MSHSML Meet 1, Event C
Study Guide

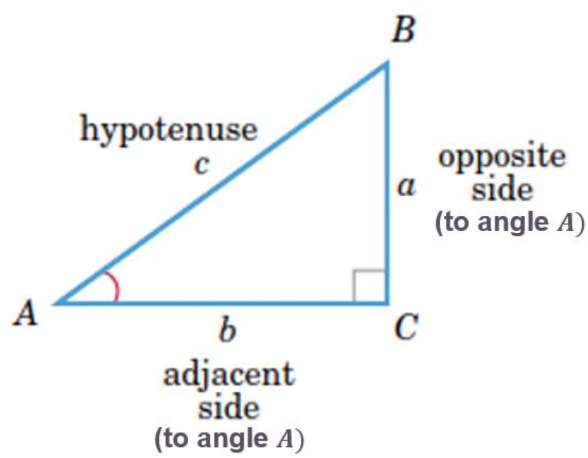
1C Elementary Trigonometry (no calculators)

Definitions and solution of right triangles
Elementary identities
Radian measure and graphs of elementary functions
Trigonometric functions of multiples of $\pi/6, \pi/4, \pi/3, \pi/2$

Definitions and solution of right triangles

...BASED ON A RIGHT TRIANGLE

For an acute angle A, we can define the trigonometric functions by looking at the ratios of the side lengths of a right triangle ABC with a right angle at C.



Sine $\sin A = \frac{a}{c} = \frac{\text{opposite side}}{\text{hypotenuse}}$	**Cosine** $\cos A = \frac{b}{c} = \frac{\text{adjacent side}}{\text{hypotenuse}}$
Tangent $\tan A = \frac{a}{b} = \frac{\text{opposite side}}{\text{adjacent side}}$	**Cotangent** $\cot A = \frac{b}{a} = \frac{\text{adjacent side}}{\text{opposite side}}$
Secant $\sec A = \frac{c}{b} = \frac{\text{hypotenuse}}{\text{adjacent side}}$	**Cosecant** $\csc A = \frac{c}{a} = \frac{\text{hypotenuse}}{\text{opposite side}}$

For angles greater than 90°, apply the right-triangle definition to a reference angle and attach the appropriate $\pm$ sign.

TECHNIQUES FOR SOLVING RIGHT TRIANGLES

Let's say $C = 90°$. There are five unknown quantities: a, b, c, A, B.

If you know…

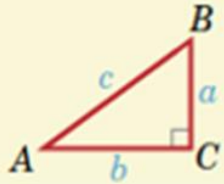
Acute angle and opposite side (say, A and a)

…you can use…

$c = \frac{a}{\sin A}$

$b = \frac{a}{\tan A}$

$B = 90° - A$



Acute angle and adjacent side (say, A and b)

$c = \frac{b}{\cos A}$

$a = b \tan A$

$B = 90° - A$

Elementary identities

PYTHAGOREAN IDENTITIES

$$\sin^2 A + \cos^2 A = 1$$

$$1 + \tan^2 A = \sec^2 A$$

$$\cot^2 A + 1 = \csc^2 A$$

CONVERTING EQUATIONS

Cosine and sine functions differ only by a phase shift.

$$\cos\theta = \sin\left(\theta + \frac{\pi}{2}\right)$$

$$\sin\theta = \cos\left(\theta - \frac{\pi}{2}\right)$$

COFUNCTION IDENTITIES

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta \qquad \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$

$$\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta \qquad \cot\left(\frac{\pi}{2} - \theta\right) = \tan\theta$$

$$\sec\left(\frac{\pi}{2} - \theta\right) = \csc\theta \qquad \csc\left(\frac{\pi}{2} - \theta\right) = \sec\theta$$

REDUCTION IDENTITIES

2nd Quadrant	$\sin(\pi - \theta) = \sin(\theta)$	$\cos(\pi - \theta) = -\cos(\theta)$
3rd Quadrant	$\sin(\pi + \theta) = -\sin(\theta)$	$\cos(\pi + \theta) = -\cos(\theta)$
4th Quadrant	$\sin(2\pi - \theta) = -\sin(\theta)$	$\cos(2\pi - \theta) = \cos(\theta)$

RECIPROCAL AND QUOTIENT IDENTITIES

$$\sin\theta = \frac{1}{\csc\theta} = \frac{\tan\theta}{\sec\theta}$$

$$\cos\theta = \frac{1}{\sec\theta} = \frac{\cot\theta}{\csc\theta}$$

$$\tan\theta = \frac{1}{\cot\theta} = \frac{\sin\theta}{\cos\theta}$$

$$\csc\theta = \frac{1}{\sin\theta} = \frac{\cot\theta}{\cos\theta}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{\tan\theta}{\sin\theta}$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$$

SYMMETRIES

Periodicity

$\sin\theta = \sin(\theta + 2k\pi)$

$\cos\theta = \cos(\theta + 2k\pi)$

$\tan\theta = \tan(\theta + k\pi)$

Even functions

Unchanged if flipped over the x-axis.

