) = \$145{,}000$$

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**Example 6.** (modified from a problem in Meet 1A ,1991-92)

Suppose you currently make $\$100{,}000$ a year and your fringe benefits (retirement, insurance, etc.) comes to $45\%$ of your salary. If you receive a $5\%$ raise at the end of the year and your fringe benefits stays at $45\%$ of your (new) salary, how much extra will the company be spending on you next year over this year.

**Solution**

Based on the answer in Example 5, this company is spending $\$145{,}000$ on you currently. If you get a $5\%$ raise it is applied on your salary only, not your salary plus fringe benefits. But then your fringe is calculated on your new salary. So your fringe and salary increase are chained.

$$p_1 = (\$100{,}000)\left(1 + \frac{5}{100}\right)\left(1 + \frac{45}{100}\right) = \$152{,}250$$

So the company will spend an additional $\$7{,}250$ next year on you.

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## 2.6 Chained Discounts Combined with Chained Increases

If the value (or cost) of an item originally equaled $d$ and increases in value (or cost) by $k_1\%$ and then by $k_2\%$ and then by another $k_3\%$ and then decreases in value (or is discounted) by $k_4\%$

and then by $k_5\%$ and then by $k_6\%$. If we are to assume that both the discounts and increases were chained, then the final value $p$ of this item would equal

$$p = d\left(1+\frac{k_1}{100}\right)\left(1+\frac{k_2}{100}\right)\left(1+\frac{k_3}{100}\right)\left(1-\frac{k_4}{100}\right)\left(1-\frac{k_5}{100}\right)\left(1-\frac{k_6}{100}\right).$$

Important Note: When calculating chained increases and/or decreases applied over time, the order in which these increases and decreases are calculated does not change the above formula.

## 2.7 Discount Coupon Combined with Sales Tax

**Example 7.**

You have a $6\%$ off coupon for a $\$50$ item you just bought. However, the state you live in applies a $6\%$ sales tax on this item. What does your total cost come to?

**Solution**

You really don't have enough information to solve this problem. In some states they apply the sales tax on the amount you pay after applying the coupon (*i.e.* the coupon and tax are chained). But other states apply the sales tax on the original cost of the item (*i.e.* additive increase and decrease).

If your state charges sales tax only on the reduced price, then you pay

$$p_1 = \left(1-\frac{6}{100}\right)\left(1+\frac{6}{100}\right)(\$50) = \$49.82$$

However, if your state charges sales tax on the original selling price, then you pay

$$p_1 = \left(1+\frac{6}{100}-\frac{6}{100}\right)(\$50) = \$50$$

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**Example 8.** (2010-11 Meet 1, Individual Event A)

I'm out for lunch at my favorite café, but I only have $\$15.00$. If the soup-and-sandwich combo I want to order costs $\$13.00$, and sales tax is $7\%$, what is the <u>minimum</u> whole-number percent-off discount coupon I must hold in my wallet to allow me to still leave an $18\%$ tip? (*Note: tax and tip are applied after the coupon, but not to each other*.)

**Solution**

The question writers were careful to spell out that both the tax and the tip were applied after the coupon but not to each other.

So we have a problem which is part additive and part chained!

The tax and tip are additive because both are applied to the same amount, namely the amount you owe after applying the coupon. So, in effect we are paying an extra $(18+7)\% = 25\%$ to cover tax and tip combined. But this $25\%$ increase is chained to whatever percent discount we will ultimately get from our coupon because we are applying this $25\%$ increase *after* applying the discount coupon.

Suppose our discount is for $x\%$ off. We want to find $x$ so that

$$13\left(1-\frac{x}{100}\right)\left(1+\frac{25}{100}\right)=15$$

Solving for $x$ we get $x = 100/13 \approx 7.69$. So to come to *exactly* $\$15$ we would need a $(100/13)\%$ off coupon. But the problem required that our coupon was an integer amount. So to make sure we don't go over $\$15$ we need at least an $8\%$ off coupon.

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## 2.8 Percent Change

If the amount of an object (for example, the number or the price) changes from $a$ to $b$, then we say that there was a

$$\left(\frac{b-a}{a}\right)\times 100\ \%$$

change in the amount of that object.

**Example 9.** (modified from a problem in Meet 1A ,1991-92)

Suppose that Elizabeth is given an annual salary increase of $6\%$ every January $1$. By what percentage will her salary increase over the next $10$ years?

**Solution**

Salary increases are chained. So if we let $p$ represent Elizabeth's salary today then her salary in $10$ years will be

$$p\left(1+\frac{6}{100}\right)^{10} = 1.79p.$$

So her percent change (increase) in salary will be

$$\left(\frac{b-a}{a}\right) \times 100\ \% = \left(\frac{1.79p - p}{p}\right) \times 100\ \% = 79\%.$$

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**Example 10.** (Indiana State Mathematics Contest, 2006, Pre-Algebra, Problem 12)

The length of a rectangle is increased by 30% and its width is decreased by 10%. This increases the area by what percent?

**Solution**

Let $x_1$ be the length of the original (old) rectangle and let $y_1$ be the width of the original (old) rectangle.

Let $x_2$ be the length of the modified (new) rectangle and let $y_2$ be the width of the modified (new) rectangle.

We are given that there was a $30\%$ increase in the length.

$30\%$ increase in length

$$\Rightarrow \left(\frac{\text{new length} - \text{old length}}{\text{old length}}\right) \times 100\ \% = 30\%$$

$$\Rightarrow \left(\frac{x_2 - x_1}{x_1}\right) \times 100\ \% = 30\%$$

$$\Rightarrow \left(\frac{x_2 - x_1}{x_1}\right) = \frac{30\%}{100\%} = \frac{30}{100} = \frac{3}{10}$$

$$\Rightarrow \frac{x_2}{x_1} - 1 = \frac{3}{10}$$

$$\Rightarrow \frac{x_2}{x_1} = 1 + \frac{3}{10}$$

$$\Rightarrow \frac{x_2}{x_1} = \frac{13}{10}$$

$$\Longrightarrow x_2 = \left(\frac{13}{10}\right) \cdot x_1.$$

We are also given that there was a 10% decrease in the width.

**Notice this negative sign!**

$$\Longrightarrow \left(\frac{\text{new width} - \text{old width}}{\text{old width}}\right) \times 100\ \% = -10\%$$

$$\Longrightarrow \left(\frac{y_2 - y_1}{y_1}\right) \times 100\ \% = -10\%$$

$$\Longrightarrow \left(\frac{y_2 - y_1}{y_1}\right) = \frac{-10\%}{100\%} = \frac{-10}{100} = \frac{-1}{10}$$

$$\Longrightarrow \frac{y_2}{y_1} - 1 = \frac{-1}{10}$$

$$\Longrightarrow \frac{y_2}{y_1} = 1 - \frac{1}{10}$$

$$\Longrightarrow \frac{y_2}{y_1} = \frac{9}{10}$$

$$\Longrightarrow y_2 = \left(\frac{9}{10}\right) \cdot y_1.$$

The problem asks us to determine the percent increase in area.

$$\%\text{ increase in area } = \frac{\text{new area } - \text{old area}}{\text{old area}} \times 100\ \%$$

$$= \frac{x_2 \cdot y_2 - x_1 \cdot y_1}{x_1 \cdot y_1} \times 100\%$$

$$= \frac{x_2 \cdot y_2 - x_1 \cdot y_1}{x_1 \cdot y_1} \times 100\%$$

$$= \frac{\left(\frac{13}{10} \cdot x_1\right) \cdot \left(\frac{9}{10} \cdot y_1\right) - x_1 \cdot y_1}{x_1 \cdot y_1} \times 100\%$$

$$= \frac{\left(\left(\frac{13}{10}\right) \cdot \left(\frac{9}{10}\right) \cdot x_1 \cdot y_1\right) - x_1 \cdot y_1}{x_1 \cdot y_1} \times 100\%$$

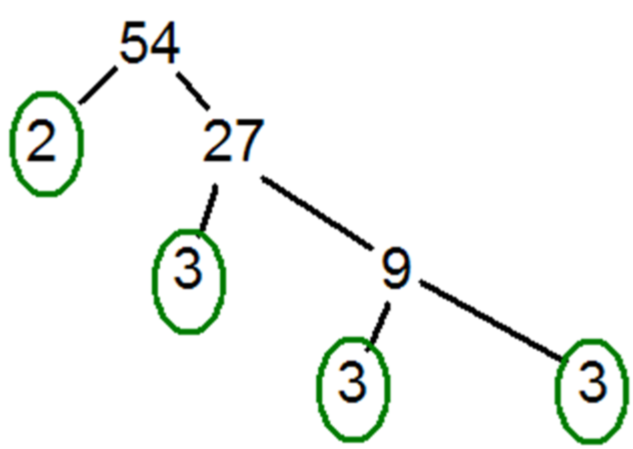
$$= \frac{\left(\left(\frac{13}{10}\right) \cdot \left(\frac{9}{10}\right) - 1\right)}{1} \times 100\%$$

$$= \left(\frac{17}{100}\right) \times 100\% = 17\%.$$

# 3 Primes and Prime Factorizations

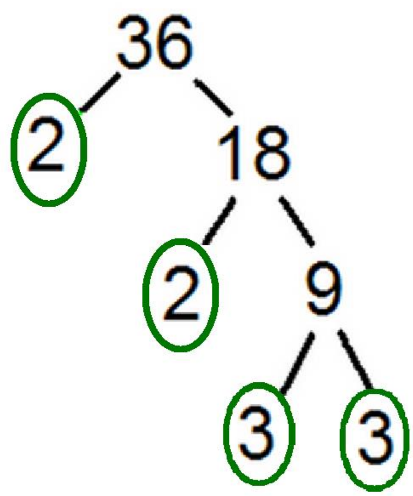
## 3.1 Factor Trees

**Example 11.** Use a factor tree to break 54 down into its prime factors.




So $54 = 2 \times 3 \times 3 \times 3 = 2^1 \times 3^3$.

**Example 12.** Use a factor tree to break 36 down into its prime factors.




So $36 = 2 \times 2 \times 3 \times 3 = 2^2 \times 3^2$.

## 3.2 Listing All Factors of a Number

**Theorem 1**

If the prime factorization of $a$ is given by

$$a = 2^{m_1} \times 3^{m_2} \times 5^{m_3} \times 7^{m_4} \times 11^{m_5} \times \cdots$$

then the set of all factors of $a$ consists of all numbers of the form

$$2^{c_1} \times 3^{c_2} \times 5^{c_3} \times 7^{c_4} \times 11^{c_5} \times \cdots$$

where $0 \leq c_1 \leq m_1$, $0 \leq c_2 \leq m_2$, $0 \leq c_3 \leq m_3$,…

**Example 13.** Make a list of all factors of $18 = 2^1 \times 3^2$.

| | | |
|---|---|---|
| $2^0 \times 3^0 = 1$ | $2^0 \times 3^1 = 3$ | $2^0 \times 3^2 = 9$ |
| $2^1 \times 3^0 = 2$ | $2^1 \times 3^1 = 6$ | $2^1 \times 3^2 = 18$ |

**Example 14.** Make a list of all factors of $24 = 2^3 \times 3^1$.

| | |
|---|---|
| $2^0 \times 3^0 = 1$ | $2^0 \times 3^1 = 3$ |
| $2^1 \times 3^0 = 2$ | $2^1 \times 3^1 = 6$ |
| $2^2 \times 3^0 = 4$ | $2^2 \times 3^1 = 12$ |
| $2^3 \times 3^0 = 8$ | $2^3 \times 3^1 = 24$ |

**Example 15.** Make a list of all factors of $120 = 2^3 \times 3^1 \times 5^1$.

| | |
|---|---|
| $2^0 \times 3^0 \times 5^0 = 1$ | $2^0 \times 3^0 \times 5^1 = 5$ |
| $2^0 \times 3^1 \times 5^0 = 3$ | $2^0 \times 3^1 \times 5^1 = 15$ |
| $2^1 \times 3^0 \times 5^0 = 2$ | $2^1 \times 3^0 \times 5^1 = 10$ |
| $2^1 \times 3^1 \times 5^0 = 6$ | $2^1 \times 3^1 \times 5^1 = 30$ |
| $2^2 \times 3^0 \times 5^0 = 4$ | $2^2 \times 3^0 \times 5^1 = 20$ |
| $2^2 \times 3^1 \times 5^0 = 12$ | $2^2 \times 3^1 \times 5^1 = 60$ |
| $2^3 \times 3^0 \times 5^0 = 8$ | $2^3 \times 3^0 \times 5^1 = 40$ |
| $2^3 \times 3^1 \times 5^0 = 24$ | $2^3 \times 3^1 \times 5^1 = 120$ |

## 3.3 Counting the Number of Divisors of $n$

**Theorem 2 (Counting the Number of Divisors of $n$)**

Let $\tau(n)$ represent the number of divisors (factors) of the positive integer $n$. If $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ is the unique prime factorization of $n$, then

$$\tau(n) = (a_1 + 1)(a_2 + 1) \cdots (a_k + 1).$$

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**Example 16.**

Applying Theorem 2 we can see that $6 = 2^1 \times 3^1$ has $(1+1) \times (1+1) = 4$ factors. These four factors are $\{1,2,3,6\}$.

