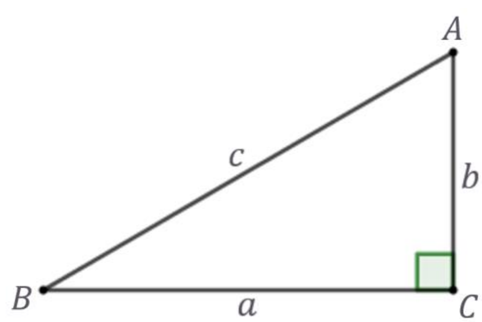
MSHSML Meet 1, Event B
Study Guide

1B Angles and Special Triangles (calculators allowed)

The Theorem of Pythagoras; familiar Pythagorean triples
Complementary, supplementary, and vertical angles
Interior and exterior angles for triangles and polygons
Angles formed by transversals cutting parallel lines
Properties of isosceles and equilateral triangles
Relationships in $30°$-$60°$-$90°$ and $45°$-$45°$-$90°$ triangles

1 The Theorem of Pythagoras; familiar Pythagorean triples



1.1 Pythagorean Theorem

Pythagorean Theorem: If ΔABC is a right triangle with hypotenuse (the longest side, opposite the right angle), then $a^2 + b^2 = c^2$.

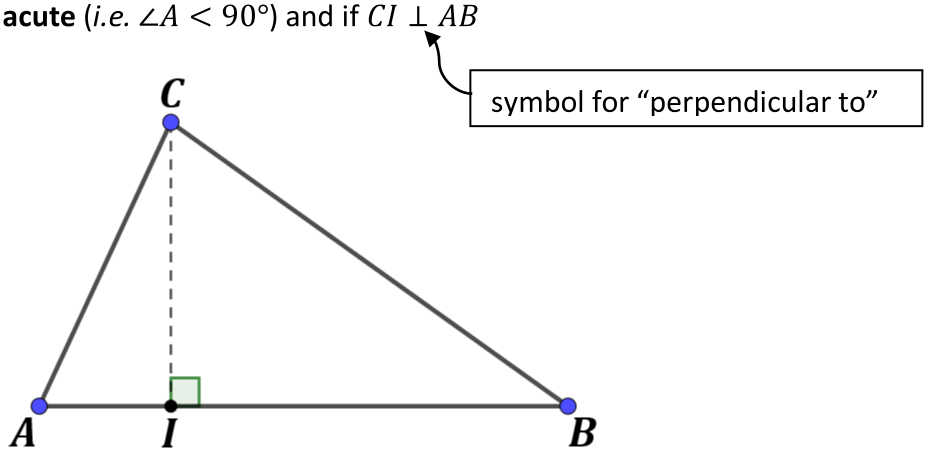
1.2 Converse of the Pythagorean Theorem

Converse of the Pythagorean Theorem: If $a^2 + b^2 = c^2$ in ΔABC, then ΔABC is a right triangle with hypotenuse (longest side) c and right angle C.

1.3 Pythagorean "Like" Theorems for Acute and Obtuse Triangles

Acute Angled Triangle Theorem

If $\angle A$ in triangle ΔABC is **acute** (*i.e.* $\angle A < 90°$) and if $CI \perp AB$



then

$$CB^2 = CA^2 + AB^2 - 2(AI)(AB).$$

<u>Proof</u>

By the Pythagorean Theorem applied to right triangles ΔAIC and ΔBIC we have

$$CB^2 = BI^2 + CI^2$$

and

$$CA^2 = AI^2 + CI^2.$$

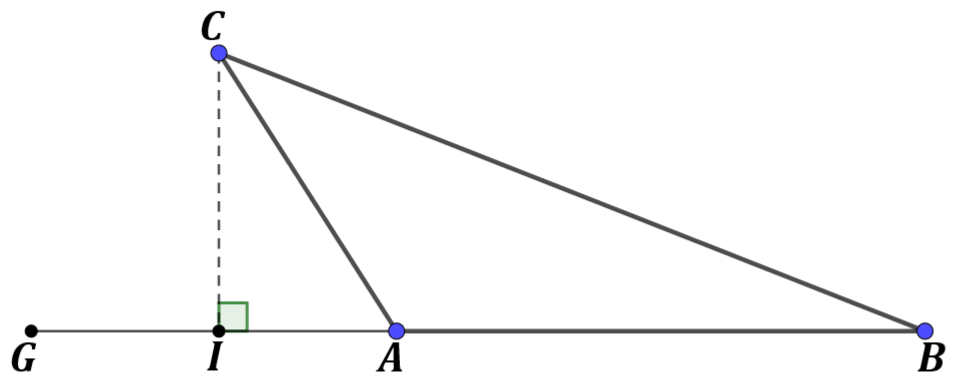
Therefore,

$$\begin{aligned} CB^2 &= BI^2 + \left(CA^2 - AI^2\right) \\ &= (AB - AI)^2 + \left(CA^2 - AI^2\right) \\ &= \left(AB^2 - 2(AI)(AB) + AI^2\right) + \left(CA^2 - AI^2\right) \\ &= CA^2 + AB^2 - 2(AI)(AB). \end{aligned}$$

■

Obtuse Angled Triangle Theorem

If $\angle A$ in triangle ΔABC is **obtuse** (*i.e.* $\angle A > 90°$), if we let side AB of ΔABC be produced (extended) to the point G and if $CI \perp BG$,



then

$$CB^2 = CA^2 + AB^2 + 2(AI)(AB).$$

The proof is analogous to the proof of the Acute Angled Triangle Theorem.

