MSHSML Meet 1, Event D
Study Guide

1D Roots of Quadratic and Polynomial Equations (no calculators)

Solution of quadratic equations by factoring, by completing the square, by formula
Complex roots of quadratic equations; the discriminant and the character of the roots
Relations between roots and coefficients
Synthetic division
Function notation

Section 1 The Algebra of Polynomials

1.1 Definitions and Terms

A polynomial $P(x)$ in the **variable** x is an expression of the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x^1 + a_0,$$

where $a_0, a_1, \ldots, a_n$, $a_n \neq 0$ are numbers in some specified set A. Usually, the set A is $\mathbb{Z}$ (the set of all integers), $\mathbb{Q}$ (the set of all rational numbers), $\mathbb{R}$ (the set of all real numbers) or $\mathbb{C}$ (the set of all complex numbers). Remember that $\mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$.

Sometimes you might see the term a_0 written as $a_0 x^0$ to make it clear that this term has the same form as the other terms.

Throughout this course we will assume $n \in \mathbb{N} = \{0,1,2, \ldots\}$.

Each individual term of the polynomial is called a **monomial** or simply a **term**. The constants a_i are called the **coefficients** of the polynomial.

By convention the terms in a polynomial are listed in descending order according to the value of the exponent of the variable x in that term. That is, by convention we would write $x^2 + 3x - 2$ instead of $3x - 2 + x^2$.

The highest exponent of x in a polynomial is called the **degree** of that polynomial. The term with the highest exponent of x is called the **leading term**. The term $a_0 = a_0 x^0$ is called the **constant term**.

By definition, the coefficient of the leading term cannot equal 0. So if the leading term is the constant term then by definition $a_0 \neq 0$. That is, $P(x) = c$ is a polynomial but only for $c \neq 0$. Note that the constant polynomial $P(x) = c = cx^0$ with $c \neq 0$ has degree 0.

There are some special names for polynomials whose degree is small.

Degree 0	Constant
Degree 1	Linear
Degree 2	Quadratic
Degree 3	Cubic
Degree 4	Quartic

Degree 5	Quintic

Polynomials are typically identified by their degree as a form of shorthand. That is, if $f(x) = 2x^4 + x^3 - 2 + 5$, then we would simply say that f is a quartic or a fourth-degree polynomial. And $g(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ would be called an n^{th} degree polynomial. In those cases, such as in $g(x)$, where the coefficients are symbolic, there is always a tacit assumption that a_n, the coefficient of the leading term, is non-zero.

Examples.

Polynomial $P(x)$	**Leading Term**	**Constant Term**	**Degree**
$3x^2 - x + 5$	$3x^2$	5	2
$\sqrt{3}x$	$\sqrt{3}x$	0	1
$4x^3 - \frac{2}{3}x - 1$	$4x^2$	-1	3
5	5	5	0

If the coefficient of the leading term equals 1, we say that the polynomial is **monic**. So $P(x) = x^2 + 3x - 1$ is said to be monic but $P(x) = 4x^2 + 3x - 1$ is not.

A **zero** of the function $f(x)$ is a number r such that $f(r) = 0$. When $f(r) = 0$, we also say that r is a **root** of the equation $f(x) = 0$.

Common Mistake

While using the terms "zero" and "root" interchangeably is common and does not lead to much, if any confusion, to be precise, remember that it's a "zero of a function" and a "root of an equation".

Finding the roots of a polynomial equation is a major topic in the theory of equations. We will devote the entire of Chapter 2 to this topic.

1.2 Arithmetic of Polynomials

• **Addition of polynomials**

combine like terms (*i.e.* add the terms where the variable is raised to the same power)

$(3x^2 - 2x + 5) + (2x^4 - x + 5) + (-x^4 + x^3 - x^2 + x + 2) = x^4 + x^3 + 2x^2 - 2x + 1$

	x^4	x^3	x^2	x^1	x^0
			3	-2	5
+	2			-1	5
	-1	1	-1	1	2
	1	1	2	-2	12
	$(= 1x^4 + 1x^3 + 2x^2 - 2x^1 + 12x^0)$				

• **Multiplication of a polynomial by a monomial**

multiply each term by that monomial

$$(2x^3) \cdot (2x^4 - x + 5) = (2 \cdot 2)(x^3x^4) - (2 \cdot 1)(x^3x) + (2 \cdot 5)(x^3) = 4x^7 - 2x^4 + 10x^3$$

$$(-1) \cdot (-x^4 + x^3 - x^2 + x + 2) = x^4 - x^3 + x^2 - x - 2$$

• **Subtraction of polynomials**

write $P(x) - F(x) - G(x)$ as $P(x) + (-1)F(x) + (-1)G(x)$. Then use the rule for multiplication of a polynomial by a monomial to simplify $(-1)F(x)$ and $(-1)G(x)$. Then use the rule for the addition of polynomials.

$$(3x^2 - 2x + 5) - (2x^4 - x + 5) - (-x^4 + x^3 - x^2 + x + 2)$$

$$= (3x^2 - 2x + 5) + (-1)(2x^4 - x + 5) + (-1)(-x^4 + x^3 - x^2 + x + 2)$$

$$= (3x^2 - 2x + 5) + (-2x^4 + x - 5) + (x^4 - x^3 + x^2 - x - 2)$$

$$= (-2 + 1)x^4 + (-1)x^3 + (3 + 1)x^2 + (-2 + 1 - 1)x + (5 - 5 - 2)$$

$$= -x^4 - x^3 + 4x^2 - 2x - 2.$$

	x^4	x^3	x^2	x^1	x^0
			3	-2	5
+	$(-1) \cdot 2$			$(-1) \cdot (-1)$	$(-1) \cdot 5$
	$(-1) \cdot (-1)$	$(-1) \cdot 1$	$(-1) \cdot (-1)$	$(-1) \cdot 1$	$(-1) \cdot 2$
	-1	-1	4	-2	-2
	$(= -x^4 - x^3 + 4x^2 - 2x - 2)$				

• **Multiplication of Two Polynomials**

distribute the first polynomial across the monomials making up the second polynomial. Then use the rule for multiplying a polynomial and a monomial.

$(x^4 - 2x^3 - 6)(2x^3 - x + 1)$

$= (x^4 - 2x^3 - 6)(2x^3) + (x^4 - 2x^3 - 6)(-x) + (x^4 - 2x^3 - 6)(1)$

$= (x^4)(2x^3) + (-2x^3)(2x^3) + (-6)(2x^3)$

$+ (x^4)(-x) + (-2x^3)(-x) + (-6)(-x)$

$+ (x^4)(1) + (-2x^3)(1) + (-6)(1)$

$= 2x^7 - 4x^6 - 12x^3 - x^5 + 2x^4 + 6x + x^4 - 2x^3 - 6$

$= 2x^7 - 4x^6 - x^5 + (2+1)x^4 + (-12-2)x^3 + 6x - 6$

$= 2x^7 - 4x^6 - x^5 + 3x^4 - 14x^3 + 6x - 6$

While the above definition is instructive for understanding what $P(x)Q(x)$ means it is not an efficient way to carry out the process. The **method of detached coefficients** shown below (with **zeroes inserted for any missing term**) is more efficient and methodical and hence less prone to error.

Notice that there is **no "carrying"** from one column to the next as there is when you use this schema in the ordinary multiplication of integers.

Consider the example,

$$(x^4 - 2x^3 - 6) \cdot (2x^3 - x + 1)$$

x^7	x^6	x^5	x^4	x^3	x^2	x^1	x^0
			1	−2	0	0	−6
				2	0	−1	1
			1	−2	0	0	−6
		−1	2	0	0	6	
	0	0	0	0	0		
2	−4	0	0	−12			
2	−4	−1	3	−14	0	6	−6

This shows that

$$(x^4 - 2x^3 - 6)(2x^3 - x + 1) = 2x^7 - 4x^6 - x^5 + 3x^4 - 14x^3 + 6x - 6.$$

● **Division of Two Polynomials**

Division Theorem

If $f(x)$ and $d(x)$ are polynomials, with $d(x) \neq 0$, then there exist unique polynomials $q(x)$ and $r(x)$, where $r(x)$ is either 0 or of degree less than the degree of $d(x)$, such that

$$f(x) = d(x) \cdot q(x) + r(x)$$

or equivalently

$$\frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}.$$

Each of these polynomials has its own name.

$f(x)$	**Dividend**
$d(x)$	**Divisor**
$q(x)$	**Quotient**
$r(x)$	**Remainder**

The division theorem is an *existence* and *uniqueness* result only. It does not actually give a method (algorithm) for finding the quotient and remainder. We need a separate algorithm for that.

Usage Note: Wherever the phrases "$d(x)$ divides $f(x)$" and "$d(x)$ is a factor of $f(x)$" are used, the phrase ending "without remainder" is to be tacitly assumed.

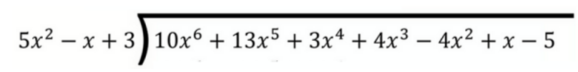
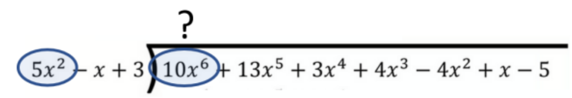
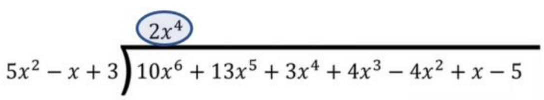
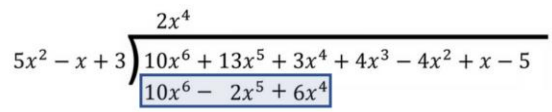
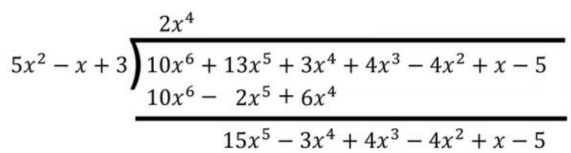
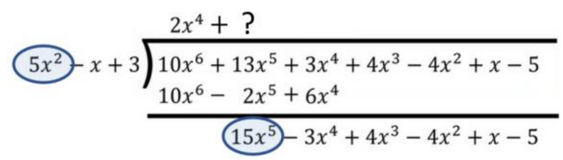
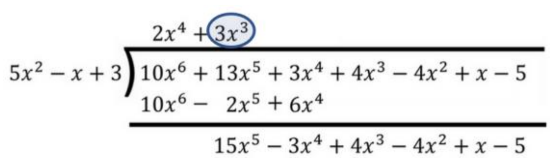
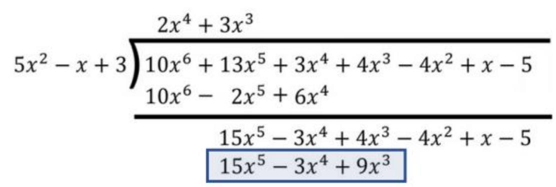
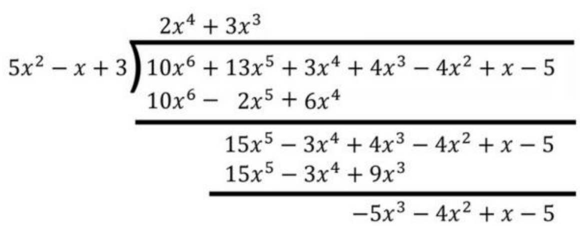
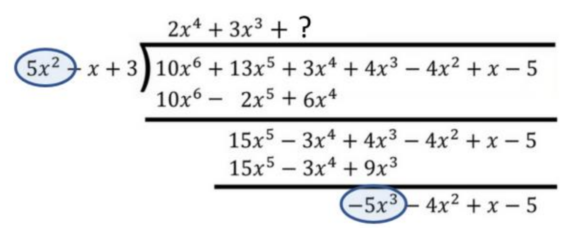
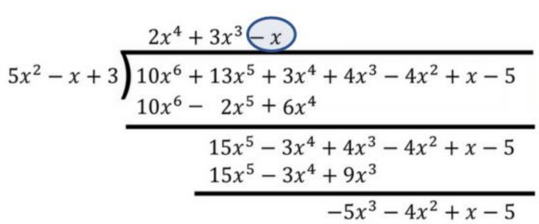
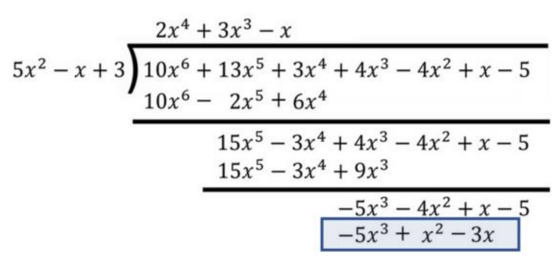
► **The Long Division Algorithm**

The method for actually finding the quotient and remainder goes by the very similar name of "the long division algorithm". The clearest approach to describing the long division algorithm is to carefully step through a particular example.

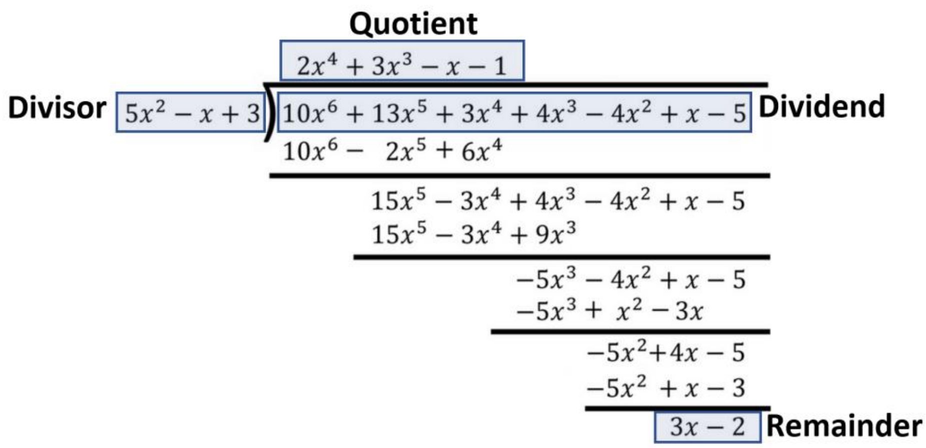
Example

Find polynomials $q(x)$ and $r(x)$, where the degree of $r(x)$ is less than two, such that

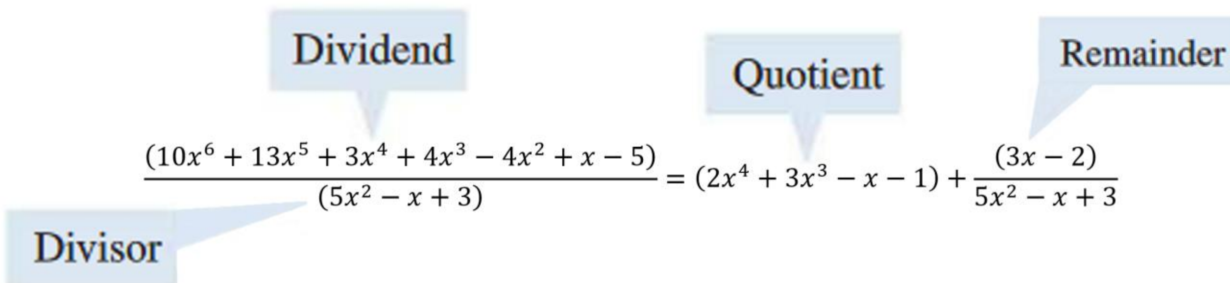
$$\frac{10x^6 + 13x^5 + 3x^4 - 4x^2 + x - 5}{5x^2 - x + 3} = q(x) + \frac{r(x)}{5x^2 - x + 3}.$$

1) Express in long division format	2) What times $5x^2$ gives $10x^6$?
	
3) $5x^2 \times 2x^4 = 10x^6$	4) Multiply each term in the divisor by $2x^4$
	
5) Draw a line and subtract	6) What times $5x^2$ gives $15x^5$?
	
7) $5x^2 \times 3x^3 = 15x^5$	8) Multiply each term in the divisor by $3x^3$
	
9) Draw a line and subtract	10) What times $5x^2$ gives $-5x^3$?
	
11) $5x^2 \times (-x) = -5x^3$	12) Multiply each term in the divisor by $-x$
	

13) Draw a line and subtract	14) What times $5x^2$ gives $-5x^2$?
$2x^4 + 3x^3 - x$ $5x^2 - x + 3 \overline{)\,10x^6 + 13x^5 + 3x^4 + 4x^3 - 4x^2 + x - 5}$ $10x^6 - 2x^5 + 6x^4$ $15x^5 - 3x^4 + 4x^3 - 4x^2 + x - 5$ $15x^5 - 3x^4 + 9x^3$ $-5x^3 - 4x^2 + x - 5$ $-5x^3 + x^2 - 3x$ $-5x^2 + 4x - 5$	$2x^4 + 3x^3 - x +$? $5x^2 - x + 3 \overline{)\,10x^6 + 13x^5 + 3x^4 + 4x^3 - 4x^2 + x - 5}$ $10x^6 - 2x^5 + 6x^4$ $15x^5 - 3x^4 + 4x^3 - 4x^2 + x - 5$ $15x^5 - 3x^4 + 9x^3$ $-5x^3 - 4x^2 + x - 5$ $-5x^3 + x^2 - 3x$ $-5x^2 + 4x - 5$
15) $5x^2 \times (-1) = -5x^2$	16) Multiply each term in the divisor by -1
$2x^4 + 3x^3 - x - 1$ $5x^2 - x + 3 \overline{)\,10x^6 + 13x^5 + 3x^4 + 4x^3 - 4x^2 + x - 5}$ $10x^6 - 2x^5 + 6x^4$ $15x^5 - 3x^4 + 4x^3 - 4x^2 + x - 5$ $15x^5 - 3x^4 + 9x^3$ $-5x^3 - 4x^2 + x - 5$ $-5x^3 + x^2 - 3x$ $-5x^2 + 4x - 5$	$2x^4 + 3x^3 - x - 1$ $5x^2 - x + 3 \overline{)\,10x^6 + 13x^5 + 3x^4 + 4x^3 - 4x^2 + x - 5}$ $10x^6 - 2x^5 + 6x^4$ $15x^5 - 3x^4 + 4x^3 - 4x^2 + x - 5$ $15x^5 - 3x^4 + 9x^3$ $-5x^3 - 4x^2 + x - 5$ $-5x^3 + x^2 - 3x$ $-5x^2 + 4x - 5$ $-5x^2 + x - 3$
17) Draw a line and subtract	18)
$2x^4 + 3x^3 - x - 1$ $5x^2 - x + 3 \overline{)\,10x^6 + 13x^5 + 3x^4 + 4x^3 - 4x^2 + x - 5}$ $10x^6 - 2x^5 + 6x^4$ $15x^5 - 3x^4 + 4x^3 - 4x^2 + x - 5$ $15x^5 - 3x^4 + 9x^3$ $-5x^3 - 4x^2 + x - 5$ $-5x^3 + x^2 - 3x$ $-5x^2 + 4x - 5$ $-5x^2 + x - 3$ $3x - 2$	STOP The long division algorithm ends when the degree of the remainder, $3x - 2$, is strictly less than the degree of the divisor, $5x^2 - x + 3$.



So, from the above long division tableau we can immediately write down that



$$\frac{(10x^6+13x^5+3x^4+4x^3-4x^2+x-5)}{(5x^2-x+3)} = (2x^4+3x^3-x-1)+\frac{(3x-2)}{5x^2-x+3}$$

Missing Terms in the Dividend

If you have any powers of x "missing" in the dividend, the long division algorithm is easier to implement if you insert placeholder 0's. For example, in the problem

$$x^3-2x+3\,\overline{)\;x^6+2x^5-x+2}$$

you will be less prone to make an error the long division algorithm if you make room for the missing x^4, x^3 and x^2 powers in your dividend

$$x^3-2x+3\,\overline{)\;x^6+2x^5 \qquad\qquad\qquad\qquad -x+2}$$

and backfill with 0's

$$x^3-2x+3\,\overline{)\;x^6+2x^5+0x^4+0x^3+0x^2-x+2}$$

as shown above. This will allow you to keep all terms with like powers (e.g. all terms with an x^4 factor) lined up in the same column.

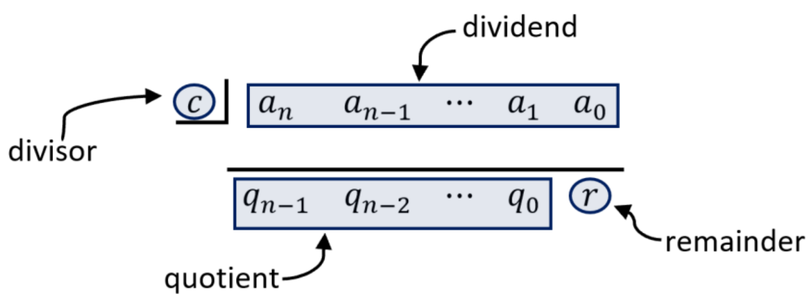
► **Synthetic Division**

According to the division algorithm, dividing the n^{th} degree polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ by the monic linear polynomial $x - c$ will result in an $(n-1)^{\text{st}}$ degree quotient polynomial $q(x) = q_{n-1}x^{n-1} + q_{n-2}x^{n-2} + \cdots + q_1 x + q_0$ and a constant remainder r.

That is,

$$\frac{f(x)}{x-c} = q(x) + \frac{r}{x-c}.$$

The synthetic division algorithm is a shortcut alternative to the standard long division algorithm in the special case of a monic linear divisor. Synthetic division condenses the entire of the polynomial division problem $f(x)/(x-c)$ into a compact schema of the form



Example 1:

For an example of synthetic division, consider $3x^3 - 6x + 2$ divided by $x - 2$.

First, if a power of x is missing from the polynomial, a term with that power and a zero coefficient must be inserted into the correct position in the polynomial. In this example, the x^2 term is missing, so we must add $0x^2$ between the cubic and linear terms:

$$3x^3 + 0x^2 - 6x + 2.$$

Next, all the variables and their exponents (x^3, x^2, x) are removed, leaving only a list of the coefficients: $3, 0, -6, 2$. These numbers form the dividend.

We form the divisor for the synthetic division using only the constant term (2) of the linear factor $x - 2$. (Note: If the divisor were $x + 2$, we would put it in the format $x - (-2)$, resulting in a divisor of -2.)

The numbers representing the divisor and the dividend are placed into a division-like configuration:

$$\begin{array}{r|rrrr} 2 & 3 & 0 & -6 & 2 \\ \hline & & & & \end{array}$$

The first number in the dividend (3) is put into the first position of the quotient area (below the horizontal line). This number is the coefficient of the x^3 term in the original polynomial.

$$\begin{array}{r|rrrr} 2 & 3 & 0 & -6 & 2 \\ \hline & 3 & & & \end{array}$$

Then this first entry in the quotient (3) is multiplied by the divisor (2) and the product is placed under the next term in the dividend (0):

$$\begin{array}{r|rrrr} 2 & 3 & 0 & -6 & 2 \\ & & 6 & & \\ \hline & 3 & & & \end{array}$$

Next the number from the dividend and the result of the multiplication are added together and the sum is put in the next position on the quotient line:

$$\begin{array}{r|rrrr} 2 & 3 & 0 & -6 & 2 \\ & & 6 & & \\ \hline & 3 & 6 & & \end{array}$$

This process is continued for all numbers making up the dividend:

1

$$\begin{array}{r|rrrr} 2 & 3 & 0 & -6 & 2 \\ & & 6 & 12 & \\ \hline & 3 & 6 & & \end{array}$$

2

$$\begin{array}{r|rrrr} 2 & 3 & 0 & -6 & 2 \\ & & 6 & 12 & \\ \hline & 3 & 6 & & \end{array}$$

3

$$\begin{array}{r|rrrr} 2 & 3 & 0 & -6 & 2 \\ & & 6 & 12 & 12 \\ \hline & 3 & 6 & 6 & \end{array}$$

4

$$\begin{array}{r|rrrr} 2 & 3 & 0 & -6 & 2 \\ & & 6 & 12 & 12 \\ \hline & \boxed{3} & \boxed{6} & \boxed{6} & \boxed{14} \end{array}$$

The result is the list $3, 6, 6, 14$. All numbers except the last become the coefficients of the quotient polynomial. Since we started with a cubic polynomial and divided it by a linear term, the quotient is a 2nd degree polynomial:

$$3x^2 + 6x + 6.$$

The last entry in the result list (14) is the remainder. The quotient and remainder can be combined into one expression:

$$\boxed{3}x^2 + \boxed{6}x + \boxed{6} + \left(\frac{\boxed{14}}{x-2}\right).$$

(Note that no division operations were performed to compute the answer to this division problem.)

To verify that this process has worked, we can multiply the quotient and the divisor and add the remainder to obtain the original polynomial:

$$(3x^2 + 6x + 6) \times (x - 2) = 3x^3 - 6x - 12$$

$$(3x^3 - 6x - 12) + 14 = 3x^3 - 6x + 2.$$

Example 2:

Find $(5x^3 - 13x^2 + 10x - 8) \div (x - 2)$ using synthetic division.

1 $\begin{array}{r\|rrrr} 2 & 5 & -13 & 10 & -8 \\ & & & & \\ \hline & & & & \end{array}$	**2** $\begin{array}{r\|rrrr} 2 & 5 & -13 & 10 & -8 \\ & & & & \\ \hline & 5 & & & \end{array}$
3 $\begin{array}{r\|rrrr} 2 & 5 & -13 & 10 & -8 \\ & & 10 & & \\ \hline & 5 & & & \end{array}$	**4** $\begin{array}{r\|rrrr} 2 & 5 & -13 & 10 & -8 \\ & & 10 & & \\ \hline & 5 & -3 & & \end{array}$
5 $\begin{array}{r\|rrrr} 2 & 5 & -13 & 10 & -8 \\ & & 10 & -6 & \\ \hline & 5 & -3 & & \end{array}$	**6** $\begin{array}{r\|rrrr} 2 & 5 & -13 & 10 & -8 \\ & & 10 & -6 & \\ \hline & 5 & -3 & 4 & \end{array}$
7 $\begin{array}{r\|rrrr} 2 & 5 & -13 & 10 & -8 \\ & & 10 & -6 & 8 \\ \hline & 5 & -3 & 4 & \end{array}$	**8** $\begin{array}{r\|rrrr} 2 & 5 & -13 & 10 & -8 \\ & & 10 & -6 & 8 \\ \hline & 5 & -3 & 4 & 0 \end{array}$

$$(5x^3 - 13x^2 + 10x - 8) \div (x - 2) = 5x^2 - 3x + 4$$

Example 3:

Find $(2y^3 + y^2 + 4y - 3) \div (y + 2)$ using synthetic division.