$\cos(-\theta) = \cos\theta$

$\sec(-\theta) = \sec\theta$

Odd functions

Unchanged if rotated $180°$. Equivalently, flipping over x-axis is the same as flipping over y-axis.

$\sin(-\theta) = -\sin\theta$

$\tan(-\theta) = -\tan\theta$

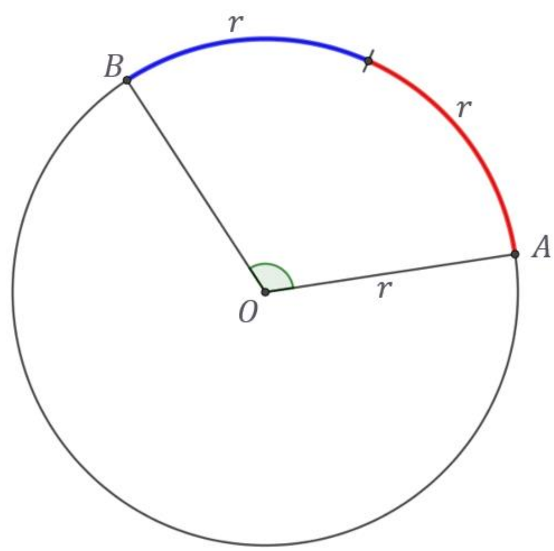
$\csc(-\theta) = -\csc\theta$

$\cot(-\theta) = -\cot\theta$

Radian measure and graphs of elementary functions

Radian – a unit of angle, equal to an angle at the center of a circle whose arc is equal in length to the radius.

$$\angle AOB = 2 \text{ radians}$$



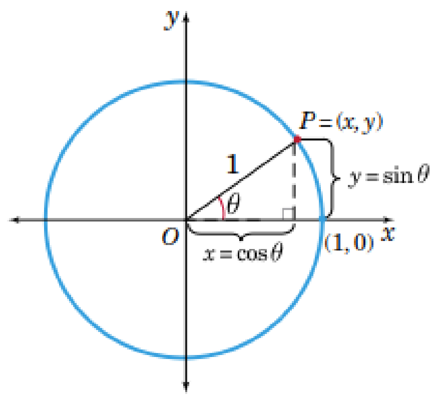
The red and blue arcs each have the same length as the radius of this circle. So Arc $\widehat{AB}$ has the same length as 2 radii. Therefore, by the definition of radians, central angle $\angle AOB$ equals 2 radians.

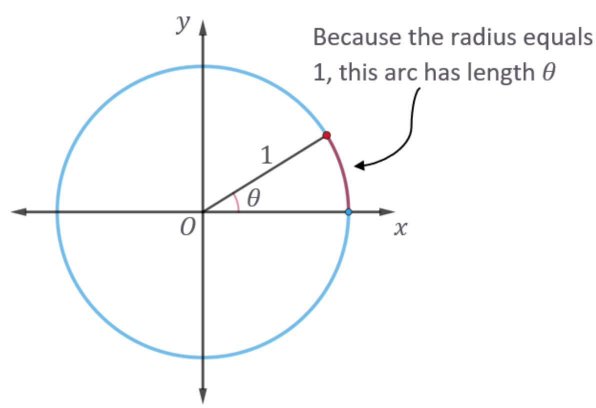
$90° = \frac{\pi}{2}$ radians	$180° = \pi$ radians	$360° = 2\pi$ radians

x degrees $= \left(\frac{\pi}{180} \cdot x\right)$ radians	x radians $= \left(\frac{180}{\pi} \cdot x\right)$ degrees

...BASED ON THE UNIT CIRCLE

Any angle θ defines a point $P = (x, y)$ on the unit circle (circle with radius 1, centered at the origin). The x coordinate is defined to be $\cos(\theta)$ and the y coordinate is defined to be $\sin(\theta)$.