Anticipating the Law of Cosines (Study Guide for Meet 3, Event C)

You might have noticed that the Acute Angled Triangle Theorem and the Obtuse Angled Triangle Theorem look suspiciously close to the Law of Cosines. In fact, these two theorems follow immediately from the Law of Cosines. In the case of $\angle A$ acute,

$$\cos(A) = \frac{AI}{AC} \Longrightarrow 2(AI)(AB) = 2(AC)(AB)\cos(A)$$

and in the case of $\angle A$ obtuse,

$$\cos(A) = -\cos(180° - A) = -\left(\frac{AI}{AC}\right) \Longrightarrow -2(AI)(AB) = 2(AC)(AB)\cos(A).$$

1.4 Pythagorean Triples

Pythagorean Triples: the three sides of a right triangle when all sides are integers. By tradition we list the triple in order from smallest to largest.

Because of the converse of the Pythagorean Theorem, any ordered triple (a, b, c) of positive integers is a Pythagorean triple if $a^2 + b^2 = c^2$.

1.4.1 Table of familiar Pythagorean Triples

A small list of familiar Pythagorean Triples would include

$(3,4,5)$	$(5,12,13)$	$(7,24,25)$
$(8,15,17)$	$(9,40,41)$	$(11,60,61)$
$(12,35,37)$	$(16,63,65)$	$(20,21,29)$

Note that multiples of Pythagorean Triples are also Pythagorean Triples. That is, if $a^2 + b^2 = c^2$, then it is also true that $(ka)^2 + (kb)^2 = (ck)^2$ for any multiple k.

1.4.2 Primitive Pythagorean Triples

A Pythagorean Triple which is not a multiple of some smaller Pythagorean Triple is called ***primitive***. Each of the Pythagorean Triples in the above table is a primitive Pythagorean Triple.

Take for example, $2 \cdot (3,4,5) = (2 \cdot 3, 2 \cdot 4, 2 \cdot 5) = (6,8,10)$. We can see that $6^2 + 8^2 = 10^2$.

Some familiar multiples of the above Pythagorean Triples (which are themselves Pythagorean Triples but not *primitive* Pythagorean triples) would include

$(6,8,10)$	$(9,12,15)$	$(12,16,20)$
$(15,20,25)$	$(10,24,26)$	$(15,36,39)$

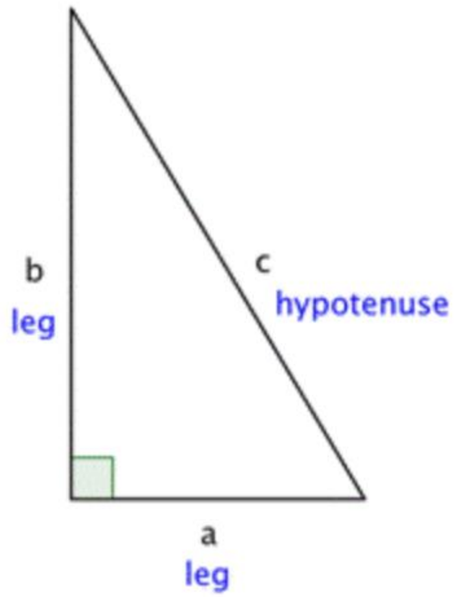
Recognizing these triples on inspection can help you to see when you have a right triangle.

1.4.3 Formula for generating *all* primitive Pythagorean Triples

Taking all triples of the form $(m^2 - n^2, 2mn, m^2 + n^2)$ for $m > n > 0$ will generate all possible primitive Pythagorean triples.

1.4.4 Generating a Pythagorean Triple with a Given Leg

Given leg a find some integers b and c such that $a^2 + b^2 = c^2$.



Case a odd. Take $b = (a^2 - 1)/2$ and $c = (a^2 + 1)/2$

Case a even. Take $b = (a^2 - 4)/4$ and $c = (a^2 + 4)/4$

1.5 Euclidean distance between two points

1.5.1 2-Dimensional Formula

Distance between (x_0, y_0) and (x_1, y_1) equals $\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$

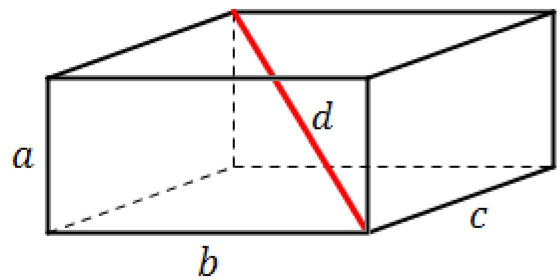
1.5.2 3-Dimensional Formula

Distance between (x_0, y_0, z_0) and (x_1, y_1, z_1) equals $\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2}$

1.5.3 Box Diagonal Formula

The length d of the diagonal going from top left to bottom right in a box with dimensions $a \times b \times c$ is

$$d = \sqrt{a^2 + b^2 + c^2}.$$



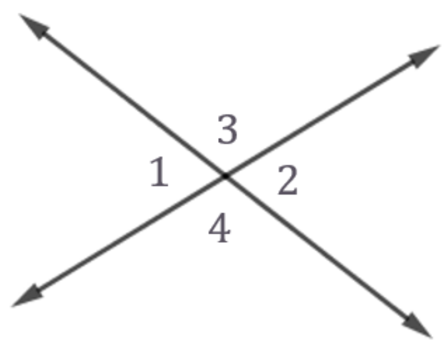
You can think of this as a 3-D generalization of the Pythagorean Theorem.