Again, applying Theorem 2 we can see that $8 = 2^3$ has $(3+1) = 4$ factors. These four factors are $\{1,2,4,8\}$.

Once again, applying Theorem 2 we can see that $60 = 2^2 \times 3^1 \times 5^1$ has $(2+1) \times (1+1) \times (1+1) = 12$ factors. These twelve factors are $\{1,2,3,4,5,6,10,12,15,20,30,60\}$.

The power of Theorem 2 really reveals itself for counting the number of factors for "large" numbers.

For example, it would be quite tedious to count the number of factors of $12600$ by writing out a complete list. By Theorem 2 we can see that $12600 = 2^3 \times 3^2 \times 5^2 \times 7^1$ has $(3+1) \times (2+1) \times (2+1) \times (1+1) = 72$ factors.

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**Theorem 3**

Here we will list several interesting consequences of Theorem 2.

$\tau(n)$ is odd if and only if $n$ is a perfect square

$\tau(n) = 2$ if and only if $n$ is a prime number

$\tau(n) = 3$ if and only if $n = p^2$ for some prime number $p$.

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**Example 17.**

Explain why $\tau(n)$ is odd if and only if $n$ is a perfect square.

**Solution**

We know from Theorem 2 that if $p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ is the prime factorization of $n$, then

$$\tau(n) = (a_1 + 1)(a_2 + 1) \cdots (a_k + 1).$$

It follows that $\tau(n)$ is odd if and only if *all* of the factors $(a_1 + 1), (a_2 + 1), \dots, (a_k + 1)$ are odd numbers. [If any of these factors is even then the product would be even.] But (for example), $(a_1 + 1)$ is odd if and only if $a_1$ is even.

So, $\tau(n)$ is odd if and only if $a_1, a_2, a_3, \ldots, a_k$ are *all* even numbers. Let $b_j = a_j/2$. Because $a_j$ is even we have that $b_j$ is an integer.

Therefore,

$$n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k} = \left(p_1^{b_1} \cdot p_2^{b_2} \cdots p_k^{b_k}\right)\left(p_1^{b_1} \cdot p_2^{b_2} \cdots p_k^{b_k}\right) = \left(p_1^{b_1} \cdot p_2^{b_2} \cdots p_k^{b_k}\right)^2.$$

That is, $n$ is a perfect square.

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## 3.4 Expressing $n$ as the product of two distinct integers

**Theorem 4 (Counting the number of possibilities)**

Let $l(n)$ equal the number of ways to express the positive integer $n$ as the product of two distinct positive integers (including $n \times 1$). Assume we do <u>not</u> want to treat $a \times b = n$ and $b \times a = n$ as different products.

Then there are two cases to consider in finding a formula for $l(n)$, the number of ways to express $n$ as the product of two distinct positive integers.

**Formula for $l(n)$.**

Case 1. $n$ is a perfect square (equivalently, if $\tau(n)$ is odd). In this case,

$$l(n) = \frac{\tau(n) + 1}{2}.$$

Case 2. $n$ is not a perfect square. In this case,

$$l(n) = \frac{\tau(n)}{2}.$$

In both cases we use $\tau(n)$ to represent the number of positive divisors of $n$.

∎

The next two examples will help to clarify where this formula for $l(n)$ comes from.

**Example 18.**

How many ways are there to write $10800$ as the product of two distinct positive integers (including $10800 \times 1$)? Assume we do not want to treat $a \times b = 10800$ and $b \times a = 10800$ as different products.

**Solution**

The prime factorization of $10800$ is $10800 = 2^4 \cdot 3^3 \cdot 5^2$. Therefore $\tau(10800)$, the number of divisors (factors) of $10800$ equals $(4+1)(3+1)(2+1) = 60$.

The set $\mathcal{D}$ of all $60$ divisors would look something like this:

$$\mathcal{D} = \{1,2,3,4,5,6, \dots ,2700,3600,5400,10800\}.$$

Clearly if $a$ and $b$ are positive integers such that $a \times b = 10800$ then $a$ and $b$ are both elements of $\mathcal{D}$.

Furthermore, each divisor in this list of $60$ numbers can be paired with a *different* divisor in this list to form a product of two positive integers equaling $10800$.

For example, the divisor $12$ can be paired with the divisor $10800/12 = 900$. This gives us $\tau(10600) = 60$ pairs of divisors whose product equals $10600$.

But this double counts each possible pair! That is, this method would count both $12 \times 900$ as well as $900 \times 12$ – which we do not want to do.

So, there are only $\tau(10600)/2 = 60/2 = 30$ ways to write $10800$ as the product of two positive integers.

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**Example 19.**

How many ways are there to write $36$ as the product of two distinct positive integers (including $36 \times 1$)? Again, assume we do not want to treat $a \times b = 10800$ and $b \times a = 10800$ as different products.

**Solution**

The prime factorization of $36$ is $36 = 2^2 \cdot 3^2$. Therefore $\tau(36)$, the number of divisors (factors) of $36$ equals $(2+1)(2+1) = 9$.

The complete set $\mathcal{D}$ of all 9 divisors would be:

$$\mathcal{D} = \{1,2,3,4,6,9,12,18,36\}.$$

Looking back at Example 18 be saw that each divisor in the list of 60 could be paired with a *different* divisor to form a product of $10800$ to form $60/2 = 30$ distinct pairs.

Is that true in this example? No. We can $(1,36), (2,18), (3,12), (4,9)$ but that leaves us trying to pair 6 with itself – which we don't want to do.

So, in this example where are $(\tau(36) - 1)/2 = (9 - 1)/2 = 4$ ways to express 36 as the product of two distinct positive integers.

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What was the critical difference between Example 18 and Example 19?

Whenever $\tau(n)$ is even, such as the case $\tau(10600) = 60$, then every number in the list of divisors can be paired with a distinct divisor from that list to form a product equaling $n$.

But when $\tau(n)$ is odd, such as the case $\tau(36) = 9$, then the median number in the list of divisors <u>cannot</u> be paired with a distinct divisor from that list to form a product equaling $n$.

So, the critical point in finding a formula for $l(n)$, the number of ways to express $n$ as the product of two distinct positive integers, was whether $\tau(n)$ is even or odd. But from Theorem 3 we know that $\tau(n)$ is odd if and only if $n$ is a perfect square. This explains how we came to Theorem 4.

**Example 20.**

How many ways can $n = 30$ be written as a product of two positive integers, including 30 and 1?

**Solution**

We see that $30 = 2^1 \cdot 3^1 \cdot 5^1 = 2 \cdot 3 \cdot 5$ and hence

$$\tau(30) = (1+1)(1+1)(1+1) = 8.$$

Furthermore 30 is not a perfect square, so

$$l(30) = \frac{\tau(30)}{2} = \frac{8}{2} = 4.$$

So there are exactly $4$ ways to write $30$ as the product of two positive integers. They are

$$30 = 1 \times 30,\ \ 30 = 2 \times 15,\ \ 30 = 3 \times 10,\ \text{and}\ \ 30 = 5 \times 6.$$

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## 3.5 Prime Factorization of $n!$

### 3.5.1 Power of $p$ in the prime factorization of $n!$

**Example 21.**

How many times does the prime $3$ occur as a factor in the prime factorization of $100!$ ?

**Solution**

Consider what $100!$ looks like.

$$100! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot\ \cdots\ \cdot 97 \cdot 98 \cdot 99 \cdot 100.$$

We can see that $3$ occurs as a factor in every number in the set $\mathcal{A} = \{3, 6, 9, 12, \dots, 99\}$. We note that $99 = 3(33)$ and this tells us that there are $33$ elements in set $\mathcal{A}$.

But $33$ is an undercount of the total number of times $3$ occurs as a factor in the prime factorization of $100!$ because $3$ occurs more than once in every number in the set $\mathcal{B} = \{9, 18, 27, \dots, 99\}$. We note that $11$ elements in the set $\mathcal{B}$.

Is $33 + 11 = 44$ the total number of times $3$ occurs as a factor in the prime factorization of $100!$ ? No. This is still an undercount because $3$ occurs more than twice in every number in the set $\mathcal{C} = \{27, 54, 81\}$.

Is $33 + 11 + 3$ the correct count? No, this is still an undercount because $3$ occurs more than three times in every number in the set $\mathcal{D} = \{81\}$. Are there any positive integers less than or equal to $100$ where $3$ occurs as a factor more than four times? No, because $3^5 > 100$.

So, the correct count is

$$N(\mathcal{A}) + N(\mathcal{B}) + N(\mathcal{C}) + N(\mathcal{D}) = 33 + 11 + 3 + 1 = 48.$$

The prime $p = 3$ occurs a total of $48$ times in the prime factorization of $100!$. Another way of saying this is that if we write out the prime factorization of $100!$,

$$100! = 2^{a_1} \cdot 3^{a_2} \cdot 5^{a_3} \cdot 7^{a_4} \cdot 11^{a_5} \cdot \ \cdots \ \cdot 89^{a_{24}} \cdot 97^{a_{25}}$$

then $a_2 = 48$.

■

The generalization of Example 21 to any prime $p$ is known as Legendre's Theorem.

**Theorem 5 (Legendre's Theorem)**

Let $n$ be a positive integer. Then the power of the prime $p \leq n$ occurring in the prime-power factorization of $n!$ is

$$\left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \cdots$$

where $\lfloor \ \rfloor$ is the notation for the *floor function* where $\lfloor x \rfloor$ is defined as the greatest integer less than or equal to $x$.

■

Note that in Example 21,

$$\left\lfloor \frac{n}{p} \right\rfloor = \left\lfloor \frac{100}{3} \right\rfloor = \lfloor 33.\overline{3} \rfloor = 33$$

$$\left\lfloor \frac{n}{p^2} \right\rfloor = \left\lfloor \frac{100}{9} \right\rfloor = \lfloor 11.\overline{1} \rfloor = 11$$

$$\left\lfloor \frac{n}{p^3} \right\rfloor = \left\lfloor \frac{100}{27} \right\rfloor = \lfloor 3.\overline{703} \rfloor = 3$$

and

$$\left\lfloor \frac{n}{p^4} \right\rfloor = \left\lfloor \frac{100}{81} \right\rfloor = \lfloor 1.23456790\overline{12345679} \rfloor = 1.$$

### 3.5.2 Number of zeros at the end of $n!$

**Example 22**

How many zeros are there at the end of $1000!$ ?

**Solution**

Suppose we write the prime factorization of $1000!$ as

$$1000! = 2^{a_2} \cdot 3^{a_3} \cdot 5^{a_5} \cdot 7^{a_7} \cdot 11^{a_{11}} \cdot \ \cdots \ \cdot 997^{a_{997}}.$$

Loosely put, $1000!$ picks up another zero on the end everytime we multiply a $2$ and a $5$.

We know from Theorem 5 that

$$a_2 = \left\lfloor \frac{1000}{2} \right\rfloor + \left\lfloor \frac{1000}{4} \right\rfloor + \left\lfloor \frac{1000}{8} \right\rfloor + \left\lfloor \frac{1000}{16} \right\rfloor + \left\lfloor \frac{1000}{32} \right\rfloor + \left\lfloor \frac{1000}{64} \right\rfloor + \cdots + \left\lfloor \frac{1000}{512} \right\rfloor = 994$$

and

$$a_5 = \left\lfloor \frac{1000}{5} \right\rfloor + \left\lfloor \frac{1000}{25} \right\rfloor + \left\lfloor \frac{1000}{125} \right\rfloor + \left\lfloor \frac{1000}{625} \right\rfloor = 249.$$

So exactly $249$ of the available $994$ "2's" in the prime factorization of $1000!$ can be paired up with a $5$ to create an extra $0$ on the end of $1000!$.

There are $\left\lfloor \frac{1000}{5} \right\rfloor + \left\lfloor \frac{1000}{25} \right\rfloor + \left\lfloor \frac{1000}{125} \right\rfloor + \left\lfloor \frac{1000}{625} \right\rfloor = 249$ zeros at the end of $1000!$

■

**Theorem 6**

There are

$$\left\lfloor \frac{n}{5} \right\rfloor + \left\lfloor \frac{n}{5^2} \right\rfloor + \left\lfloor \frac{n}{5^3} \right\rfloor + \cdots + \left\lfloor \frac{n}{5^k} \right\rfloor$$

zeros are there at the end of $n!$ where $k$ is that integer such that $5^k \leq n < 5^{k+1}$.

■

Why? It is necessarily true that the power of $2$ is greater than or equal to the power of $5$ in the prime factorization of $n!$. So, the number of $5$'s is the limiting factor in how many $(2 \cdot 5)$ pairs that can be formed by the factors in $n!$.