1 $\begin{array}{r\|rrrr} -2 & 2 & 1 & 4 & -3 \\ & & & & \\ \hline & & & & \end{array}$	**2** $\begin{array}{r\|rrrr} -2 & 2 & 1 & 4 & -3 \\ & & & & \\ \hline & 2 & & & \end{array}$
3 $\begin{array}{r\|rrrr} -2 & 2 & 1 & 4 & -3 \\ & & -4 & & \\ \hline & 2 & & & \end{array}$	**4** $\begin{array}{r\|rrrr} -2 & 2 & 1 & 4 & -3 \\ & & -4 & & \\ \hline & 2 & -3 & & \end{array}$
5 $\begin{array}{r\|rrrr} -2 & 2 & 1 & 4 & -3 \\ & & -4 & 6 & \\ \hline & 2 & -3 & & \end{array}$	**6** $\begin{array}{r\|rrrr} -2 & 2 & 1 & 4 & -3 \\ & & -4 & 6 & \\ \hline & 2 & -3 & 10 & \end{array}$
7 $\begin{array}{r\|rrrr} -2 & 2 & 1 & 4 & -3 \\ & & -4 & 6 & -20 \\ \hline & 2 & -3 & 10 & \end{array}$	**8** $\begin{array}{r\|rrrr} -2 & 2 & 1 & 4 & -3 \\ & & -4 & 6 & -20 \\ \hline & 2 & -3 & 10 & -23 \end{array}$

$$(2y^3 + y^2 + 4y - 3) \div (y + 2) = y^2 - 3y + 10 - \left(\frac{23}{x + 2}\right)$$

Why use the Synthetic Division Algorithm?

The synthetic division algorithm has several advantages over the long division algorithm.

(1) It is more compact. This may be hard to see in the above examples because I wrote out each step as its own diagram. But in actual practice you would do all your work in a single diagram. The entire of $(2y^3 + y^2 + 4y - 3) \div (y + 2)$ is completed in the single diagram

$$\begin{array}{r|rrrr} -2 & 2 & 1 & 4 & -3 \\ & & -4 & 6 & -20 \\ \hline & 2 & -3 & 10 & -23 \end{array}$$

Additionally, synthetic division is more compact because you do not include the factor y^k with each coefficient.

(2) The long division algorithm uses division and subtract. In contrast, the synthetic division algorithm uses multiplication and addition. Most people (myself included) can do mental multiplication and addition faster and more accurately than they can mental division and subtraction.

Why use the Long Division Algorithm?

The synthetic division algorithm as explained above *only* applies when the divisor is a monic linear polynomial (*i.e.* a polynomial of the form $x - a$ or $x + a$). Therefore, to this point there is no choice to the long division algorithm when the divisor is not a monic linear polynomial.

Is it possible to generalize the synthetic division algorithm beyond monic linear polynomial divisors? Technically, yes. But the clear-cut advantages of synthetic division over long division for monic linear divisors do not all carry over to higher degree divisors. The article "Synthetic Division for Nonlinear Factors", Kurt Reimann, *The Mathematics Teacher*, Vol. 73, No. 3, March 1980, gives a highly readable account of the extended synthetic division algorithm. The wiki page on synthetic division, https://en.wikipedia.org/wiki/Synthetic_division also describes the extended algorithm.

Remainder Theorem

If a polynomial $f(x)$ is divided by $(x - a)$ then the remainder is a constant and equals $f(a)$.

Factor Theorem

$(x - a)$ is a factor of the polynomial $f(x)$ if and only if $f(a) = 0$.

Extended Factor Theorem

If $r_1, r_2, \ldots, r_k$ are zeros of the polynomial $f(x)$, then $(x - r_1)(x - r_2) \cdots (x - r_k)$ divides $f(x)$. That is,

$$f(x) = (x - r_1)(x - r_2) \cdots (x - r_k) \cdot q(x)$$

for some quotient polynomial $q(x)$.

Multiplicity of a Root of a Polynomial Equation

If the polynomial

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

can be written as

$$f(x) = (x - r)^k s(x)$$

for some polynomial $s(x)$ such that $s(r) \neq 0$, then we say that the root r of $f(x) = 0$ has multiplicity k.

A root with multiplicity 1 is called a **simple** root.

Fundamental Theorem of Algebra

The fundamental theorem of algebra states that an n^{th} degree polynomial has exactly n roots in the complex numbers if we count multiplicities as separate roots.

Because the set of real numbers are a subset of the set of complex numbers, the fundamental theorem of algebra implies that an n^{th} degree polynomial can have **at most $\boldsymbol{n}$ real** roots.

Equality of Two Polynomials Theorem

Two polynomials are equal **for all $\boldsymbol{x}$** if and only if their coefficients are equal term by term, that is, if the coefficients of the monomials of the same degree are equal.

For example, if $3x^4 - 2x^3 + 5 = ax^5 + bx^4 + cx^3 + dx^2 + ex^1 + f$ for *all* x, then $a = 0, b = 3, c = -2, d = 0, e = 0$ and $f = 5$.

This is often referred to as simply "***equating coefficients***".

Expressing a Polynomial in Terms of its Roots

If $r_1, r_2, \dots, r_n$ are the n roots of the n^{th} degree polynomial equation $f(x) = 0$ where

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

then

$$f(x) = a_n(x - r_1)(x - r_2)\cdots(x - r_n).$$

Sum and Product of the Roots

If $r_1, r_2, \dots, r_n$ are the n roots of the n^{th} degree polynomial equation $f(x) = 0$ where

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

then

$$r_1 + r_2 + \cdots + r_n = -\frac{a_{n-1}}{a_n}$$

and

$$r_1 \cdot r_2 \cdots r_n = (-1)^n \frac{a_0}{a_n}.$$

These are special cases of *Viète's Theorem* which we will discuss at length in Section 3.

1.3 Remainder Theorem Applications

The first two exercises below show that in some problems the easier way of finding the remainder of $f(x)/(x-a)$ is by constructing it through synthetic division. And other times it will be easier to find the remainder via the Remainder Theorem.

Exercise 1.1

Find the remainder when $x^8 - 15x^7 + 46x^6 + 68x^5 + 94x^4 + 15x^3 + 9x^2 - 3x - 26$ is divided by $x - 8$.

Solution

Here is a situation where using synthetic division is easier than using the Remainder Theorem.

8	1	-15	46	68	94	15	9	-3	-26
		8	-56	-80	-96	-16	-8	8	40
	1	-7	-10	-12	-2	-1	1	5	14

Remainder $= 14$.

■

Exercise 1.2

Find the remainder when $f(x) = x^{46} - 15x^5 + 9x^3 + 8x + 5$ is divided by $x + 1$.

Solution

Here is a situation where using the Remainder Theorem is easier than using synthetic division.

$$\text{Remainder} = f(-1) = (-1)^{46} - 15(-1)^5 + 9(-1)^3 + 8(-1) + 5$$
$$= 1 + 15 - 9 - 8 + 5$$
$$= 4.$$

■

Exercise 1.3 (Winning Solutions, Lozansky and Rousseau)

Find the remainder when $x^{81} + x^{49} + x^{25} + x^9 + x$ is divided by $x^3 - x$.

Solution

Let $f(x) = x^{81} + x^{49} + x^{25} + x^9 + x$. By the division theorem

$$\frac{f(x)}{x^3 - x} = \frac{f(x)}{x(x-1)(x+1)} = q(x) + \frac{r(x)}{x(x-1)(x+1)}$$

or alternatively,

$$f(x) = x(x-1)(x+1)q(x) + r(x)$$

where $r(x)$ has degree 2 or less. That is, the general form for $r(x) = ax^2 + bx + c$ for some constants a, b and c.

Therefore,

$$f(0) = 0(0-1)(0+1)q(0) + (a \cdot 0^2 + b \cdot 0 + c) = c$$

$$f(1) = 1(1-1)(1+1)q(1) + (a \cdot 1^2 + b \cdot 1 + c) = a + b + c$$

$$f(-1) = (-1)((-1)-1)\big((-1)+1\big)q(-1) + (a \cdot (-1)^2 + b \cdot (-1) + c) = a - b + c.$$

But we also have that

$$f(0) = 0^{81} + 0^{49} + 0^{25} + 0^9 + 0 = 0$$

$$f(1) = 1^{81} + 1^{49} + 1^{25} + 1^9 + 1 = 5$$

$$f(-1) = (-1)^{81} + (-1)^{49} + (-1)^{25} + (-1)^9 + (-1) = -5.$$

So $c = f(0) = 0$

$$a + b = f(1) = 5$$

$$a - b = f(-1) = -5.$$

From the second equation, $a = b - 5$. Therefore $5 = a + b = (b - 5) + b = 2b - 5$. Hence $b = 5$. Therefore $a = 0$. So $r(x) = 0x^2 + 5x + 0 = 5x$.

■

Exercise 1.4 (Winning Solutions, Lozansky and Rousseau)

Find the remainder when $x^{60} - 1$ is divided by $x^3 - 2$.

Solution

After a few iterations of the long division algorithm for $f_1(x)/g_1(x) = (x^{60} - 1)/(x^3 - 2)$ you can start to see a pattern forming involving 2^k. You can get to the final answer by carefully tracking the number of iterations needed to reach the remainder, but there is a much cleaner way of doing this problem.

Let $y = x^3$ and consider the problem $f_2(y)/g_2(y) = (y^{20} - 1)/(y - 2)$. By the remainder theorem, the remainder of this new problem equals the constant $f_2(2) = 2^{20} - 1$. That is, for some quotient polynomial $q_2(y)$,

$$y^{20} - 1 = (y - 2)q_2(y) + (2^{20} - 1).$$

If we replace y with x^3 in this **identity** we have

$$x^{60} - 1 = (x^3 - 2)q_2(x^3) + (2^{20} - 1).$$

> Reminder:
>
> **Identity** – an equality relation satisfied by all values of the variable where that relation is defined. *e.g.* $4x + 2 = 2(2x + 1)$.
>
> **Conditional Equation** – an equality relation satisfied by *some* but not all values of the variable where that relation is defined. *e.g.* $4x + 2 = 6$.
>
> **Contradiction** – an equality relation not satisfied by *any* values of the variable where that relation is defined. *e.g.* $4x + 2 = 4x$.

Therefore,

$$x^{60} - 1 = (x^3 - 2)q_1(x) + (2^{20} - 1)$$

where we let $q_1(x) = q_2(x^3)$. We note that $q_1(x)$ is a polynomial in the variable x. So, it follows from the division theorem that the constant $2^{20} - 1$ is also the remainder of $(x^{60} - 1)/(x^3 - 2)$.

∎

$x - b$ is a factor of $f(x) - f(b)$

Exercise 1.5

a) Show that for every polynomial $f(x)$, $x - b$ is a factor of $f(x) - f(b)$.

b) Show that if all the coefficients of the polynomial $f(x)$ are integers and if a and b are integers, then the number $a - b$ divides the number $f(a) - f(b)$.

Solution (part a)

From the division theorem and the remainder theorem we have

$$f(x) = (x - b)q(x) + f(b).$$

From this equation we have

$$f(x) - f(b) = (x - b)q(x).$$

This shows that $(x - b)$ is a factor of the polynomial $f(x) - f(b)$.

Solution (part b)

► First, let's establish that under the given conditions the quotient $q(x)$ will also have to have integer coefficients.

You can see this must be the case if you think about dividing $f(x)$ by $x - b$ using synthetic division.

Let's look at an example. Suppose you divide $f(x) = c_3x^3 + c_2x^2 + c_1x + c_0$ by $x - b$ where c_3, c_2, c_1, c_0 and b are all integers.

b	c_3	c_2	c_1	c_0
		c_3b	$c_2b + c_3b^2$	$c_1b + c_2b^2 + c_3b^3$
	c_3	$c_2 + c_3b$	$c_1 + c_2b + c_3b^2$	$c_0 + c_1b + c_2b^2 + c_3b^3$

In this case, $q(x) = c_3x^2 + (c_2 + c_3b)x + (c_1 + c_2b + c_3b^2)$.

Is the coefficient of x^2 in $q(x)$ an integer? Yes, because we required that c_3 is an integer.

Is $(c_2 + c_3 b)$, the coefficient of x in $q(x)$ an integer? Yes, because c_2, c_3 and b are all integers it follows that the product $c_3 b$ of two integers is still an integer and the sum $c_2 + c_3 b$ of two integers is still an integer.

In the same way we can see that the constant term $(c_1 + c_2 b + c_3 b^2)$ is also an integer.

► Secondly, note that if the polynomial $f(x)$ has all integer coefficients then $f(a)$ is necessarily an integer for any integer a. Similarly for $q(a)$. So, for integers a and b we have that $f(a) - f(b)$ is an integer, $(a - b)$ is an integer and $q(a)$ is an integer.

Now we already know that $f(x) - f(b) = (x - b)q(x)$ for all values of x. If we let $x = a$ we get $f(a) - f(b) = (a - b)q(a)$.

That is, the product of the two integers $(a - b)$ and $q(a)$ equals the integer $\big(f(a) - f(b)\big)$. So obviously, the number $a - b$ divides $f(a) - f(b)$.

■

Remainder when dividing by $(x - c)(x - d)$

Exercise 1.6a

Suppose that when the polynomial $f(x)$ is divided by $x - 2$ it leaves a remainder of 3 and when divided by $x - 5$ it leaves a remainder of 4. What is the remainder when $f(x)$ is divided by the product $(x - 2)(x - 5)$? Assume that the degree of $f(x)$ is at least two.

Exercise 1.6b

Suppose that when the polynomial $f(x)$ is divided by $x - c$ it leaves a remainder of r_1 and when divided by $x - d$ it leaves a remainder of r_2. What is the remainder when $f(x)$ is divided by the product $(x - c)(x - d)$? Assume that the degree of $f(x)$ is at least two.

Solution (part a)

By the division theorem

$$\frac{f(x)}{(x-2)(x-5)} = q(x) + \frac{r(x)}{(x-2)(x-5)}$$

or alternatively,

$$f(x) = (x-2)(x-5)q(x) + r(x)$$

where $r(x)$ has degree 1 or less. That is, the general form for $r(x) = ax + b$ for some constants a and b.

Therefore,

$$f(2) = (2-2)(2-5)q(2) + r(2) = r(2) = 2a + b$$

and

$$f(5) = (5-2)(5-5)q(5) + r(5) = r(5) = 5a + b.$$

Solving the system

$$f(2) = 2a + b$$
$$f(5) = 5a + b$$

for a and b we find

$$a = \frac{f(5) - f(2)}{5-2}, \quad b = \frac{5f(2) - 2f(5)}{5-2}.$$

Therefore, the remainder on dividing $f(x)$ by the product $(x-2)(x-5)$ equals

$$r(x) = ax + b = \left(\frac{f(5) - f(2)}{5-2}\right)x + \left(\frac{5f(2) - 2f(5)}{5-2}\right).$$

We know from the remainder theorem that the remainder when the polynomial $f(x)$ is divided by $x - c$ is just $f(c)$. So from the information given, $f(2) = 3$ and $f(5) = 4$.

Therefore,

$$r(x) = \left(\frac{4-3}{5-2}\right)x + \left(\frac{5(3) - 2(4)}{5-2}\right) = \frac{x+7}{3}.$$

■

Solution (part b)

We can mimic the work in Part a. By the division theorem

$$\frac{f(x)}{(x-c)(x-d)} = q(x) + \frac{r(x)}{(x-c)(x-d)}$$

or alternatively,

$$f(x) = (x-c)(x-d)q(x) + r(x)$$

where $r(x)$ has degree 1 or less. That is, the general form for $r(x) = ax + b$ for some constants a and b. Therefore,

$$f(c) = (c - c)(c - d)q(c) + r(c) = r(c) = ac + b = r_1$$

and

$$f(d) = (d - c)(d - d)q(d) + r(d) = r(d) = ad + b = r_2.$$

Solving the system

$$r_1 = ac + b$$
$$r_2 = ad + b$$

for a and b we find

$$a = \frac{r_2 - r_1}{d - c}, \quad b = \frac{dr_1 - cr_2}{d - c}.$$

Therefore, the remainder on dividing $f(x)$ by the product $(x - c)(x - d)$ equals

$$r(x) = \left(\frac{r_2 - r_1}{d - c}\right)x + \left(\frac{dr_1 - cr_2}{d - c}\right).$$

■

Exercise 1.7 (Golden Algebra, Bali)

Find the values of a and b so that $(x - 3)$ and $(x - 1)$ will divide $2x^4 - 7x^3 + ax + b$ with no remainder.

Solution

By the division theorem, there exists some (unique) quotient polynomial $q(x)$ and remainder polynomial $r(x)$ where $\deg r(x) < \deg q(x)$ and

$$2x^4 - 7x^3 + ax + b = (x - 3)(x - 1)q(x) + r(x).$$

If we find a and b such that $r(x) = 0$, then

$$2x^4 - 7x^3 + ax + b = (x - 3) \cdot (x - 1) \cdot q(x)$$

which shows $x - 3$ and $x - 1$ are both factors of $2x^4 - 7x^3 + ax + b$. But this is just another way of saying that $x - 3$ and $x - 1$ will both divide $2x^4 - 7x^3 + ax + b$ with no remainder.

We note that $(x-3)(x-1) = x^2 - 4x + 3$. By polynomial long division

$$
\begin{array}{r|rrrrr}
 & & & 2x^2 & +x & -2 \\
\cline{2-6}
x^2-4x+3 & 2x^4 & -7x^3 & +0x^2 & +ax & +b \\
 & 2x^4 & -8x^3 & +6x^2 & & \\
\cline{2-6}
 & & x^3 & -6x^2 & +ax & +b \\
 & & x^3 & -4x^2 & +3x & \\
\cline{3-6}
 & & & -2x^2 & +(a-3)x & +b \\
 & & & -2x^2 & +8x & -6 \\
\cline{4-6}
 & & & & (a-11)x & +(b+6)
\end{array}
$$

That is,

$$
\begin{aligned}
2x^4 - 7x^3 + ax + b &= (x-3)\cdot(x-1)\cdot q(x) + r(x) \\
&= (x-3)\cdot(x-1)\cdot(2x^2 + x - 2) + \big((a-11)x + (b+6)\big).
\end{aligned}
$$

So, it suffices to pick the value(s) of a and b that make

$$r(x) = \big((a-11)x + (b+6)\big) = 0x + 0 \text{ for all } x.$$

For two polynomials to be equal for all x they must agree coefficient by coefficient. That is

$$(a-11) = 0 \text{ and } (b+6) = 0.$$

The only possible values for a and b are thus $a = 11$ and $b = -6$.

■

1.4 Polynomial Transformations

1.4.1 Polynomial Expansion I

Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$. Find the coefficients $A_0, A_1, \ldots, A_n$ so that

$$f(x) = A_n(x-c)^n + A_{n-1}(x-c)^{n-1} + \cdots + A_1(x-c)^1 + A_0$$

for a given constant c.

This is often referred to as writing $f(x)$ as a polynomial in the variable $x - c$ or just as a polynomial in $x - c$. We will demonstrate the algorithm for obtaining this expansion through examples.

Example

Express the polynomial $f(x) = 2x^3 + x^2 - x - 3$ as a polynomial in $x - 2$.

Solution

The problem is asking us to find the coefficients a, b, c and d such that

$$f(x) = a(x-2)^3 + b(x-2)^2 + c(x-2) + d.$$

Let $g(x) = a(x-2)^3 + b(x-2)^2 + c(x-2) + d$. We can see immediately that $\boldsymbol{a = 2}$ in order that the leading coefficients of $f(x) = g(x)$ agree.

We can also see that $g(x) = (x-2)\left(2(x-2)^2 + b(x-2) + c\right) + d = (x-2)q_1(x) + d$. So d is the remainder in the quotient $g(x)/(x-2)$. But $f(x) = g(x)$ so d is the remainder in the quotient $f(x)/(x-2)$. Using synthetic division, we see that the remainder equals 15. That is, $\boldsymbol{d = 15}$.

2	2	1	−1	−3
		4	10	18
	2	5	9	**15**

We also see from this division that $q_1(x) = 2(x-2)^2 + b(x-2) + c = 2x^2 + 5x + 9$. And we can write $q_1(x)$ in the form $q_1(x) = (x-2)\big((x-2) + b\big) + c = (x-2)q_2(x) + c$ with $q_2(x) = (x-2) + b$. So c is the remainder in the quotient $q_1(x)/(x-2)$, where $q_1(x) = 2x^2 + 5x + 9$. Using synthetic division we see that $\boldsymbol{c = 27}$ and that $q_2(x) = (x-2) + b = 2x + 9$.

2	2	5	9
		4	18
	2	9	**27**

So b is the remainder when $q_2(x)$ is divided by $x - 2$. Using synthetic division we see that $\boldsymbol{b = 13}$.

2	2	9
		4
	2	**13**

Therefore, we have

$$2x^3 + x^2 - x - 3 = 2(x-2)^3 + 13(x-2)^2 + 27(x-2) + 15.$$

■

1.4.2 Polynomial Expansion II

Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$. Find the coefficients $A_0, A_1, \ldots, A_n$ so that

$$f(x) = A_0 + A_1(x - c_1) + A_2(x - c_1)(x - c_2) + \cdots + A_n(x - c_1)(x - c_2)\cdots(x - c_n)$$

for a given set of constants $c_1, c_2, \ldots, c_n$.

We will demonstrate the algorithm for obtaining this expansion through examples.

Example

Find constants A_0, A_1, A_2, A_3 such that

$$x^3 - 1 = A_0 + A_1(x-1) + A_2(x-1)(x-2) + A_3(x-1)(x-2)(x-3)$$

for all x.

Solution

We can see by inspection that the coefficient of x^3 on the right-hand side of this proposed identity equals 1 and that the coefficient of x^3 on the left-hand side will be A_3 because the only term involving x^3 is the product $A_3(x-1)(x-2)(x-3)$ whose x^3 term will obviously be A_3x^3 after multiplying this out.

So $A_3 = 1$ by inspection (see *equating coefficients*, page 15)

If we divide the right-hand side of the proposed identity by $x - 1$ we get

$$\frac{A_0 + A_1(x-1) + A_2(x-1)(x-2) + A_2(x-1)(x-2)(x-3)}{x-1}$$

$$= \Big(A_1 + A_2(x-2) + A_2(x-2)(x-3)\Big) + \frac{A_0}{x-1}$$

which has the form

$$\text{(quotient)} + \frac{\text{(remainder)}}{\text{(divisor)}}.$$

That is A_0 equals the remainder and $A_1 + A_2(x-2) + A_2(x-2)(x-3)$ is the quotient after dividing the right hand side by the divisor $(x-1)$. If we divide the left hand side by $(x-1)$ we get a remainder of 0 and a quotient of $x^2 + x + 1$.

1	1	0	0	−1
		1	1	1
	1	1	1	0

Therefore, $A_0 = 0$ and equating the two quotients we have

$$x^2 + x + 1 = A_1 + A_2(x-2) + A_3(x-2)(x-3)$$

for all x. Now we can proceed as before, only this time dividing both sides by $(x-2)$. On the right-hand side we get a remainder of A_1 and a quotient of $A_2(x-2) + A_3(x-2)(x-3)$. On the left-hand side we get a remainder of 7 and a quotient of $x + 3$.

2	1	1	1
		2	6
	1	3	7

Therefore, $A_1 = 7$ and we are left with the identity $x + 3 = A_2(x-2) + A_3(x-2)(x-3)$.

Finally, dividing both sides by $(x-3)$ and equating the remainders, we get $A_2 = 6$.

3	1	3
		3
	1	6

So, we have determined $A_0 = 0, A_1 = 7, A_2 = 6$ and $A_3 = 1$. Hence,

$$x^3 - 1 = 0 + 7 \cdot (x-1) + 6 \cdot (x-1)(x-2) + 1 \cdot (x-1)(x-2)(x-3).$$

■

1.4.3 Translation of the Roots

Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be an n^{th} degree polynomial where the roots of $f(x) = 0$ are $r_1, r_2, \ldots., r_n$.

For all real numbers c (positive and negative) define $g(x) = f(x+c)$. Then the roots of $g(x) = 0$ are $r_1 - c, r_2 - c, \ldots, r_n - c$.

Sometimes this is worded as "diminishing the roots by c" but c could also be negative so diminishing the roots by a negative c which, of course, is equivalent to *increasing* the roots by c.

Proof

By definition $f(r_1) = f(r_2) = \cdots = f(r_n) = 0$. It follows that $g(x) = f(x+c) = 0$ if and only if $x + c \in \{r_1, r_2, \ldots, r_n\}$ if and only if $x \in \{r_1 - c, r_2 - c, \ldots, r_n - c\}$.

That is,

$$f(x+c) = a_n(x+c)^n + a_{n-1}(x+c)^{n-1} + \cdots + a_1(x+c) + a_0 = 0$$

has roots which are c less than the roots of

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0.$$

However, this only partly solves the problem of translating the roots. We don't want an answer in in the form

$$g(x) = f(x+c) = a_n(x+c)^n + a_{n-1}(x+c)^{n-1} + \cdots + a_1(x+c) + a_0.$$

We want to find the coefficients $b_0, b_1, \ldots, b_n$ such that

$$\begin{aligned} g(x) = f(x+c) &= a_n(x+c)^n + a_{n-1}(x+c)^{n-1} + \cdots + a_1(x+c) + a_0 \\ &= b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x^1 + b_0. \end{aligned}$$

Recall that in the earlier transformation section we titled the "Polynomial Expansion I" we dealt with a very similar problem. In that problem we wanted to find the coefficients $A_0, A_1, \ldots, A_n$ so that

$$f(x) = A_n(x-c)^n + A_{n-1}(x-c)^{n-1} + \cdots + A_1(x-c)^1 + A_0.$$

Briefly stated, in that section we developed an algorithm that for a given polynomial $f(x)$ would find that polynomial $g(x)$ such that

$$f(x) = g(x - c).$$

While in our current problem for a given polynomial $f(x)$ we want to find that polynomial $g(x)$ such that

$$f(x + c) = g(x).$$

This equation, $f(x + c) = g(x)$, is an identity. It must be true for *all* x. So it must be true for $x = y - c$ for any generic value. Plugging in $y - c$ for x we get $f\big((y - c) + c\,\big) = g(y - c)$. That is $f(y) = g(y - c)$.