2 Complementary, supplementary, and vertical angles

Complementary angles – two angles whose measures have the sum 90

Supplementary angles – two angles whose measures have the sum 180

Vertical angles – two angles whose sides form two pairs of opposite rays, $\angle 1$ and $\angle 2$ are vertical angles, as are $\angle 3$ and $\angle 4$ in the figure below.



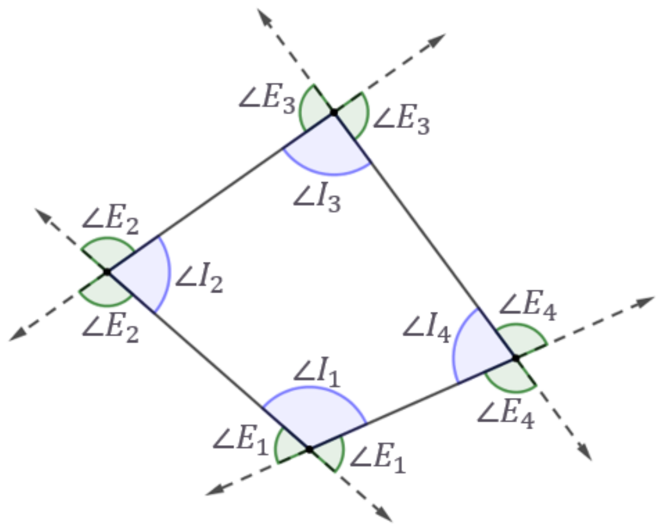
Theorem Vertical angles are equal: $\angle 1 = \angle 2$ and $\angle 3 = \angle 4$ in the above figure.

3 Interior and exterior angles for triangles and polygons

Sum of all interior angles of any n-sided polygon: $(n-2)180°$.

Sum of all exterior angles of any ***convex*** n-sided polygon: $360°$.

(Note: The "sum of all exterior angles" formula assumes you are including just one of the two equivalent exterior angles at each vertex in the sum.)



The blue angles are interior angles. The green angles are the associated exterior angles. The above results about the sum of interior and exterior angles tells us that

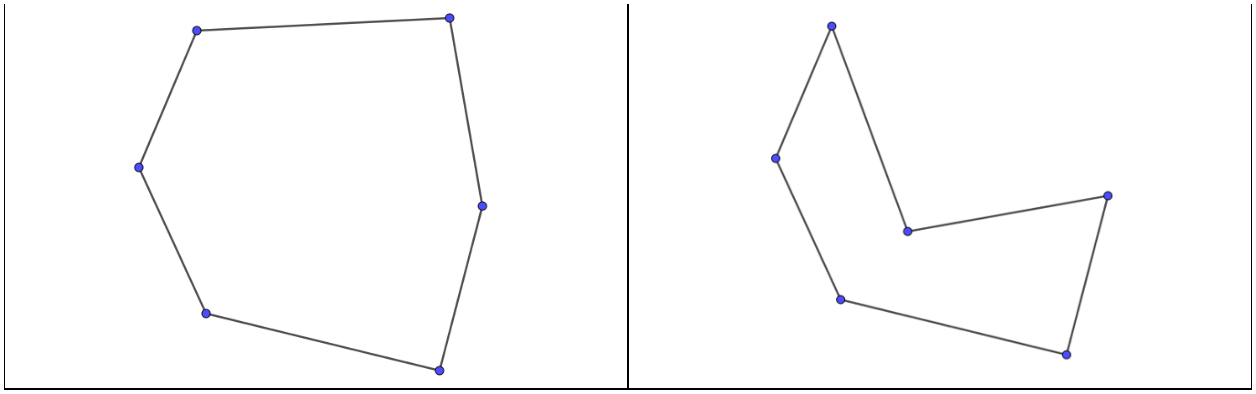
$$\angle I_1 + \angle I_2 + \angle I_3 + \angle I_4 = (4-2)180°$$

$$\angle E_1 + \angle E_2 + \angle E_3 + \angle E_4 = 360°$$

Convex / Concave

Convex means it has no "inward pointing" angles. The measure of an inward pointing angle is necessarily greater than $180°$ so a convex polygon is a polygon where all interior angles are less than $180°$. A polygon that is not convex is called ***concave***.

Convex Polygon: each interior angle $< \mathbf{180°}$	Concave Polygon: some interior angle $> \mathbf{180°}$

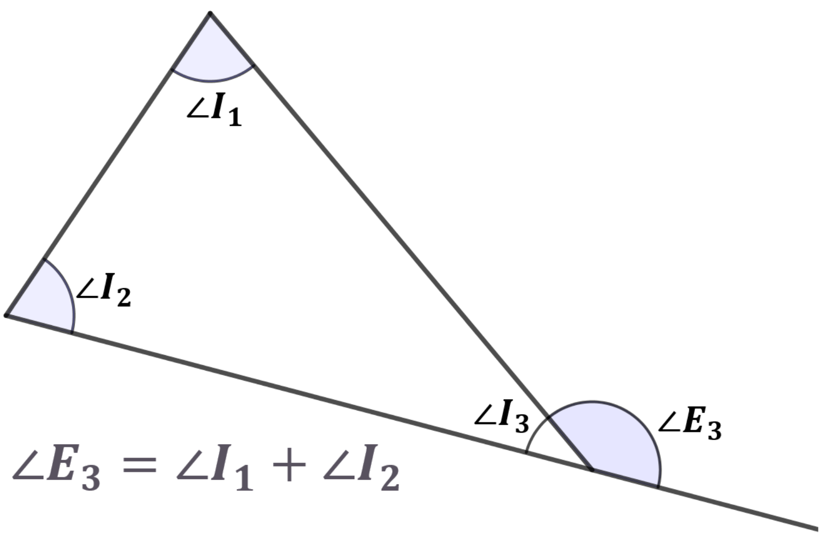


Regular

A polygon is called ***regular*** if all of its interior angles are equal. It follows that each interior angle of a regular n-sided polygon has measure: $(n-2)180°/n$.