# 4 Least common multiple, greatest common divisor

## 4.1 LCM

The least common multiple (lcm) of two or more integers, which are not all zero, is the smallest positive integer that is divisible by each of the integers. Note: The least common multiple is also known as the least common denominator (lcd).

## 4.2 GCD

The greatest common divisor (gcd) of two or more integers, which are not all zero, is the largest positive integer that divides each of the integers. Note: The greatest common divisor is also known as the greatest common factor (gcf).

**Example 23.**

Find $\gcd(54, 24)$, the greatest common divisor of the integers $54$ and $24$.

**Solution**

The prime factorization of $54$ is $54 = 2^1 \cdot 3^3$ and the prime factorization of $24$ is $24 = 2^3 \cdot 3^1$.

The divisors of $54 = 2^1 \cdot 3^3$ must have the form $2^a \cdot 3^b$ where $a \in \{0,1\}$ and $b \in \{0,1,2,3\}$. So there are $2 \times 4 = 8$ divisors of $54$. They are $\{1, 2, 3,6,9, 18, 27, 54\}$.

The divisors of $24 = 2^3 \cdot 3^1$ must have the form $2^c \cdot 3^d$ where $c \in \{0,1,2,3\}$ and $b \in \{0,1\}$. So there are $4 \times 2 = 8$ divisors of $24$. They are $\{1, 2, 3, 4,6,8, 12, 24\}$.

The numbers shared by these two lists are the common divisors of $54$ and $24$. There are $\{1,2,3,6\}$.

The greatest of the common divisors of $54$ and $24$ is $6$. That is, $\gcd(54,24) = 6$.

■

**Example 24.**

Find $\mathrm{lcm}(54, 24)$, the least common multiple of the integers $54$ and $24$.

**Solution**

The prime factorization of $54$ is $54 = 2 \cdot 3^3$ and the prime factorization of $24$ is $24 = 2^3 \cdot 3$.

The positive integer multiples of $54$ are $\{54, 108, 162, \mathbf{216}, 270, 324, \dots\}$

The positive integer multiples of $24$ are $\{24, 48, 72, 96, 120, 144, 168, 192, \mathbf{216}, 240, \dots\}$.

The numbers shared by these two lists are the common multiples of $54$ and $24$.

The smallest of the common multiples of $54$ and $24$ is $216$.

$\blacksquare$

## 4.3 GCD, LCM and Prime Factorizations

**Theorem 7**

For positive integers $a$ and $b$, suppose

$$a = (p_1)^{a_1}(p_2)^{a_2}\cdots(p_n)^{a_n} \quad \text{and} \quad b = (p_1)^{b_1}(p_2)^{b_2}\cdots(p_n)^{b_n}$$

where $p_1, p_2, \dots, p_n$ are distinct prime numbers and $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n$ are nonnegative (possibly zero). Then

$$\gcd(a, b) = (p_1)^{\min(a_1,b_1)}(p_2)^{\min(a_2,b_2)}\cdots(p_n)^{\min(a_n,b_n)}$$

and

$$\operatorname{lcm}(a, b) = (p_1)^{\max(a_1,b_1)}(p_2)^{\max(a_2,b_2)}\cdots(p_n)^{\max(a_n,b_n)}.$$

For example, consider the prime factorizations of $54 = 2^1 \cdot 3^3$ and $24 = 2^3 \cdot 3^1$. Applying the above theorem,

$$\gcd(54,24) = 2^{\min(1,3)} \cdot 3^{\min(3,1)} = 2^1 \cdot 3^1 = 6$$

and

$$\operatorname{lcm}(54,24) = 2^{\max(1,3)} \cdot 3^{\max(3,1)} = 2^3 \cdot 3^3 = 8 \cdot 27 = 216.$$

$\blacksquare$

## 4.4 Extended GCD, LCM and Prime Factorizations

**Theorem 8**

We can extend Theorem 7 to any number of components in a natural way. If positive integers $a, b, c$ and $d$ have prime factorizations

$$a = 2^{r_1} \times 3^{r_2} \times 5^{r_3} \times 7^{r_4} \times 11^{r_5} \times \cdots$$

$$b = 2^{s_1} \times 3^{s_2} \times 5^{s_3} \times 7^{s_4} \times 11^{s_5} \times \cdots$$

$$c = 2^{t_1} \times 3^{t_2} \times 5^{t_3} \times 7^{t_4} \times 11^{t_5} \times \cdots$$

$$d = 2^{w_1} \times 3^{w_2} \times 5^{w_3} \times 7^{w_4} \times 11^{w_5} \times \cdots$$

then

$$\gcd(a, b, c, d) = 2^{\min\{r_1, s_1, t_1, w_1\}} \times 3^{\min\{r_2, s_2, t_2, w_2\}} \times 5^{\min\{r_3, s_3, t_3, w_3\}} \times 7^{\min\{r_4, s_4, t_4, w_4\}} \times \cdots$$

and

$$\operatorname{lcm}(a, b, c, d) = 2^{\max\{r_1, s_1, t_1, w_1\}} \times 3^{\max\{r_2, s_2, t_2, w_2\}} \times 5^{\max\{r_3, s_3, t_3, w_3\}} \times 7^{\max\{r_4, s_4, t_4, w_4\}} \times \cdots$$

■

**Example 25.**

Find $\operatorname{lcm}(2,4,5,6,12)$.

**Solution**

First, we find the prime factorization of 2, 4, 5, 6 and 12.

$$\begin{aligned} 2 &= 2^1 \\ 4 &= 2^2 \\ 5 &= 5^1 \\ 6 &= 2^1 \times 3^1 \\ 12 &= 2^2 \times 3^1 \end{aligned}$$

So,

$$\begin{aligned} &\operatorname{lcm}(2,4,5,6,12) \\ &\quad = \operatorname{lcm}(2^1, 2^2, 5^1, 2^1 \times 3^1, 2^2 \times 3^1) \\ &\quad = 2^{\max\{1,2,0,1,2\}} \times 3^{\max\{0,0,0,1,1\}} \times 5^{\max\{0,0,1,0,0\}} \end{aligned}$$

$$= 2^2 \times 3^1 \times 5^1 = 60.$$

■

Theorems 7 and 8 give a straightforward and *apparently* simple way of finding gcd's and lcm's for any number of components. The reason for the caveat "apparently" simple is the approach depends on having the prime factorization of each component number.

This is fine for "smaller" numbers such as those used in the above examples but is a roadblock for "larger" numbers. Attempting to find $\gcd(457, 213, 1447)$ using Theorem 8 is not a time efficient approach in a contest setting because of the difficulty in finding the prime factorization of each of these three component numbers.

Theorems 9, 10 and 11 which follow below, with special attention to Theorem 9 (The Euclidean Algorithm) introduce a route for finding gcd's and lcm's with any number of components *without* having to find the prime factorization of each component number.

## 4.5 The Division Algorithm (Division with remainder)

An important part of understanding the Euclidean Algorithm is the ***division algorithm*** which we explain below.

Consider the process of dividing $54$ by $24$. Obviously $24$ goes into $54$ twice with a remainder of $6$. We would write this as

$$\frac{54}{24} = 2 + \frac{6}{24} \quad \text{or} \quad 54 = 2(24) + 6.$$

| |
|---|
| The division algorithm is the result that for all positive integers $a$ and $b$ with $a \geq b$ there always exists a unique integer $m$ such that $a = mb + r$ with $0 \leq r < b$. |

Examples Illustrating the Division Algorithm

| $\boldsymbol{a, b, a \geq b}$ | $\boldsymbol{a = mb + r}$ | $\boldsymbol{0 \leq r < b}$ |
|---|---|---|
| $a = 17,\ b = 3$ | $17 = 5(3) + 2$ | $0 \leq 2 < 3$ |
| $a = 31,\ b = 12$ | $31 = 2(12) + 6$ | $0 \leq 6 < 12$ |
| $a = 311,\ b = 42$ | $311 = 7(42) + 17$ | $0 \leq 17 < 42$ |

| | | |
|---|---|---|
| $a = 2208, \ b = 113$ | $2208 = 19(113) + 61$ | $0 \leq 61 < 113$ |

The Division Algorithm states that this middle column is the *only* way to write $a$ in the form $a = mb + r$ with $0 \leq r < b$.

## 4.6 The Euclidean Algorithm

**Theorem 9 (Euclidean Algorithm for gcd$(a, b)$)**

Suppose $a$ and $b$ are positive integers with $a \geq b$. Suppose we divide $b$ into $a$ with remainder and get

$$a = mb + r \text{ with } 0 \leq r < b.$$

Then

$$\gcd(a, b) = \begin{cases} \gcd(b, r) & r > 0 \\ b & r = 0. \end{cases}$$

If $r \neq 0$ then we can repeat the Euclidean algorithm for the positive integers $b > r$.

■

**Example 26.**

When we divide $54$ by $24$ with remainder we get $54 = 2(24) + 6$. Therefore by the Euclidean Algorithm,

$$\gcd(54,24) = \gcd(24,6).$$

But it is easy to see that $\gcd(24,6) = 6$. So, $\gcd(54,24) = 6$.

■

**Example 27.**

Apply the Euclidean Algorithm (repeatedly) to find $\gcd(1800,168)$.

**Solution**

$$1800 = 10(168) + \quad 120$$

$$168 = 1(120) + \quad 48$$

$$120 = 2(48) + \quad 24$$

$$48 = 2(24) + 0$$

$$\gcd(1800{,}168) = \gcd(168{,}120) = \gcd(120{,}48) = \gcd(48{,}24) = 2.$$

■

## 4.7 lcm($a$,$b$) as a function of gcd($a$,$b$)

**Theorem 10**

$$\text{lcm}(a, b) = \frac{a \cdot b}{\gcd(a, b)}$$

and

$$\gcd(a, b) \cdot \text{lcm}(a, b) = a \cdot b.$$

■

**Caution**: It is tempting to assume that we can directly extend Theorem 10 to $\text{lcm}(a, b, c, \dots, z)$. Unfortunately, when we have more than two variables,

$$\text{lcm}(a, b, c, \dots, z) \neq \frac{a \cdot b \cdot c \cdot \ \cdots \cdot z}{\gcd(a, b, c, \dots, z)}.$$

**Example 28.**

Find lcm$(54{,}24)$.

**Solution**

Suppose we already have found that $\gcd(54{,}24) = 6$. Then

$$\text{lcm}(54,24) = \frac{54 \cdot 24}{6} = \frac{1296}{6} = 216.$$

■

## 4.8 Finding gcd($a$,$b$,$c$, … ) and lcm($a$,$b$,$c$,… ) without prime factorizations

We have already mentioned that attempting to find gcd$(457, 213, 1447)$ using Theorem 8 is not a time efficient approach in a contest setting because of the difficulty in finding the prime factorization of each of these three component numbers. But our go to alternative, the Euclidean Algorithm, is not set up to calculate the gcd except for the case of two variables, *i.e.* gcd $(a, b)$. To add to the problem, we have already remarked in the above "Caution" that even if we had $\gcd(a, b, c, \dots)$ we cannot use that to find $\text{lcm}(a, b, c, \dots)$.

Theorem 11 which follows below gives us a way to circumvent these problems.

**Theorem 11**

$$\gcd(a, b, c) = \gcd(\gcd(a, b), c)$$

and

$$\text{lcm}(a, b, c) = \text{lcm}(\text{lcm}(a, b), c).$$

That is, for finding the gcd or lcm for a list of more than two numbers you can break the problem down into finding the gcd or lcm a pair of numbers at a time.

■

**Example 29.**

Find gcd$(457, 213, 1447)$.

**Solution**

Applying Theorem 11 we have

$$\gcd(457, 213, 1447) = \gcd\big(\gcd(457,213), 1447\big).$$

Now we can use the Euclidean Algorithm to find $\gcd(457,213)$.

$$
\begin{aligned}
457 &= 2(213) + 31 \\
213 &= 7(31) + 14 \\
31 &= 2(14) + 3 \\
14 &= 4(3) + 2 \\
3 &= 1(2) + 1 \\
2 &= 2(1) + 0.
\end{aligned}
$$

Therefore, $\gcd(457{,}213) = 1$ by the Euclidean Algorithm. Hence,

$$\gcd(457, 213, 1447) = \gcd\big(\mathbf{gcd(457, 213)}, 1447\big) = \gcd(\mathbf{1}, 1447).$$

We can see by inspection that $\gcd(1{,}1447) = 1$. Hence, $\gcd(457, 213, 1447) = 1$.

■

**Example 30.**

Find $\operatorname{lcm}(457{,}213{,}1447)$.