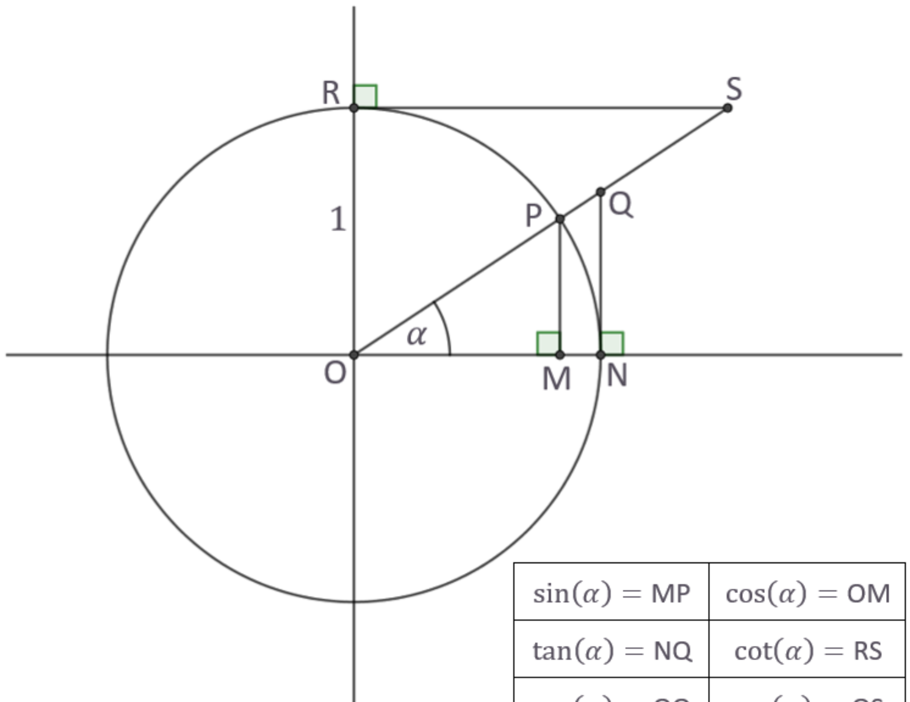




Sine $\sin(\theta) = y$	**Cosine** $\cos(\theta) = x$
Tangent $\tan(\theta) = \frac{y}{x}$	**Cotangent** $\cot(\theta) = \frac{x}{y}$
Secant $\sec(\theta) = \frac{1}{x}$	**Cosecant** $\csc(\theta) = \frac{1}{y}$

Note that $\tan(\theta) = y/x$ equals the slope of the line $\overline{OP}$.

Because θ and $\theta + 2k\pi$ define the same point on the unit circle, all trigonometric functions are periodic with a period of 2π (sin, cos, sec, csc), or π (tan, cot).



$\sin(\alpha) = \text{MP}$	$\cos(\alpha) = \text{OM}$
$\tan(\alpha) = \text{NQ}$	$\cot(\alpha) = \text{RS}$
$\sec(\alpha) = \text{OQ}$	$\csc(\alpha) = \text{OS}$

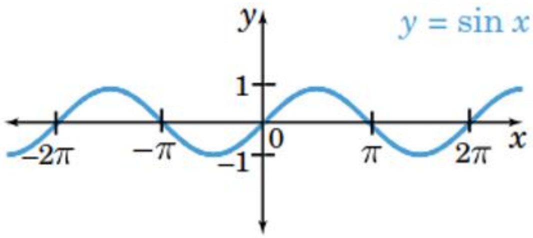
To see why, note that

$\sin(\alpha) = \frac{\text{MP}}{\text{OP}} = \frac{\text{MP}}{1} = \text{MP}$	$\cos(\alpha) = \frac{\text{OM}}{\text{OP}} = \frac{\text{OM}}{1} = \text{OM}$
$\tan(\alpha) = \frac{\text{NQ}}{\text{ON}} = \frac{\text{NQ}}{1} = \text{NQ}$	$\sec(\alpha) = \frac{\text{OQ}}{\text{ON}} = \frac{\text{OQ}}{1} = \text{OQ}$

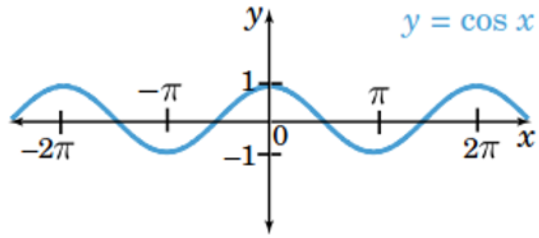
Then note that $\angle RSO = \alpha$ because $\angle MOP$ and $\angle RSO$ are opposite interior angles of the transversal OS cutting the parallel lines ON and RS. Based on the angle $\alpha = \angle RSO$ we have