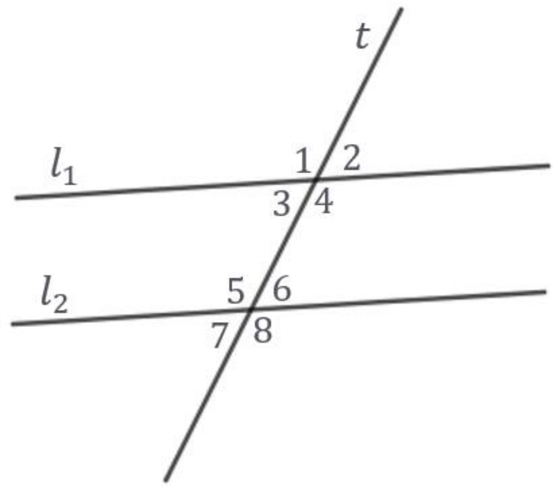
Exterior Angle Theorem (for a Triangle)

The measure of an exterior angle of a triangle equals the sum of the measures of the two remote (or non-adjacent) interior angles of that triangle.



4 Angles formed by transversals cutting parallel lines

Transversal – a line which intersects two other lines, all in the same plan. Line t in the diagram below is a transversal.



If lines l_1 and l_2 are parallel (denoted $l_1 \parallel l_2$), $\angle 1 = \angle 4 = \angle 5 = \angle 8$ and $\angle 2 = \angle 3 = \angle 6 = \angle 7$.

If lines l_1 and l_2 are parallel (denoted $l_1 \parallel l_2$), $\angle 4$ and $\angle 6$ are supplementary (add to 180). Also $\angle 3$ and $\angle 5$ are supplementary.

The converses of these results are also true. That is,

If $\angle 1 = \angle 4 = \angle 5 = \angle 8$ and $\angle 2 = \angle 3 = \angle 6 = \angle 7$, then lines l_1 and l_2 are parallel.

If $\angle 4$ and $\angle 6$ are supplementary and $\angle 3$ and $\angle 5$ are supplementary, then lines l_1 and l_2 are parallel.

Terminology used with transversals.

Corresponding Angles – any pair of angles, such as $\angle 1$ and $\angle 5$ in the above diagram, which are in corresponding positions relative to the eight angles formed by a transversal.

Likewise, in the above diagram the pair $\angle 2$ and $\angle 6$ are corresponding angles. As are the angle pair $\angle 3$ and $\angle 5$ and the angle pair $\angle 4$ and $\angle 8$.

Exterior Angles: $\angle 1, \angle 2, \angle 7, \angle 8$

Interior Angles: $\angle 3, \angle 4, \angle 5, \angle 6$

Same Side Angles: a pair of angles on the same side of the transversal

Alternate Side Angles: a pair of angles on opposite sides of the transversal.

So, for example, the pair $\angle 4$ and $\angle 5$ are referred to as alternate side interior angles.

5 Properties of isosceles and equilateral triangles

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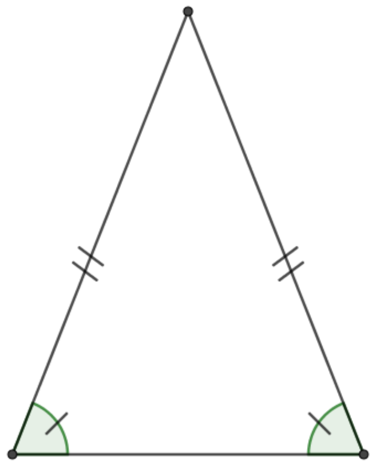
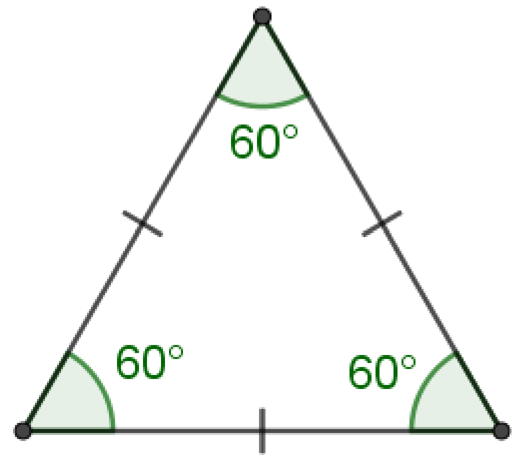
<tr><th>Isosceles</th><th>Equilateral ≡ Equiangular</th></tr>
<tr><td></td><td>
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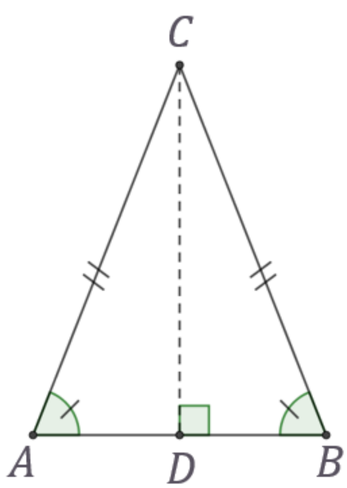
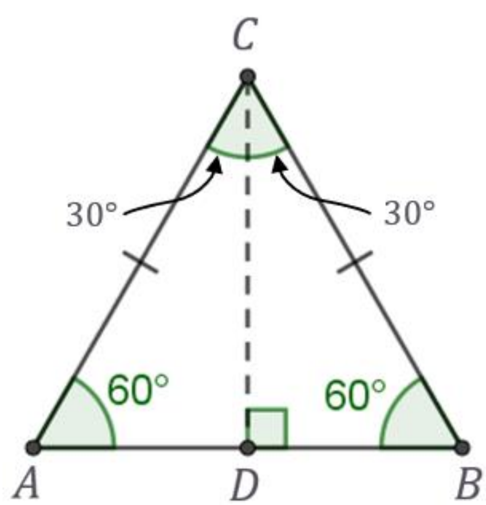
<tr><td>