But of course, the method for finding the coefficients of $g(y)$ such that $f(y) = g(y - c)$ is exactly the algorithm for "Polynomial Expansion I".

So, the algorithm for translating the roots has two parts.

First. Find the coefficients $b_0, b_1, \dots, b_n$ such that $f(x) = g(x - c) = b_n(x - c)^n + b_{n-1}(x - c)^{n-1} + \cdots + b_1(x - c) + b_0$ using the "Polynomial Expansion I" algorithm.

Second. Use those same coefficients $b_0, b_1, \dots, b_n$ to solve the new problem

$$f(x + c) = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x^1 + b_0.$$

Let's step through a couple of examples to solidify this new algorithm.

Example

Let $f(x) = x^3 + 2x^2 + x + 80$. Let r_1, r_2, r_3 be the roots of $f(x) = 0$. Find the polynomial $g(x) = b_3x^3 + b_2x^2 + b_1x + b_0$ such that the roots of $g(x) = 0$ are $r_1 - 5, r_2 - 5, r_3 - 5$.

Solution

As we saw in the above discussion on this problem, the roots of $f(x + 5) = 0$ will be $r_1 - 5$, $r_2 - 5$ and $r_3 - 5$ where r_1, r_2 and r_3 are the roots of $f(x) = 0$.

In that same discussion we found that the coefficients b_0, b_1, b_2, b_3 such that

$$f(x + 5) = (x + 5)^3 + 2(x + 5)^2 + (x + 5) + 80 = b_3x^3 + b_2x^2 + b_1x + b_0$$

are the *very same coefficients* b_0, b_1, b_2, b_3 such that

$$f(x) = x^3 + 2x^2 + x + 80 = b_3(x - 5)^3 + b_2(x - 5)^2 + b_1(x - 5) + b_0.$$

But the problem stated in this form is the very problem answered by the "Polynomial Expansion I" algorithm.

We can find b_3 by inspection because the only place x^3 can show up is in $b_3(x - 5)^3$ and we can see by inspection that the coefficient of x^3 in $b_3(x - 5)^3 = b_3(x - 5)(x - 5)(x - 5)$ is just $b_3(1) = b_3$. So b_3 is determined by matching it with the coefficient of x^3 in the left hand polynomial, which we see equals 1. Therefore, $b_3 = 1$.

Now we will start the repeated synthetic division part of the procedure.

5	1	2	1	80
		5	35	180
	1	7	36	**260**

So $b_0 = 260$.

5	1	7	36
		5	60
	1	12	**96**

So $b_1 = 96$.

5	1	12
		5
	1	**17**

So $b_2 = 17$.

We have just shown that

$$x^3 + 2x^2 + x + 80 = (\mathbf{1})(x-5)^3 + (\mathbf{17})(x-5)^2 + (\mathbf{96})(x-5) + (\mathbf{260}).$$

So, the polynomial

$$g(x) = (\mathbf{1})x^3 + (\mathbf{17})x^2 + (\mathbf{96})x + (\mathbf{260}) = x^3 + 17x^2 + 96x + 260$$

is that polynomial such that the roots of $g(x) = 0$ are exactly 5 less than the roots of $f(x) = x^3 + 2x^2 + x + 80 = 0$.

■

Check!

Let's actually find the roots of these polynomials, *just to be sure*. By the rational root theorem if $x^3 + 2x^2 + x + 80 = 0$ has any rational roots they would have to be integers and they would have to be divisors of 80. The divisors of 80 are $\pm1, \pm2, \pm4, \pm5, \pm8, \pm10, \pm16, \pm20, \pm40$ and ±80. That is a *lot* of potential roots to have to check and all that work may be a waste of time if none of these root candidates actually ends up being a root.

This is a good example to motivate finding some improvements on the rational root theorem that can speed up the process. We will do just that in a later section when we take up rational roots in detail.

But for now, in the interest of my time and your patience, I will just tell you that you will find that $x = -5$ is a root

-5	1	2	1	80
		-5	15	-80
	1	-3	16	**0**

and that the quotient equals $x^2 - 3x + 16$. Because this is a quadratic, we can use the quadratic formula to find the remaining two roots.

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(16)}}{2(1)} = \frac{3 \pm \sqrt{-55}}{2}.$$

If we haven't made any mistakes then we should find that the roots of $x^3 + 17x^2 + 96x + 260 = 0$ are

$$-5 - 5 = -10 \text{ and } \frac{3 \pm \sqrt{-55}}{2} - 5 = \frac{-7 \pm \sqrt{-55}}{2}.$$

Let's check that!

-10	1	17	96	260
		-10	-70	-260
	1	7	26	**0**

and using the quadratic formula to examine the quotient $x^2 + 7x + 26$, we find the remaining two roots are

$$x = \frac{-7 \pm \sqrt{7^2 - 4(1)(26)}}{2(1)} = \frac{-7 \pm \sqrt{-55}}{2}.$$

So it worked!

■

1.4.4 Scaling the Roots

Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be an n^{th} degree polynomial with integer coefficients. Suppose the roots of $f(x) = 0$ are $r_1, r_2, \dots, r_n$.

For all positive integers c the polynomial $g(x) = c^n f(x/c)$ has integer coefficients and the roots of $g(x) = 0$ are $cr_1, cr_2, \dots, cr_n$.

Proof

By definition $f(r_1) = f(r_2) = \cdots = f(r_n) = 0$. Consider the polynomial $f(x/c), c \neq 0$. Clearly $f(x/c) = 0$ if and only if $x/c \in \{r_1, r_2, \dots, r_n\}$ if and only if $x \in \{cr_1, cr_2, \dots, cr_n\}$.

That is the roots of $f(x/c) = 0$ are exactly the roots of $f(x)$ multiplied by c. But $f(x/c)$ no longer will have integer coefficients

$$f\left(\frac{x}{c}\right) = a_n\left(\frac{x}{c}\right)^n + a_{n-1}\left(\frac{x}{c}\right)^{n-1} + \cdots + a_1\left(\frac{x}{c}\right) + a_0.$$

However, for all $c \neq 0$, $f(x/c) = 0$ if and only if $c^n f(x/c) = 0$. That is, the roots of $c^n f(x/c) = 0$ are still c times the roots of $f(x) = 0$.

Let's examine the polynomial $g(x) = c^n f(x/c)$ more closely.

$$\begin{aligned} g(x) = c^n f\left(\frac{x}{c}\right) &= c^n\left(a_n\left(\frac{x}{c}\right)^n + a_{n-1}\left(\frac{x}{c}\right)^{n-1} + \cdots + a_1\left(\frac{x}{c}\right) + a_0\right) \\ &= c^n\left(\frac{a_n x^n}{c^n} + \frac{a_{n-1}x^{n-1}}{c^{n-1}} + \cdots + \frac{a_1 x}{c} + a_0\right) \\ &= a_n x^n + c^1 a_{n-1}x^{n-1} + c^2 a_{n-2}x^{n-2} + \cdots + c^{n-1}a_1 x + c^n a_0. \end{aligned}$$

We can see that $g(x)$ has integer coefficients because the a_j's and c are integers. So

$$g(x) = c^n f\left(\frac{x}{c}\right)$$

has integer coefficients and the roots of $g(x) = 0$ are c times the roots of $f(x) = 0$.

Exercise 1.8

Let $f(x) = 5x^4 + 9x^2 - 11$. Find a polynomial $g(x)$ with integer coefficients such that the roots of $g(x) = 0$ are exactly the roots of $f(x) = 0$ multiplied by 2.

Solution

Applying the above result, we have that the polynomial equation

$$g(x) = 5x^4 + 2^1 \cdot 0x^3 + 2^2 \cdot 9x^2 + 2^3 \cdot 0x^1 - 2^4 \cdot 11 = 0$$

has integer coefficients and the roots of $g(x) = 0$ are exactly twice as large as the roots of $f(x) = 5x^4 + 9x^2 - 11 = 0$. On simplifying we can see that the roots of $g(x) = 5x^4 + 36x^2 - 176 = 0$ are exactly twice as large as the roots of $f(x) = 5x^4 + 9x^2 - 11 = 0$.

■

1.4.5 Negation of the Roots

Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be an n^{th} degree polynomial such that $f(x) = 0$ has roots $r_1, r_2, \dots, r_n$. Then the polynomial $g(x) = (-1)^n f(-x) = 0$ will have roots $-r_1, -r_2, \dots, -r_n$.

This is the special case of scaling with $c = -1$.

1.4.6 Reciprocal of the Roots

Reciprocal Roots

If $r_1, r_2, \dots, r_n$ are the roots of $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$, then

$$\frac{1}{r_1}, \frac{1}{r_2}, \dots, \frac{1}{r_n}$$

are the roots of

$$g(x) = x^n f\left(\frac{1}{x}\right) = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n.$$

Exercise 1.9 (Source: MSHSML 1T076)

Each zero of $f(x) = ax^3 + bx^2 + cx + 7$ is one more than the reciprocal of a zero of $g(x) = x^3 + x^2 - 5x + 2$. Determine a, b, and c.

Solution

$$g(x) = x^3 + x^2 - 5x + 2$$

$$h(x) = x^3\left(g\left(\frac{1}{x}\right)\right) = 2x^3 - 5x^2 + x + 1$$

$$f(x) = h(x-1) = 2(x-1)^3 - 5(x-1)^2 + (x-1) + 1$$
$$= 2x^3 - 11x^2 + 17x - 7$$

■

Exercise 1.10 (Source: MSHSML SD042)

The roots of $a_3x^3 + a_2x^2 + a_1x + a_0 = 0$ are $-\frac{4}{3}$, 2, and $\frac{7}{2}$. What are the roots of $a_0y^3 + a_1y^2 + a_2y + a_3 = 0$?

Solution

It follows immediately from the reciprocal roots transformation that the roots of

$$a_0y^3 + a_1y^2 + a_2y + a_3 = 0$$

are the reciprocals of $-\frac{4}{3}$, 2, and $\frac{7}{2}$, the roots of $a_3x^3 + a_2x^2 + a_1x + a_0 = 0$.

So, the answers are

$$-\frac{3}{4}, \frac{1}{2}, \text{and } \frac{2}{7}.$$

■

Exercise 1.11 (Source: MSHSML SD032)

The roots of $f(x) = a_2x^2 + a_1x + a_0 = 0$ are r and s. What, in terms of r and s, are the roots of $a_0t^2 - a_1t + a_2 = 0$?

Solution

In general, the roots of $f(-x) = 0$ will be the negatives of the roots of $f(x)$. So

$$f(-x) = a_2(-x)^2 + a_1(-x) + a_0 = a_2x^2 - a_1x + a_0 = 0$$

has roots $-r$ and $-s$.

The roots of

$$x^2f\left(-\frac{1}{x}\right) = a_0x^2 - a_1x + a_2 = 0$$

are the reciprocals of the roots of $f(-1/x) = 0$. Therefore, the roots of $a_0x^2 - a_1x + a_2 = 0$ are

$$\frac{1}{-r} \text{ and } \frac{1}{-s}.$$

■

1.4.7 Squaring the Roots

Squaring the Roots

If $r_1, r_2, \dots, r_n$ are the roots of $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = 0$, then the roots of the polynomial

$$g(x) = (-1)^n f(\sqrt{x}) f(-\sqrt{x}) = 0$$

are

$$r_1^2, r_2^2, \dots, r_n^2.$$

There are two parts to understanding this result. First, will $g(x)$ as defined above necessarily be a polynomial? Afterall a polynomial cannot have terms raised to the one-half power. And second, will the roots of $g(x) = 0$ really be the squares of the roots of $f(x) = 0$?

To verify that the answer to both questions is “Yes” begin by looking writing both $f(x)$ in terms of its roots.

$$f(x) = a_n(x - r_1)(x - r_2) \cdots (x - r_n).$$

Therefore,

$$f(-x) = a_n(-x - r_1)(-x - r_2) \cdots (-x - r_n)$$

and

$$f(x)f(-x) = a_n^2 \Big((x - r_1)(-x - r_1)\Big)\Big((x - r_2)(-x - r_2)\Big) \cdots \Big((x - r_n)(-x - r_n)\Big)$$

$$= a_n^2 \Big((-1)(x - r_1)(x + r_1)\Big)\Big((-1)(x - r_2)(x + r_2)\Big) \cdots \Big((-1)(x - r_n)(x + r_n)\Big)$$

$$= a_n^2(-1)^n(x^2 - r_1^2)(x^2 - r_2^2) \cdots (x^2 - r_n^2).$$

Now define

$$g(x) = a_n^2(x - r_1^2)(x - r_2^2) \cdots (x - r_n^2).$$

We can see that

(i) $g(x)$ is a polynomial

and

(ii) the roots of $g(x) = 0$ are $r_1^2, r_2^2, \dots, r_n^2$.

We can also see that

$$g(x^2) = (x^2 - r_1^2)(x^2 - r_2^2) \cdots (x^2 - r_n^2) = (-1)^n f(x) f(-x).$$

But

$$g(x^2) = (-1)^n f(x) f(-x) \Longrightarrow g(x) = (-1)^n f(\sqrt{x}) f(-\sqrt{x}).$$

■

Exercise 1.12

Let $f(x) = x^3 - 4x^2 + x + 6$ and suppose the roots of $f(x) = 0$ are r_1, r_2 and r_3.

(a) Find a polynomial whose roots are r_1^2, r_2^2 and r_3^2.

(b) Find a polynomial whose roots are $r_1^2 + 1, r_2^2 + 1$ and $r_3^2 + 1$.

(c) Find a polynomial whose roots are

$$\frac{1}{r_1^2 + 1}, \frac{1}{r_2^2 + 1} \text{ and } \frac{1}{r_3^2 + 1}.$$

(d) Find a polynomial whose roots are

$$\frac{4}{r_1^2 + 1}, \frac{4}{r_2^2 + 1} \text{ and } \frac{4}{r_3^2 + 1}.$$

Solutions

(a) From the "Squaring the Roots" transformation, if we take

$$g(x) = (-1)^3 f(\sqrt{x}) f(-\sqrt{x})$$

then $g(x) = 0$ will have roots r_1^2, r_2^2 and r_3^2.

The easiest way to find $(-1)^3 f(\sqrt{x}) f(-\sqrt{x})$ is to find $(-1)^3 f(x) f(-x)$ and then replace x^2 with x. After simplification, we will find that

$$\begin{aligned}(-1)^3 f(x)f(-x) &= (-1)^3(x^3 - 4x^2 + x + 6)(-x^3 - 4x^2 - x + 6)\\ &= x^6 - 14x^4 + 49x^2 - 36\\ &= (x^2)^3 - 14(x^2)^2 + 49(x^2) - 36.\end{aligned}$$

Therefore,

$$g(x) = (-1)^3 f\left(\sqrt{x}\right) f\left(-\sqrt{x}\right) = x^3 - 14x^2 + 49x - 36.$$

Verify.

In this problem we can verify our work because the polynomial $f(x) = x^3 - 4x^2 + x + 6$ was chosen to factor nicely. In particular,

$$x^3 - 4x^2 + x + 6 = (x-2)(x-3)(x+1).$$

Therefore, if our work in (a) is correct we should find that

$$\begin{aligned}g(x) = x^3 - 14x^2 + 49x - 36 &= (x - 2^2)(x - 3^2)(x - (-1)^2)\\ &= (x-4)(x-9)(x-1).\end{aligned}$$

And after grinding out this polynomial multiplication we will see that this is, in fact, the case.

(b) From the "Translation of the Roots" transformation, the roots of $h(x) = g(x-1) = 0$ will each be 1 greater than the roots of $g(x) = 0$. After simplification, we find that

$$\begin{aligned}h(x) = g(x-1) &= (x-1)^3 - 14(x-1)^2 + 49(x-1) - 36\\ &= x^3 - 17x^2 + 80x - 100.\end{aligned}$$

Verify.

If our work in (b) is correct we should find that

$$\begin{aligned}h(x) = x^3 - 17x^2 + 80x - 100 &= \big(x - (4+1)\big)\big(x - (9+1)\big)\big(x - (1+1)\big)\\ &= (x-5)(x-10)(x-2).\end{aligned}$$

And after grinding out this polynomial multiplication we will see that this is, in fact, the case.

(c) From the "Reciprocal Roots" transformation, the roots of $v(x) = x^3 h(1/x) = 0$ will be the reciprocals of the roots of $h(x) = 0$.

$$v(x) = x^3 h(1/x) = -100x^3 + 80x^2 - 17x + 1.$$

Verify.
If our work in (c) is correct we should find that

$$\begin{aligned} -100x^3 + 80x^2 - 17x + 1 &= -100\left(x - \frac{1}{5}\right)\left(x - \frac{1}{10}\right)\left(x - \frac{1}{2}\right) \\ &= (-1)(5x - 1)(10x - 1)(2x - 1). \end{aligned}$$

And after grinding out this polynomial multiplication we will see that this is, in fact, the case.

(d) From the "Scaling the Roots" transformation, the roots of $w(x) = 4^3 v(x/4) = 0$ will be 4 times the roots of $v(x) = 0$.

$$w(x) = 4^3\left(-100\left(\frac{x}{4}\right)^3 + 80\left(\frac{x}{4}\right)^2 - 17\left(\frac{x}{4}\right) + 1\right) = -100x^3 + 320x^2 - 272x + 64.$$

Verify.
If our work in (d) is correct we should find that

$$-100x^3 + 320x^2 - 272x + 64 = -100\left(x - \frac{4}{5}\right)\left(x - \frac{4}{10}\right)\left(x - \frac{4}{2}\right)$$

And after grinding out this polynomial multiplication we will see that this is, in fact, the case.

■

1.5 Equating Coefficients of Polynomials Applications

We have previously stated (see page 15) the "Equality of Two Polynomials Theorem" which states that two polynomials are equal *for all* x if and only if their coefficients are equal term by term.

Consider the following exercise.

Exercise 1.13 (MSHSML 1D154)

Let $f(x) = 2x^2 - bx + 7$ and $g(x) = 2(x-c)^2 - 43$. If $f(x)$ and $g(x)$ are equal for all values of x, determine exactly all possible values of b.

Solution

Recall that in general, $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0$ for all $-\infty < x < \infty \Leftrightarrow a_k = b_k, k = 0,1,2,\dots,n$. (i.e. equate matching coefficients).

$$2x^2 - bx + 7 = 2(x-c)^2 - 43 = 2x^2 - 4cx + (2c^2 - 43)$$

$\therefore\ -b = -4c, 7 = 2c^2 - 43 \Longrightarrow c^2 = 25, b = 4c$

$\therefore (b,c) = (20,5)$ or $(b,c) = (-20,-5)$.

■

Exercise 1.14 (Source: *Competition Algebra*, Xing Zhou)

Find $a + b + c$ if $2, -3$ and 5 are roots of $x^4 + ax^2 + bx + c = 0$.

Solution

$$\begin{aligned} &x^4 + ax^2 + bx + c \\ &\quad = (x-2)(x+3)(x-5)(x-r) \\ &\quad = x^4 - (r+4)x^3 + (4r-11)x^2 + (11r+30)x - 30r \end{aligned}$$

In order for these polynomials to match coefficient by coefficient, the coefficient of x^3 must be 0 in both cases. This implies that $r + 4 = 0 \Longrightarrow r = -4$. In this case, we can see that

$$a = 4r - 11 = -16 - 11 = -27$$

$$b = 11r + 30 = -44 + 30 = -14$$

$$c = -30r = 120.$$

Therefore, $a + b + c = -27 - 14 + 120 = 79$.

■

1.6 Fitting a Polynomial to Data

Suppose you are given that some unknown quadratic $f(x) = ax^2 + bx + c$ goes through the three points (x_1, y_1), (x_2, y_2), and (x_3, y_3). How can you find the unknown coefficients a, b and c?

One method is to set up the three equations $f(x_1) = y_1, f(x_2) = y_2$ and $f(x_3) = y_3$ and then solve this system of equations simultaneously. This is generally a time-consuming approach.

The standard "short cut" is known as the **Lagrange interpolation formula**. I will put the details of this approach in the appendix to this chapter but will not discuss it here because there is a lesser known algorithm, **Newton's method**, that is faster and easier to carry out. I will illustrate Newton's method for finding the quadratic from three points taken from that quadratic, but it should be clear from this how to generalize to fitting higher degree polynomials to more data points from that polynomial.

Newton's Method

Suppose (x_1, y_1), (x_2, y_2), and (x_3, y_3) are three points on the unknown quadratic $f(x)$. Then

$$f(x) = c_0 + c_1(x - x_1) + c_2(x - x_1)(x - x_2)$$

for some constants c_0, c_1 and c_2 which can be determined from the three equations $f(x_1) = y_1, f(x_2) = y_2$ and $f(x_3) = y_3$.

The generalization to four or more points is immediate. Suppose (x_1, y_1), (x_2, y_2), (x_3, y_3) and (x_4, y_4) are four points on the unknown cubic $f(x)$. Then

$$f(x) = c_0 + c_1(x - x_1) + c_2(x - x_1)(x - x_2) + c_3(x - x_1)(x - x_2)(x - x_3)$$

for some constants c_0, c_1, c_2 and c_3 which can be determined from the four equations $f(x_1) = y_1, f(x_2) = y_2, f(x_3) = y_3$ and $f(x_4) = y_4$.

Exercise 1.15 (MSHSML 1D133)

Let $f(x)$ be a quadratic polynomial. If it is known that $f(0) = 1, f(1) = 2$ and $f(2) = 0$, determine $f(-1)$ exactly.

Solution

We will start by finding $f(x)$ using Newton's Method. The starting form is

$$f(x) = c_0 + c_1(x-0) + c_2(x-0)(x-1).$$

Therefore,

$$1 = f(0) = c_0 + c_1(0-0) + c_2(0-0)(0-1) = c_0 \Rightarrow c_0 = 1$$

$$2 = f(1) = 1 + c_1(1-0) + c_2(1-0)(1-1) = 1 + c_1 \Rightarrow c_1 = 1$$

$$0 = f(2) = 1 + 1(2-0) + c_2(2-0)(2-1) = 3 + 2c_2 \Rightarrow c_2 = -3/2.$$

Hence

$$f(x) = 1 + x - \frac{3}{2}x(x-1) = -\frac{3}{2}x^2 + \frac{5}{2}x + 1.$$

Now we can find $f(-1)$ by substitution.

$$f(-1) = \left(-\frac{3}{2}\right)(-1)^2 + \left(\frac{5}{2}\right)(-1) + 1 = -3.$$

■

The feature with Newton's method that makes it so nice is the resulting system of equations will always be a **triangular system**. That is, the system of equations will have the form,

$$a_1x = b_1$$

$$a_2x + a_3y = b_2$$

$$a_4x + a_5y + a_6z = b_3$$

$$\vdots$$

In such a system of equations, you can immediately find x from the first equation. Then you plug this solution for x into the second equation and immediately solve for y. Then plug these solutions for x and y into the third equation and immediately solve for z. etc.

Why does Newton's method work?

Consider again the above exercise. We picked c_0, c_1 and c_2 to force

$$f(x) = c_0 + c_1(x-0) + c_2(x-0)(x-1)$$

to equal contain the points $(0,1), (1,2)$ and $(2,0)$. But as a consequence of the Equality of Two Polynomials Theorem, there can only be one quadratic that contains three **non-collinear** points (points that are not all on the same line).

So, the polynomial $f(x)$ determined by Newton's method must be the unique polynomial containing the given points.

Exercise 1.16

Let $f(x)$ be a cubic polynomial. It is known that $f(1) = 2$, $f(3) = 24$, $f(-1) = -4$ and $f(-2) = -31$. Find $f(2)$ exactly.