$\cot(\alpha) = \frac{\text{RS}}{\text{OR}} = \frac{\text{RS}}{1} = \text{RS}$	$\csc(\alpha) = \frac{\text{OS}}{\text{OR}} = \frac{\text{OS}}{1} = \text{OS}.$

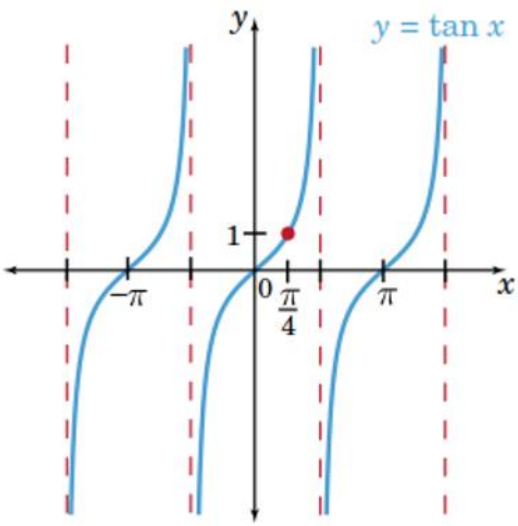
SINE



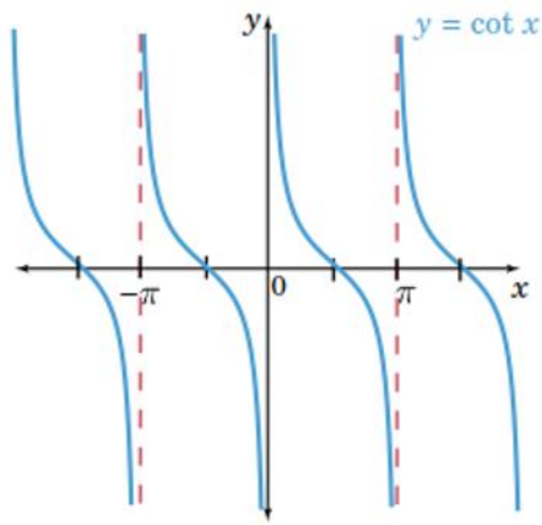
COSINE



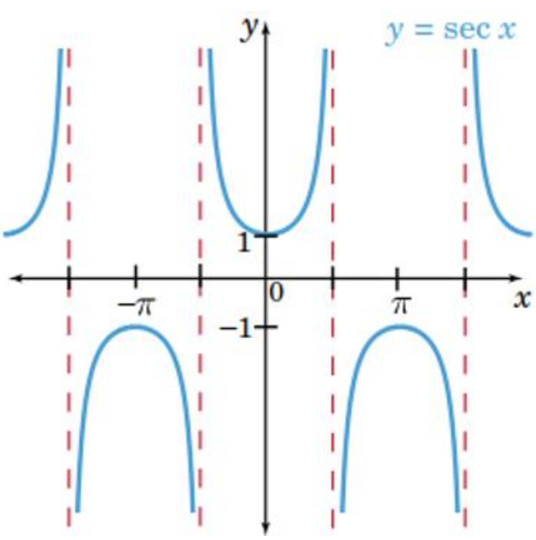
TANGENT



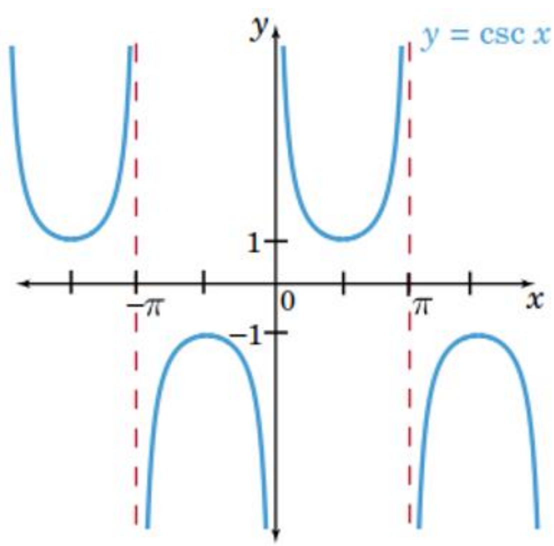
COTANGENT



SECANT



COSECANT



GRAPHING $y = A \sin B(x - h) + k$ AND $y = A \cos B(x - h) + k$

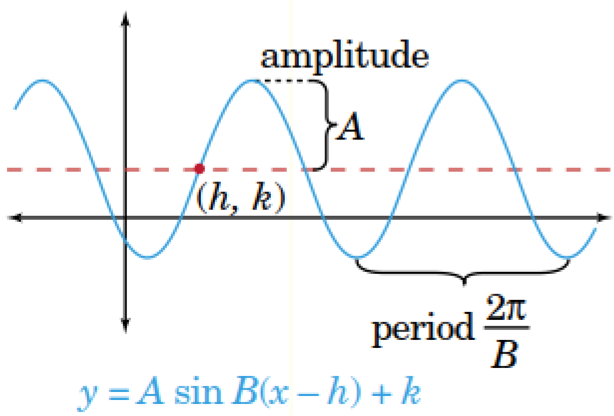
$|A|$ is the **amplitude.**

k is the is the **average value:** halfway between the maximum and the minimum value of the function.