Altitude $\overline{CD}$:
(1) meets the base at a right angle
(2) bisects the apex angle at C so $\widehat{ACD} = \widehat{BCD}$
(3) bisects the base so $\overline{AD} = \overline{BD}$
(4) splits the original isosceles triangle into two congruent halves so $\Delta ADC \cong \Delta BDC$.</td><td>

Altitude $\overline{CD}$:
(1) meets the base at a right angle
(2) bisects the apex angle at C so $\widehat{ACD} = \widehat{BCD} = 30°$
(3) bisects the base so $\overline{AD} = \overline{BD}$
(4) splits the original isosceles triangle into two congruent halves so $\Delta ADC \cong \Delta BDC$.</td></tr>
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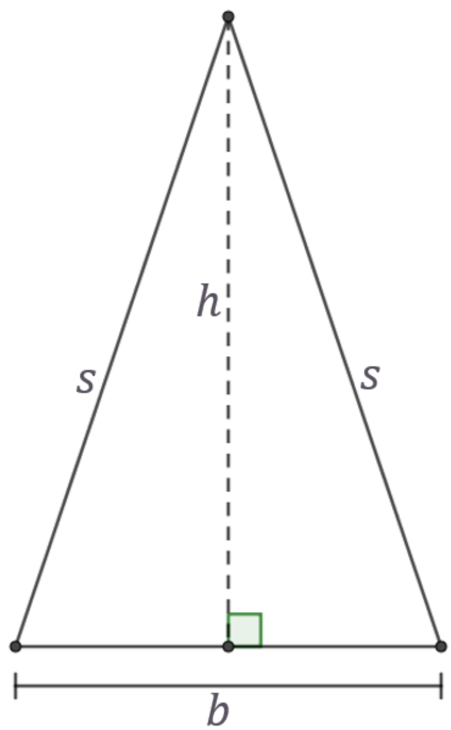

For an isosceles triangle with common sides s, base b and base height h, we have

$$\text{Area of } \Delta = \frac{b}{4}\sqrt{4s^2 - b^2}$$

$$h = \frac{\sqrt{4s^2 - b^2}}{s}$$
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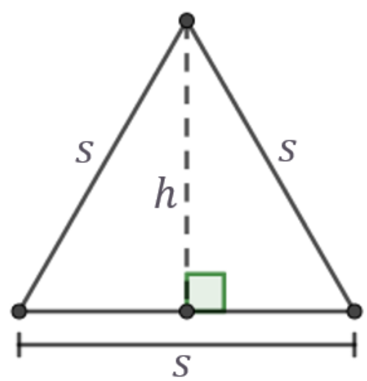

For an equilateral triangle with common side s and with height (altitude) h, we have

$$\text{Area of } \Delta = \frac{\sqrt{3}}{4}s^2 = \frac{h^2}{\sqrt{3}}$$

$$h = \frac{\sqrt{3}}{2}$$
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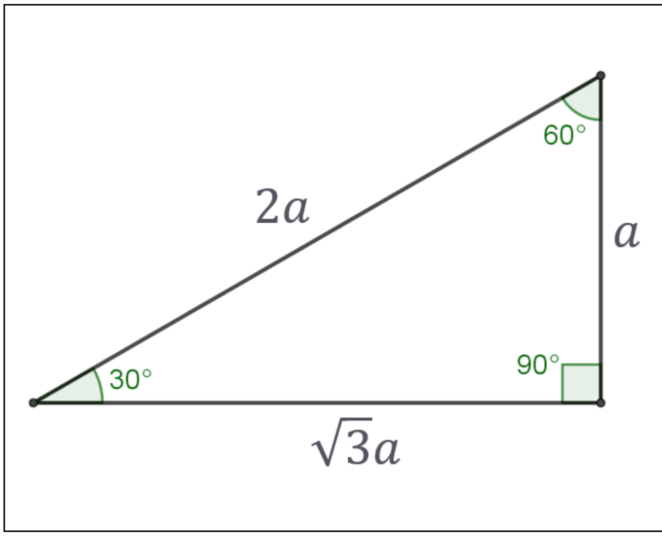
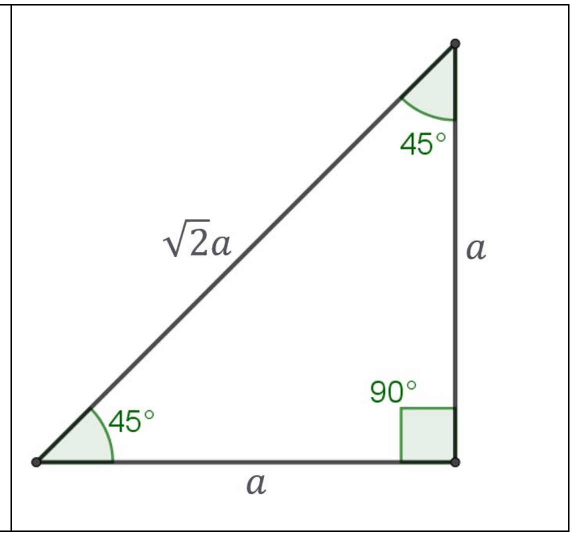
6 Relationships in 30°-60°-90° and 45°-45°-90° triangles