Solution

Our four points are $(1,2), (3,24), (-1,-4)$ and $(-2,-31)$. Writing $f(x)$ in the form of Newton's method we have

$$f(x) = c_0 + c_1(x-1) + c_2(x-1)(x-3) + c_3(x-1)(x-3)(x+1).$$

Therefore,

$$\begin{aligned} 2 = f(1) &= c_0 + c_1(1-1) + c_2(1-1)(1-3) + c_3(1-1)(1-3)(1+1) \\ &= c_0 \Rightarrow c_0 = 2 \end{aligned}$$

$$\begin{aligned} 24 = f(3) &= 2 + c_1(3-1) + c_2(3-1)(3-3) + c_3(3-1)(3-3)(3+1) \\ &= 2 + 2c_1 \Rightarrow c_1 = 11 \end{aligned}$$

$$\begin{aligned} -4 = f(-1) &= 2 + 11(-1-1) + c_2(-1-1)(-1-3) + c_3(-1-1)(-1-3)(-1+1) \\ &= 2 - 22 + c_2(-2)(-4) + 0 = -20 + 8c_2 \Rightarrow c_2 = 2 \end{aligned}$$

$$\begin{aligned} -31 = f(-2) &= 2 + 11(-2-1) + 2(-2-1)(-2-3) + c_3(-2-1)(-2-3)(-2+1) \\ &= 2 - 33 + 30 - 15c_3 \Rightarrow c_3 = 2. \end{aligned}$$

So,

$$\begin{aligned} f(x) &= 2 + 11(x-1) + 2(x-1)(x-3) + 2(x-1)(x-3)(x+1) \\ &= 2x^3 - 4x^2 + x + 3. \end{aligned}$$

Therefore,

$$f(2) = 2(2^3) - 4(2^2) + 2 + 3 = 5.$$

■

Section 2 Solving Polynomials Equations

2.1 Factoring Polynomials

Special Case Factoring Formulas

Difference of two squares	$a^2 - b^2 = (a+b)(a-b)$
Difference of two cubes	$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$
Sum of two cubes	$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$
Difference of two powers	$a^n - b^n = (a-b)\big(a^{n-1} + a^{n-2}b + a^{(n-3)}b^2 + \cdots + ab^{n-2} + b^{n-1}\big)$
Sum of two odd powers	$a^n + b^n = (a+b)\big(a^{n-1} - a^{n-2}b + a^{(n-3)b^2} - \ldots + a^2b^{n-3} - ab^{n-2} + b^{n-1}\big)$ $n = 3,5,7,9,\ldots$

Three ancillary results which are sometimes useful when factoring are:

(i) $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

(ii) $(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

(iii) $\big(x - (c+d)\big)\big(x - (-c+d)\big)\big(x - (c-d)\big)\big(x - (-c-d)\big)$
$= x^4 - 2(c^2 + d^2)x^2 + (c^2 - d^2)^2$

Maximal Factoring of $x^n - y^n, n \geq 4$

Let $n = a \cdot b$ where a is the smallest prime divisor of n. Then,

$x^n - y^n = (x^b)^a - (y^b)^a$

$$= (x^b - y^b)\Big((x^b)^{a-1} + (x^b)^{a-2}(y^b) + (x^b)^{a-3}(y^b)^2 + \cdots + (x^b)^1(y^b)^{a-2} + (y^b)^{a-1}\Big)$$

Maximal Factoring of $x^n + y^n, n \geq 4$

Let $n = a \cdot b$ where a is the smallest odd prime divisor of n. Then,

$$x^n + y^n = (x^b)^a + (y^b)^a$$
$$= (x^b + y^b)\Big((x^b)^{a-1} - (x^b)^{a-2}(y^b) + (x^b)^{a-3}(y^b)^2 - \cdots - (x^b)(y^b)^{a-2} + (y^b)^{a-1}\Big)$$

- If n has no odd prime factors then $x^n + y^n$ does not factor.
- The reason for wanting a in $n = a \cdot b$ to be small is that it will b large in the factors $(x^b - y^b)$ or $(x^b + y^b)$ which will allow for as much follow up factoring of $(x^b - y^b)$ and $(x^b + y^b)$ as possible.
- In particular, if you are confronted with a polynomial that is both a difference of squares and a difference of cubes you should factor it as a difference of squares first to ensure the most complete factorization.

Factoring by Adding and Subtracting the Same Term

Transforming to a Difference of Squares

Examples

$$x^4 + 4 = x^4 + (\mathbf{4x^2}) + 4 - (\mathbf{4x^2})$$
$$= (x^4 + 4x^2 + 4) - 4x^2$$
$$= (x^2 + 2)^2 - (2x)^2$$
$$= (x^2 + 2 - 2x)(x^2 + 2 + 2x)$$

$$x^2 + 2x^2y^2 + 9y^4 = x^4 + 2x^2y^2 + (\mathbf{4x^2y^2}) + 9y^4 - (\mathbf{4x^2y^2})$$
$$= (x^4 + 6x^2y^2 + 9y^4) - 4x^2y^2$$

$$= (x^2 + 3y^2)^2 - (2xy)^2$$
$$= (x^2 + 3y^2 - 2xy)(x^2 + 3y^2 + 2xy)$$

$$x^4 + x^2 + 1 \quad = x^4 + x^2 + (\boldsymbol{x^2}) + 1 - (\boldsymbol{x^2})$$
$$= (x^4 + 2x^2 + 1) - x^2$$
$$= (x^2 + 1)^2 - x^2$$

Factoring by Grouping

Examples

$$x^3 - 3x + 2 \quad = (x^3) + (2 - 3x)$$
$$= (x^3 \mathbf{- 1}) + (2 - 3x \mathbf{+ 1})$$
$$= (x^3 - 1) + (3 - 3x)$$
$$= (x - 1)(x^2 + x + 1) - 3(x - 1)$$
$$= (x - 1)(x^2 + x + 1 - 3)$$
$$= (x - 1)(x^2 + x - 2)$$

$$2x^3 + 3x^2 + 3x + 1 \quad = (\boldsymbol{x^3}) + (\boldsymbol{x^3} + 3x^2 + 3x + 1)$$
$$= x^3 + (x + 1)^3$$
$$= (x + x + 1)(x^2 - x(x + 1) + (x + 1)^2)$$
$$= (2x + 1)(x^2 + x + 1)$$

$$2a^2 - 12b^2 + 3bd - 5ab - 9bc - 6ac + 2ad$$
$$= (2a^2 - 5ab - 12b^2) + (3bd - 9bc - 6ac + 2ad)$$
$$= (2a + 3b)(a - 4b) + (2a + 3b)(d - 3c)$$
$$= (2a + 3b)(a - 4b - 3c + d)$$

$$x^5 + x + 1$$
$$= x^5 - \boldsymbol{x^2} + \boldsymbol{x^2} + x + 1$$

$$= (x^5 - x^2) + (x^2 + x + 1)$$
$$= x^2(x^3 - 1) + (x^2 + x + 1)$$
$$= x^2(x - 1)(x^2 + x + 1) + (x^2 + x + 1)$$
$$= (x^2 + x + 1)(x^2(x - 1) + 1)$$
$$= (x^2 + x + 1)(x^3 - x^2 + 1)$$

First Find the Form. Then Find the Coefficients.

Factor $21xy + 49x - 3y^2 - 7y$.

(1) We have four terms, namely $21xy, 49x, -3y^2$ and $-7y$. So, the form $(a + b)(c + d)$ should work.

(2) We have a y^2 term which tells us we need $(ay + \quad)(cy + \quad)$.

(3) We do not have a constant term. This tells us that we **<u>do not</u>** want to consider something like $(ay + \text{constant}_1)(cy + \text{constant}_2)$ because this leads to a constant term when you multiply $\text{constant}_1 \times \text{constant}_2$.

(4) We do not have an x^2 term which tells us we **<u>do not</u>** want to consider something like $(ay + bx)(cy + dx)$ because this leads to an x^2 term when you multiply $(bx) \cdot (dx) = (b \cdot d)x^2$.

So, the form of the factorization of $21xy + 49x - 3y^2 - 7y$ should be $(ay + bx)(cy + d)$ where a, b, c and d are constants.

Now focus on the coefficients. *If* there are integers than will satisfy this factorization then we can find them by equating the coefficients of like terms in the identity

$$21xy + 49x - 3y^2 - 7y = (ay + bx)(cy + d) = (bc)xy + (bd)x + (ac)y^2 + (ad)y.$$

$$bd = 49 \Longrightarrow b = d = \pm 7$$

$$bc = 21 \Longrightarrow c = \pm 3$$

$$ac = -3 \Longrightarrow a = \pm 1$$

$$ad = -7 \Longrightarrow d = \pm 7$$

If we take $b = d = 7$, then $c = 3$ and $a = -1$.

$$21xy + 49x - 3y^2 - 7y = (ay + bx)(cy + d) = (-y + 7x)(3y + 7)$$

which is an identity.

Factoring by Substitution

Example 1

Factor $x^6 + 19x^3 - 216$.

Solution

Make the substitution $y = x^3$. Then we can see that

$$\begin{aligned} x^6 + 19x^3 - 216 &= y^2 + 19y - 216 \\ &= (y-8)(y+27) \\ &= (x^3 - 2^3)(x^3 + 3^3) \\ &= (x-2)(x^2+2x+4)(x+3)(x^2-3x+9). \end{aligned}$$

■

Example 2

Factor $(3x^2 + 5x - 7)^2 + 14(3x^2 + 5x) - 53$.

Solution

Make the substitution $y = 3x^2 + 5x - 7$. Then we have

$$\begin{aligned} y^2 + 14(y-7) - 53 &= y^2 + 14y + 45 \\ &= (y+9)(y+5) \\ &= (3x^2+5x-7+9)(3x^2+5x-7+5) \\ &= (3x^2+5x+2)(3x^2+5x-2) \\ &= (3x+2)(x+1)(3x+1)(x-2). \end{aligned}$$

■

Exercise 2.1

Fully factor $x^{12} - y^{12}$.

Solution

$x^{12} - y^{12}$

$= (x^6)^2 - (y^6)^2$ 2 is the smallest prime factor of 12

$= \left(x^6 - y^6\right)\left(x^6 + y^6\right)$ Difference of two squares

$x^6 - y^6$

$= (x^3)^2 - (y^3)^2$ 2 is the smallest prime factor of 6

$= \left(x^3 - y^3\right)\left(x^3 + y^3\right)$ Difference of two squares

$x^6 + y^6$

$= (x^2)^3 + (y^2)^3$ 3 is the smallest odd prime factor of 6

$= (x^2 + y^2)(x^4 - x^2y^2 + y^2)$ Sum of two cubes

$x^3 - y^3$

$= (x - y)(x^2 + xy + y^2)$ Difference of two cubes

$x^3 + y^3$

$= (x + y)(x^2 - xy + y^2)$ Sum of two cubes

$x^4 - x^2y^2 + y^2$

$= (x^4 - x^2y^2 + (\mathbf{3x^2y^2}) + y^2) - (\mathbf{3x^2y^2})$ Adding and subtracting like factor

$= (x^4 + 2x^2y^2 + y^2) - 3x^2y^2$

$= (x^2 + y^2)^2 - \left(\sqrt{3}xy\right)^2$

$= \left(x^2 + y^2 - \sqrt{3}xy\right)\left(x^2 + y^2 + \sqrt{3}xy\right)$ Difference of two squares

Piecing these results together, we have

$$x^{12} - y^{12} = (x - y)(x^2 + xy + y^2)(x + y)(x^2 - xy + y^2)(x^2 + y^2) \cdot \left(x^2 + y^2 - \sqrt{3}xy\right)\left(x^2 + y^2 + \sqrt{3}xy\right).$$

■

Exercise 2.2

Find integers a and b such that $ax^5 + bx^4 + 1$ factors into $x^2 - x - 1$ and a cubic polynomial with integer coefficients.

Solution

$$ax^5 + bx^4 + 1 = (x^2 - x - 1)(cx^3 + dx^2 + ex + f)$$

We can immediately see that $c = a$ and $f = -1$ which reduces the problem to two unknowns.

$$ax^5 + bx^4 + 1 = (x^2 - x - 1)(ax^3 + dx^2 + ex - 1).$$

Now we need to equate coefficients of the like terms.

$$ax^5 + bx^4 + 1 = ax^5 + (d - a)x^4 + (e - d - a)x^3 - (e + d + 1)x^2 + (1 - e)x + 1$$

$$1 - e = 0 \Longrightarrow e = 1$$

$$e + d + 1 = 0 \Longrightarrow d = -e - 1 = -2$$

$$e - d - a = 0 \Longrightarrow a = e - d = 1 - (-2) = 3$$

$$b = d - a = -2 - 3 = -5.$$

This shows

$$3x^5 - 5x^4 + 1 = (x^2 - x - 1)(3x^3 - 2x^2 + x - 1).$$

■

2.2 Factoring Integer Valued Polynomials

Up to now, we have discussed real-valued polynomials, polynomials whose input and output are the real numbers. The general purpose for factoring such polynomials is that it is easier to find the zeros of a polynomial when it is in factored form. Consider the problem of finding all real numbers x such that $x^2 = (\sqrt{2} + \sqrt{3})x - \sqrt{6}$.

The first step is rewrite the equation as $x^2 - (\sqrt{2} + \sqrt{3})x + \sqrt{6} = 0$ with all terms on the left side and 0 on the right side of the equation in anticipation of applying the **zero product principle** which tells us that for numbers a and b if $ab = 0$ then $a = 0$ or $b = 0$ (or both equal 0).

Factoring the left-hand side, we have $(x - \sqrt{2})(x - \sqrt{3}) = 0$ and by the zero-product principle, we see that x can be either of the real numbers $\sqrt{2}$ or $\sqrt{3}$.

In this section (only) we will consider **integer-valued polynomials**, polynomials in any number of variables whose value is an integer for all integer values of those variables. Integer-valued polynomial equations where we only consider integer solutions are known as **Diophantine equations** and are an important topic in the field of number theory.

There is more freedom in factoring integer-valued polynomials because Diophantine equations can be solved with constants other than 0 on the right-hand side. The following exercises will illustrate the idea.

Exercise 2.3 (AIME, 1987, Problem 5)

Find $3x^2y^2$ if x and y are integers such that $y^2 + 3x^2y^2 = 30x^2 + 517$.

Solution

If we bring the $30x^2$ to the right-hand side we have

$$y^2 + 3x^2y^2 - 30x^2 = 517.$$

If we somehow "just magically knew" to subtract 10 from both sides the left-hand we are left with

$$y^2 + 3x^2y^2 - 30x^2 - 10 = 517 - 10 = 507.$$

We will return to how one can know to magically subtract 10 from both sides but from here we can solve the problem.

The constant 507 on the right-hand side illustrates the contrast with factoring a real-valued polynomial versus an integer valued polynomial. In the case of a real-valued polynomial we would need to have 0 on the right-hand side so we could employ the zero-product principle.

The expression on the left-side factors into $(3x^2 + 1)(y^2 - 10)$ and the prime factorization of the constant of the right-side is $3 \cdot 13 \cdot 13$. To make it easier to see what is going on let's use the label $m = 3x^2 + 1$ and $n = y^2 - 10$ so that

$$m \cdot n = 3 \cdot 13 \cdot 13.$$

Because the right-hand side is positive, we can conclude that m and n are both positive or both negative. But $m = 3x^2 + 1$ is always positive. Therefore, the only possibility is that both m and n are positive.

It follows that the only possible values for either m and n are

$$\{3^0 13^0, 3^1 13^0, 3^0 13^1, 3^1 13^1, 3^0 13^2, 3^1 13^2\} = \{1,3,13,39,169,507\}.$$

Consider the case of $n = y^2 - 10 \in \{1,3,13,39,169,507\}$. It follows that $y^2 = n + 10$ must be a perfect square. We can see that

$$\begin{aligned} 1 + 10 &= 11, &&\text{is not a perfect square} \\ 3 + 10 &= 13, &&\text{is not a perfect square} \\ 3 + 10 &= 13, &&\text{is not a perfect square} \\ 13 + 10 &= 23, &&\text{is not a perfect square} \\ 39 + 10 &= 49, &&\text{is a perfect square} \\ 169 + 10 &= 179, &&\text{is not a perfect square} \\ 507 + 10 &= 517, &&\text{is not a perfect square.} \end{aligned}$$

Therefore $n = 39 = 3 \cdot 13$. This implies that $m = 13$. Therefore

$$m = 3x^2 + 1 = 13 \Longrightarrow x^2 = 4$$

and

$$n = y^2 - 10 = 39 \Longrightarrow y^2 = 49.$$

Thus,

$$3x^2y^2 = 3 \cdot 4 \cdot 49 = 588.$$

■

Simon's Favorite Factoring Trick

Now let's get back to the magic of subtracting 10 from both sides. How did I know to do this? On math contest sites such as AoPS (https://artofproblemsolving.com/) and Math Stack Exchange (https://math.stackexchange.com/) the idea goes by the name "**Simon's Favorite Factoring Trick**" or just **SFFT**.

> The general idea of SFFT is that if you have a Diophantine equation of the form
>
> $$af(x)g(y) + abf(x) + cg(y) = d$$
>
> in the variables x and y then add the constant bc to both sides. In this way the left-hand side will factor as
>
> $$(af(x) + c)(g(y) + b) = d + bc.$$
>
> By writing $d + bc$ in terms of its prime factors you can find $f(x)$ and $g(y)$ by considering cases of splitting the prime factors between the two factors on the left-hand side.
>
> The process of adding bc to both sides is called "completing the rectangle".

In the problem we just solved we had

$$(3)(x^2)(y^2) + (3)(-10)(x^2) + (1)(y^2) = 517.$$

So, our $a = 3, b = -10, c = 1$ in the general form $af(x)g(y) + abf(x) + cg(y)$. Therefore we need to add $bc = -10$ to both sides.

Exercise 2.4

Find the length and width of a rectangle whose area is numerically equal to its perimeter. Assume the length and width are integers.

Solution

Let x be the length and let y be the width. Then the story line translates to $xy = 2x + 2y$. We can write this as $xy - 2x - 2y = 0$. Using SFFT we want to add $(-2)(-2) = 4$ to both sides. Doing this we have

$$xy - 2x - 2y + 4 = 4$$
$$(x-2)(y-2) = 2 \cdot 2.$$

Therefore $(x-2) \in \{1,2,4\}$ and similarly $(y-2) \in \{1,2,4\}$. So, there are three possibilities:

$$(x-2, y-2) = \{(1,4), (2,2), (4,1)\}$$

or

$$(x, y) = \{(3,6), (4,4), (6,3)\}.$$

■

Exercise 2.5 (AMC 12B, 2007, Problem 23)

How many non-congruent right triangles with positive integer leg lengths have areas that are numerically equal to 3 times their perimeters?

Solution

Let x and y be the length of the two legs of the right triangle where x and y are positive integers. Then the hypotenuse has length $\sqrt{x^2 + y^2}$ and the perimeter will equal $x + y + \sqrt{x^2 + y^2}$. The area of a right triangle equals $(1/2)xy$.

So, we have

$$\frac{1}{2}xy = 3\left(x + y + \sqrt{x^2 + y^2}\right)$$

$$xy = 6\left(x + y + \sqrt{x^2 + y^2}\right)$$

$$6\sqrt{x^2 + y^2} = xy - 6x - 6y$$

$$36(x^2 + y^2) = x^2y^2 + 36(x + y)^2 - 12xy(x + y)$$

$$36x^2 + 36y^2 = x^2y^2 + 36x^2 + 36y^2 + 72xy - 12x^2y - 12xy^2$$

$$x^2y^2 + 72xy - 12x^2y - 12xy^2 = 0$$

$$xy + 72 - 12y - 12x = 0$$

$$xy - 12x - 12y = -72.$$

Now we want to use Simon's Favorite Factoring Trick to "complete the rectangle" on the left-hand side by add $(-12)(-12) = 144$ to both sides of this equation.

$$xy - 12x - 12y + 144 = -72 + 144.$$

Now we can factor the left-hand side and we can find the prime factorization of the constant on the right-hand side.

$$(x - 12)(y - 12) = 72 = 2^3 \cdot 3^2.$$

Now we want to look at all ways to split these prime factors between the $(x - 12)$ and $(y - 12)$. In general, 2^a3^b has $(a + 1)(b + 1)$ factors. That is, 72 has $(3 + 1)(2 + 1) = 12$ distinct factors so there are 12 distinct values for $(x - 12)$. But half of these will lead to congruent right triangles.

$$(x - 12, y - 12) = \{(1,72), (2,36), (3,24), (4,18), (6,12), (8,9)\}$$

with the remaining possibilities such as $(9,8), (12,6)$, etc. yielding only congruent triangles. Therefore,

$$(x, y) = \{(13,84), (14,48), (15,36), (16,30), (18,24), (20,21)\}.$$

We can see there are 6 possible (x, y) pairs for the legs which would give us 6 non-congruent right triangles.

■

Simon's Favorite Factoring Trick is only one of many factoring approaches to solving Diophantine equations. The book Indeterminate Equation by Xing Zhou is a very readable reference for getting a good foundation and lots of problem-solving practice on the subject.

But from here on down in these notes we are jumping back into the assumption that our variable(s) can assume any real number (not just integers) unless otherwise stated.

2.3 Substitution

"Hidden" quadratic equations such as $3x^4 - 5x^2 - 2 = 0$ are familiar examples where we need to make a substitution to solve the equations. In this problem if we substitute $y = x^2$ then we get

$$3y^2 - 5y - 2 = (y-2)(3y+1) = 0$$

$$\Rightarrow y = 2, y = -1/3$$

$$\Rightarrow x = \pm\sqrt{2} \text{ and } x = \pm\frac{1}{\sqrt{-3}} = \mp\frac{\sqrt{3}}{3}i.$$

In this section we will build on this idea by looking at examples where the needed substitution is even more hidden. All of these examples come from the book *The Substitution Method* by Yongcheng Chen which is a great resource for learning for high school math contest preparation.

A common thread with problems where a substitution is most effective is the presence of somewhat similar expressions appearing in multiple spots in the equation. Noticing and taking full advantage of this is a critical part in figuring out what substitution will be effective in a particular problem.

Exercise 2.6

Find all real values of x that satisfy $(16x^2 - 9)^3 + (9x^2 - 16)^3 = (25x^2 - 25)^3$.

Solution

The very first thing you should see in this problem is that hoping for a lot of cancellation if you multiply everything out should be a backup plan at best. It will take a lot of time just to find out if it happens.

The occurrence of the coefficients 16 and 9 occurring in both of the first two terms is a clue of some sort towards finding a faster approach. Also notice the coefficient 25 in the third term is $16 + 9$. Another clue.

If we let $a = 16x^2 - 9$ and $b = 9x^2 - 16$ then $a + b = 25x^2 - 25$, which occurs in our third term. Making this substitution we have

$$a^3 + b^3 = (a + b)^3.$$

Now you can see much clearer how multiplying out the expression on the right-hand side is going to get you the cancellation we need.

$$a^3 + b^3 = (a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

$$3a^2b + 3ab^2 = 0$$

$$3ab(a + b) = 0.$$

From here the Zero Product Principle tells us that $a = 0$ or $b = 0$ or $a + b = 0$. Now we can easily solve the original problem.

$$a = 16x^2 - 9 = 0 \Longrightarrow x^2 = \frac{9}{16} \Longrightarrow x = \pm\frac{3}{4}.$$

$$b = 9x^2 - 16 \Longrightarrow x^2 = \frac{16}{9} \Longrightarrow x = \pm\frac{4}{3}$$

$$a + b = 25x^2 - 25 = 0 \Longrightarrow x^2 = 1 \Longrightarrow x = \pm 1.$$

■

Exercise 2.7

Find the smallest value of x that satisfies $(3x^2 - 2x + 1)(3x^2 - 2x - 7) + 12 = 0$.

Solution

You might hope that if you multiple this out you will get a hidden quadratic such as we saw in the opening discussion of this section. Unlike in Exercise 2.6 it won't take too long to check. But unfortunately, nothing good comes out of it.

Don't overlook the similarity in the two factors. They both include $3x^2 - 2x$. So, try the substitution $y = 3x^2 - 2x$. The problem reduces to a solvable quadratic equation.

$$(y + 1)(y - 7) + 12 = 0$$

$$y^2 - 6y + 5 = 0$$

$$(y-5)(y-1)=0 \Longrightarrow y=5 \text{ or } y=1.$$

In terms of the original variable x these solutions become

$3x^2-2x=5$	$3x^2-2x=1$
$3x^2-2x-5=0$	$3x^2-2x-1=0$
$(3x-5)(x+1)=0$	$(3x+1)(x-1)=0$
$x=5/3$ or $x=-1$	$x=-1/3$ or $x=1$

The smallest of these four solutions is $x=-1$.

■

2.4 Palindromic and Anti-Palindromic Polynomials

In this section we will see that not all substitutions are visually obvious. We will consider a class of polynomials in x where the not so obvious substitution $y=x+\frac{1}{x}$ will reduce the power of a polynomial of degree $2n$ down to a polynomial of degree n.

Let $f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$. Then $f(x)$ is **palindromic** if $a_j=a_{n-j}$ for $j=1,2,\ldots,n$.

Examples:

$ax^4+bx^3+cx^2+bx+a$

$2x^4-5x-x^2-5x+2$

x^2+5x+1

$ax^5+bx^4+cx^3+cx^2+bx+a$

$2x^5+3x^4-2x^3-2x^2+3x+2$

x^3+1

Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$. Then $f(x)$ is **anti-palindromic** if either (i) n is odd and $a_j = -a_{n-j}$ for $j = 1,2,\dots,n$ or (ii) n is even, $a_{n/2} = 0$ and $a_j = -a_{n-j}$ for $j = 1,2,\dots,n$.

Examples:

$ax^5 + bx^4 + c^3 - cx^2 - bx - a$

$2x^5 - 5x^4 - 3x^3 + 3x^2 + 5x - 2$

$x^3 - 4x^2 + 4x - 1$

$ax^4 + bx^3 + \mathbf{0}x^2 - bx - a$

$-2x^4 - 5x^3 + 5x + 2$

$x^2 - 1$

Properties

- If $f(x)$ is palindromic and if $c \neq 0$ is a root of $f(x) = 0$, then $x = 1/c$ must also be a root of $f(x) = 0$.

- If $f(x)$ is palindromic and n is odd, then $(x+1)$ is a factor of $f(x)$ and $g(x) = f(x)/(x+1)$ is an even degree palindromic polynomial.

(i)	$\dfrac{2x^5 + 3x^4 - 2x^3 - 2x^2 + 3x + 2}{x+1} = 2x^4 + x^3 - 3x^2 + x + 2$
(ii)	$\dfrac{x^3 + 1}{x+1} = x^2 - x + 1$

- If $f(x)$ is anti-palindromic then $(x-1)$ is a factor of $f(x)$ and $g(x) = f(x)/(x-1)$ is a palindromic polynomial.

(i)	$\dfrac{2x^5 - 5x^4 - 3x^3 + 3x^2 + 5x - 2}{x-1} = 2x^4 - 3x^3 - 6x^2 - 3x + 2$

(ii)	$\dfrac{x^3 - 4x^2 + 4x - 1}{x - 1} = x^2 - 3x + 1$
(iii)	$\dfrac{-2x^4 - 5x^3 + 5x + 2}{x - 1} = -2x^3 - 7x^2 - 7x - 2$
(iv)	$\dfrac{x^2 - 1}{x - 1} = x + 1$

- Every palindromic polynomial $f(x)$ of degree $2n$ can be written in the form

$$f(x) = x^n g\left(x + \frac{1}{x}\right)$$

where $g(y)$ is a polynomial in y of degree n.

Example 1.

Let $f(x) = 2x^4 - 4x^3 + 5x^2 - 4x + 2$. Find the polynomial $g(y)$ such that $f(x) = xg(y)$ with $y = x + \frac{1}{x}$.