**Solution**

Applying Theorem 11 we have

$$\operatorname{lcm}(457{,}213{,}1447) = \operatorname{lcm}\big(\operatorname{lcm}(457{,}213), 1447\big)$$

We can apply Theorem 10 to find $\operatorname{lcm}(457{,}213)$ because it only involves two variables. By Theorem 10 we have

$$\operatorname{lcm}(457{,}213) = \frac{457 \cdot 213}{\gcd(457{,}213)}.$$

We can use the Euclidean Algorithm to find $\gcd(457{,}213)$. We already found in Example 29 that $\gcd(457{,}213) = 1$. Therefore,

$$\operatorname{lcm}(457{,}213) = \frac{457 \cdot 213}{\gcd(457{,}213)} = (457)(213) = 97341.$$

Hence,

$$\text{lcm}(457, 213, 1447) = \text{lcm}\big(97341{,}1447\big) = \frac{(97341)(1447)}{\gcd(97341{,}1447)}.$$

Now we use the Euclidean Algorithm to find $\gcd(97341{,}1447)$.

$$\begin{aligned}
97341 &= 67(1447) + 392\\
1447 &= 3(392) + 271\\
392 &= 1(271) + 121\\
271 &= 2(121) + 29\\
121 &= 4(29) + 5\\
29 &= 5(5) + 4\\
5 &= 1(4) + 1\\
4 &= 4(1) + 0
\end{aligned}$$

Thus, by the Euclidean Algorithm, $\gcd(97341{,}1447) = 1$. Hence,

$$\begin{aligned}
\text{lcm}(457, 213, 1447) &= \text{lcm}\big(97341{,}1447\big)\\
&= \frac{(97341)(1447)}{\gcd(97341{,}1447)}\\
&= (97341)(1447)\\
&= (457)(213)(1447).
\end{aligned}$$

■

## 4.9 $ax + by = \gcd(a,b)$

### 4.9.1 Bezout's Lemma

**Theorem 12**

*Bezout's Lemma* states that for all integers $a$ and $b$ there exist integers $x$ and $y$ such that

$$ax + by = \gcd(a, b).$$

■

**Examples Illustrating Bezout's Lemma**

| $(a, b)$ | $\gcd(a, b)$ | $(x_0, y_0)$ | Linear Combination $ax_0 + by_0 = \gcd(a, b)$ |
|---|---|---|---|
| $(12,9)$ | $\gcd(12,9) = 3$ | $(1, -1)$ | $12(1) + 9(-1) = 3$ |
| $(35,14)$ | $\gcd(35,14) = 7$ | $(1, -2)$ | $35(1) + 14(-2) = 7$ |
| $(36,20)$ | $\gcd(36,20) = 4$ | $(-1,2)$ | $36(-1) + 20(2) = 4$ |
| $(13,5)$ | $\gcd(13,5) = 1$ | $(2, -5)$ | $13(2) + 5(-5) = 1$ |
| $(45,14)$ | $\gcd(45,14) = 1$ | $(5, -16)$ | $45(5) + 14(-16) = 1$ |

There are two additional results associated with Bezout's Lemma.

### 4.9.2 The possible values of $ax + by$

**Theorem 13**

For any given integers $a, b$ and $c$ the equation $ax + by = c$ has an integer solution $(x, y)$ if and only if $c$ is a multiple of $\gcd(a, b)$. That is, $c = k \cdot \gcd(a, b)$ for some integer $k$.

■

### 4.9.3 Finding a solution of $ax + by = \gcd(a,b)$

**Reversing the Euclidean Algorithm**

Bezout's Lemma (Theorem 12) only guarantees the existence of some $(x_0, y_0)$ such that $ax_0 + by_0 = \gcd(a, b)$. It does not tell us how to find $(x_0, y_0)$. Fortunately, there are several techniques known for constructing a solution $(x_0, y_0)$. In Example 32 we will illustrate the technique of working backwards from the steps taken in the Euclidean Algorithm for finding $\gcd(a, b)$.

**Example 31.**

Find $\gcd(500{,}222)$, the greatest common divisor of $500$ and $222$, using the Euclidean Algorithm.

**Solution**

Express $a$ as $a = cb + r$ with $0 \leq r < b$. Then

$$\gcd(a,b) = \begin{cases} \gcd(b,r) & r > 0 \\ \quad b & r = 0. \end{cases}$$

The Euclidean Algorithm is the process of repeating this until a remainder of $r = 0$ is reached.

$$500 = 2(222) + 56 \Longrightarrow \gcd(500{,}222) = \gcd(222{,}56)$$

$$222 = 3(56) + 54 \Longrightarrow \gcd(222{,}56) = \gcd(56{,}54)$$

$$56 = 1(54) + 2 \Longrightarrow \gcd(56{,}54) = \gcd(54{,}2)$$

$$54 = 27(2) + 0 \Longrightarrow \gcd(54{,}2) = 2.$$

So, we have the following string of equalities.

$$\gcd(500{,}222) = \gcd(222{,}56) = \gcd(56{,}54) = \gcd(54{,}2) = 2.$$

■

**Example 32.**

Find integers $x$ and $y$ such that $500x + 222y = \gcd(500{,}222)$ by reversing the Euclidean Algorithm.

**Solution**

We showed in Example 31 that $\gcd(500{,}222) = 2$ through the steps

| | | | |
|---|---|---|---|
| Step 1 | $500$ | $=$ | $2(222) + 56$ |
| Step 2 | $222$ | $=$ | $3(56) + 54$ |
| Step 3 | $56$ | $=$ | $1(54) + 2$ |
| Step 4 | $54$ | $=$ | $27(2) + 0$ |

| | |
|---|---|
| $2 = 56 - 1(54)$ | Solve for 2, the GCD, in Step 3. |
| $2 = 56 - 1\big(222 - 3(56)\big)$ | Solve for 54 in Step 2 and substitute. |
| $2 = \big(500 - 2(222)\big) - 1\Big(222 - 3\big(500 - 2(222)\big)\Big)$ | Solve for 56 in Step 1 and substitute. |
| $2 = 500(1+3) + 222(-2-1-7)$ <br> $= 500(4) + 222(-9)$ | Simplify. |

So,

$$500(4) + 222(-9) = 2 = \gcd(500,222).$$

■

**Example 33. (MSHSML 2006-2007, Test 1A, Problem 4)**

If $d$ is the greatest common divisor of 399 and $959$, then it is possible to find integers $r$ and $s$ so that $d = 399r + 959s$. Find $d, r$, and $s$.

**Solution**

By the Euclidean Algorithm

$$\begin{aligned} 959 &= 2(399) + 161 \\ 399 &= 2(161) + 77 \\ 161 &= 2(77) + 7 \\ 77 &= 11(7) + 0 \end{aligned}$$

Hence $\gcd(959,399) = 7$. By working the Euclidean Algorithm backwards we find

$$\begin{aligned} 7 &= 161 - 2(77) \\ &= 161 - 2\big(399 - 2(161)\big) \\ &= (5)(161) - 2(399) \\ &= (5)\big(959 - 2(399)\big) - 2(399) \\ &= 5(959) - 12(399) \end{aligned}$$

So $399(-12) + 959(5) = 7$ is a solution to $399r + 959s = \gcd(399,959)$. That is, the solution to this problem is $d = 7, r = -12, s = 5$.

■

### 4.9.4 Finding a solution of $ax + by = k \cdot \gcd(a,b)$

In the previous section we learned how to reverse the steps of the Euclidean Algorithm when finding $\gcd(a, b)$.

Suppose $x = x_0$, $y = y_0$ is a solution to $ax + by = \gcd(a, b)$. Then by multiplying both sides of this equation by the constant $k$ we get

$$a(kx_0) + b(ky_0) = k \cdot \gcd(a, b).$$

That is, if $(x, y) = (x_0, y_0)$ is a solution to $ax + by = \gcd(a, b)$, then $(x, y) = (kx_0, ky_0)$ is a solution to $ax + by = k \cdot \gcd(a, b)$.

### 4.9.5 All solutions of $ax + by = k \cdot \gcd(a,b)$

**Theorem 14**

If $(x, y) = (x_0, y_0)$ is any particular solution to $ax + by = k \cdot \gcd(a, b)$ then

$$\boxed{(x, y) = \left( x_0 + \left(\frac{b}{\gcd(a, b)}\right) n,\ \ y_0 - \left(\frac{a}{\gcd(a, b)}\right) n \right), \quad n = 0, \pm 1, \pm 2, \ldots}$$

gives the set of all possible solutions to $ax + by = k \cdot \gcd(a, b)$.

■

Pay close attention to the form of the answer in Theorem 14 above. There are two things about this formula that you need to be careful to notice. First is that the location of $a$ and $b$ in

$$\left( x_0 + \left(\frac{\boldsymbol{b}}{\gcd(a, b)}\right) n,\ \ y_0 - \left(\frac{\boldsymbol{a}}{\gcd(a, b)}\right) n \right)$$

might seem backwards, but this is correct. Secondly, notice that we ***add*** the extra term to $x_0$ but we ***subtract*** the extra term from $y_0$.

$$\left(x_0 + \left(\frac{b}{\gcd(a,b)}\right)n,\ \ y_0 - \left(\frac{a}{\gcd(a,b)}\right)n\right).$$

**Example 34.**

Find an expression for $x$ and $y$ that shows all possible integer solutions $x$ and $y$ such that $500x + 222y = \gcd(500{,}222)$.

**Solution**

The result we need is that if $x_0$ and $y_0$ is any particular solution to $500x + 222y = \gcd(500{,}222)$ then

$$x = x_0 + \left(\frac{222}{\gcd(500{,}222)}\right)n$$

and

$$y = y_0 - \left(\frac{500}{\gcd(500{,}222)}\right)n$$

with $n = 0, \pm 1, \pm 2, \pm 3, \ldots$ will be the set of all possible solutions to $500x + 222y = \gcd(500{,}222)$.

We found in Example 31 that $\gcd(500{,}222) = 2$ and in Example 32 that $x = 4$ and $y = -9$ is a solution to $500x + 222y = \gcd(500{,}222) = 2$.

Therefore,

$$x = 4 + \left(\frac{222}{2}\right)n = 4 + 111n$$

and

$$y = -9 - \left(\frac{500}{2}\right)n = -9 - 250n$$

with $n = 0, \pm 1, \pm 2, \ldots$ is the set of all possible solutions to $500x + 222y = \gcd(500{,}222)$.

■

**Exercise 35. (MSHSML 2001-2002, Test 1A, Problem 4)**

(a) Find an integer solution to $13x + 29y = 48$.

(b) Find an expression for all solutions to $13x + 29y = 48$.

(c) Find the three lattice points (points with integer coordinates) closest to the origin that satisfy $13x + 29y = 48$.

**Solution**

**(a)**

$$\begin{aligned} 29 &= 2(13) + 3 \\ 13 &= 4(3) + 1 \\ 3 &= 3(1) + 0 \end{aligned}$$

Therefore, $\gcd(29,13) = 1$. We can reverse the above steps to find a solution to $13x + 29y = \gcd(29,13) = 1$.

$$\begin{aligned} 1 &= 13 - 4(3) \\ &= 13 - 4\big(29 - 2(13)\big) \\ &= 13(9) - 29(4). \end{aligned}$$

That is, $x = 9$ and $y = -4$ is an integer solution to $13x + 29y = \gcd(29,13) = 1$. Therefore,

$$13(9 \cdot 48) - 29(4 \cdot 48) = 1 \cdot 48.$$

That is, $x = 9(48) = 432$ and $y = -4(48) = -192$ is an integer solution to $13x + 29y = 48$.

**(b)** By Theorem 14, if $(x_0, y_0)$ is a solution to $13x + 29y = 48$ then

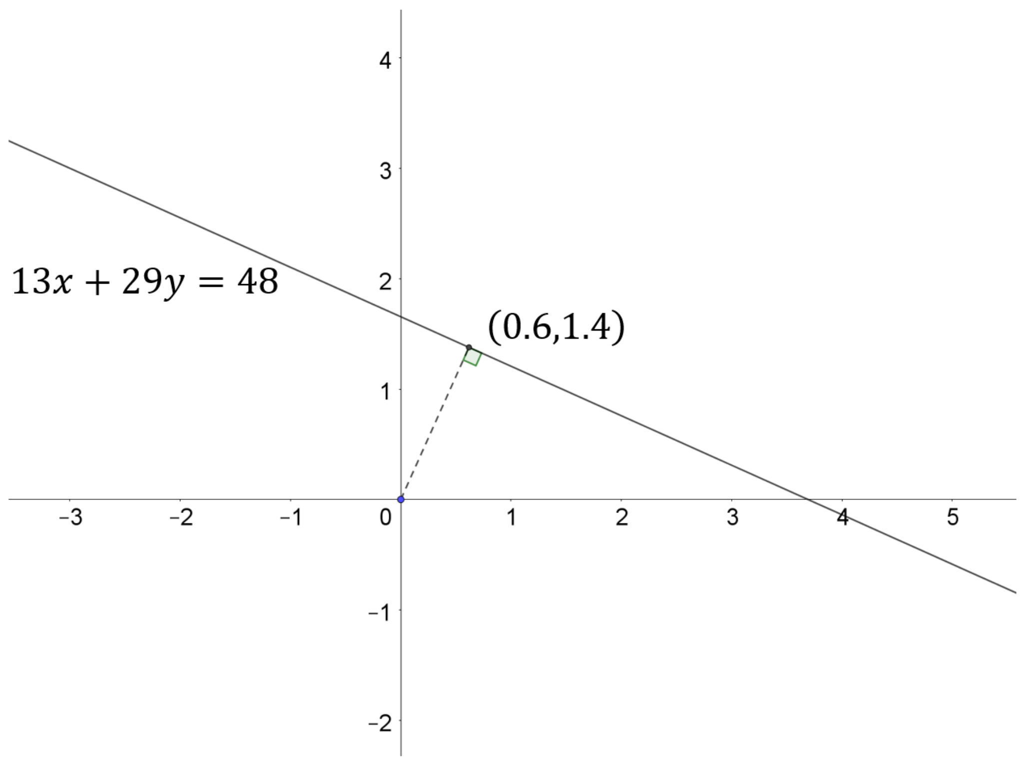
$$(x,y) = \left( x_0 + \left(\frac{b}{\gcd(a,b)}\right)n,\ \ y_0 - \left(\frac{a}{\gcd(a,b)}\right)n \right), \quad n = 0, \pm 1, \pm 2, \ldots$$

is the set of all possible solutions to $13x + 29y = 48$. Therefore,

$$(x,y) = \left(432 + \left(\frac{29}{1}\right)n, -192 - \left(\frac{13}{1}\right)n\right) = (432 + 29n, -192 - 13n)$$

with $n = 0, \pm 1, \pm 2, \ldots$ gives us the set of all possible solutions to $13x + 29y = 48$.