Therefore,

$\sin 30° = \dfrac{a}{2a} = \dfrac{1}{2}$	$\sin 60° = \dfrac{\sqrt{3}a}{2a} = \dfrac{\sqrt{3}}{2}$	$\sin 45° = \dfrac{a}{\sqrt{2}a} = \dfrac{\sqrt{2}}{2}$
$\cos 30° = \dfrac{\sqrt{3}a}{2a} = \dfrac{\sqrt{3}}{2}$	$\cos 60° = \dfrac{a}{2a} = \dfrac{1}{2}$	$\cos 45° = \dfrac{a}{\sqrt{2}a} = \dfrac{\sqrt{2}}{2}$
$\tan 30° = \dfrac{\sin 30°}{\cos 30°} = \dfrac{\sqrt{3}}{3}$	$\tan 60° = \dfrac{\sin 60°}{\cos 60°} = \sqrt{3}$	$\tan 45° = \dfrac{\sin 45°}{\cos 45°} = 1$

7 Acute, Right or Obtuse Triangle

If a triangle has sides a, b and c with longest side c, then

$$a^2 + b^2 < c^2 \implies \text{triangle is acute}$$

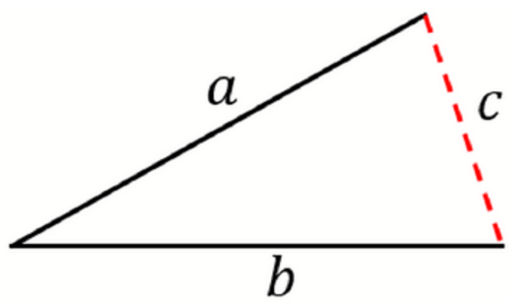
$$a^2 + b^2 = c^2 \implies \text{triangle is right}$$

$$a^2 + b^2 > c^2 \implies \text{triangle is obtuse.}$$

8 Missing Third Side of a Triangle

If side lengths a and b are known, the missing third side length c must satisfy the inequalities

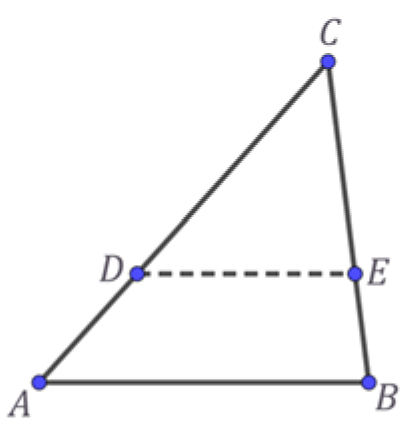
$$|a - b| < c < |a + b|$$



9 Some Geometry Theorems Useful for Test 1B

Note: The theorems that follow are most commonly needed in Test 2B (as opposed to Test 1B) but occasionally you will need to use these results in Test 1B so I've included them.

9.1 Triangle Proportionality Theorem (*a.k.a.* the Side Splitter Theorem)

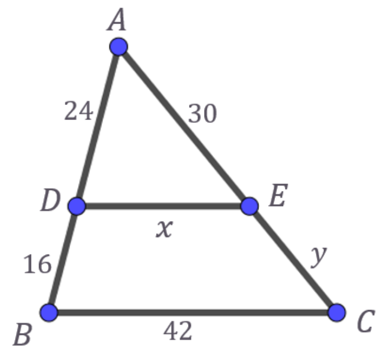


$$\frac{AD}{CD} = \frac{BE}{CE} \text{ if and only if } DE \parallel AB$$

Notation: $DE \parallel AB$ means $\overline{DE}$ and $\overline{AB}$ are parallel.

Example

Find x and y assuming $DE \parallel BC$.



Solution

By the Side Splitter Theorem

$$\frac{24}{16} = \frac{30}{y} \Longrightarrow y = \frac{30 \cdot 16}{24} = 20.$$

Caution!

The Side Splitter is for sides only. We cannot use the side splitter theorem (at least not directly) to find base x.

But we can use similar triangles! We note that corresponding angles in ΔADE and ΔABC are the same (AD and AC are transversals and hence $\angle ADE = \angle ABC$ and $\angle AED = \angle ACB$.