Solution

We see that $f(x)$ is palindromic of degree $2n = 2(1)$. So, divide $f(x)$ by $x^n = x^1$.

$$f(x) = x\left(\frac{f(x)}{x}\right) = x\left(\frac{3x^2 - 2x + 3}{x}\right) = x\left(3x - 2 + \frac{3}{x}\right) = x\left(3\left(x + \frac{1}{x}\right) - 2\right) = xg(y)$$

with $g(y) = 3y - 2$ and $y = x + \frac{1}{x}$.

Example 2.

Let $f(x) = 2x^4 - 4x^3 + 5x^2 - 4x + 2$. Find the polynomial $g(y)$ such that $f(x) = x^2 g(y)$ with $y = x + \frac{1}{x}$.

Solution

We see that $f(x)$ is palindromic of degree $2n = 2(2)$. So, divide $f(x)$ by $x^n = x^2$.

$$f(x) = x^2\left(\frac{f(x)}{x^2}\right) = x^2\left(\frac{2x^4 - 4x^3 + 5x^2 - 4x + 2}{x^2}\right)$$

$$= x^2\left(2x^2 - 4x + 5 - 4\left(\frac{1}{x}\right) + 2\left(\frac{1}{x^2}\right)\right)$$

$$= x^2\left(2\left(x^2 + \frac{1}{x^2}\right) - 4\left(x + \frac{1}{x}\right) + 5\right)$$

$$= x^2\left(2\left(\left(x + \frac{1}{x}\right)^2 - 2\right) - 4\left(x + \frac{1}{x}\right) + 5\right) \text{ using recursion identity}$$

$$= x^2\left(2\left(x + \frac{1}{x}\right)^2 - 4\left(x + \frac{1}{x}\right) + 1\right)$$

$$= x^2 g(y)$$

where $g(y) = 2y^2 - 4y + 1$ and $y = x + \frac{1}{x}$.

Recursion Identity
$$x^n + \frac{1}{x^n} = \left(x^{n-1} + \frac{1}{x^{n-1}}\right)\left(x + \frac{1}{x}\right) - \left(x^{n-2} + \frac{1}{x^{n-2}}\right).$$
$$x^2 + \frac{1}{x^2} = \left(x^1 + \frac{1}{x^1}\right)\left(x + \frac{1}{x}\right) - \left(x^0 + \frac{1}{x^0}\right)$$ $$= \left(x + \frac{1}{x}\right)^2 - 2$$

Example 3.

Let $f(x) = x^6 + x^3 + 1$. Find the polynomial $g(y)$ such that $f(x) = x^3 g(y)$ with $y = x + \frac{1}{x}$.

Solution

We see that $f(x)$ is palindromic of degree $2n = 2(3)$. So, divide $f(x)$ by $x^n = x^3$.

$$f(x) = x^6 + x^3 + 1$$

$$= x^3\left(\frac{f(x)}{x^3}\right) = x^3\left(\frac{x^6 + x^3 + 1}{x^3}\right)$$

$$= x^3\left(x^3 + 1 + \left(\frac{1}{x^3}\right)\right)$$

$$= x^3\left(\left(x^3 + \frac{1}{x^3}\right) + 1\right)$$

$$= x^3\left(\left(x+\frac{1}{x}\right)^3 - 3\left(x+\frac{1}{x}\right)+1\right) \text{ using recursion identity}$$

$$= x^3 g\left(x+\frac{1}{x}\right)$$

where $g(y) = y^3 - 3y + 1$.

Recursion Identity
$$x^n + \frac{1}{x^n} = \left(x^{n-1}+\frac{1}{x^{n-1}}\right)\left(x+\frac{1}{x}\right) - \left(x^{n-2}+\frac{1}{x^{n-2}}\right).$$
$$x^3+\frac{1}{x^3} = \left(x^2+\frac{1}{x^2}\right)\left(x+\frac{1}{x}\right) - \left(x^1+\frac{1}{x^1}\right)$$ $$= \left(\left(x+\frac{1}{x}\right)^2 - 2\right)\left(x+\frac{1}{x}\right) - \left(x+\frac{1}{x}\right)$$ $$= \left(x+\frac{1}{x}\right)^3 - 3\left(x+\frac{1}{x}\right)$$

Theorem A

If $f(x)$ is palindromic then

$$f(x) = x^n f\left(\frac{1}{x}\right).$$

Proof

$$x^n f\left(\frac{1}{x}\right) = x^n\left(a_n\left(\frac{1}{x}\right)^n + a_{n-1}\left(\frac{1}{x}\right)^{n-1} + \cdots + a_1\left(\frac{1}{x}\right) + a_0\right)$$

$$= a_n + a_{n-1}x + \cdots + a_1 x^{n-1} + a_0 x^n$$

$$= a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} + a_n x^n$$

$$= f(x).$$

■

Theorem B

If $f(x)$ is palindromic and if $c \neq 0$ is a root of $f(x) = 0$, then $\frac{1}{c}$ is a root of $f(x) = 0$.

Proof

$$0 = f(c) = c^{n-1} f\left(\frac{1}{c}\right) \Rightarrow f\left(\frac{1}{c}\right) = 0.$$

■

Consequently, apart from the cases of $c = 1$ and $c = -1$ which are their own reciprocals, every root of a palindromic polynomial is paired with a distinct reciprocal.

Exercise 2.8 (Source: MSHSML 1T982)

Find a value of x between 0 and 1 such that $x^4 - 4x^3 + 5x^2 - 4x + 1 = 0$. Write your answer in the form $\frac{a+b\sqrt{c}}{2}$, where a, b, c are integers. (Idea: Put the equation in the form $(x + 1/x)^2 + A(x + 1/x) + B = 0.$)

Solution

$$x^4 - 4x^3 + 5x^2 - 4x + 1 = 0$$

$$(x^4 + 1) - 4x(x^2 + 1) + 5x^2 = 0$$

$$\left(\frac{x^4 + 1}{x^2}\right) - \left(\frac{4x}{x}\right)\left(\frac{x^2 + 1}{x}\right) + \left(\frac{5x^2}{x^2}\right) = \frac{0}{x^2}$$

$$\left(x^2 + \frac{1}{x^2}\right) - 4\left(x + \frac{1}{x}\right) + 5 = 0$$

$$\left(x^2 + 2 + \frac{1}{x^2}\right) - 4\left(x + \frac{1}{x}\right) + (5 - 2) = 0$$

$$\left(x + \frac{1}{x}\right)^2 - 4\left(x + \frac{1}{x}\right) + 3 = 0$$

$$y^2 - 4y + 3 = 0$$

$$(y - 3)(y - 1) = 0$$

where $y = \left(x + \frac{1}{x}\right)$.

$$x+\frac{1}{x}=3 \text{ or } x+\frac{1}{x}=1.$$

We note that

$$x+\frac{1}{x}=a \Leftrightarrow x^2-ax+1=0$$

so, we have the two equations

$$x^2-3x+1=0 \text{ or } x^2-x+1=0.$$

$$x=\frac{3\pm\sqrt{5}}{2} \quad \text{or} \quad x=\frac{1\pm\sqrt{1-4}}{2}.$$

Remember we are required to find a root in $[0,1]$. We note that

$0<\frac{3-\sqrt{5}}{2}<1$	Acceptable answer
$\frac{3+\sqrt{5}}{2}>1$	Not an acceptable answer
$\frac{1-\sqrt{1-4}}{2}$	Not an acceptable answer because it is not a real number
$\frac{1+\sqrt{1-4}}{2}$	Not an acceptable answer because it is not a real number

So, the only possible case solution is

$$x=\frac{3-\sqrt{5}}{2}.$$

■

Exercise 2.9 (Source: MSHSML 3A843)

Given that $p=\left(x+\frac{1}{x}\right)^2$, express $x^4+x^2+\frac{1}{x^2}+\frac{1}{x^4}$ in terms of p.

Solution

Using the recursion identity

$$x^n + \frac{1}{x^n} = \left(x^{n-1} + \frac{1}{x^{n-1}}\right)\left(x + \frac{1}{x}\right) - \left(x^{n-2} + \frac{1}{x^{n-2}}\right)$$

it follows that

$$\begin{aligned} x^2 + \frac{1}{x^2} &= \left(x^1 + \frac{1}{x^1}\right)\left(x + \frac{1}{x}\right) - \left(x^0 + \frac{1}{x^0}\right) \\ &= \left(x + \frac{1}{x}\right)^2 - 2 \end{aligned}$$

and

$$\begin{aligned} x^3 + \frac{1}{x^3} &= \left(x^2 + \frac{1}{x^2}\right)\left(x + \frac{1}{x}\right) - \left(x^1 + \frac{1}{x^1}\right) \\ &= \left(\left(x + \frac{1}{x}\right)^2 - 2\right)\left(x + \frac{1}{x}\right) - \left(x + \frac{1}{x}\right) \\ &= \left(x + \frac{1}{x}\right)^3 - 3\left(x + \frac{1}{x}\right) \end{aligned}$$

and

$$\begin{aligned} x^4 + \frac{1}{x^4} &= \left(x^3 + \frac{1}{x^3}\right)\left(x + \frac{1}{x}\right) - \left(x^2 + \frac{1}{x^2}\right) \\ &= \left(\left(x + \frac{1}{x}\right)^3 - 3\left(x + \frac{1}{x}\right)\right)\left(x + \frac{1}{x}\right) - \left(\left(x + \frac{1}{x}\right)^2 - 2\right) \\ &= \left(x + \frac{1}{x}\right)^4 - 4\left(x + \frac{1}{x}\right)^2 + 2. \end{aligned}$$

Therefore,

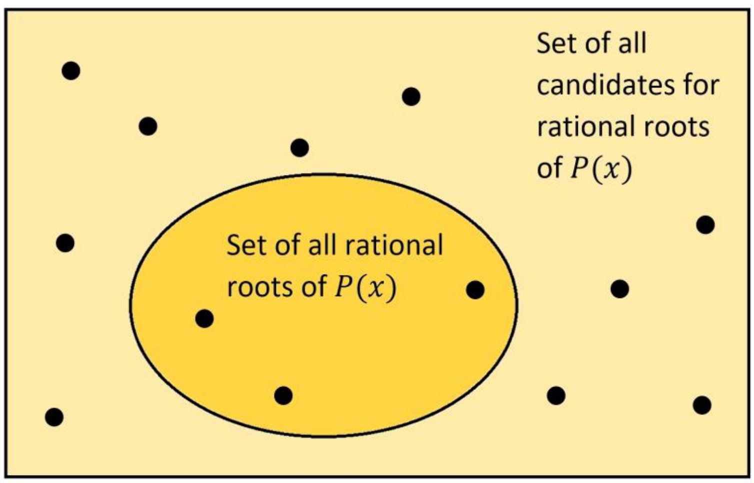
$$\begin{aligned} x^4 + x^2 + \frac{1}{x^2} + \frac{1}{x^4} &= \left(x^4 + \frac{1}{x^4}\right) + \left(x^2 + \frac{1}{x^2}\right) \\ &= \left(\left(x + \frac{1}{x}\right)^4 - 4\left(x + \frac{1}{x}\right)^2 + 2\right) + \left(x + \frac{1}{x}\right)^2 - 2 \\ &= \left(x + \frac{1}{x}\right)^2\left(\left(x + \frac{1}{x}\right)^2 - 3\right) = p(p-3) \end{aligned}$$

where $p = \left(x + \frac{1}{x}\right)^2$.

■

2.5 Finding Rational Roots of a Polynomial

In this section we will see how to build a finite set of rational numbers that we will call our "candidates" for being rational roots of a polynomial equation $P(x) = 0$ with integer coefficients. Unfortunately, not all candidates will actually turn out to be a root. But on the bright side, all of the actual rational roots of $P(x)$ will belong to our set of candidates.



The tool for constructing our set of candidates is the **rational root theorem**. Let

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

represent a polynomial *with integral coefficients* where as usual we assume $a_n \neq 0$.

Rational Root Theorem

If the reduced rational number p/q (*i.e.* p and q relatively prime) is a root of $P(x) = 0$, then p is a factor of a_0 and q is a factor of a_n.

The Integer Root Theorem is the special case of the rational root theorem for monic polynomials (*i.e.* the leading coefficient $a_n = 1$).

Integral Root Theorem
If $a_n = 1$ in $P(x)$, then all rational roots of $P(x) = 0$ must be integral factors of a_0.

Let's look at an example to help solidify this result. Let

$$P(x) = 10x^4 - 23x^3 - 8x^2 + 46x - 24.$$

Our first point is to note that this polynomial has integer coefficients, so the rational root theorem does apply.

In the notation of the rational root theorem, $p = 24$ and $q = 10$. The set of factors of $p = 24$ is $A = \{\pm1, \pm2, \pm3, \pm4, \pm6, \pm8, \pm12, \pm24\}$. The set of factors of $q = 10$ is $B \in \{\pm1, \pm2, \pm5, \pm10\}$.

We can construct the set of candidate rational roots of $P(x) = 0$ by picking, in all possible ways, a numerator from the set A and a denominator from the set B. The set A contains 16 integers and the set B contains 8 integers. So, our set of candidates will $8 \times 16 = 128$ rational numbers. Some of these will be duplicates once you divide out any common factors (such as $4/2$ reduces to $2/1$ which shouldn't be checked twice). Remember that the rational root theorem only applies when your numerator and denominator have been reduced to have no common factors.

The point being it can be prohibitively time consuming during a contest *to check even a small fraction* of these candidates – unless you are able *to substantially trim down this list* in ways that are much more time efficient than checking each candidate root individually.

The remainder of this section is an investigation in the ways to trim down the list of candidate rational roots in a time effective way.

Dickson's Test

Let

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

represent a polynomial with integral coefficients where $a_n \neq 0$. For any relatively prime integers p and q and for any integer m such that $P(m) \neq 0$, p/q is not a root of $P(x) = 0$ if $qm - p$ is not a divisor of $P(m)$.

You won't find Dickson's test in any recent textbooks. I've given this result the name Dickson's Test myself because it doesn't have a name and the earliest source I could find for it was in Leonard Dickson's (University of Chicago) textbook *Elementary Theory of Equations* published in 1914. The result is rediscovered from time to time in mathematics journals but the only textbook's I have seen in discussed were all textbooks on the theory of equations published in the 1950's or earlier.

I think the two examples I give below will more than justify its utility.

Example 1

Let $P(x) = x^3 - 12x^2 + 47x - 60$. According to the Integer Root Theorem (applies because the leading coefficient equals 1), the set of candidate rational roots of $P(x) = 0$ are the set of all factors of $60 = 2^2 \cdot 3 \cdot 5$, namely $\{\pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6, \pm 10, \pm 12, \pm 15, \pm 20, \pm 30, \pm 60\}$.

Let's trim this list of 24 candidate roots by applying Dickson's Test for the case $m = 1$. In this situation, p/q cannot be a root if $qm - p = 1(1) - p = 1 - p$ does not divide $P(1) = 1^3 - 12(1^2) + 47(1) - 60 = -24$. $P(1) \neq 0$ so Dickson's Test applies for $m = 1$.

Note: It is a little easier to mentally check if $p - 1$ divides 24. This is equivalent to checking if $1 - p$ divides -24 because in general the statements $\pm a$ divides $\pm b$ are equivalent for all ways of choosing the pluses and minuses.

$$p \in \{-1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6, \pm 10, \pm 12, \pm 15, \pm 20, \pm 30, \pm 60\}$$

Rule out all candidates p if $p - 1$ does not divide 24. Note we have already ruled out $p = 1$ because $P(1) \neq 0$.

So we can rule out $p = 6$ (because $6 - 1 = 5$ does not divide 24), $p = -4$ (because $-4 - 1 = -5$ does not divide 24), $p = -6$ (because $-6 - 1 = -7$ does not divide 24), $p = \pm 10, p = \pm 12, p = \pm 15, p = \pm 20, p = \pm 30$, and $p = \pm 60$.

Very time efficient! We ruled out 12 candidates almost instantly. The remaining candidates are $\{-1, \pm 2, \pm 3, 4, \pm 5\}$.

Now pick a different value for m and eliminate some more candidates. Let's consider the case $m = -1$. $P(-1) = (-1)^3 - 12(-1)^2 + 47(-1) - 60 = -120 \neq 0$ so that Dickson's Test applies for $m = -1$. Also we can rule out $p = -1$ because $P(-1) \neq 0$.

By Dickson's Test, p/q cannot be a root if $qm - p = 1(-1) - p = -1 - p$ does not divide $P(-1) = -120$. Equivalently, we can check if $p + 1$ divides $120 = 2^4 \cdot 3 \cdot 5$.

Rule out all (remaining) candidates p from $\{\pm 2, \pm 3, 4, \pm 5\}$ if $p + 1$ does not divide 120.

Whoops! No help here because $p+1$ divides 120 for all of the candidates p remaining.

Now pick a different value for m and eliminate some more candidates. Let's consider the case $m=2$. $P(2)=(2)^3-12(2)^2+47(2)-60=-6\neq 0$ so that Dickson's Test applies for $m=2$. Also we can rule out $p=2$ because $P(2)\neq 0$.

By Dickson's Test, p/q cannot be a root if $qm-p=1(2)-p=2-p$ does not divide $P(2)=-6$. Equivalently, we can check if $p-2$ divides 6 for the remaining candidates $p\in\{-2,\pm 3,4,\pm 5\}$.

So we can rule out $p=-2$ (because $-2-2=-4$ does not divide 6), $p=-3$ (because $-3-2=-5$ does not divide 6), and $p=-5$.

Now our candidate list is reduced to just $\{3,4,5\}$.

So, in a matter of moments we have reduced our list of 24 candidates down to a list of 3 candidates. A big time saver!

At this point the list of remaining candidates is small enough that the most time effective next step is to directly check if $P(x)=0$ for $x=3,4$ or 5. You will find that $P(3)=P(4)=P(5)=0$ so this is the complete set of roots of $P(x)=x^3-12x^2+47x-60=0$.

Example 2

Let $P(x)=7x^3-2x^2+4x-6$. According to the rational root test the candidate roots are

$$\frac{p}{q}\in\left\{\pm\frac{1}{1},\pm\frac{2}{1},\pm\frac{3}{1},\pm\frac{6}{1},\pm\frac{1}{7},\pm\frac{2}{7},\pm\frac{3}{7},\pm\frac{6}{7}\right\}.$$

Let's use Dickson's Test with $m=1$ to trim down this list of candidates.

We note that $P(1)=3\neq 0$ so Dickson's Test applies for $m=1$. Also, it tells us that 1 is not a root of $P(x)=0$.

By Dickson's test with $m=1$, p/q is not a root if $qm-p=q-p$ does not divide $P(1)=3$.

Check $p/q=2/1$. Does $q-p=1-2=-1$ divide 3? Yes. Therefore, we cannot rule it out 2 as a root.

Check $p/q=-2/1$. Does $q-p=1-(-2)=3$ divide $P(1)=3$? Yes. So, we cannot rule out -2 as a root.

Continuing on, we find

$p/q = 3/1$	$q - p = 1 - 3 = -2$	does not divide $P(1) = 3$ (so 3 is not a root)
$p/q = -3/1$	$q - p = 4$	4 does not divide 3 (so 4 is not a root)
$p/q = 6/1$	$q - p = -5$	-5 does not divide 3
$p/q = -6/1$	$q - p = 7$	7 does not divide 3
$p/q = 1/7$	$q - p = 6$	6 does not divide 3
$p/q = -1/7$	$q - p = 8$	8 does not divide 3
$p/q = 2/7$	$q - p = 5$	5 does not divide 3
$p/q = -2/7$	$q - p = 9$	9 does not divide 3
$p/q = 3/7$	$q - p = 4$	4 does not divide 3
$p/q = -3/7$	$q - p = 10$	10 does not divide 3
$p/q = 6/7$	$q - p = 1$	1 DOES divide 3. So, we CANNOT rule out $6/7$ as a rational root.
$p/q = -6/7$	$q - p = 13$	13 does not divide 3

So, we have whittled down our candidate rational root pool from

$$\left\{\pm\frac{1}{1}, \pm\frac{2}{1}, \pm\frac{3}{1}, \pm\frac{6}{1}, \pm\frac{1}{7}, \pm\frac{2}{7}, \pm\frac{3}{7}, \pm\frac{6}{7}\right\}$$

to just the three candidates $\{-2, 2, 6/7\}$.

Let's further trim down this list by using $m = -1$ with Dickson's test. We note that $P(-1) = -7 - 2 - 4 - 6 = -19 \neq 0$ so Dickson's test applies for $m = -1$ and also -1 is not a root.

$p/q = -2/1$	$q + p = -1$	-1 DOES divide -19. So, we CANNOT rule out $-2/1$ as a rational root.
$p/q = 2/1$	$q + p = 3$	3 does not divide -19 (so 2 is not a root)
$p/q = 6/7$	$q + p = 13$	13 does not divide -19 (so $6/7$ is not a root)

At this point we have reduced our pool of candidate rational roots to just $\{-2\}$. Let's use synthetic division to check if $P(-2) \stackrel{?}{=} 0$.

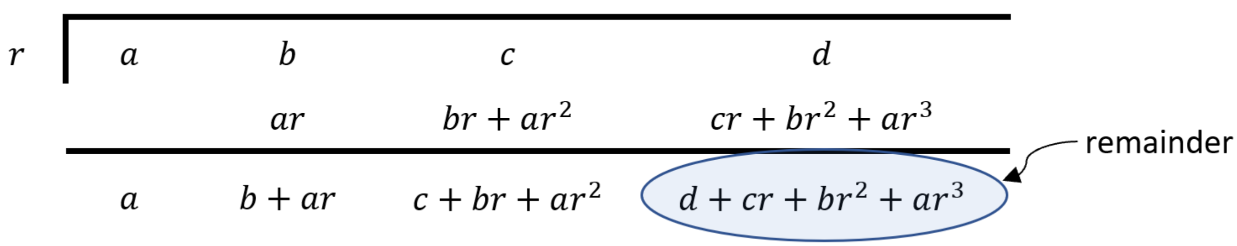
-2	7	-2	4	-6
		-14	32	-72
	7	-16	36	-78

We see $P(-2) = -78 \neq 0$ so -2 is not a root. So $P(x) = 7x^3 - 2x^2 + 4x - 6$ has no rational roots.

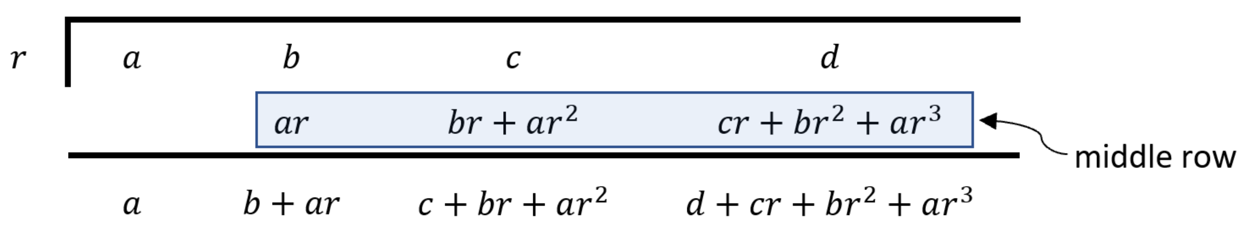
Page & Chosid's Shortened Synthetic Division Test

Even with the effectiveness of Dickson's test there will be situations where you want or need to use first principles to see if the rational number p/q is a root of $P(x) = 0$. That is, to check if $P(p/q) = 0$ either by the remainder theorem or by direct enumeration of $P(p/q)$.

The synthetic division tableau for $P(x)/(x - r)$ for $P(x) = ax^3 + bx^2 + cx + d$ is shown below. The remainder theorem states that r is a root of $P(x) = 0$ if and only if the remainder in the division $P(x)/(x - r)$ equals 0. That is, if and only if $d + cr + br^2 + ar^3 = 0$.



The Page and Chosid Test says that if *any* of the numbers in the middle row of the synthetic division tableau for $P(x)/(x - r)$ is not an integer for some rational number r, then r is not a root of $P(x) = 0$.



The time saving aspect of this test is that the tableau is filled in going from left to right so you can immediately interrupt the synthetic division process of finding the remainder as soon as any number in this middle row appears that is not an integer.

Consider the following example:

Use the remainder theorem to show that $P(2/5) \neq 0$ and hence the candidate rational root $2/5$ is not a root of $P(x) = 15x^5 - 13x^4 - 23x^3 - 13x^2 + 7x^2 + 8x - 24 = 0$.

$\frac{2}{5}$	15	-13	-23	-13	7	8	-24
		6	$-\frac{14}{5}$	$-\frac{258}{25}$	$-\frac{1166}{125}$	$-\frac{582}{625}$	$\frac{8836}{3125}$
	15	-7	$-\frac{129}{5}$	$-\frac{583}{25}$	$-\frac{291}{125}$	$\frac{4418}{625}$	$-\frac{66164}{3125}$

By the remainder theorem, $P(2/5) = -66164/3125$ which does not equal 0. Therefore $2/5$ is not a root of $P(x) = 0$.

What the Page & Chosid test tells us is that all the work filling in the synthetic division tableau after getting the non-integer $(-14/5)$ in the middle row was unnecessary!

$\frac{2}{5}$	15	-13	-23	-13	7	8	-24
		6	$-\frac{14}{5}$				
	15	-7					

Once we observe the non-integer $(-14/5)$ we can stop and immediately assert that $r = 2/5$ is not a root of $P(x) = 0$.

One point of clarification. The converse of the Page & Chosid test is not true. It is <u>not</u> true that r is a root if all the numbers in the middle row are integers. But if you do find all the numbers in the middle row to be integers it is a simple matter to check if the remainder equals 0 or not.

Also note that the numbers in the middle row will always be integers when r is an integer and the coefficients of $P(x)$ are integers because the calculation of the middle row will only involve the multiplication and addition of integers (which will always be an integer).

However, the Page & Chosid test can be very useful when r is *not* an integer. As our one example above shows, as soon as a non-integer occurs in the middle row the calculations from that point on get more and more time consuming to complete.