**(c)** If we sketch $13x + 29y = 48$ we can see that the point on this line that is closest to the origin is (approximately) $(0.6, 1.4)$. But what part (c) is asking for are lattice points (both coordinates are integers) that are close to the origin.



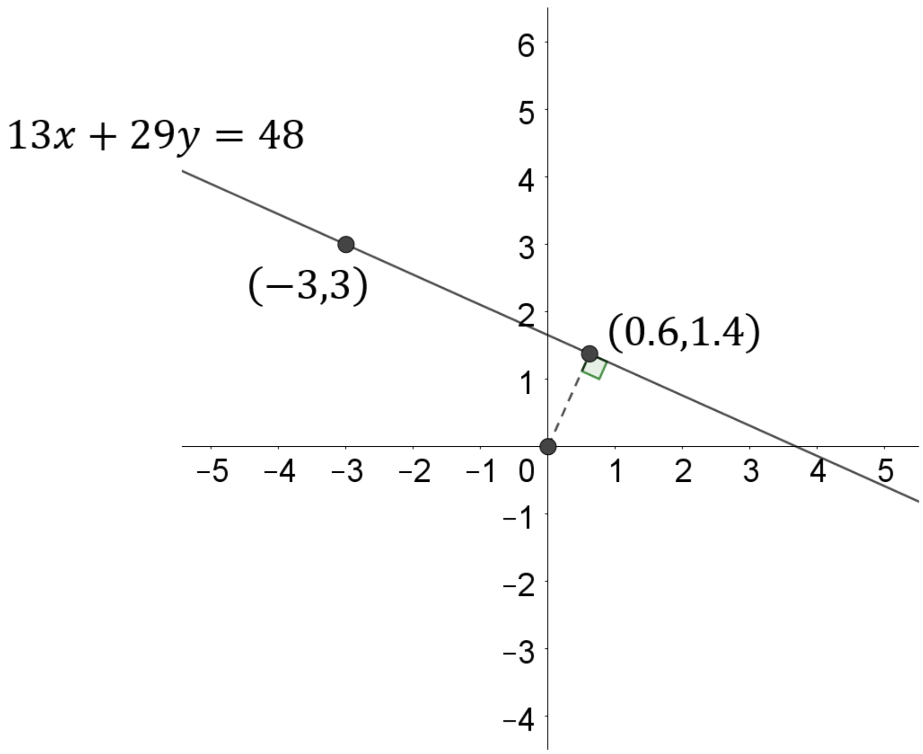

We have shown that all lattice points on this line have the form $(432 + 29n, -192 - 13n)$ for some integer $n$. So, we want to $(432 + 29n, -192 - 13n) \approx (0.6,1.4)$.

$$432 + 29n = 0.6 \Rightarrow n = \frac{0.6 - 432}{29} = -14.9$$

and

$$-192 - 13n = 1.4 \Rightarrow n = \frac{1.4 + 192}{-13} = -14.9.$$

But $n$ has to be an integer. The nearest integer to $-14.9$ is $-15$. Taking $n = -15$ in the general formula $(432 + 29n, -192 - 13n)$ for the lattice points on the line $13x + 29y = 48$ we get $\big(432 + 29(-15), -192 - 13(-15)\big) = (-3,3)$.




What other lattice points on the line $13x + 29y = 48$ are close to the origin? We need to consider other values of $n$ close to $-15$. Try $n = -13, -14, -16$ and $n = -17$.

$$\big(432 + 29(-13), -192 - 13(-13)\big) = (55, -23)$$
$$\big(432 + 29(-14), -192 - 13(-14)\big) = (26, -10)$$
$$\big(432 + 29(-15), -192 - 13(-15)\big) = (-3,3)$$
$$\big(432 + 29(-16), -192 - 13(-16)\big) = (-32,16)$$
$$\big(432 + 29(-17), -192 - 13(-17)\big) = (-61,29)$$

| **Lattice Point $(x, y)$** | **Distance to the origin** $= \sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + y^2}$ |
|---|---|
| $(55, -23)$ | 59.6 |

| | |
|---|---|
| $(26,-10)$ | $27.9$ |
| $(-3,3)$ | $4.2$ |
| $(-32,16)$ | $35.8$ |
| $(-61,29)$ | $67.6$ |

So the three lattice points on the line $13x + 29y = 48$ that are closest to the origin are $(-3,3)$, $(26,-10)$ and $(-32,16)$.

■

## 4.10 GCD of Two Polynomials

First, let's take some examples to clarify what we mean by the terms "common divisor" and "greatest" as applied to polynomials.

We say polynomial $h(x)$ is a ***common divisor*** of polynomials $f(x)$ and $g(x)$ if both $\frac{f(x)}{h(x)}$ and $\frac{g(x)}{h(x)}$ leave no remainder polynomial when you perform these polynomial divisions.

We say $h(x)$ is the ***greatest common divisor*** of polynomials $f(x)$ and $g(x)$ if $h(x)$ is a common divisor and the power of $h(x)$ (the highest exponent of $x$) is greater than the power of any other common divisor.

When finding the GCD of two integers it was very easy to spot all the common divisors if we wrote out the prime factorization of each integer. Can we express polynomials in a way to make it very easy to spot all the common divisors? Yes.

> Analogous to factoring an integer into the product of primes is the idea of factoring a polynomial into the product of linear and irreducible quadratic polynomials.

Recall that a quadratic polynomial is irreducible (or more precisely, irreducible over the real numbers) if its two roots are imaginary numbers.

A quick way to check to see if the quadratic $ax^2 + bx + c$ is irreducible is to check if its discriminant is negative. That is, if $b^2 - 4ac < 0$.

To say that a polynomial is "completely factored" means that it has been factored into the product of linear and irreducible quadratic polynomials.

Note: If you have a *reducible* quadratic factor then you have to factor that quadratic polynomial into the product of two linear polynomials before you can say the polynomial is "completely factored".

Now consider the two completely factored polynomials

$$f(x) = (x-2)(2x-3)(x^2+2)(x^2+3)$$

and

$$g(x) = (x-2)(2x-3)(x+5)(x^2+2).$$

Because these two polynomials have been completely factored we can easily see that $(x-2)$, $(2x-3)$ and $(x^2+2)$ are common factors of the polynomials $f(x)$ and $g(x)$ and that

$$\gcd\Big(f(x), g(x)\Big) = (x-2)(2x-3)(x^2+2) = 2x^4 - 7x^3 + 10x^2 - 14x + 12.$$

But what would you do if you were asked to find the greatest common factor of the same two polynomials $f(x)$ and $g(x)$ but given to you in expanded form

$$f(x) = 2x^6 - 7x^5 + 16x^4 - 35x^3 + 42x^2 - 42x + 36$$

and

$$g(x) = 2x^5 + 3x^4 - 25x^3 + 36x^2 - 58x + 60$$

instead of the completely factored form? Trying to start the process by first completely factoring $f(x)$ and $g(x)$ would be extremely time consuming! This is analogous to the problem we faced when we wanted to find the GCD of two large integers $a$ and $b$. Finding the prime factorization of large integers is also extremely time consuming.

Recall that this was our motivation for introducing the Euclidean Algorithm for integers. But is there an Euclidean Algorithm for polynomials? YES.

In fact, it is the same algorithm except that we use polynomial division instead of the division of integers.

To make my typing job easier let me demonstrate with $f(x) = x^3 - 9x^2 - x + 105$ and $g(x) = x^3 - 3x^2 - 25x - 21$.

**Example 36.**

Find the GCD of the polynomials $f(x) = x^3 - 9x^2 - x + 105$ and $g(x) = x^3 - 3x^2 - 25x - 21$.

**Solution**

Let's start by carrying out the polynomial division $f(x)/g(x)$.

$$\begin{array}{r|rrrr} & 1 & & & \\ \hline x^3 - 3x^2 - 25x - 21 & x^3 & -9x^2 & -x & +105 \\ & x^3 & -3x^2 & -25x & -21 \\ \hline & & -6x^2 & 24x & 126 \end{array}$$

or

$$x^3 - 9x^2 - x + 105 = 1(x^3 - 3x^2 - 25x - 21) + (-6x^2 + 24x + 126).$$

Recall that the key idea in the Euclidean Algorithm for finding the GCD of two integers $a \geq b > 0$ was that if we write $a$ in the form $a = mb + r$ for some integers $m$ and $b$ with $0 \leq r < b$, then

$$\gcd(a, b) = \begin{cases} \gcd(b, r) & r > 0 \\ b & r = 0. \end{cases}$$

This key idea is also true for polynomials. Therefore,

$$x^3 - 9x^2 - x + 105 = 1(x^3 - 3x^2 - 25x - 21) + (-6x^2 + 24x + 126)$$

implies that

$$\begin{aligned} &\gcd(x^3 - 9x^2 - x + 105, x^3 - 3x^2 - 25x - 21) \\ &\qquad = \gcd(x^3 - 3x^2 - 25x - 21, -6x^2 + 24x + 126). \end{aligned}$$

Now continue.

$$
\begin{array}{r|rrrr}
 & -(1/6)x & -1/6 & & \\
\hline
-6x^2+24x+126 & x^3 & -3x^2 & -25x & -21 \\
 & x^3 & -4x^2 & -21x & \\
\hline
 & & x^2 & -4x & -21 \\
 & & x^2 & -4x & -21 \\
\hline
 & & 0 & 0 & 0
\end{array}
$$

or

$$x^3 - 3x^2 - 25x - 21 = \left(-\frac{1}{6}x - \frac{1}{6}\right)(-6x^2 + 24x + 126) + 0.$$

Notice that we have a remainder of $0$. By the Euclidean Algorithm, but applied to polynomials, our GCD is the last non-zero remainder. Namely,

$$\gcd(x^3 - 9x^2 - x + 105, x^3 - 3x^2 - 25x - 21) = -6x^2 + 24x + 126.$$

Check!

$$x^3 - 9x^2 - x + 105 = \left(\frac{-1}{6}\right)(-6x^2 + 24x + 126)(x - 5)$$

and

$$x^3 - 3x^2 - 25x - 21 = \left(-\frac{1}{6}\right)(-6x^2 + 24x + 126)(x + 1).$$

So, the polynomial $h(x) = -6x^2 + 24x + 126$ is a common divisor of $f(x) = x^3 - 9x^2 - x + 105$ and $g(x) = x^3 - 3x^2 - 25x - 21$.

Is it the *greatest* common divisor? The polynomial $h(x)$ has power $2$. For any common divisor to be "greater" it would have to have power $3$. But the only cubic polynomial that can divide the cubic polynomial $f(x)$ is the polynomial $f(x)$ itself. Similarly for $g(x)$. And $f(x) \neq g(x)$.

So, there cannot be a "greater" common divisor. That is, $h(x) = -6x^2 + 24x + 126$ is the GCD.

Recall that the GCD of two integers is unique. Is the GCD of two polynomials unique? No. Notice that $h^*(x) = (-1/6)h(x) = x^2 - 24x - 21$ is a common divisor of power two.

More generally, $h^*(x) = c \cdot h(x)$ is also "the" GCD of the polynomials $f(x)$ and $g(x)$.