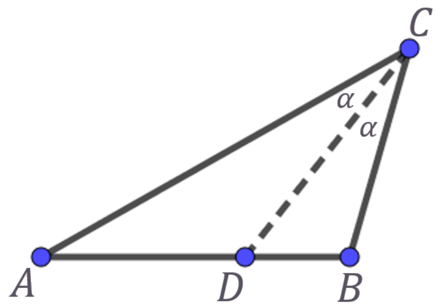
Therefore, by similar triangles

$$\frac{24}{24+16} = \frac{x}{42} \Longrightarrow x = 25.2.$$

9.2 Angle Bisector Theorems

9.2.1 Interior Angle Bisector Theorem

If $\overline{CD}$ bisects interior angle $\angle ACB$ of triangle ΔABC

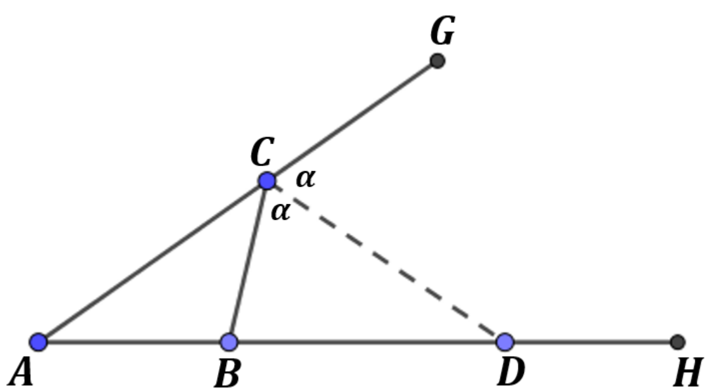


then

$$\frac{CA}{CB} = \frac{DA}{DB} = \frac{Area(\Delta CAD)}{Area(\Delta CBD)}.$$

9.2.2 External Angles Bisector Theorem

Let sides AC and AB of ΔABC be produced (extended) to the points G and H respectively. If the bisector of exterior angle $\angle BCG$ of triangle ΔABC meets line AH at point D,



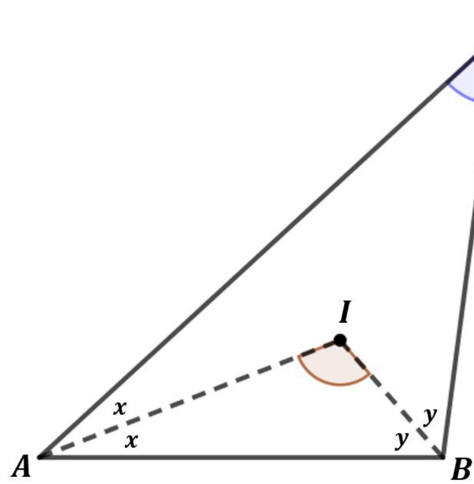
then

$$\frac{CA}{CB} = \frac{DA}{DB}.$$

9.3 Intersection of Angle Bisectors Theorems

9.3.1 Bisector of Interior Angles of a Triangle

If the bisectors of interior angles $\angle CAB$ and $\angle CBA$ of triangle ΔABC meet at point I,

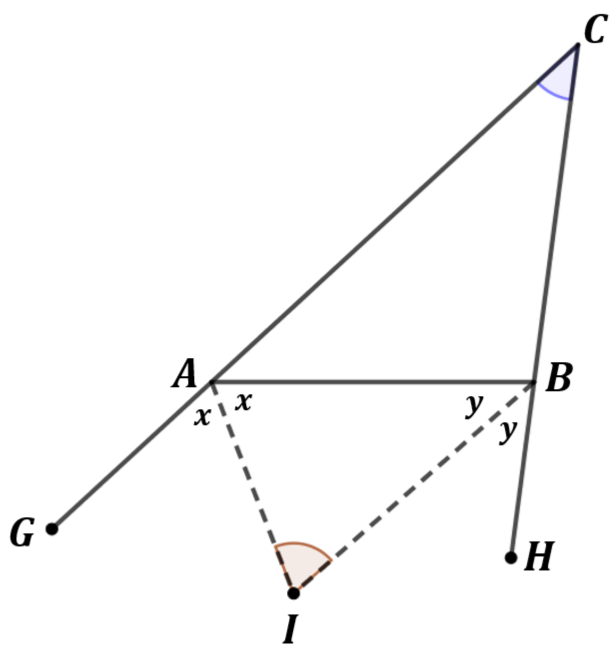


then

$$\angle AIB = 90° + \frac{1}{2}(\angle ACB)$$

9.3.2 Bisector of Exterior Angles of a Triangle

Let sides CA and CB of ΔABC be produced (extended) to points G and H respectively. If the bisectors of exterior angles $\angle GAB$ and $\angle HBA$ of triangle ΔABC meet at point I,

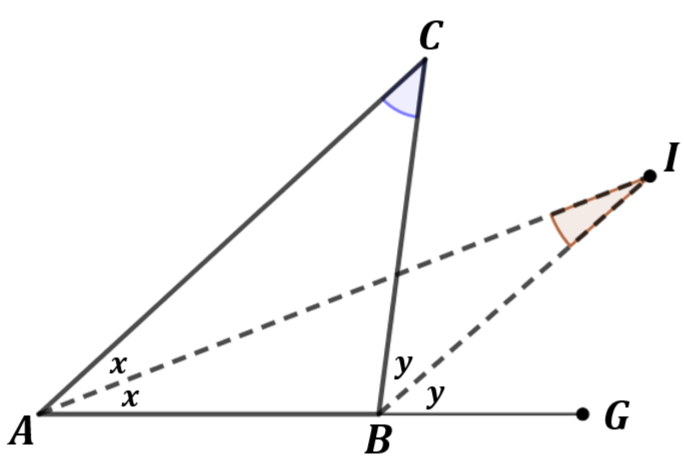


then

$$\angle AIB = 90° - \frac{1}{2}(\angle ACB).$$

9.3.3 Bisector of an Interior and an Exterior Angle of a Triangle

Let side AB of ΔABC be produced (extended) to the point G. If the bisector of interior angle $\angle CAB$ and exterior angle $\angle CBG$ of triangle ΔABC meet at point I,