The only place I've ever seen the Page & Chosid test mentioned is in their original journal article, "Synthetic Division Shortened", Warren Page and Leo Chosid, *The Two-Year College Mathematics Journal*, Vol. 12, No. 5, November 1981, pages 334-336.

2.6 Deflation

Suppose $r_1, \ldots, r_n$ are the n roots of an n^{th} degree polynomial $f(x)$. Then by the division algorithm for polynomials we know there exists a polynomial $q(x)$ of degree $n-1$ such that $f(x) = (x - r_1)q(x)$.

Now consider the root r_2. It is certainly true that either $r_2 \neq r_1$ or $r_2 = r_1$.

Case 1. $\boldsymbol{r_2 \neq r_1}$

Because r_2 is a root we know $f(r_2) = 0$. But we also have that

$$0 = f(r_2) = (r_2 - r_1) \cdot q(r_2).$$

As $r_2 - r_1 \neq 0$ by our supposition, it follows that $q(r_2) = 0$. This means that $x = r_2$ is necessarily a root of $q(x)$.

Case 2. $\boldsymbol{r_2 = r_1}$.

This means that r_1 is (at least) a double root of $f(x)$. That is,

$$f(x) = (x - r_1)(x - r_1)q^*(x)$$

for some quotient $q^*(x)$. But it is still true that $f(x) = (x - r_1)q(x)$. Therefore,

$$(x - r_1)q(x) = (x - r_1)(x - r_1)q^*(x)$$

or

$$q(x) = (x - r_1)q^*(x).$$

Hence,

$$q(r_2) = (r_2 - r_1)q^*(r_2) = 0$$

because we are in the situation with $r_2 - r_1$.

So, we can conclude that in all cases $q(r_2) = 0$.

Generally speaking, the idea of **deflation** is that once we have determined that r is a root of the polynomial equation $f(x) = 0$ and if $q(x)$ is that quotient such that $f(x) = (x - r)q(x)$, then the remaining roots $r_2, \ldots, r_n$ of $f(x)$ constitute the $n-1$ roots of $q(x)$.

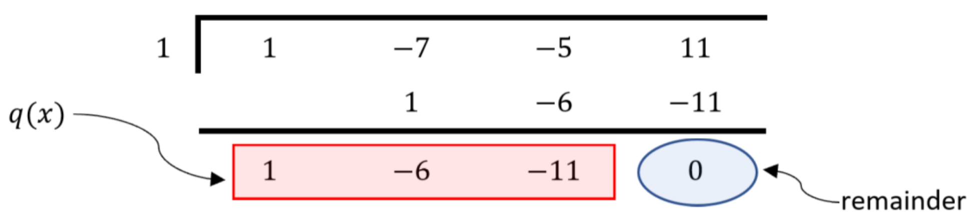
Therefore, it is sufficient (and easier) to find all roots of the lower degree polynomial $q(x)$.

Example

Let $f(x) = x^3 - 7x^2 - 5x + 11$. Find all three roots of $f(x) = 0$.

Solution

Because our constant coefficient is prime and because our leading coefficient equals 1, the list of candidates for rational roots of $f(x) = 0$ is necessarily small. In particular any rational root of $f(x) = 0$ would have to belong to the set $\{\pm 1, \pm 11\}$. If we start checking at $x = 1$ we get



a remainder of 0 so $x = 1$ is one of the roots. At this point the deflation principle tells us we can find the remaining two roots of the cubic equation $f(x) = 0$ by finding the two roots of the quadratic $q(x) = 0$.

I have constructed this problem so that there are no more rational roots. But that's OK at this point because we can use the quadratic formula to find the two roots of $q(x) = x^2 - 6x - 11 = 0$.

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(-11)}}{2(1)} = \frac{6 \pm \sqrt{80}}{2} = 3 \pm 2\sqrt{5}.$$

So, the two roots of $q(x) = 0$ are $3 \pm 2\sqrt{5}$ which means that the remaining two roots of $f(x) = 0$ are $3 \pm 2\sqrt{5}$.

2.7 Bounding the Roots

We've seen that Dickson's test and Page & Chosid's test can considerably trim down a list of candidate rational roots. Another step in that direction is to establish upper and lower bounds for roots (rational or otherwise).

> ► **Upper Bound Theorem**
>
> Let $f(x)$ be any polynomial with real coefficients with a nonzero leading coefficient. If $k > 0$ and if all the numbers in the bottom row of the synthetic division of $f(x)$ by $(x - k)$ are of the same sign or zero, then no real root of $f(x) = 0$ is greater than k.

Example

Let $f(x) = 4x^5 - 16x^4 + 9x^3 + 35x^2 - 51x + 18$. By the rational root theorem $x = 6$ is a candidate rational root. Suppose we used synthetic division to check if $x - 6$ divides $f(x)$.

6	4	-16	9	35	-51	18
		24	48	342	2262	13266
	4	8	57	377	2211	13284

At this point we notice that all the numbers in the bottom row are positive and so by the upper bound theorem we can conclude that there cannot be *any* real roots (rational or not) larger than 6.

■

It is pretty easy to see where this comes from. We'll use this example to explain. From the above tableau we can see that

$$f(x) = 4x^5 - 16x^4 + 9x^3 + 35x^2 - 51x + 18$$

$$= (4x^4 + 8x^3 + 57x^2 + 377x + 2211)(x - 6) + 13284$$

$$= q(x)(x - 6) + 13284$$

Now imagine evaluating $f(x)$ for some value of $x = x_0 > 6$. In this case

$$q(x_0) = 4x_0^4 + 8x_0^3 + 57x_0^2 + 377x_0 + 2211 > 0$$

because we multiplying a positive x_0 with positive coefficients in every term. And obviously $(x_0 - 6)$ is positive because we are assuming $x_0 > 6$. So

$$f(x_0) = q(x_0)(x_0 - 6) + 13284 > 0$$

which means that x_0 is not a root.

■

> ► **Lower Bound Theorem**
>
> Let $f(x)$ be any polynomial with real coefficients and a nonzero leading coefficient. If $k < 0$ and if all the numbers in the bottom row of the synthetic division of $f(x)$ by $(x - k)$ alternate in sign, then no real root of $f(x) = 0$ is smaller than k. Note that a "zero" must be given either a + sign or a – sign; it cannot be ignored.

Example

Let $f(x) = 4x^5 - 16x^4 + 9x^3 + 35x^2 - 51x + 18$. By the rational root theorem $x = -2$ is a candidate rational root. Suppose we used synthetic division to check if $x - (-2) = x + 2$ divides $f(x)$.

-2	4	-16	9	35	-51	18
		-8	48	-114	158	-214
	4	-24	57	-79	107	-196

At this point we notice that all the numbers in the bottom row alternate from positive to negative and so by the lower bound theorem we can conclude that there cannot be *any* real roots (rational or not) smaller than -2.

■

Justifying the lower bound theorem is a little more complicated than the upper bound theorem because there are cases to consider.

I will step through the justification in this example and leave it to you to follow up that the very same type of argument can be made for all other possible cases (such as when the power of the leading term in $q(x)$ is odd and/or when the coefficient the terms in the bottom row alternative starting with a negative number instead of a positive number.)

From the above tableau we can see that

$$f(x) = 4x^5 - 16x^4 + 9x^3 + 35x^2 - 51x + 18$$

$$= (4x^4 - 24x^3 + 57x^2 - 79x + 107)(x + 2) - 196$$

$$= q(x)(x + 2) - 196$$

Now imagine evaluating $f(x)$ for some value of $x = x_0 < -2$. In this case

$$q(x_0) = 4x_0^4 - 24x_0^3 + 57x_0^2 - 9x_0 + 107 > 0$$

because when we take the negative x_0 to an even power it switches from negative to positive. But notice that all the even powered terms have a positive coefficient. So, we will be multiplying two positive numbers.

And when we take the negative x_0 to an odd power it remains negative. But notice that all the odd powered terms have a negative coefficient. So, we will be multiplying two negative numbers and getting a positive number out of that.

$$q(x_0) = 4x_0^4 + (-24x_0^3) + 57x_0^2 + (-9x_0) + 107$$

So, we taking the sum of a string of positive numbers. Hence the total is positive.

Now we can see that in this example

$$f(x_0) = q(x_0)(x_0 + 2) - 196 < 0$$

for every $x_0 < -2$ because $q(x_0) > 0$ and $(x_0 + 2) < 0$ which makes their product negative. Then subtracting a negative number keeps the total negative.

Therefore $f(x) = 0$ cannot have any roots less than $x = -2$.

■

► There are two related results that fallout from the above arguments without having to do any division.

All nonnegative coefficients ⟹ No positive roots

If $f(x)$ has all nonnegative coefficients, then $f(x)$ cannot have any positive roots.

Alternating coefficients ⟹ No negative roots

If the coefficients of $f(x)$ alternate from positive to negative (or vice versa), then $f(x)$ cannot have any negative roots.

2.8 Conjugate Roots

Algebra I and II textbooks as well as precalculus textbooks cover the irrational conjugate root theorem and the complex conjugate root theorem which are given below.

Irrational Conjugate Root Theorem

If $f(x)$ is a polynomial with coefficients in $\mathbb{Q}$ (the set of rational numbers) and if $a + \sqrt{b}$ is a root of $f(x)$ for some rational numbers a and b such that b is not a perfect square, then $a - \sqrt{b}$ is also a root of $f(x)$.

Complex Conjugate Root Theorem

If $f(x)$ is a polynomial with coefficients in $\mathbb{R}$ (the set of real numbers) and if $a + bi$ is a root of $f(x)$ for some real numbers a and b such that $b \neq 0$, then $a - bi$ is also a root of $f(x)$.

Both of these results make their way on to math contests when, for example, you are given a fourth-degree polynomial $f(x)$ with rational coefficients and the information that $2 - \sqrt{3}$ is a root of $f(x) = 0$ and then you are asked to find the remaining three roots.

The idea is to immediately realize that $2 + \sqrt{3}$ must also be a root and then to use deflation to whittle down $f(x)$ into a quadratic which can be handled with the quadratic formula.

This all looks completely straightforward – but there is more to the story.

What if the contest problem said that $\sqrt{2} - \sqrt{3}$ was one of the roots instead of $2 - \sqrt{3}$? Go back to the irrational conjugate root theorem and read it carefully. It specifically requires that have to know that $a - \sqrt{b}$ is one root where a and b are *both rational numbers*. Is $\sqrt{2}$ rational? NO.

The general problem of finding conjugate roots is more involved than we will get into here but the following two results are useful.

Extended Irrational Conjugate Root Theorem

If $f(x)$ is a polynomial with coefficients in $\mathbb{Q}$ (the set of rational numbers) and if $\sqrt{a}+\sqrt{b}$ is a root of $f(x)$ for some rational numbers a and b such that none of a, b and ab are perfect squares then $\sqrt{a}-\sqrt{b}$, $-\sqrt{a}+\sqrt{b}$ and $-\sqrt{a}-\sqrt{b}$ are also roots of $f(x)$.

Mixed Irrational & Complex Conjugate Root Theorem

If $f(x)$ is a polynomial with coefficients in $\mathbb{Q}$ (the set of rational numbers) and if $\sqrt{a}+bi$ is a root of $f(x)$ for some rational numbers a and b such that a is not a perfect square and $b \neq 0$, then $\sqrt{a}-bi$, $-\sqrt{a}+bi$ and $-\sqrt{a}-bi$ are also roots of $f(x)$.

When using either of the above two results it is use to remember "ancillary result three" on page 42. Recall from that factoring formula that for general c and d we have the identity:

$$\begin{aligned}&\big(x-(c+d)\big)\big(x-(-c+d)\big)\big(x-(c-d)\big)\big(x-(-c-d)\big)\\&\quad= x^4-2(c^2+d^2)x^2+(c^2-d^2)^2.\end{aligned}$$

In particular we have

$$\begin{aligned}m_1(x) &= \Big(x-\big(\sqrt{a}+\sqrt{b}\big)\Big)\Big(x-\big(\sqrt{a}-\sqrt{b}\big)\Big)\Big(x-\big(-\sqrt{a}+\sqrt{b}\big)\Big)\Big(x-\big(-\sqrt{a}-\sqrt{b}\big)\Big)\\ &= x^4-2(a+b)x^2+(a-b)^2.\end{aligned}$$

and

$$\begin{aligned}m_2(x) &= \Big(x-\big(\sqrt{a}+bi\big)\Big)\Big(x-\big(\sqrt{a}-bi\big)\Big)\Big(x-\big(-\sqrt{a}+bi\big)\Big)\Big(x-\big(-\sqrt{a}-bi\big)\Big)\\ &= x^4-2(a-b^2)x^2+(a+b^2)^2.\end{aligned}$$

Exercise 2.10

Find all solutions to the equation $3x^5-4x^4-42x^3+56x^2+27x-36=0$ if one root is given to be $\sqrt{2}+\sqrt{5}$. (Source: *Golden Algebra*, N.P. Bali)

Solution

We can apply the above extended irrational conjugate root theorem because the coefficients of $f(x) = 3x^5 - 4x^4 - 42x^3 + 56x^2 + 27x - 36$ are all rational and none of $a = 2, b = 5$ or $ab = 10$ is a perfect square. So, from the information that $\sqrt{2} + \sqrt{5}$ is a root of $f(x) = 0$ we can also be guaranteed that $\sqrt{2} - \sqrt{5}$, $-\sqrt{2} + \sqrt{5}$ and $-\sqrt{2} - \sqrt{5}$ are roots of $f(x) = 0$.

So, we already know 4 of the 5 roots! We can find the last root by *deflation*, that is by dividing $f(x)$ by the minimal polynomial $m_1(x)$ of $\sqrt{2} + \sqrt{5}$. From our above discussion, we know that

$$m_1(x) = \left(x - (\sqrt{2} + \sqrt{5})\right)\left(x - (\sqrt{2} - \sqrt{5})\right)\left(x - (-\sqrt{2} + \sqrt{5})\right)\left(x - (-\sqrt{2} - \sqrt{5})\right)$$

$$= x^4 - 2(2+5)x^2 + (2-5)^2 = x^4 - 14x^2 + 9.$$

And by dividing $f(x) = 3x^5 - 4x^4 - 42x^3 + 56x^2 + 27x - 36$ by $m_1(x)$ we find the remaining factor of $f(x)$ is $(3x - 4)$.

	$3x$	-4				
$x^4 - 14x^2 + 9$	$3x^5$	$-4x^4$	$-42x^3$	$56x^2$	$27x$	-36
	$3x^5$		$-43x^3$		$27x$	
		$-4x^4$		$56x^2$		-36
		$-4x^4$		$-56x^2$		-36
						0

So, the fifth root of $f(x) = 0$ is $x = 4/3$.

But did we miss a chance for a shortcut for finding the fifth root?

Don't forget about the *Sum and Product of the Roots* result (see page 15). It can be very useful.

Because the sum of *all five* roots of $3x^5 - 4x^4 - 42x^3 + 56x^2 + 27x - 36 = 0$ equals

$$-a_{n-1}/a_n = -(-4)/3 = 4/3$$

and because the sum of the first four roots equals

$$\left(\sqrt{2} + \sqrt{5}\right) + \left(\sqrt{2} - \sqrt{5}\right) + \left(-\sqrt{2} + \sqrt{5}\right) + \left(-\sqrt{2} - \sqrt{5}\right) = 0$$

it follows immediately that the fifth root must equal $4/3$.

■

Exercise 2.11

For purposes of this exercise, suppose you were unaware of the extended irrational conjugate root theorem used in Exercise 2.10. How would you find a polynomial (not necessarily the polynomial of minimal degree) with rational coefficients such that $\sqrt{2} + \sqrt{5}$ is a root of that polynomial?

Solution

Obviously, the polynomial $x - \left(\sqrt{2} + \sqrt{5}\right) = 0$ has $\sqrt{2} + \sqrt{5}$ as a root but not all the coefficients are rational. But we can "kill off" the square roots by repeated squaring. The resulting equation will acquire extra roots in the process but will keep the required $\sqrt{2} + \sqrt{5}$ as a root.

$$\begin{aligned}
x - \left(\sqrt{2} + \sqrt{5}\right) &= 0 \\
x &= \sqrt{2} + \sqrt{5} \\
x^2 &= \left(\sqrt{2} + \sqrt{5}\right)^2 = 2 + 5 + 2\sqrt{10} \\
x^2 - 7 &= 2\sqrt{10} \\
(x^2 - 7)^2 &= \left(2\sqrt{10}\right)^2 = 40 \\
x^4 - 14x^2 + 49 - 40 &= 0 \\
x^4 - 14x^2 + 9 &= 0.
\end{aligned}$$

■

We've seen this polynomial before. This is just the polynomial of minimum degree for $\sqrt{2}+\sqrt{5}$ from Exercise 2.10.

Will "working back up" from a single root like we just did in Exercise 2.11 *always* lead to the polynomial of minimum degree that contains a given root? Unfortunately, not.

But it *often* will. It is possible to construct examples where it fails to find the polynomial of minimum degree. Nevertheless, it works often enough that it is a standard approach for finding a good candidate for a polynomial of minimum degree with a given root and then to use other techniques to verify that the resulting polynomial is irreducible and hence minimal. These "other techniques" are often part of a college level course in abstract algebra.

Exercise 2.12

Find the polynomial of lowest degree with rational coefficients that has $\sqrt{2}-i$ as a root.

Solution

This problem is specifically asking for the minimal polynomial of $\sqrt{2}-i$ with rational coefficients. We can apply the Mixed Irrational & Complex Conjugate Root Theorem because the given root is of the form $\sqrt{a}+bi$ where $a=2$ is not a perfect square and $b=-1$ does not equal 0.

According to that theorem, the minimal polynomial of $\sqrt{a}+bi$ is

$$\begin{aligned} m_2(x) &= \left(x-\left(\sqrt{a}+bi\right)\right)\left(x-\left(\sqrt{a}-bi\right)\right)\left(x-\left(-\sqrt{a}+bi\right)\right)\left(x-\left(-\sqrt{a}-bi\right)\right) \\ &= x^4-2(a-b^2)x^2+(a+b^2)^2. \end{aligned}$$

Applying this general result to this specific problem we have

$$\begin{aligned} m_2(x) &= \left(x-\left(\sqrt{2}-i\right)\right)\left(x-\left(\sqrt{2}+i\right)\right)\left(x-\left(-\sqrt{2}-i\right)\right)\left(x-\left(-\sqrt{2}+i\right)\right) \\ &= x^4-2(2-(-1)^2)x^2+(2+(-1)^2)^2 \\ &= x^4-2x^2+9. \end{aligned}$$

Would we get to the same answer if we used the "working back up" approach of Exercise 2.20? The answer is "Yes". We note that

$$\begin{aligned} x-\left(\sqrt{2}-i\right) &= 0 \\ x &= \sqrt{2}-i \\ x^2 &= \left(\sqrt{2}-i\right)^2 = 2-2\sqrt{2}i-1 \\ x^2-1 &= -2\sqrt{2}i \end{aligned}$$

$$(x^2-1)^2 = \left(-2\sqrt{2}i\right)^2 = -8$$
$$x^4 - 2x^2 + 1 + 8 = 0$$
$$x^4 - 2x^2 + 9 = 0.$$

■

Disparate* Parts

We already know that two polynomials are equal for all x if and only if the coefficients of like powers are equal. For example, $2x^2 - 3x + 5 = ax^2 + bx + c$ for all x if and only if $a = 2, b = -3$ and $c = 5$.

For polynomials, terms with different powers are disparate parts. They cannot be combined in a single part. For example, we cannot simplify $2x^2 + 5x$ into a single term. For the same reason it makes no sense to say $2x^2 = 5x$ for all x. That just isn't possible. We can say that $2x^2$ and $5x$ are **disparate** parts.

There are other mathematical situations with disparate parts. For example, rational and irrational parts and also real and imaginary parts. We have the following two often useful results.

Rational and Irrational Parts are Disparate

Suppose $a + b\sqrt{c} = d + e\sqrt{f}$ where a, b, d and e are rational numbers while $\sqrt{c}$ and $\sqrt{f}$ are irrational numbers. Then it follows that $a = d, b = e$ and $c = f$.

Real and Imaginary Parts are Disparate

Suppose $a + bi = c + di$ where a, b, c and d are real numbers with $b \neq 0$ and $d \neq 0$ and $i = \sqrt{-1}$. Then it follows that $a = c$ and $b = d$.

*Disparate: things so unalike that there is no basis for comparison, *i.e.* apples and oranges.

Exercise 2.13

If $\sqrt{28 - 10\sqrt{3}}$ is a root of the quadratic equation $x^2 + ax + b = 0$ and if both a and b are rational numbers, find the value of ab. (Source: *Problem Solving Using Viète's Theorem*, Yongcheng Chen)

Solution

It is tempting to apply the irrational conjugate root theorem right away and conclude that if $\sqrt{28-10\sqrt{3}}$ is a root of the given quadratic equation then so is $-\sqrt{28-10\sqrt{3}}$.

But if you do that you will end up with the *wrong* answer.

Let's check *all* the conditions that make the irrational conjugate root theorem applicable.

Does the polynomial have rational coefficients? Yes, we are given that.

Is the given root irrational? Yes.

Is the given root of the form $c + d\sqrt{e}$ for some rational numbers $c, d,$ and e? **No**.

In this problem $e = 28 - 10\sqrt{3}$ is not a rational number as required by the theorem.

But it is *sometimes* possible to get the root into the necessary form so that we can apply the irrational conjugate root theorem.

Are there rational numbers c, d and e such that $\sqrt{28-10\sqrt{3}} = c + d\sqrt{e}$? Let's look.

$$\sqrt{28-10\sqrt{3}} = c + d\sqrt{e}$$

$$\Rightarrow \left(\sqrt{28-10\sqrt{3}}\right)^2 = \left(c + d\sqrt{e}\right)^2$$

$$28 - 10\sqrt{3} = c^2 + 2cd\sqrt{e} + d^2e = (c^2 + d^2e) + 2cd\sqrt{e}.$$

Now look again at the immediately previous result "Rational and Irrational Parts are Disparate". From that result we know that

$$28 - 10\sqrt{3} = (c^2 + d^2e) + 2cd\sqrt{e}$$

if and only if

$$c^2 + d^2e = 28,\ 2cd = -10, \text{ and } e = 3.$$

From $2cd = -10$ we have that $d = -5/c$. Plugging this fact and the fact that $e = 3$ into the first equation we have

$$c^2 + \left(\frac{-5}{c}\right)^2 \cdot 3 = 28.$$

Is there a rational value for c that satisfies this equation?

$$c^2 + \left(\frac{-5}{c}\right)^2 \cdot 3 = 28 \Rightarrow c^2 + \frac{75}{c^2} - 28 = 0$$

$$\Rightarrow c^4 - 28c^2 + 300 = 0.$$

But this is a "hidden" quadratic equation. Letting $y = c^2$ we have

$$y^2 - 28y + 75 = 0$$

which we can solve.

$$y = \frac{-(-28) \pm \sqrt{(-28)^2 - 4(1)(75)}}{2} = \frac{28 \pm \sqrt{484}}{2} = \frac{28 \pm 22}{2} = 14 \pm 11$$

$$= 3 \text{ or } 25.$$

Therefore $c = \sqrt{y} = \sqrt{3}$ or 5. Only $c = 5$ is rational. But let's check to make sure this satisfies the original equation (*i.e.* is not an extraneous solution).

$$5^2 + \left(\frac{-5}{5}\right)^2 \cdot 3 \stackrel{?}{=} 28.$$

Yes, it does. So, we have just shown that

$$\sqrt{28 - 10\sqrt{3}} = 5 - \sqrt{3}$$

which *is* an irrational number of the form $c + d\sqrt{e}$ for rational numbers c, d, e. Therefore, the irrational conjugate root theorem does apply now and we can say that $5 + \sqrt{3}$ is also a root of the given quadratic equation. Therefore,

$$x^2 + ax + b = \left(x - \left(5 + \sqrt{3}\right)\right)\left(x - \left(5 - \sqrt{3}\right)\right) = x^2 - 10x + 22.$$

Matching like coefficients we have that $a = -10$ and $b = 22$. Therefore, $ab = -220$.

■

2.9 Discriminant of a Quadratic Polynomial

For the general quadratic function $f(x) = ax^2 + bx + c$, the **discriminant** is defined by $\boldsymbol{\Delta = b^2 - 4ac}$. (The discriminant is usually denoted by the capital Greek Delta Δ.)

The importance of Δ is in the information it yields about the nature of the two roots r_1 and r_2.

Theorem 1

Let $f(x) = ax^2 + bx + c$ and let $\Delta = b^2 - 4ac$. Then

$\Delta > 0 \Leftrightarrow f(x) = 0$ has two distinct real roots.
$\Delta = 0 \Leftrightarrow f(x) = 0$ has one real root with multiplicity two.
$\Delta < 0 \Leftrightarrow f(x) = 0$ has two complex roots, which are complex conjugates.

Theorem 2

Let $f(x) = ax^2 + bx + c$ and let $\Delta = b^2 - 4ac$. Then $f(x) = 0$ has two distinct rational roots if and only if all of the following
conditions are met.

(i) $\Delta > 0$
(ii) $\sqrt{\Delta}$ is a rational number
(iii) a, b and c are rational numbers
(iv) $a \neq 0$.

Both of these theorems follow almost immediately by looking at the formula for the two roots r_1 and r_2 of the quadratic equation $ax^2 + bx + c = 0$ from the quadratic formula:

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Theorem 3

Let $f(x) = ax^2 + bx + c$ and let $\Delta = b^2 - 4ac$. If r_1 and r_2 represent the two roots of the general quadratic equation $f(x) = 0$, then

$$\Delta = a^2(r_1 - r_2)^2.$$

The proof of Theorem 3 follows easily from substituting the above formulas for r_1 and r_2:

$$a^2(r_1 - r_2)^2 = a^2\left(\left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right) - \left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)\right)^2$$

$$= a^2\left(\frac{2\sqrt{b^2 - 4ac}}{2a}\right)^2 = b^2 - 4ac.$$

Exercise 2.14

Find the value of the discriminant of a monic quadratic polynomial if the absolute difference in the two roots equals 2. (Remember that **monic** means that the leading coefficient equals 1.)