■

# 5 Fractions to add and express as the quotient of two relatively prime integers

## 5.1 Divisibility Shortcuts

Learning a few divisibility rules can help identify and speed up the process of cancelling common factors in a numerator and denominator.

| |
|---|
| An integer is divisible by **3** provided **the sum of its digits is divisible by 3**. |
| An integer is divisible by **4** provided **the number formed by the last two digits is divisible by 4**. |
| An integer is divisible by **6** provided **it is divisible by 2 and by 3.** |
| An integer is divisible by **8** provided **the number formed by the last two digits is divisible by 8**. |
| An integer is divisible by **9** provided **the sum of its digits is divisible by 9**. |

## 5.2 Reducing Fractions

Multiplying the numerator and denominator by the LCM of all denominators helps to simplify fractions.

$$\frac{\frac{5}{63}+\frac{3}{35}}{\frac{7}{45}+\frac{5}{18}} = \frac{\left(\frac{5}{63}+\frac{3}{35}\right)}{\left(\frac{7}{45}+\frac{5}{18}\right)} \cdot \frac{(\mathbf{2 \cdot 3^2 \cdot 5 \cdot 7})}{(\mathbf{2 \cdot 3^2 \cdot 5 \cdot 7})} = \frac{50+54}{98+175} = \frac{104}{273} = \frac{8}{21}.$$

Note: $\operatorname{lcm}(63,35,45,18) = \mathbf{2 \cdot 3^2 \cdot 5 \cdot 7}$.

- Remember that the rules of the test require you to simplify all fractions to the point where the numerator and denominator are relatively prime (have no common factors).

### 5.3 Ratios

If you are told objects are distributed into three piles in a $2:5:6$ ratio, this means that the first pile gets the fraction $2/(2+5+6)$ of the objects (and the second pile gets the fraction $5/13$ and the third piles gets the fraction $6/13$ of the objects.

### 5.4 Ordering fractions

For positive numbers $a, b, c$ and $d$, $\dfrac{a}{b} < \dfrac{c}{d} \Leftrightarrow ad < bc$.

For positive numbers $a, b, c$ and $d$, $\dfrac{a}{b} > \dfrac{c}{d} \Leftrightarrow ad > bc$.

# 6 Complex Fractions and Continued Fractions

To simplify a finite continued fraction (as in the example below) start at the bottom and work up.

$$5-\cfrac{1}{4-\cfrac{1}{3-\cfrac{1}{2-\cfrac{1}{1}}}} = 5-\cfrac{1}{4-\cfrac{1}{3-\cfrac{1}{1}}} = 5-\cfrac{1}{4-\cfrac{1}{2}} = 5-\cfrac{1}{\cfrac{7}{2}} = 5-\frac{2}{7} = \frac{33}{7}.$$

To simplify an infinite continued fraction, identified as $x$ in the example below, look for a way to rewrite a "smaller" part of the fraction in terms of the same $x$. Then solve for $x$.

$$x \quad = \quad \cfrac{3}{2+\cfrac{3}{2+\cfrac{3}{2+\cfrac{3}{\ddots}}}} \quad = \quad \cfrac{3}{2+\left(\cfrac{\mathbf{3}}{\mathbf{2}+\cfrac{\mathbf{3}}{\mathbf{2}+\cfrac{\mathbf{3}}{\ddots}}}\right)} \quad = \quad \frac{3}{2+\boldsymbol{x}}$$

That is,

$$x = \frac{3}{2+x} \Rightarrow x(2+x) = 3 \Rightarrow x^2+2x-3=0 \Rightarrow (x+3)(x-1)=0.$$

So, $x=-3$ and $x=1$. But $x=-3$ is an extraneous solution (a false solution that satisfies the final step of the derivation but does not satisfy the original problem) so it does not count as a solution. (i.e. toss out $x=-3$ because $x$ is clearly positive)

So,

$$x = \cfrac{3}{2 + \cfrac{3}{2 + \cfrac{3}{2 + \cfrac{3}{\ddots}}}} = 1.$$

## 6.1 Expand a number into continued fraction form

**Example 37**.

The fraction $\frac{37}{13}$ can be written in the form $2 + \cfrac{1}{x + \cfrac{1}{y + \cfrac{1}{z}}}$ where $x, y$ and $z$ are positive integers. Find the values of $(x, y, z)$.

**Solution**

Step 1. Express $37/13$ in the form $q + r/13$ where $q$ and $r$ are positive integers and $r < 13$ (*i.e.* integer quotient with remainder form). A result called the **remainder theorem** says that there will always be a $q$ and $r$ as described above.

$$\frac{37}{13} = 2 + \frac{11}{13}.$$

Step 2. Rewrite the fraction $r/13$ as $1/(13/r)$.

$$\frac{11}{13} = \frac{1}{\frac{13}{11}}$$

Step 3. It follows by our requirement that $r < 13$ that $13/r > 1$. So we can carry out Step 1 on $13/r$.

$$\frac{13}{11} = 1 + \frac{2}{11}.$$

Summarizing our work up to this point we have

$$\frac{37}{13} = 2 + \frac{11}{13} = 2 + \frac{1}{\frac{13}{11}} = 2 + \cfrac{1}{1 + \frac{2}{11}}.$$

Step 4. Continue in this pattern.

$$\frac{2}{11} = \frac{1}{\frac{11}{2}} = \frac{1}{5+\frac{1}{2}}.$$

$$\frac{37}{13} = 2 + \frac{1}{1+\left(\frac{2}{11}\right)} = 2 + \frac{1}{1+\left(\frac{1}{5+\frac{1}{2}}\right)} = 2 + \frac{1}{1+\frac{1}{5+\frac{1}{2}}}.$$

Final step. Compare and identify $(x, y, z)$.

$$\frac{37}{13} = 2 + \frac{1}{x+\frac{1}{y+\frac{1}{z}}} = 2 + \frac{1}{\mathbf{1}+\frac{1}{\mathbf{5}+\frac{1}{\mathbf{2}}}}$$

So $(x, y, z) = (1,5,2)$. Note that the **process stops** when we reach a **remainder of 1**.

■

**Example 38.**

The fraction $\frac{18}{11}$ can be written in the form $2 - \frac{1}{x+\frac{1}{y-\frac{1}{z}}}$ where $x, y$ and $z$ are positive integers. Find the values of $(x, y, z)$.

**Solution**

The new twist is the presence of minus $(-)$ signs in the above form. Similar to our first step in the last example we now need to $18/11$ in the form $q - r/11$ where $q$ and $r$ are positive integers and $r < 11$ (*i.e.* integer quotient with remainder form). The **remainder theorem** mentioned above also guarantees that this is always possible.

$$\frac{18}{11} = 2 - \frac{3}{11}.$$

Step 2. Continue as in Example 1.

$$\frac{18}{11} = 2 - \frac{3}{11} = 2 - \frac{1}{\frac{11}{3}} = 2 - \frac{1}{3+\frac{2}{3}} = 2 - \frac{1}{3+\frac{1}{\frac{3}{2}}} = 2 - \frac{1}{3+\frac{1}{2-\frac{1}{2}}}.$$

So $(x, y, z) = (3,2,2)$.

■

## 6.2 Summary Result

In general, if we want a plus $(+)$ sign we can construct

$$\frac{a}{b} = q + \frac{r}{b} \text{ for some integers } a.b, q \text{ and } r$$

with $r < b < a$.

And if we want a minus $(-)$ sign we can construct

$$\frac{a}{b} = q - \frac{r}{b} \text{ for some (different) integers } a, b, q \text{ and } r$$

with $r < b < a$.

# 7 Number Bases; Change of Base

$235_7$ represents 235 in base 7 and equals $2(7^2) + 3(7^1) + 5(7^0) = 98 + 21 + 5 = 124_{10}$ or just $124$ in base $10$. In general, if no base subscript is attached to a number it is assumed that you are using base $10$ notation. In base 7 we only use the digits $\{0,1,2,3,4,5,6\}$. The next number in the base seven sequence would be $10_7 = 1(7^1) + 0(7^0) = 7 + 0 = 7_{10}$.

## 7.1 Converting from Base $b$ to Base 10

**Example 39.**

Find the base 10 representation of $3201_6$.

**Solution**

$$\begin{aligned} 3201_6 &= 3(6^3) + 2(6^2) + 0(6^1) + 1(6^0) \\ &= 3(216) + 2(36) + 0(6) + 1(1) \\ &= 648 + 72 + 0 + 1 \\ &= 721 \end{aligned}$$

That is, $3201_6 = 721_{10} = 721$.

■

**Example 40.**

Find the base 10 representation of $54401_7$.

**Solution**

$$\begin{aligned} 54401_7 &= 5(7^4) + 4(7^3) + 4(7^2) + 0(7^1) + 1(7^0) \\ &= 5(2401) + 4(343) + 4(49) + 0(7) + 1(1) = 13574 \end{aligned}$$

That is,

$$54401_7 = 13574_{10} = 13574.$$

■

## 7.2 Converting from Base 10 to Base $b$ : Bottom Up "Short Cut" Method

**Example 41.**

Find the base 5 representation of 1073.

**Answer**

$$1073 = 1(5^4) + 3(5^3) + 2(5^2) + 4(5^1) + 3(5^0) = 13243_5.$$

**Solution**

Divide 1073 by 5 with remainder. That is, express 1073 in the form $1073 = 5 \cdot d_1 + r_1$ where $r_1 \in \{0,1,2,3,4\}$.

$$1073 = 214(5) + 3$$

Now divide $d_1$ by 5 with remainder. That is, express $d_1 = 214$ in the form $214 = 5 \cdot d_2 + r_2$ where $r_2 \in \{0,1,2,3,4\}$.

$$214 = 42(5) + 4$$

Now divide $d_2$ by 5 with remainder. That is, express $d_2 = 42$ in the form $42 = 5 \cdot d_3 + r_3$ where $r_3 \in \{0,1,2,3,4\}$.

$$42 = 8(5) + 2.$$

Continue like this until you reach $d_k < 5$ and the line $d_k = 0(5) + r_{k+1}$.

$$8 = 1(5) + 3$$
$$1 = 0(5) + 1.$$

The remainders (shown in red), reading from the bottom up, reveal the digits of the base five representation of $1073$.

■

**Example 42.**

Find the base 2 representation of $1073$.

**Solution**

$$\begin{aligned}
1073 &= 536(2) + 1 \\
536 &= 268(2) + 0 \\
268 &= 134(2) + 0 \\
134 &= 67(2) + 0 \\
67 &= 33(2) + 1 \\
33 &= 16(2) + 1 \\
16 &= 8(2) + 0 \\
8 &= 4(2) + 0 \\
4 &= 2(2) + 0 \\
2 &= 1(2) + 0 \\
1 &= 0(2) + 1
\end{aligned}$$

The remainders (shown in red), reading from the bottom up, reveal the digits of the base two representation of $1073$. That is,

$$1073 = 10000110001_2.$$

■

**Example 43.**

Find the base 7 representation of 1073.

**Solution**

$$\begin{aligned} 1073 &= 153(7) + 2 \\ 153 &= 21(7) + 6 \\ 21 &= 3(7) + 0 \\ 3 &= 0(7) + 3 \end{aligned}$$

Therefore,

$$1073 = 3062_7.$$

■

**Example 44**.

Write the base-ten number 140 in base 15. (Source: 2012-13, Meet 1, Event A)

**Solution**

$$\begin{aligned} 140 &= 9(15) + 5 \\ 9 &= 0(15) + 9 \end{aligned}$$

Therefore,

$$140 = 95_{15}.$$

■

## 7.3 Converting from Base $a$ to Base $b$ $(a \neq 10, b \neq 10)$

**Example 45.**

Find the base-nine number that is equivalent to $245_6$. (Source: 2016-17, Meet 1, Event A)

**Solution**

First convert from base 6 to base 10. Then convert from base 10 to base 9

$$245_6 = 2(6^2) + 4(6^1) + 5(6^0) = 101$$

$$101 = 11(9) + 2$$
$$11 = 1(9) + 2$$
$$1 = 0(9) + 1$$

Therefore

$$245_6 = 122_9.$$

■

**Example 46.**

Express using base nine the integer which is written 54321 using base six. (Source: 2007-08, Meet 1, Team Event)

**Solution**

$$54321_6 = 5(6^4) + 4(6^3) + 3(6^2) + 2(6^1) + 1(6^0) = 7465$$

$$7465 = 829(9) + 4$$
$$829 = 92(9) + 1$$
$$92 = 10(9) + 2$$
$$10 = 1(9) + 1$$
$$1 = 0(9) + 1$$

Therefore,

$$54321_6 = 7465 = 11214_9.$$

■

## 7.4 Converting from Base $a$ to Base $a^2$ and vice versa (without going through Base 10)

Note: You certainly can do these problems by going through base 10 but the procedure below is a nice shortcut.