$\frac{2\pi}{B}$ is the **period.** There are B cycles in every interval of length 2π; so $\frac{B}{2\pi}$ is the **frequency**.

h is the **phase shift,** or how far the beginning of the cycle is from the y-axis.

The basic shape of the function will stay the same. The sine curve will start at (h, k) as though it were the origin and go up if A is positive (down if A is negative). A cosine curve will start at (h, k) at the crest if A is positive (trough if A is negative).

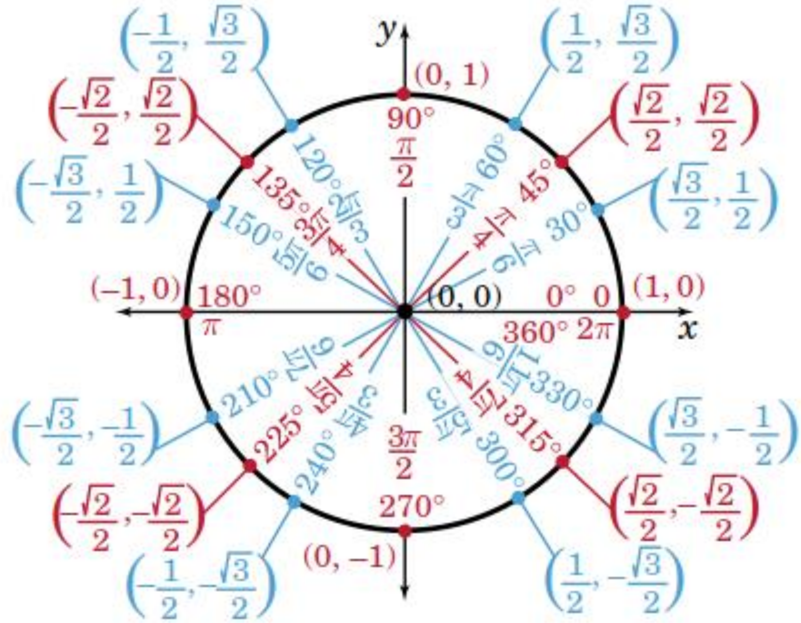


Trigonometric functions of multiples of $\pi/6, \pi/4, \pi/3, \pi/2$

SPECIAL TRIGONOMETRIC VALUES

θ (°)	θ (rad)	sin θ	cos θ	tan θ	csc θ	sec θ	cot θ
0°	0	$0 = \frac{\sqrt{0}}{2}$	1	0	undefined	1	undefined
30°	$\frac{\pi}{6}$	$\frac{1}{2} = \frac{\sqrt{1}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$
90°	$\frac{\pi}{2}$	$1 = \frac{\sqrt{4}}{2}$	0	undefined	1	undefined	0
120°	$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	-2	$-\frac{\sqrt{3}}{3}$
135°	$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	-1	$\sqrt{2}$	$-\sqrt{2}$	-1
150°	$\frac{5\pi}{6}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3}$	2	$-\frac{2\sqrt{3}}{3}$	$-\sqrt{3}$
180°	π	0	-1	0	undefined	-1	undefined
210°	$\frac{7\pi}{6}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	-2	$-\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
225°	$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	1	$-\sqrt{2}$	$-\sqrt{2}$	1
240°	$\frac{4\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\sqrt{3}$	$-\frac{2\sqrt{3}}{3}$	-2	$\frac{\sqrt{3}}{3}$
270°	$\frac{3\pi}{2}$	-1	0	undefined	-1	undefined	0
300°	$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\sqrt{3}$	$-\frac{2\sqrt{3}}{3}$	2	$-\frac{\sqrt{3}}{3}$
315°	$\frac{7\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	-1	$-\sqrt{2}$	$\sqrt{2}$	-1
330°	$\frac{11\pi}{6}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3}$	-2	$\frac{2\sqrt{3}}{3}$	$-\sqrt{3}$
360° = 0°	$2\pi = 0$	0	1	0	undefined	1	undefined

Common angles and the points they define on the unit circle



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