Solution

We are given that $a = 1$ in the above-mentioned quadratic and that $|r_1 - r_2| = 2$. Therefore, from Theorem 3 above,

$$\Delta = a^2(r_1 - r_2)^2 = a^2|r_1 - r_2|^2 = 1^2 \cdot 2^2 = 4.$$

■

Exercise 2.15 (Source: MSHSML 1D153)

The quadratic function $f(x) = 3x^2 + kx + 5$, where k is an integer, does not have any real roots. What is the greatest possible value of k?

Solution

The quadratic equation $ax^2 + bx + c = 0$ has no real roots if and only if its discriminant, $b^2 - 4ac$, is negative. Therefore, $k^2 - 4(3)(5) < 0 \Leftrightarrow k^2 < 60 \Leftrightarrow -\sqrt{60} < k < \sqrt{60}$. The largest integer value of k in this range is $k = 7$.

■

Exercise 2.16 (Source: MSHSML 1D122)

What is the **greatest** integer c for which the quadratic polynomial $5x^2 + 11x + c$ has two distinct rational roots?

Solution

For the roots to be rational, the discriminant must be a perfect square of a rational number. For the roots to be rational and distinct, the discriminant must be a perfect square of a rational number greater than 0.

The discriminant equals $\Delta = b^2 - 4ac = 11^2 - 4(5)c = 121 - 20c$. Notice that when $c = 6$ then $\Delta = 1^2$. But c cannot be greater than 6 because then $\Delta < 0$ and cannot be a perfect square. So, the greatest integer c equals 6.

■

Exercise 2.17 (Source: MSHSML 1D124)

Alec found that a quadratic polynomial $f(x)$ had zeroes at -1 and -11. He forgot the values of the polynomial's coefficients, but remembered that its discriminant was 64 and that $f(10)$ was less than $f(-10)$. Calculate $f(5)$.

Solution

We are given that -1 and -11 are roots of $f(x)$ and that $f(x)$ is a quadratic. It follows that

$$f(x) = a(x+1)(x+11) = ax^2 + 12ax + 11a$$

for some a. It follows that the discriminant of $f(x)$ equals $(12a)^2 - 4(a)(11a) = 100a^2$. The problem states that the discriminant equals 64. Therefore

$$a^2 = \frac{64}{100} \Longrightarrow a = \pm\frac{4}{5}.$$

It follows that

$$f(10) = \left(\pm\frac{4}{5}\right)(10+1)(10+11) = \pm\frac{924}{5}$$

and

$$f(-10) = \left(\pm\frac{4}{5}\right)(-10+1)(-10+11) = \mp\frac{36}{5}.$$

It follows that the only way for $f(10) < f(-10)$ is when $f(10)$ is negative and $f(-10)$ is positive. This can only happen if $a < 0$. Therefore $a = -4/5$.

Hence,

$$f(5) = -\frac{4}{5}(5+1)(5+11) = -\frac{384}{5}.$$

■

Exercise 2.18 (Source: AMC 12, 2005)

There are two values of a for which the equation $4x^2 + ax + 8x + 9 = 0$ has only one solution for x. What is the sum of these values of a?

Solution

Because we want the a's where there is only one solution to x, the discriminant has to be 0. That is,

$$(a+8)^2 - 4(4)(9) = a^2 + 16a - 80 = 0.$$

$$(a-4)(a+20) = 0$$

$$a = 4, a = -16.$$

Hence the sum of these values of a is -16.

■

Exercise 2.19

Find the value of a for which the equation $(1+a)x^2 - 2(1+3a)x + (1+8a) = 0$ has equal roots.

Solution

For a quadratic to have equal roots its discriminant must equal 0. That is,

$$\left(-2(1+3a)\right)^2 - 4(1+a)(1+8a) = 0$$

$$4(1+6a+9a^2) - 4(1+9a+8a^2) = 0$$

$$(36-32)a^2 + (24-36)a + (4-4) = 0$$

$$4a^2 - 12a = 0$$

$$a = 0, a = 3.$$

■

Exercise 2.20 (Source: 50 Lectures for AMC, Volume 2, Chen and Chen)

Find the possible values of k if $f(x) = kx^2 - 2x + k < 0$ for all real x.

Solution

If $f(x) < 0$ for all real x then $f(x)$ cannot have any real zeros. So, it must have two nonreal zeros. But this implies $\Delta < 0$. That is, $\Delta = (-2)^2 - 4(k)(k) = 4 - 4k^2 < 0$.

But

$$4 - 4k^2 < 0 \Leftrightarrow 1 - k^2 < 0 \Leftrightarrow k^2 > 1 \Leftrightarrow k < -1 \text{ or } k > 1.$$

We also know that that as x approaches infinity that $f(x)$ will either approach $\pm\infty$ depending on the sign of k. If k is positive then $f(x)$ approaches $+\infty$ and if k is negative then $f(x)$ approaches $-\infty$ as x approaches ∞. But $f(x) < 0$ for all real x implies we must be in the latter case where k is negative.

Therefore, the possible value of k must be all $k < -1$.

■

Exercise 2.21 (Source: 50 Lectures for AMC, Volume 2, Chen and Chen)

Find the possible values of k if $f(x) = kx^2 + (k-1)x + (k-1) < 0$ for all real x.

Solution

If $f(x) < 0$ for all real x then $\Delta < 0$ and $k < 0$.

Here is a repeat of the argument for that initially conclusion made in Exercise 2.20.

> If $f(x) < 0$ for all real x then $f(x)$ cannot have any real zeros. So, it must have two nonreal zeros. But this implies $\Delta < 0$. We also know that that as x approaches infinity that $f(x)$ will either approach $\pm\infty$ depending on the sign of k. If k is positive then $f(x)$ approaches $+\infty$ and if k is negative then $f(x)$ approaches $-\infty$ as x approaches ∞. But $f(x) < 0$ for all real x implies we must be in the latter case where k is negative.

For this problem

$$\begin{aligned}\Delta = b^2 - 4ac &= (k-1)^2 - 4(k)(k-1) \\ &= (k-1)(k-1-4k) = (k-1)(-3k-1).\end{aligned}$$

So, we know that $\Delta = (k-1)(-3k-1) < 0$ and $k < 0$. However,

$$\begin{aligned}&\Delta = (k-1)(-3k-1) < 0 \\ &\quad \Leftrightarrow (k-1 < 0 \text{ and } -3k-1 > 0) \text{ or } (k-1 > 0 \text{ and } -3k-1 < 0) \\ &\quad \Leftrightarrow \left(k < 1 \text{ and } k < -\frac{1}{3}\right) \text{ or } \left(k > 1 \text{ and } k > -\frac{1}{3}\right) \\ &\quad \Leftrightarrow \left(k < -\frac{1}{3}\right) \text{ or } (k > 1).\end{aligned}$$

Thus, we have that

$$\big((k-1)(-3k-1)<0\big) \text{ and } (k<0)$$

$$\Leftrightarrow \left(\left(k<-\frac{1}{3}\right) \text{ or } (k>1)\right) \text{ and } (k<0)$$

$$\Leftrightarrow \left(\left(k<-\frac{1}{3}\right) \text{ and } k<0\right) \text{or } \big((k>1) \text{ and } (k<0)\big)$$

$$\Leftrightarrow \left(k<-\frac{1}{3}\right) \text{or } (\emptyset) \;\Leftrightarrow\; k<-\frac{1}{3}.$$

■

Exercise 2.22 (Source: 50 Lectures for AMC, Volume 2, Chen and Chen)

Find all possible values of k if $f(x)=x^2+3kx+k^2-k+\frac{1}{4}$ can be express as the square of an expression that is linear in x.

Solution

Suppose that for some a and b we have

$$f(x)=x^2+3kx+k^2-k+\frac{1}{4}=(ax+b)^2.$$

Then $f(x)=0 \Leftrightarrow x=-b/a$ and $x=-b/a$. That is the root $x=-b/a$ is a repeated root.

But will can only get a repeated root if $\Delta=0$. So we know that

$$\Delta=(3k)^2-4(1)\left(k^2-k+\frac{1}{4}\right)=0$$

$$\Leftrightarrow 9k^2-4k^2+4k-1=0$$

$$\Leftrightarrow 5k^2+4k-1=0$$

$$\Leftrightarrow (5k-1)(k+1)=0$$

$$\Leftrightarrow k=1/5 \text{ or } k=-1.$$

■

Exercise 2.23 (Source: 50 Lectures for AMC, Volume 2, Chen and Chen)

Find all real values of x and y such that $x^2-4xy+5y^2+2x-8y+5=0$.

Solution

We can rewrite this equation in x and y as a quadratic equation in y

$$5y^2 - 4(x+2)y + (x^2 + 2x + 5) = 0.$$

Because we are limited to real values of y the discriminant of this quadratic in y must be nonnegative. That is $\Delta \geq 0$.

$$\Delta = \left(-4(x+2)\right)^2 - 4(5)(x^2 + 2x + 5) \geq 0$$

$$\Leftrightarrow (16x^2 + 64x + 64) - (20x^2 + 40x + 100) \geq 0$$

$$\Leftrightarrow -4x^2 + 24x - 36 \geq 0$$

$$\Leftrightarrow x^2 - 6x + 9 \leq 0$$

$$\Leftrightarrow (x-3)^2 \leq 0$$

$$\Leftrightarrow x - 3 = 0$$

$$\Leftrightarrow x = 3.$$

So, the only real valued possibility for x is $x = 3$. Now we can plug $x = 3$ back into the defining equation $5y^2 - 4(x+2)y + (x^2 + 2x + 5) = 0$ to get

$$5y^2 - 20y + 20 = 0$$

$$\Leftrightarrow y^2 - 4y + 4 = 0$$

$$\Leftrightarrow (y-2)^2 = 0$$

$$\Leftrightarrow y = 2.$$

So, the only real valued pair for (x, y) is $(x, y) = (3,2)$.

■

Exercise 2.24 (Source: 50 Lectures for AMC, Volume 2, Chen and Chen)

Find all possible integers k such that $f(x) = kx^2 + (2k+3)x + 1 = 0$ has rational roots.

Solution

The quadratic equation $f(x) = 0$ has rational roots if and only if the coefficients of $f(x)$ are rational and its discriminant is the square of a rational number.

Because we are assuming that k is an integer, it follows that the coefficients of $f(x) = kx^2 + (2k+3)x + 1$ are necessarily integers (and hence rational).

The discriminant of $f(x)$ equals $\Delta = (2k+3)^2 - 4(k)(1) = 4k^2 + 8k + 9$ and the requirement is that $\Delta = m^2$ for some positive rational number m.

However, $\Delta = 4k^2 + 8k + 9$ is always an integer for integer k. Therefore m must be a positive integer.

Therefore, we are searching for integer values of k that satisfy the quadratic equation (in the variable k)

$$g(k) = 4k^2 + 8k + 9 - m^2 = 0$$

where m is an integer. Now we are requiring that the discriminant Δ_2 of $g(k)$ must be the square of an integer (*i.e.* a square number). We note that

$$\Delta_2 = 8^2 - 4(4)(9 - m^2) = 16(m^2 - 5).$$

So, the requirement is that Δ_2 is a square number. And as $16 = 4^2$, Δ_2 can only be a square number if the factor $m^2 - 5$ is a square number.

So, it has come down to $m^2 - 5 = n^2$ for some positive integer n. Equivalently, for some positive integers m and n, we require

$$m^2 - n^2 = 5$$

$$(m-n)(m+n) = 5.$$

Because $m^2 - n^2 = 5 > 0$ and m and n are positive, it is necessary that $m > n$. So, the only positive integers $m - n$ and $m + n$ whose product equals 5 are $m - n = 1$ and $m + n = 5$.

Solving simultaneously, we have $m = 1 + n$ and $5 = (1+n) + n = 2n + 1$. Hence $n = 2$ and $m = 3$.

Plugging $m = 3$ back into the requirement that

$$\Delta = 4k^2 + 8k + 9 = m^2$$

we find that

$$4k^2 + 8k + 9 = 9$$

$$k(4k + 8) = 0$$

$$k = 0 \text{ or } k = -2.$$

But taking $k = 0$ would give $f(x) = 0x^2 + 3x + 1$ which only has a single rational root and the statement of the problem requires $f(x)$ have rational roots (plural). So $k \neq 0$.

Therefore, the only possibility is $k = -2$.

In this case we have

$$f(x) = -2x^2 + (-4 + 3)x + 1 = -2x^2 - x + 1 = (-2x + 1)(x + 1).$$

Therefore $f(x) = 0$ has the two rational roots $x = 1/2$ and $x = -1$.

■

Exercise 2.25 (Source: 1966 AHSME, Problem 23)

If x is real and if $4y^2 + 4xy + x + 6 = 0$, find the set of all x for which y is real.

Solution

Viewing this equation as a quadratic in y (with x as a parameter), then the discriminant Δ becomes

$$\Delta = b^2 - 4ac = (4x)^2 - 4(4)(x + 6) = 16(x^2 - x - 6) = 16(x - 3)(x + 2).$$

The requirement that y is real tells us that $\Delta \geq 0$. This can only happen when the factors $(x - 3)$ and $(x + 2)$ are both positive or both negative.

That is,

$$(x - 3 \leq 0 \text{ and } x + 2 \leq 0) \text{ or } (x - 3 \geq 0 \text{ and } x + 2 \geq 0)$$

$$(x \leq 3 \text{ and } x \leq -2) \text{ or } (x \geq 3 \text{ and } x \geq -2)$$

$$(x \leq -2) \text{ or } (x \geq 3).$$

■

Exercise 2.26 (Source: 50 Lectures for AMC, Volume 2, Chen and Chen)

Find the possible values of k if $f(x) = 2(k + 1)x^2 + 4kx + 3k - 2$ can be expressed as a square of a linear expression of x.

Solution

Suppose that for some a and b we have

$$f(x) = 2(k+1)x^2 + 4kx + 3k - 2 = (ax+b)^2.$$

Then $f(x) = 0 \Leftrightarrow x = -b/a$ and $x = -b/a$. That is the root $x = -b/a$ is a repeated root.

But will can only get a repeated root if $\Delta = 0$. So we know that

$$\Delta = -8k^2 - 8k + 16 = 0$$

$$\Leftrightarrow k^2 + k - 2 = 0$$

$$\Leftrightarrow (k-1)(k+2) = 0$$

$$\Leftrightarrow k = 1 \text{ or } k = -2.$$

■

2.10 Greatest Common Divisor of Two Polynomials

In *Study Guide for Meet 1, Event A* we worked through results and solved problems dealing with the greatest common divisor (GCD) and the least common multiple (LCM)of two ***integers***. In this section we will extend these ideas for the purpose of finding the greatest common divisor and the least common multiple of two ***polynomials***.

> The **greatest common divisor of polynomials** $f(x)$ and $g(x)$ is the monic polynomial of largest degree which divides both $f(x)$ and $g(x)$.
>
> The **least common multiple of polynomials** $f(x)$ and $g(x)$ is the monic polynomial of smallest degree that has $f(x)$ and $g(x)$ as factors.

We will use $\gcd\big(f(x), g(x)\big)$ and $\operatorname{lcm}\big(f(x), g(x)\big)$ to denote the greatest common divisor and the least common multiple, respectively, of the polynomials $f(x)$ and $g(x)$.

If f and g are easily factored or already factored, then finding the $\gcd\big(f(x), g(x)\big)$ and/or the $\operatorname{lcm}\big(f(x), g(x)\big)$ is straightforward.

Example 1

$$\gcd(-8x^2 + 22x - 15, 4x^2 - 9x + 5)$$

$$= \gcd\big((-2x+3)(4x-5), (4x-5)(x-1)\big)$$

$$= \frac{4x-5}{4} = x - \frac{5}{4}$$

and

$$\begin{aligned}
&\operatorname{lcm}(\,-8x^2+22x-15,4x^2-9x+5) \\
&\quad=\operatorname{lcm}\big((-2x+3)(4x-5),(4x-5)(x-1)\big) \\
&\quad=\frac{(-2x+3)(4x-5)(x-1)}{-8} \\
&\quad=x^3-\frac{15x^2}{4}+\frac{37x}{8}-\frac{15}{8}.
\end{aligned}$$

Example 2

$$\gcd\big((2x-1)(x+5)^2(x-3)^4,(x+5)^3(x-3)^3\big)=(x+5)^2(x-3)^3.$$

and

$$\operatorname{lcm}\big((2x-1)(x+5)^2(x-3)^4,(x+5)^3(x-3)^3\big)=(2x-1)(x+5)^2(x-3)^4.$$

As a general rule *factoring is hard*. Fortunately, the Euclidean Algorithm ***for polynomials*** is allows us to find the gcd of two polynomials *without factoring*.

Euclidean Algorithm for gcd of Polynomials

Consider two polynomials

$$f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$$

and

$$g(x)=b_tx^t+b_{t-1}x^{t-1}+\cdots+b_1x+b_0$$

with real coefficients and with $a_n\neq 0$, $b_t\neq 0$, $t\leq n$. Let's consider $f(x)/g(x)$.

The division algorithm tells us there exists some quotient polynomial $q(x)$ and some remainder polynomial $r(x)$ with degree less than the degree of $g(x)$ such that

$$\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)}$$

or equivalently

$$f(x) = q(x) \cdot g(x) + r(x).$$

In this case, the Euclidean algorithm depends on noticing that

$$\boxed{\gcd\big(f(x), g(x)\big) = \gcd\big(r(x), g(x)\big).}$$

Why is this true? The gist of the argument is that if we let $d(x) = \gcd\big(f(x), g(x)\big)$, then $d(x)$ must divide by $f(x)$ and $g(x)$ with no remainder. That is, $f(x) = d(x)m_1(x)$ and $g(x) = d(x)m_2(x)$ for some quotient polynomials $m_1(x)$ and $m_2(x)$.

But in this case,

$$\begin{aligned} r(x) = f(x) - q(x) \cdot g(x) &= \big(d(x)m_1(x)\big) - q(x)\big(d(x)m_2(x)\big) \\ &= d(x)\big(m_1(x) - q(x) \cdot m_2(x)\big). \end{aligned}$$

So $d(x)$ must divide $r(x)$. On the other hand, if $d(x)$ divides both $r(x)$ and $g(x)$, that is, if $r(x) = d(x)m_3(x)$ and $g(x) = d(x)m_2(x)$, then

$$\begin{aligned} f(x) &= q(x) \cdot g(x) + r(x) \\ &= q(x)\big(d(x)m_2(x)\big) + d(x)m_3(x) \\ &= d(x)\big(q(x)m_2(x) + m_3(x)\big) \end{aligned}$$

which tells us that $d(x)$ must also divide $f(x)$.

How does this help us find $\gcd\big(f(x), g(x)\big)$? We started with the assumption that $\deg f(x) \geq \deg g(x)$ and we know from the division algorithm that $\deg g(x) > \deg r(x)$. So the result

$$\gcd\big(f(x), g(x)\big) = \gcd\big(r(x), g(x)\big)$$

has allowed us to replace $f(x)$ with a polynomial of lower degree. By the definition of gcd it is obvious that $\gcd\big(r(x), g(x)\big) = \gcd\big(g(x), r(x)\big)$.

So now we consider the ratio $g(x)/r(x)$. Again, by the division algorithm,

$$g(x) = r(x)q(x) + r_2(x)$$

for some quotient and remainder polynomials $q(x)$ and $r_2(x)$ and hence iterating on the same idea as before we get that

$$\gcd\big(f(x), g(x)\big) = \gcd\big(r(x), g(x)\big) = \gcd\big(g(x), r(x)\big) = \gcd\big(r_2(x), r(x)\big)$$

where the $\deg r(x) > \deg r_2(x)$.

Continuing like this we will necessarily reach a result of the form $\gcd(w(x), 0)$ for some polynomial $w(x)$.

But at this point we can stop because $w(x)$ divides both $w(x)$ and 0 and nothing greater than $w(x)$ can possibly divide $w(x)$.

So, if $w(x)$ is monic, then $\gcd(w(x), 0) = w(x)$. If $w(x)$ is not monic then $\gcd(w(x), 0) = w(x)/a$ where a is the coefficient of the leading term of $w(x)$.

So, for example, if we end up at $\gcd(3x - 2,0)$ our final answer would be

$$\gcd(3x - 2,0) = \frac{3x - 2}{3} = x - (2/3).$$

The process of going from

$$\gcd\big(f(x), g(x)\big) \longrightarrow \gcd\big(r(x), g(x)\big) \longrightarrow \cdots \longrightarrow \gcd(w(x), 0)$$

Is called the Euclidean Algorithm.

$$= \gcd\big(r(x), g(x)\big) = \gcd\big(g(x), r(x)\big) = \gcd\big(r_2(x), r(x)\big).$$

One last "trick" before we go through some examples.

$\gcd(f(x), g(x)) = \gcd\big(k_1 f(x), k_2 g(x)\big)$ for all nonzero k_1, k_2.

That is, the **gcd of two polynomials is not changed if either or both polynomials are multiplied by constants**. This follows immediately from the requirement that our gcd must be monic. <u>But it is a useful result for keeping fractions out of the division process.</u>

Exercise 1.27

Find $\gcd(x^2 - 3x + 2, x^2 - 4x + 3)$

Solution

The fastest way to solve this problem is to just factor each of these polynomials and pull out the largest common factor.

$$\gcd(x^2 - 3x + 2, x^2 - 4x + 3) = \gcd\big((x - 1)(x - 2), (x - 1)(x - 3)\big) = x - 1.$$

But to get practice using the Euclidean algorithm we note that

$$
\begin{array}{r|rrr}
 & 1 & & \\
\hline
x^2-4x+3 & x^2 & -3x & +2 \\
 & x^2 & -4x & +3 \\
\hline
 & & x & -1
\end{array}
$$

$\Longrightarrow \gcd(x^2-3x+2, x^2-4x+3) = \gcd(x^2-4x+3, x-1)$

$$
\begin{array}{r|rrr}
 & x & -3 & \\
\hline
x-1 & x^2 & -4x & +3 \\
 & x^2 & -x & \\
\hline
 & & -3x & +3 \\
 & & -3x & +3 \\
\hline
 & & & 0
\end{array}
$$

$\Longrightarrow \gcd(x^2-4x+3, x-1) = \gcd(x-1, 0) = x-1.$

Exercise 1.28

Find $\gcd(x^5-2x^4-2x^3+8x^2-7x+2, x^4-4x+3)$

Solution

$$
\begin{array}{r|rrrrrr}
 & x & -2 & & & & \\
\hline
x^4-4x+3 & x^5 & -2x^4 & -2x^3 & +8x^2 & -7x & +2 \\
 & x^5 & & & -4x^2 & +3x & \\
\hline
 & & -2x^4 & -2x^3 & +12x^2 & -10x & +2 \\
 & & -2x^4 & & & +8x & -6 \\
\hline
 & & & -2x^3 & +12x^2 & -18x & +8
\end{array}
$$

$\Longrightarrow \gcd(x^5-2x^4-2x^3+8x^2-7x+2, x^4-4x+3)$

$= \gcd(x^4 - 4x + 3, -2x^3 + 12x^2 - 18x + 8)$

$= \gcd(x^4 - 4x + 3, x^3 - 6x^2 + 9x - 4)$ (by dividing the polynomial on the right by the constant -2)

$$
\begin{array}{r|rrrrr}
 & x & +6 & & & \\
\hline
x^3 - 6x^2 + 9x - 4 & x^4 & & & -4x & +3 \\
 & x^4 & -6x^3 & +9x^2 & -4x & \\
\hline
 & & 6x^3 & -9x^2 & & +3 \\
 & & 6x^3 & -36x^2 & 54x & -24 \\
\hline
 & & & 27x^2 & -54x & +27
\end{array}
$$

$\Longrightarrow \gcd(x^4 - 4x + 3, x^3 - 6x^2 + 9x - 4)$

$= \gcd(x^3 - 6x^2 + 9x - 4, 27x^2 - 54x + 27)$

$= \gcd(x^3 - 6x^2 + 9x - 4, x^2 - 2x + 1)$ (by dividing the polynomial on the right by the constant 27)

$$
\begin{array}{r|rrrr}
 & x & -4 & & \\
\hline
x^2 - 2x + 1 & x^3 & -6x^2 & +9x & -4 \\
 & x^3 & -2x^2 & +x & \\
\hline
 & & -4x^2 & +8x & -4 \\
 & & -4x^2 & +8x & -4 \\
\hline
 & & & & 0
\end{array}
$$

$\Longrightarrow \gcd(x^3 - 6x^2 + 9x - 4, x^2 - 2x + 1) = \gcd(x^2 - 2x + 1, 0) = x^2 - 2x + 1.$

Exercise 1.29

Find $\gcd(x^4 + x^2 + 1, x^2 + 1)$

Solution

$$
\begin{array}{r|lll}
 & x^2 & & \\ \hline
x^2+1 & x^4 & +x^2 & +1 \\
 & x^4 & +x^2 & \\ \hline
 & & & 1
\end{array}
$$

$\Rightarrow \gcd(x^4 + x^2 + 1, x^2 + 1) = \gcd(x^2 + 1,1)$

$$
\begin{array}{r|ll}
 & x^2 & +1 \\ \hline
1 & x^2 & +1 \\
 & x^2 & \\ \hline
 & & +1 \\
 & & +1 \\ \hline
 & & 0
\end{array}
$$

$\Rightarrow \gcd(x^2 + 1,1) = \gcd(1,0) = 1$

Relatively Prime Polynomials

When the gcd of two polynomials equals 1 we say the two polynomials are ***relatively prime***. Therefore, we can say that $x^4 + x^2 + 1$ and $x^2 + 1$ are relatively prime.

Finding Common Zeros of Two Polynomials

If θ is a zero of both polynomials $f(x)$ and $g(x)$, then θ is necessarily a zero of $\gcd\big(f(x), g(x)\big)$. So, to find all common zeros of polynomials $f(x)$ and $g(x)$ it suffices to find all zeros of $\gcd\big(f(x), g(x)\big)$.