**Example 47.**

Using binary notation (base 2), let $N = 11110101$. Write $N$ in octal notation (base 8). (Source: 1998-99, Meet 1, Event A)

### Solution

Procedure: We want to go from base 2 to base $2^{\mathbf{3}}$. Group the base 2 number into sets of $\mathbf{3}$. Putting in leading 0's as needed to make each a full group.

$$11|110|101 = 011|110|101$$

added in a leading 0

Now convert each base 2 group of three.

$$011_2 = 0(2^2) + 1(2^1) + 1(2^0) = 3$$
$$110_2 = 1(2^2) + 1(2^1) + 0(2^0) = 6$$
$$101_2 = 1(2^2) + 0(2^1) + 1(2^0) = 5$$

These are the digits of our base 8 conversion.

$$11110101_2 = 365_8$$

Here is why it works.

$$11110101_2 = 1(2^7) + 1(2^6) + 1(2^5) + 1(2^4) + 0(2^3) + 1(2^2) + 0(2^1) + 1(2^0)$$

$$= \big(0(2^8) + 1(2^7) + 1(2^6)\big) + \big(1(2^5) + 1(2^4) + 0(2^3)\big) + \big(1(2^2) + 0(2^1) + 1(2^0)\big)$$

$$= 2^6\big(0(2^2) + 1(2^1) + 1(2^0)\big) + 2^3\big(1(2^2) + 1(2^1) + 0(2^0)\big) + 2^0\big(1(2^2) + 0(2^1) + 1(2^0)\big)$$

$$= (2^3)^2\big(0(2^2) + 1(2^1) + 1(2^0)\big) + (2^3)^1\big(1(2^2) + 1(2^1) + 0(2^0)\big) + (2^3)^0\big(1(2^2) + 0(2^1) + 1(2^0)\big)$$

$$= \big(0(2^2) + 1(2^1) + 1(2^0)\big) \cdot 8^2 + \big(1(2^2) + 1(2^1) + 0(2^0)\big) \cdot 8^1 + \big(1(2^2) + 0(2^1) + 1(2^0)\big) \cdot 8^0$$

$$= 3(8^2) + 6(8^1) + 5(8^0) = 365_8.$$

■

**Example 48.**

Find the base 9 equivalent to $12012111_3$.

**Solution**

We are converting from base 3 to base $3^{\mathbf{2}}$ so form groups of size **2** from right to left adding leading 0's if necessary to form a complete group of size 2.

$$12|01|21|11$$

Convert each group (base 3).

$$\begin{aligned} 12_3 &= 1(3^1) + 2(3^0) = 5 \\ 01_3 &= 0(3^1) + 1(3^0) = 1 \\ 21_3 &= 2(3^1) + 1(3^0) = 7 \\ 11_3 &= 1(3^1) + 1(3^0) = 4 \end{aligned}$$

These are the digits of our base 9 equivalent.

$$12012111_3 = 5174_9.$$

■

**Example 49.**

Find the base 3 equivalent to $5174_9$.

**Solution**

We are converting from base $3^{\mathbf{2}}$ to base 3 so find the 2 digit base 3 equivalent of each digit in $5174_9$. Include a leading 0 if necessary to make the equivalent a two digit number.

$$5 = 1(3^1) + 2(3^0) = 12_3$$

$$1 = 1(3^0) = 1_3 = 01_3 \text{ (put in the leading 0 as needed)}$$

$$7 = 2(3^1) + 1(3^0) = 21_3$$

$$4 = 1(3^1) + 1(3^0) = 11_3$$

These are the digits of our base 3 equivalent.

$$5174_9 = 12|01|21|11_3 = 12012111_3.$$

■

## 7.5 Converting in a generic base

**Example 50.**

In base $b$, $c^2$ is written $10$. How do you write $b^2$ in case $c$? (Source: 1999-2000, State Tournament, Event A)

**Solution**

"$c^2$ is written $10$ in base $b$" $\Longrightarrow$ $c^2 = 1b^1 + 0b^0$. But $1b^1 + 0b^0 = b$.

$$\Longrightarrow c^2 = b$$
$$\Longrightarrow c^4 = b^2$$
$$\Longrightarrow b^2 = c^4 = 1c^4 + 0c^3 + 0c^2 + 0c^1 + 0c^0$$
$$\Longrightarrow b^2 = 10000_c$$

■

## 7.6 Finding the base $b$ such that …

**Example 51.**

In what base $b$ does the integer $63_b$ equal $117_{10}$? (Source: 2009-10, State Tournament, Event A)

**Solution**

$$63_b = 6b^1 + 3b^0 = 6b + 3 = 117 \Longrightarrow 6b = 114 \Longrightarrow b = 19.$$

■

**Example 52.**

In some number base $b$, the number $121$ is equal to the decimal (base-10) number $324$. Calculate $b$. (Source: 2012-13, Meet 1, Team Event)

**Solution**

$$\begin{aligned} 121_b = 1b^2 + 2b^1 + 1 &= 324 \\ \Longrightarrow (b+1)^2 &= 324 \\ \Longrightarrow b + 1 &= \sqrt{324} = 18 \\ \Longrightarrow b &= 17. \end{aligned}$$

■

**Example 53.**

A certain integer is represented base 5 by $40142_5$ and base $b$ by $1583_b$. Find $b$. (Source: 2001-02, State Tournament, Event A)

**Solution**

$$40142_5 = 4(5^4) + 0(5^3) + 1(5^2) + 4(5^1) + 2(5^0) = 2547$$

$$1583_b = 1(b^3) + 5(b^2) + 8(b^1) + 3(b^0)$$

$\Longrightarrow b^3 + 5b^2 + 8b + 3 = 2547$

$\Longrightarrow b^3 + 5b^2 + 8b - 2544 = 0$

$\Longrightarrow b$ must divide $2544 = 2^4 \cdot 3 \cdot 54$ (rational root theorem).

Synthetic division shows $b = 8$ is not big enough

$$\begin{array}{c|cccc} 8 & 1 & 5 & 8 & -2544 \\ \hline & & 8 & 104 & 896 \\ \hline & 1 & 13 & 112 & -1648 \end{array}$$

but that the next largest potential root is $b = 12$ is in fact a root.

| | | | | |
|---|---|---|---|---|
| 12 | 1 | 5 | 8 | −2544 |
| | | 12 | 204 | 2544 |
| | 1 | 17 | 112 | 0 |

We can also see from this division that

$$b^3 + 5b^2 + 8b - 2544 = (b - 12)(b^2 + 17b + 112)$$

and because $b^2 + 17b + 112$ is an irreducible quadratic (the discriminant is negative) there are no other real roots. So $b = 12$ is the only possible answer.

■

**Example 54.**

Let $N$ be a number in base $b$ such that $N_b = 14_b \cdot 17_b$. What is the greatest base $b$ for which $N_b$ would be written with "2" as its left-most digit? (Source: 2013-14, Meet 1, Team Event)

**Solution**

First note that $b \geq 8$ or $17_b$ could not be a base $b$ number. Now

$$N_b = 14_b \cdot 17_b \Longrightarrow N = (b + 4)(b + 7) = b^2 + 11b + 28.$$

To get a sense for what is going on think base 10 for a moment. In this case ($b = 10$) we get

$$b^2 + 11b + 28 = 100 + 110 + 28 = 228_{10}$$

and we observe the left most digit is a "2" as required. In particular, it was necessary that $110 + 28 \geq 100$ so that our hundreds digit could increase from 1 to 2. In the general case this means that we must choose $b$ such that

$$11b + 28 \geq b^2.$$

Solving this inequality we will find that $b \leq 13$. Checking $b = 13$ we see that

$$N = 13^2 + 11(13) + 28 = 340 = 2(13^2) + 0(13^1) + 2(13^0) = 202_{13}.$$

Therefore $b = 13$.

■

## 7.7 Disguised polynomial factorization problems

**Example 55.**

The integer $N = 10100$ is expressed using base $b > 1$. Express $N$ as a product of two integers, expressed as polynomials in $b$, that are both greater than 1. (Source: 2005-06, Meet 1, Event A)

**Solution**

$$\begin{aligned} 10100_b &= 1(b^4) + 0(b^3) + 1(b^2) + 0(b^1) + 0(b^0) \\ &= b^4 + b^2 \\ &= b^2(b^2 + 1) \end{aligned}$$

Note: $b > 1$ implies $b^2 > 1$ and $b^1 + 1 > 1$ as required.

■

**Example 56.**

Expressed using base $b > 3$, the integer $M = 231$. Write $M$ as a product of two integers, also expressed using base $b$. (Note carefully, you are not being asked to express them as polynomials in $b$, but as integers, just as $M$ is expressed.) (Source: 2005-06, State Tournament, Event D)

**Solution**

$$\begin{aligned} M = 231_b &= 2(b^2) + 3(b^1) + 1(b^0) \\ &= 2b^2 + 3b + 1 \end{aligned}$$

$$= (2b+1)(b+1)$$
$$= (21_b)(11_b).$$

■

## 7.8 Addition in Base $b$

**Example 57.**

Find $315_6 + 153_6$.

**Solution**

It can help to write out a base 6 addition table when you first learning to add in a different base.

Base 6 Addition Table

| $+$ | $0_6$ | $1_6$ | $2_6$ | $3_6$ | $4_6$ | $5_6$ |
|---|---|---|---|---|---|---|
| $0_6$ | 0 | 1 | 2 | 3 | 4 | 5 |
| $1_6$ | 1 | 2 | 3 | 4 | 5 | 10 |
| $2_6$ | 2 | 3 | 4 | 5 | 10 | 11 |
| $3_6$ | 3 | 4 | 5 | 10 | 11 | 12 |
| $4_6$ | 4 | 5 | 10 | 11 | 12 | 13 |
| $5_6$ | 5 | 10 | 11 | 12 | 13 | 14 |

Using this table we can see that

$$\begin{array}{cccc} & {\scriptstyle 1} & {\scriptstyle 1} & \\ & 3 & 1 & 5 \\ + & 1 & 5 & 3 \\ \hline & 5 & 1 & 2 \end{array}$$

That is, $315_6 + 153_6 = 512_6$.

■

**Example 58.**

Given the following summation in base 6

$$\begin{array}{r} b\ 3\ 4_6 \\ +\ a\ a\ c_6 \\ \hline a\ 0\ b\ a_6 \end{array}$$

find the sum of $a + b + c$ in base 6. (Source: Greater New Haven Mathematics League, 2009)

**Solution**

$$\begin{array}{cccc} & b & 3 & 4 \\ + & a & a & c \\ \hline a & 0 & b & a \end{array}$$

Forces that $a = 1$ because we can carry at most 1 in adding two numbers in the right most column of the addition.

$$\begin{array}{cccc} & b & 3 & 4 \\ + & 1 & 1 & c \\ \hline 1 & 0 & b & 1 \end{array}$$

After substituting for $a$ we can see that $c = 3$ from the left most column of addition.

$$\begin{array}{cccc} & & 1 & \\ & b & 3 & 4 \\ + & 1 & 1 & 3 \\ \hline 1 & 0 & b & 1 \end{array}$$

Now we can see that $b = 5$ from the second column from the left of addition, noticing that we also had to carry 1 from the previous column.

$$\begin{array}{cccc} & 5 & 3 & 4 \\ + & 1 & 1 & 3 \\ \hline 1 & 0 & 5 & 1 \end{array}$$

Therefore,

$$(a+b+c)_6 = (1+5+3)_6 = 13_6.$$

■

**Example 59.**

Add the following binary numbers and express the sum as a number in base three. (Source: 2000-01, Meet 4, Event C)

$$\begin{array}{ccccccccc} & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ + & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ \hline \end{array}$$

**Solution**

$$\begin{array}{ccccccccc} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \\ & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ + & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ \hline 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \end{array}$$

$$101110100_2 = 2^8 + 2^6 + 2^5 + 2^2 = 372$$

$$372 = 3^5 + 3^4 + 3^3 + (2)\cdot 3^2 + 3^1 + (0)\cdot 3^0 = 111210_3$$

■

# 8 Decimals, repeating decimals

## 8.1 Theorem 15

Every repeating decimal can be expressed in the form $a/b$ where $a$ and $b$ are integers.

■

## 8.2 Theorem 16

A fraction $a/b$, where $a$ and $b$ are relatively prime integers, is terminating $\Longleftrightarrow$ the prime factorization of $b$ only contains 2's and/or 5's.

■

**Example 60.**

Convert $0.38\overline{427} = 0.38427427427\ldots$ into a rational number.

**Solution**

Let $s = 0.38\overline{427} = 0.38427427427\ldots$

Then,

$$\begin{aligned} 100000s &= 38427.427427427\ldots \\ 100s &= \quad\;\; 38.427427427\ldots \end{aligned}$$

and

$$\begin{aligned} 100000s - 100s &= 38427 - 38 \\ 99900s &= 38389 \end{aligned}$$

So,

$$s = 0.38\overline{427} = \frac{38389}{99900}.$$

■

## 8.3 Basimals

Numbers of the form $(0.a_1a_2a_3\cdots)_{10} = \frac{a_1}{10} + \frac{a_2}{10^2} + \frac{a_3}{10^3} + \cdots$ with $a_j \in \{0,1,2,3,\ldots,9\}$ are called decimals.