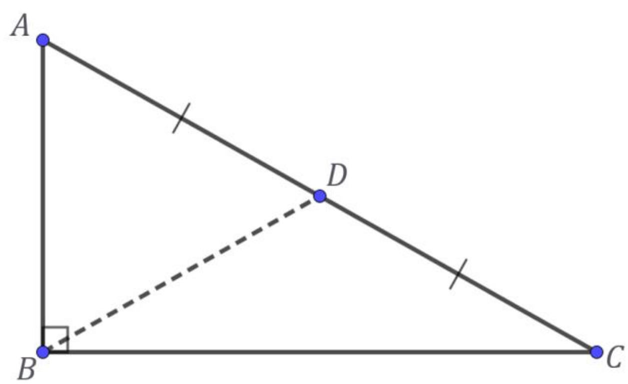


then

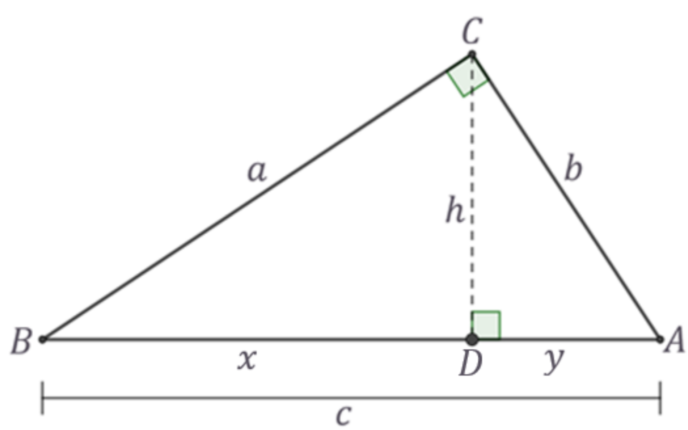
$$\angle AIB = \frac{1}{2}(\angle ACB).$$

9.4 Median to Hypotenuse Theorem

The median drawn from a right angle to the hypotenuse in a right triangle is half as long as the hypotenuse. That is, $BD = AC/2$.

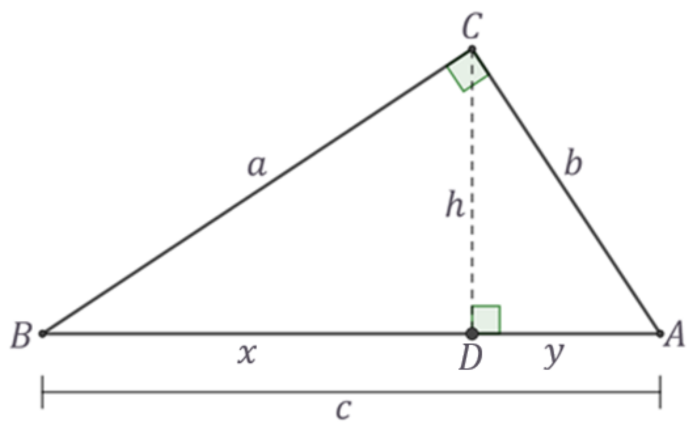


9.5 Altitude to Hypotenuse Theorem (Three Similar Triangles)



$$\Delta ACB \sim \Delta ADC \sim \Delta CDB$$

9.6 Geometric Means Theorem



$$h^2 = x \cdot y$$

$$a^2 = x \cdot c$$

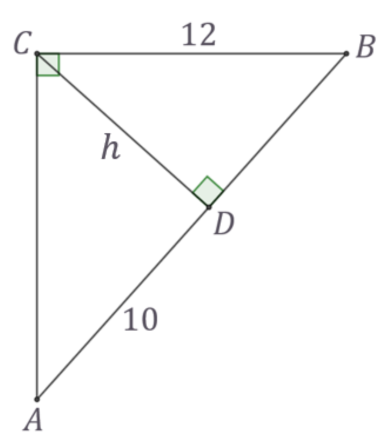
$$b^2 = y \cdot c$$

$$ab = ch$$

$$\frac{1}{h^2} = \frac{1}{a^2} + \frac{1}{b^2}$$

Example

Find h.



Solution

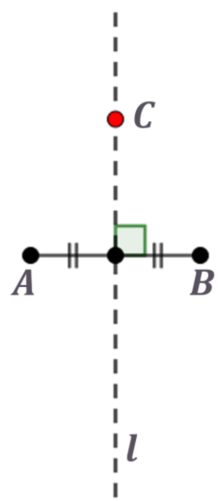
Let $x = \overline{BD}$. Then by the Geometric Means Theorem above we have

$$h^2 = 10x \text{ and } 12^2 = x(10 + x).$$

$$\begin{aligned} 12^2 = x(10 + x) &\Longrightarrow x^2 + 10x - 144 = 0 \\ &\Longrightarrow (x - 8)(x + 18) = 0 \\ &\Longrightarrow x = 8 \text{ or } x = -18. \end{aligned}$$

But x cannot be negative so x must equal 8. Therefore, $h^2 = 10x = 80$ and $h = 4\sqrt{5}$.

9.7 Perpendicular Bisector Theorem and It's Converse



- If point C is on the perpendicular bisector l of line segment $\overline{AB}$, then point C is equidistant from the endpoints A and B of line segment $\overline{AB}$.

- If a point C is equidistant from the endpoints A and B of line segment $\overline{AB}$, then point C is on the perpendicular bisector l of line segment $\overline{AB}$.