Analogously, we define the term ***basimals*** for non-base 10 numbers of this form. That is, a basimal is a number of the form $(0.a_1a_2a_3\cdots)_k = \frac{a_1}{k} + \frac{a_2}{k^2} + \frac{a_3}{k^3} + \cdots$ with $a_j \in \{0,1,2,\ldots,k-1\}$.

■

### 8.3.1 Converting Basimals

**Example 61.**

Convert the basimal number $0.234_6$ to a fraction in base 10.

**Solution**

We have

$$0.234_6 = \frac{2}{6^1} + \frac{3}{6^2} + \frac{4}{6^3}.$$

So,

$$0.234_6 = \frac{2}{6^1} + \frac{3}{6^2} + \frac{4}{6^3} = \frac{2(6^2) + 3(6) + 4}{6^3} = \frac{72 + 18 + 4}{216} = \frac{94}{216}.$$

■

### 8.3.2 Converting a Repeating Basimal Number

**Example 62.**

Convert $0.\overline{234}_6$ to a fraction in base $10$.

**Solution**

Let $x = \overline{234}_6$. Then $1000_6 \cdot x = (1000_6) \cdot \left(\overline{234}_6\right) = 234.\overline{234}_6$

Therefore,

$$1000_6 \cdot x - x = 234.\overline{234}_6 - 0.\overline{234}_6 = 234_6.$$

$$(1000_6 - 1_6)x = 234_6$$

$$(555_6)x = 234_6$$

$$x = \frac{234_6}{555_6}.$$

Now separately convert $234_6$ and $555_6$ to base $10$.

$$234_6 = 2 \cdot 6^2 + 3 \cdot 6^1 + 4 \cdot 6^0 = 72 + 18 + 4 = 94_{10}$$

$$555_6 = 5 \cdot 6^2 + 5 \cdot 6^1 \cdot 5 \cdot 6^0 = 5(6^2 + 6^1 + 6^0) = 6^3 - 1 = 216 - 1 = 215_{10}$$

Note:

$$s = 6^2 + 6^1 + 6^0$$

$$6s = 6^3 + 6^2 + 6^1$$

$$\therefore \quad 5s = 6^3 - 6^0 = 6^3 - 1$$

Therefore,

$$0.\overline{234}_6 = \frac{234_6}{555_6} = \frac{94_{10}}{215_{10}}.$$

■

**Example 63.**

Convert $0.\overline{31}_5$ to a fraction in base 10.

**Solution**

Let $x = 0.\overline{31}_5$. Then $100_5 \cdot x = 100_5 \cdot \overline{31}_5 = 31.\overline{31}_5$. Therefore

$$100_5 x - x = 31.\overline{31}_5 - 0.\overline{31}_5 = 31_5$$

$$(100_5 - 1_5)x = 31_5$$

$$44_5 \cdot x = 31_5$$

$$x = \frac{31_5}{44_5} = \frac{3(5^1) + 1(5^0)}{4(5^1) + 4(5^0)} = \frac{16_{10}}{24_{10}} = \frac{16}{24} = \frac{2}{3}.$$

■

### 8.3.3 Converting a Decimal to a Basimal Number

**Example 64.**

Convert $\frac{3}{5} = .6$ to a basimal in base 7.

**Solution**

In analogy to how we can express an integer in base $10$ to an integer in base $7$ we ask what is the largest fraction $\frac{k}{7}$, $k = 0,1,2,3,4,5,6$ that is less than or equal to $\frac{3}{5}$.

$$\frac{4}{7} \approx 0.57 < \frac{3}{5} = .6 < \frac{5}{7} \approx .71$$

So, the first digit must be $4$.

Now repeat this process by finding the largest fraction $\frac{k}{7^2}$, $k = 0,1,2, \dots ,6$ that is less than or equal to $\frac{3}{5} - \frac{4}{7} = \frac{1}{35} \approx 0.029$. We note that

$$\frac{1}{49} \approx 0.020 < \frac{1}{35} \approx 0.028 < \frac{2}{49} \approx 0.041.$$

So, the second digit must be $1$.

This is already getting tedious! Fortunately, there is a very simple short cut procedure.

#### 8.3.3.1 Introducing a Short Cut Approach

(1) Multiple the base ten decimal by the base you want to convert to. In this case, the base is $7$.

$$0.6 \cdot 7 = 4.2.$$

The units digit is the first digit in the basimal representation in base 7.

$$0.4$$

(2) Multiple just the decimal part of the above product (*i.e.* the 0.2 from the product 4.2) by 7.

$$0.2 \cdot 7 = 1.4$$

The units digit in this product is the second digit in the basimal representation in base 7.

$$0.41$$

(3) Repeat

$$0.6 \cdot 7 = 4.2 \Longrightarrow 0.4$$

$$0.2 \cdot 7 = 1.4 \Longrightarrow 0.41$$

$$0.4 \cdot 7 = 2.8 \Longrightarrow 0.412$$

$$0.8 \cdot 7 = 5.6 \Longrightarrow 0.4125$$

$$\vdots$$

We can see that this basimal will continuously repeat the pattern 4125 after this. That is,

$$0.6_{10} = 0.\overline{4125}_7 \,.$$

■

Check!

$$x = 0.\overline{4125}_7$$

$$10000_7 \cdot x = 4125.\overline{4125}_7$$

$$10000_7 \cdot x - x = 4125_7$$

$$6666_7 \cdot x = 4125_7$$

$$x = \frac{4125_7}{6666_7} = \frac{4 \cdot 7^3 + 1 \cdot 7^2 + 2 \cdot 7^1 + 5 \cdot 7^0}{6 \cdot 7^3 + 6 \cdot 7^2 + 6 \cdot 7^1 + 6 \cdot 7^0} = \frac{1440}{2400} = 0.6$$

■

## 8.4 Repetends

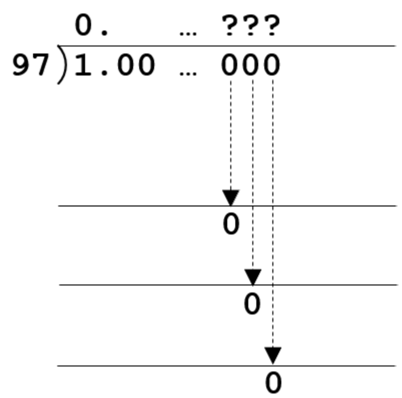
**Example 65.**

When expanded as a decimal, the fraction $1/97$ has a ***repetend*** (the repeating part of the decimal) of $96$ digits that start right after the decimal point. Find the last three digits $CBA$ of the repetend.

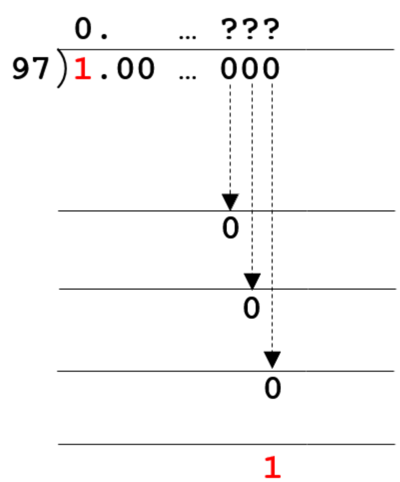
**Solution**

We need to reverse engineer the standard long-division algorithm.

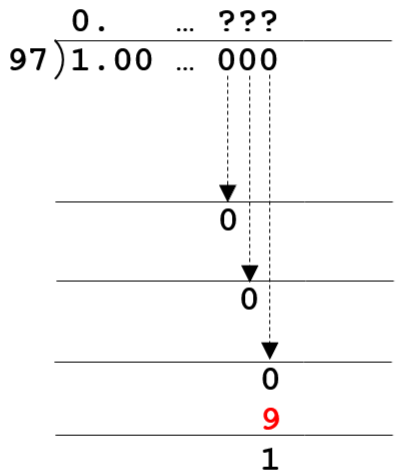
We want to determine the last three digits in the repetend so draw a three step long-division grid with the three "dropped" $0$'s in place.



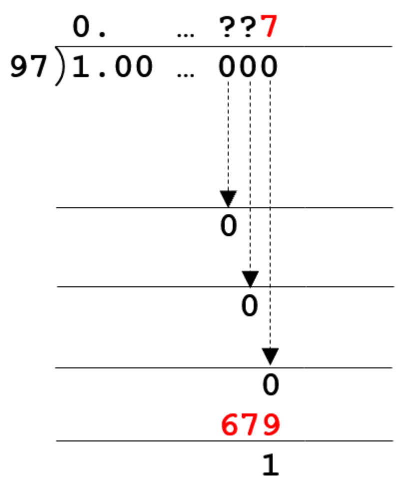

The quotient will restart its repeating pattern when the remainder is the same as the dividend. That is, when the remainder equals $1$.



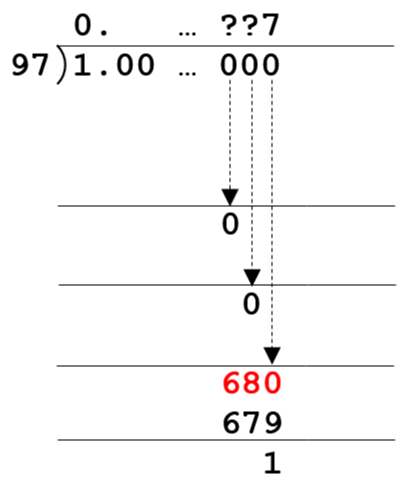
The last digit in this row must be a $9$ in order to leave a difference of $1$ $(10 - 9 = 1)$.



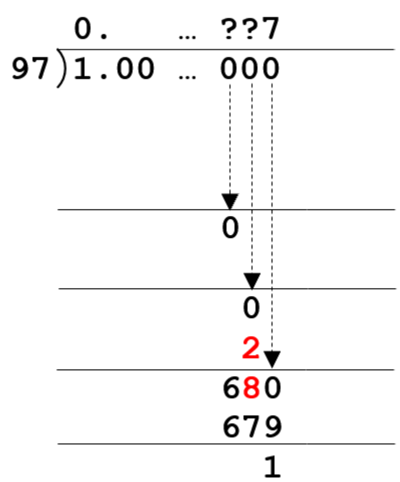

The third $? = 7$ because $97 \times ?$ must end in a $9$ and the only digit times $7$ that ends in a $9$ is $7$ $(7 \times 7 = 4\mathbf{9})$. Now that we have the third $? = 7$ we can see that $97 \times ? = 97 \times 7 = 679$.




In order to leave a difference of $1$ we have to subtract the $679$ from $680$.




The last digit in the next row up must be a $2$ in order to leave a difference of $8$ $(10 - 2 = 8)$.



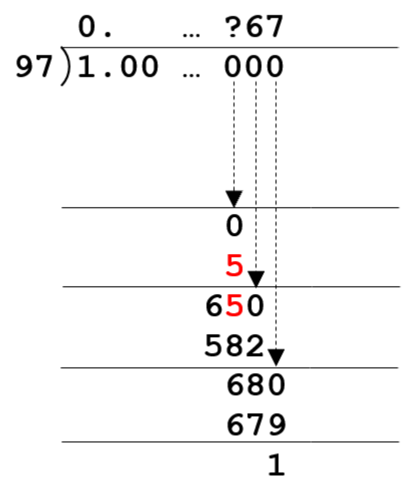

The next $? = 6$ because $97 \times ?$ must end in a $2$ and the only digit times $7$ that ends in a $2$ is $6$ $(7 \times 6 = 42)$. Now that we have the next $? = 2$ we have $97 \times ? = 97 \times 6 = 582$.

```
  0.    … ?67
97)1.00 … 000

          0

           0
        582
         680
         679
           1
```

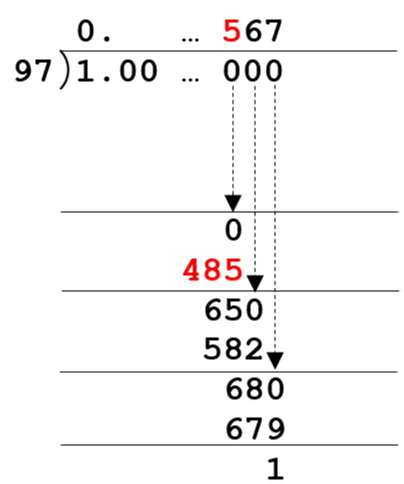
In order to leave a difference of $68$ we have to subtract the $582$ from $650$.

```
  0.    … ?67
97)1.00 … 000

          0

         650
         582
         680
         679
           1
```

The last digit in the next row up must be a $5$ in order to leave a difference of $5$ $(10 - 5 = 5)$.




The first $? = 5$ because $97 \times ?$ must end in a $5$ and the only digit times $7$ that ends in a $5$ is $5$ $(7 \times 5 = 35)$. Now that we have the next $? = 5$ we have $97 \times ? = 97 \times 5 = 485$.




Of course, we could continue to work backwards but the question only asked for the last three digits of the repetend.

We have reversed engineered (worked backwards) the long division process to find out that the last three digits of the repetend are $567$.

■