Exercise 1.30 (Source: *Golden Algebra*, N.P. Bali)

Find the common zeros of the two polynomials $x^4 + 3x^3 - 5x^2 - 6x - 8$ and $x^4 + x^3 - 9x^2 + 10x - 8$.

Solution

As given above, it suffices to find zeros of $\gcd\big(f(x), g(x)\big)$. So our first step is to use the Euclidean Algorithm to find $\gcd\big(f(x), g(x)\big)$.

$$
\begin{array}{r|rrrrrl}
 & 1 & & & & & \\
\hline
x^4 + x^3 - 9x^2 + 10x - 8 & x^4 & +3x^3 & -5x^2 & -6x & -8 & \\
 & x^4 & +x^3 & -9x^2 & +10x & -8 & \\
\hline
 & & \cancel{2x^3} & \cancel{+4x^2} & \cancel{-16x} & \cancel{0} & \text{(Divide} \\
 & & \mathbf{x^3} & \mathbf{+2x^2} & \mathbf{-8x} & \mathbf{0} & \text{by 2)}
\end{array}
$$

$$
\begin{array}{r|rrrrr}
 & x & -1 & & & \\
\hline
x^3 + 2x^2 - 8x & x^4 & +x^3 & -9x^2 & +10x & -8 \\
 & x^4 & +2x^3 & -8x^2 & & \\
\hline
 & & -x^3 & -x^2 & +10x & -8 \\
 & & -x^3 & -2x^2 & +8x & \\
\hline
 & & & x^2 & +2x & -8
\end{array}
$$

$$
\begin{array}{r|rrr}
 & x & & \\
\hline
x^2 + 2x - 8 & x^3 & +2x^2 & -8x \\
 & x^3 & +2x^2 & -8x \\
\hline
 & & & 0
\end{array}
$$

So

$$\gcd\big(f(x), g(x)\big) = x^2 + 2x - 8.$$

Solving $\gcd\big(f(x), g(x)\big) = 0$ we can find the common roots of f and g.

$$x^2 + 2x - 8 = 0 \Leftrightarrow (x+4)(x-2) = 0 \Leftrightarrow x = -4,\ x = 2.$$

■

Exercise 1.31 (Source: MSHSML 1T012)

Find the polynomial $P(x) = a_n x^n + \cdots a_1 x + a_0$ of least possible degree that is a multiple of each of the three polynomials below, and which has integral coefficients with a greatest common divisor of 1.

$$r(x) = 4x^2 + 2x - 2, \quad s(x) = 12x^2 + 2x - 4, \quad t(x) = 9x^3 + 15x^2 + 6x$$

Solution

First, to be clear, when the asks for that polynomial $P(x)$ with integral coefficients *with a greatest common divisor* of 1, it means that the greatest common divisor of the integral coefficients of $P(x)$ must equal 1.

It follows that $\boldsymbol{P(x) = k \cdot \mathbf{lcm}(r(x), s(x), t(x))}$ where k is the necessary to make all the coefficients of $P(x)$ integers and with a greatest common divisor of 1.

Each of these polynomials factors nicely. We note that

$$r(x) = (2x + 2)(2x - 1) = 2(x + 1)(2x - 1)$$

$$s(x) = (4x - 2)(3x + 2) = 2(2x - 1)(3x + 2)$$

$$t(x) = x(3x + 3)(3x + 2) = 3x(x + 1)(3x + 2).$$

Therefore,

$$\mathrm{lcm}(r(x), s(x), t(x)) = \frac{1}{6} \cdot x(x + 1)(2x - 1)(3x + 2) = \frac{1}{6}(6x^4 + 7x^3 - x^2 - 2x).$$

where the constant $1/6$ is necessary to make the right-hand side polynomial monic. It follows by inspection that

$$P(x) = 6x^4 + 7x^3 - x^2 - 2x.$$

2.11 Polynomials with Non-Simple Roots

Recall that a **simple** root (see page 14) is a root with multiplicity 1 and then consider the following problem (Source: MSHSML 1T926):

> Determine the roots of the equation $x^3 - 6x^2 + 6\sqrt{3}x - 4 = 0$ if you know that one of the roots is a double root.

How do we take advantage of the information that the polynomial equation $f(x) = 0$ has some non-simple root?

If let r represent the unknown non-simple root of $f(x) = 0$, then the answer is hidden in the polynomial division $f(x)/(x - r)$, (which we can easily find using synthetic division).

By the division theorem (see page 6) and the remainder theorem (see page 14) taken together we know there exists a unique **quotient polynomial** $q(x)$ such that

$$\frac{f(x)}{x-r} = q(x) + \frac{f(r)}{x-r}.$$

But saying r is a root of $f(x) = 0$ is another way of saying $f(r) = 0$. So in this case,

$$\frac{f(x)}{x-r} = q(x).$$

But remember we are assuming that r is a non-simple root (*i.e.* multiplicity of at least 2) which means that $f(x)$ is also divisible by $(x - r)^2$. That is, $f(x)/(x - r)^2$ has no remainder.

Now

$$\frac{f(x)}{(x-r)^2} = \frac{\left(\frac{f(x)}{x-r}\right)}{x-r} = \frac{q(x)}{x-r}.$$

Therefore, we can also see that $q(x)/(x - r)$ has no remainder.

Now once again, using the division theorem in conjunction with the remainder theorem, there exists a unique quotient polynomial $w(x)$ such that

$$\frac{q(x)}{x-r} = w(x) + \frac{q(r)}{x-r}.$$

Therefore, $q(r) = 0$ because $q(x)/(x - r)$ has no remainder. We have just proven the following theorem.

Quotient Remainder Theorem

If r is a non-simple root of the equation $f(x) = 0$ and if by polynomial division

$$\frac{f(x)}{x-r} = q(x) + \frac{f(r)}{x-r}$$

then $f(r) = 0$ and $q(r) = 0$.

Exercise 1.32 (Source: MSHSML 1T926)

Determine the roots of the equation $x^3 - 6x^2 + 6\sqrt{3}x - 4 = 0$ if you know that one of the roots is a double root.

Solution

To take advantage of the quotient remainder theorem, we need to evaluate

$$\frac{x^3 - 6x^2 + 6\sqrt{3}x - 4}{x - r}$$

for some generic number r.

r	1	-6	$6\sqrt{3}$	-4
		r	$-6r + r^2$	$6\sqrt{3}r - 6r^2 + r^3$
	1	$-6 + r$	$6\sqrt{3} - 6r + r^2$	$-4 + 6\sqrt{3}r - 6r^2 + r^3$

Therefore,

$$\frac{f(x)}{x - r} = \frac{x^3 - 6x^2 + 6\sqrt{3}x - 4}{x - r} = q(x) + \frac{f(r)}{x - r}$$

where

$$q(x) = x^2 + (-6 + r)x + \left(6\sqrt{3} - 6r + r^2\right).$$

By the above theorem we can know that

$$q(r) = r^2 + (-6 + r)r + \left(6\sqrt{3} - 6r + r^2\right) = 0$$

and

$$f(r) = r^3 - 6r^2 + 6\sqrt{3}r - 4 = 0.$$

It is the equation $q(r) = 0$ that is most useful to us in this problem because it is a quadratic equation and hence we can easily solve for r.

Simplifying we have

$$q(r) = 3r^2 - 12r + 6\sqrt{3}.$$

Using the quadratic formula to solve $q(r) = 0$ we see that

$$r = \frac{-(-12) \pm \sqrt{(-12)^2 - 4(3)\left(6\sqrt{3}\right)}}{2(3)} = 2 \pm \sqrt{4 - 2\sqrt{3}}\,.$$

An expression such as $\sqrt{4 - 2\sqrt{3}}$ is called a "**nested root**" because it has a square root inside or nested in another square root. The process of pulling this expression apart so it does not have a root inside another root is called "**denesting**".

For now, I will just state that

$$\sqrt{4 - 2\sqrt{3}} = \sqrt{3} - 1.$$

At the end of this problem we will discuss a general procedure for denesting expressions of the form $\sqrt{a \pm b\sqrt{c}}$ for rational numbers a, b and c and we will justify the above result.

So for now,

$$r = 2 \pm \sqrt{4 - 2\sqrt{3}} = 2 \pm \sqrt{\left(\sqrt{3} - 1\right)^2} = 2 \pm \left(\sqrt{3} - 1\right).$$

If we plug both of these solutions back into the original equation $x^3 - 6x^2 + 6\sqrt{3}x - 4 = 0$ we can see that only $2 + \left(\sqrt{3} - 1\right) = 1 + \sqrt{3}$ is a solution and the other case, $2 - \left(\sqrt{3} - 1\right) = 3 - \sqrt{3}$ is extraneous.

So we know that the cubic equation $x^3 - 6x^2 + 6\sqrt{3}x - 4 = 0$ has the double root $1 + \sqrt{3}$.

We could find the third solution by *deflation* (see page 70). That is, by calculating

$$\frac{x^3 - 6x^2 + 6\sqrt{3}x - 4}{\left(x - \left(1 + \sqrt{3}\right)\right)\left(x - \left(1 + \sqrt{3}\right)\right)}.$$

But it will be much quicker to use the above "Sum of the Roots" result (see page 15).

By that result, $r_1 + r_2 + r_3 = -b/a = -(-6)/1 = 6$.

So,

$$\left(1 + \sqrt{3}\right) + \left(1 + \sqrt{3}\right) + r_3 = 6 \Longrightarrow r_3 = 4 - 2\sqrt{3}.$$

So, the three roots of $x^3 - 6x^2 + 6\sqrt{3}x - 4 = 0$ are $\{1 + \sqrt{3}, 1 + \sqrt{3}, 4 - 2\sqrt{3}\}$.

■

How to Denest a Nested Root

The key is the following *Nested Roots Identity*:

$$\sqrt{a \pm b\sqrt{c}} = \sqrt{\frac{a + \sqrt{a^2 - b^2c}}{2}} \pm \sqrt{\frac{a - \sqrt{a^2 - b^2c}}{2}}.$$

Clearly, if $\sqrt{a^2 - b^2c}$ is a rational number then neither term on the left-hand side of this identity will be nested and the left-hand will be in denested form.

It is less obvious, but true, that the ***only*** situation where

$$\sqrt{a \pm b\sqrt{c}} = \sqrt{x} \pm \sqrt{y}$$

for some rational numbers a, b, c, x and y with $a > 0$ and $c > 0$ is if

$$\sqrt{a^2 - b^2c}$$

is a rational number.

Can we use the above identity to "denest" the expression $\sqrt{a - b\sqrt{c}} = \sqrt{4 - 2\sqrt{3}}$? In this problem we have $a = 4 > 0$, $b = 2$, $c = 3$ and

$$\sqrt{a^2 - b^2 c} = \sqrt{4^2 - 2^2 \cdot 3} = 2,$$

which is a rational number (more specifically, an integer). Therefore, we can use the above identity to denest our problem.

In particular, we have,

$$\sqrt{4 - 2\sqrt{3}} = \sqrt{\frac{a + \sqrt{a^2 - b^2 c}}{2}} - \sqrt{\frac{a - \sqrt{a^2 - b^2 c}}{2}}$$

$$= \sqrt{\frac{4 + 2}{2}} - \sqrt{\frac{4 - 2}{2}} = \sqrt{3} - 1$$

as we stated previously (without justification).

■

2.12 General Form for the Quotient Polynomial at r

In the previous section we saw the importance of the unique quotient polynomial $q(x)$ in the division problem

$$\frac{f(x)}{x - r} = q(x) + \frac{f(r)}{x - r}$$

in the case where it is known (or assumed) that r in a non-simple root of $f(x) = 0$. But if we look back at the last exercise the only part of $q(x)$ that we actually needed was the value of $q(x)$ when $x = r$. That is, $q(r)$.

It would save time if we could find $q(r)$ without first having to find $q(x)$. Is this possible? Look again at $f(r)$ and $q(r)$ in the previous exercise

$$f(r) = r^3 - 6r^2 + 6\sqrt{3}r - 4$$

$$q(r) = 3r^2 - 12r + 6\sqrt{3}.$$

Do you see a relationship between $f(r)$ and $q(r)$? It is probably not obvious from just a single example but there is a general relationship that allows us to easily find $q(r)$ directly from $f(r)$.

Here are a few more examples.

$\boldsymbol{f(r)}$	$\boldsymbol{q(r)}$
$-4r^5 + 2r - 1$	$-20r^4 + 2$
$2r^3 - r^2$	$6r^2 - 2r$
$8r^4 - 7r^2 + 2r - 5$	$32r^3 - 14r + 2$

Now do you see the relation? In each of these examples notice that for each term of the form $a \cdot r^n$ in $f(r)$ there is a term $\boldsymbol{n} \cdot a \cdot r^{\boldsymbol{n-1}}$ in $q(r)$.

Term in $\boldsymbol{f(r)}$	Associated Term in $\boldsymbol{q(r)}$
r^3	$3 \cdot r^{3-1} = 3r^2$
$6 \cdot r^2$	$6 \cdot 2 \cdot r^{2-1} = 12r$
$6\sqrt{3} \cdot r^1$	$6\sqrt{3} \cdot 1 \cdot r^{1-1} = 6\sqrt{3} \cdot r^0 = 6\sqrt{3}$
$4 = 4 \cdot r^0$	$4 \cdot 0 \cdot r^{0-1} = 0$

Let's put this relationship in functional form and give it the notation $f'(x)$.

Definition: $\boldsymbol{f'(x)}$

If $f(x)$ is the n^{th} degree polynomial

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_3x^3 + a_2x^2 + a_1x^1 + a_0x^0$$

then we will define $f'(x)$ as the $(n-1)^{st}$ degree polynomial

$$f'(x) = \boldsymbol{n} \cdot a_n x^{\boldsymbol{n-1}} + \boldsymbol{(n-1)} \cdot a_{n-1}x^{\boldsymbol{n-2}} + \cdots + \boldsymbol{3} \cdot a_3x^{\boldsymbol{2}} + \boldsymbol{2} \cdot a_2x^{\boldsymbol{1}} + \boldsymbol{1} \cdot a_1x^{\boldsymbol{0}} + \boldsymbol{0} \cdot a_0x^{\boldsymbol{-1}}.$$

The function $f'(x)$ is called the **derivative** of $f(x)$.

Aside: When you take a first course in calculus you will study the properties and applications of $f'(x)$, the **derivative** of $f(x)$, in considerable depth.

Tying this back to what we have already shown, we can now say that if the polynomial $q(x)$ is defined thought the polynomial division

$$\frac{f(x)}{x-r} = q(x) + \frac{f(r)}{x-r}$$

then $q(r)$ equals $f'(x)$ when $x = r$, which is denoted by $f'(r)$.

Using this new notation, we can restate the Quotient Remainder Theorem (see page 101) and avoid the reference to the quotient polynomial $q(x)$ altogether.

Non-Simple Root Theorem

If r is a non-simple root of the equation $f(x) = 0$, then $f(r) = 0$ and $f'(r) = 0$.

Exercise 2.33 (Source: MSHSML 1D944)

There are two choices of k for which $f(x) = x^3 - 4x^2 - 3x + k = 0$ has a double root. Find them.

Solution

Let r denote the double root of $f(x) = 0$. Then by the Non-Simple Root Theorem, $f(r) = 0$ and $f'(r) = 0$, where $f'(x) = 3x^2 - 8x - 3$.

Solving the quadratic equation $f'(r) = 3r^2 - 8r - 3 = 0$ by the quadratic theorem, we find that

$$r = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(3)(-3)}}{2(3)} = \frac{8 \pm \sqrt{100}}{6} = \frac{8 \pm 10}{6}.$$

So $r = 3$ or $r = -1/3$. Now we need to back solve for k in $f(3) = 0$ and $f(-1/3) = 0$.

$$f(3) = 27 - 36 - 9 + k = 0 \Longrightarrow k = -18.$$

$$f(-1/3) = -\frac{1}{27} - \frac{4}{9} + 1 + k = 0 \Rightarrow k = -\frac{14}{27}.$$

■

Exercise 2.34 (Source: MSHSML 1T145)

What is the integer value for k for which the cubic polynomial $f(x) = x^3 - 5x^2 + kx + 9 = 0$ has two equal rational roots?

Solution

Let r denote the double root of $f(x) = 0$. Then by the Non-Simple Root Theorem (see page 107), $f(r) = 0$ and $f'(r) = 0$, where $f'(x) = 3x^2 - 10x + k$.

Solving for k in the equation $f'(r) = 3r^2 - 10r + k = 0$ we find $k = 10r - 3r^2$. We can now plug this formula for k back into the equation $f(r) = 0$, to get

$$9 + (10r - 3r^2)r - 5r^2 + r^3 = -2r^3 + 5r^2 + 9 = 0.$$

This is a *cubic equation* which we don't have any easy way of solving in general. However, in this problem we are given that r is rational.

By the rational root theorem (see page 63) the possible rational roots of $f(x) = x^3 - 5x^2 + kx + 9 = 0$ are $\pm1, \pm3, \pm9$. So r has to be one of these numbers. Let's check and see which, if any, of these possible values for r satisfies the requirement that $f(r) = -2r^3 + 5r^2 + 9 = 0$.

$\boldsymbol{r}$	$\boldsymbol{-2r^3 + 5r^2 + 9 \stackrel{?}{=} 0}$
1	12
-1	16
3	0
-3	108
9	-1044
-9	1872

This shows that $r = 3$. But remember that $k = 10r - 3r^2$. Therefore,

$$k = 10(3) - 3(3^2) = 30 - 27 = 3.$$

■

Exercise 2.35 (Source: MSHSML 2T872)

For what values of k will the equation $3x^3 - 6x^2 - 5x + k = 0$ have two equal roots?

Solution

Let r denote the double root of $f(x) = 0$. Then by the Non-Simple Root Theorem (see page 107), $f(r) = 0$ and $f'(r) = 0$, where $f'(x) = 3x^2 - 12x - 5$.

Because $f'(r) = 0$ is a quadratic equation in r we can easily solve for r.

$$9r^2 - 12r - 5 = (3r - 5)(3r + 1) = 0 \Rightarrow r = \frac{5}{3} \text{ and } r = -\frac{1}{3}.$$

So, the double root r of $f(x) = 0$ can occur at either $r = 5/3$ or $r = -1/3$.

Now we need to find the value of k at each of these values of r.

$$f\left(\frac{5}{3}\right) = 3\left(\frac{5}{3}\right)^3 - 6\left(\frac{5}{3}\right)^2 - 5\left(\frac{5}{3}\right) + k = 0 \Rightarrow k = \frac{100}{9}$$

$$f\left(-\frac{1}{3}\right) = 3\left(-\frac{1}{3}\right)^3 - 6\left(-\frac{1}{3}\right)^2 - 5\left(-\frac{5}{3}\right) + k \Rightarrow k = -\frac{8}{9}.$$

■

Exercise 2.36 (Source: MSHSML 1D904)

One root of $f(x) = 4x^3 - 8x^2 + cx + d = 0$ is -1. The other two roots are equal. Find d.

Solution

Method 1.

Using synthetic division to divide $f(x)$ by $x + 1$.

-1	4	-8	c	d
		-4	12	$-c - 12$
	4	-12	$c + 12$	$d - c - 12$

From this tableau we can see that

$$\frac{f(x)}{x+1} = (4x^2 - 12x + c + 12) + \frac{(d-c-12)}{x+1}.$$

Also, we know that the remainder must equal zero because $x = -1$ is given to be a root of $f(x) = 0$. Therefore $d - c - 12 = 0$ or equivalently $d - c = 12$.

It follows that

$$\begin{aligned} f(x) &= (x+1)(4x^2 - 12x + c + 12) + \frac{0}{x+1} \\ &= (x+1)(4x^2 - 12x + c + 12). \end{aligned}$$

Let $g(x) = 4x^2 - 12x + c + 12$. By deflation (see page 70) we know that the remaining two roots of $f(x)$ are the two roots of $g(x)$.

But the statement of the problem says that "the other two roots are equal". That is, $g(x) = 0$ has a double root.

Let r denote the double root of $g(x) = 0$. Then by the Non-Simple Root Theorem (see page 107), $g(r) = 0$ and $g'(r) = 0$, where $g'(x) = 8x - 12$.

Solve for r in $g'(r) = 8r - 12 = 0$ we have $r = 3/2$.

But we know that $g(r) = 0$, so

$$g(3/2) = 4\left(\frac{3}{2}\right)^2 - 12\left(\frac{3}{2}\right) + c + 12 = 0.$$

Simplifying and solving for c we have

$$9 - 18 + c + 12 = 0$$

$$c = -3.$$

We have already established that $d - c = 12$. Therefore, $d = 12 + c = 12 - 3 = 9$.

Method 2.

Let r represent the unknown double root. By the Sum and Product Theorem (see page 15), we know that

$$r + r + (-1) = -(-8)/4 = 2 \qquad (1)$$

$$r \cdot r \cdot (-1) = -d/4. \qquad (2)$$

Solving for r in (1) we have $2r = 3$ or $r = 3/2$. Solving for d in (2) we have

$$d = 4r^2 = 4(3/2)^2 = 9.$$

■

2.13 The Role of GCDs in Locating Non-Simple Roots

In the previous section we established the non-simple root theorem (if r is a non-simple root of the equation $f(x) = 0$, then $f(r) = 0$ and $f'(r) = 0$).

But by the factor theorem (see page 14) this means that $(x - r)$ is a factor of the polynomial $f(x)$ and also the polynomial $f'(x)$.

Now recall our previous result for "Finding Common Zeros of Two Polynomials" (see page 98)

> If θ is a zero of both polynomials $f(x)$ and $g(x)$, then θ is necessarily a zero of $\gcd\big(f(x), g(x)\big)$. So, to find all common zeros of polynomials $f(x)$ and $g(x)$ it suffices to find all zeros of $\gcd\big(f(x), g(x)\big)$.

We can therefore state the following result:

GCD of $f(x)$ and $f'(x)$

If r is a non-simple root of $f(x) = 0$ then r is a root (but perhaps only a simple root) of the polynomial equation $\gcd\big(f(x), f'(x)\big) = 0$.

Actually, there is more we can say.

GCD of $f(x)$ and $f'(x)$, Extended

For all integer m greater than or equal to 1, r occurs m times as a root of the polynomial equation $\gcd\big(f(x), f'(x)\big) = 0$ if and only if r occurs m times as a root of the polynomial equation $f'(x) = 0$ and $m + 1$ times as a root of the polynomial equation $f(x) = 0$.

Additionally, the polynomial equation $f(x) = 0$ has no repeated roots if and only if $\gcd\big(f(x), f'(x)\big) = 1$ for all x.

We will use these two extended results but we will not proof them here. The proof requires a few results from basic calculus so we will not consider that here.

Exercise 2.37 (Source: *Golden Algebra*, N.P. Bali)

Find all solutions to the equation $f(x) = x^4 - 6x^3 + 12x^2 - 10x + 3 = 0$ given that not all of its roots are distinct.

Solution

We are given that "not all roots of $f(x) = 0$ are distinct". This is just another way of saying that $f(x) = 0$ has (at least one) non-simple root.

Let r be a non-simple root of $f(x) = 0$.

But we just established that if r is a non-simple root of $f(x) = 0$ then r is a root (but perhaps only a simple root) of $\gcd\big(f(x), f'(x)\big) = 0$.

Now we can use the Euclidean algorithm to find $\gcd\big(f(x), f'(x)\big)$ where

$$f(x) = x^4 - 6x^3 + 12x^2 - 10x + 3$$

and

$$f'(x) = 4x^3 - 18x^2 + 24x - 10.$$

	x	$+1$				
~~$4x^3 - 18x^2 + 24x - 10$~~	~~x^4~~	~~$-6x^3$~~	~~$+12x^2$~~	~~$-10x$~~	~~$+3$~~	**(Mult. by 2)**
$\mathbf{2x^3 - 9x^2 + 12x - 5}$	$\mathbf{2x^4}$	$\mathbf{-12x^3}$	$\mathbf{+24x^2}$	$\mathbf{-20x}$	$\mathbf{+6}$	
(Divide by 2)	$2x^4$	$-9x^3$	$+12x^2$	$-5x$		
		~~$-3x^3$~~	~~$+12x^2$~~	~~$-15x$~~	~~$+6$~~	**(Mult. by $-2/3$)**
		$\mathbf{2x^3}$	$\mathbf{-8x^2}$	$\mathbf{+10x}$	$\mathbf{-4}$	
		$2x^3$	$-9x^2$	$+12x$	-5	
			x^2	$-2x$	$+1$	

	$2x$	-5		
$x^2 - 2x + 1$	$2x^3$	$-9x^2$	$+12x$	-5
	42	$-4x^2$	$+2x$	
		$-5x^2$	$+10x$	-5
		$-5x^2$	$+10x$	-5
				0

Therefore,

$$\gcd\big(f(x), f'(x)\big) = x^2 - 2x + 1 = (x-1)^2.$$

So 1 occurs twice as a root of the polynomial equation $\gcd\big(f(x), f'(x)\big) = 0$. Therefore, from our above extended conclusions about $\gcd\big(f(x), f'(x)\big)$ we can conclude that 1 occurs **three times** as a root of $f(x) = 0$.

So, **three of the four** roots of $f(x)$ are 1,1 and 1.

As we have noted before, we can find the fourth root by finding $f(x)/(x-1)^3$ and using the principle of deflation (see page 70).

In particular,

	x	-3			
$x^3 - 3x^2 + 3x - 1$	x^4	$-6x^3$	$12x^2$	$-10x$	3
	x^4	$-3x^3$	$3x^2$	$-x$	
		$-3x^3$	$9x^2$	$-9x$	3
		$-3x^3$	$9x^2$	$-9x$	3
					0

Therefore, the fourth root of $f(x) = 0$ is 3.

However, taking advantage of the "Sum of the Roots" result (see page 15) is a quicker method for finding the fourth root. By that result, the sum of *all four* roots equals $-(-6) = 6$. That is, if we take d as our missing fourth root, then $1 + 1 + 1 + d = 6$. So, we can immediately see that $d = 3$.

Either way, we have determined that $x = 1,1,1,3$ are the four roots of the quartic equation $f(x) = x^4 - 6x^3 + 12x^2 - 10x + 3 = 0$.

■

2.14 Non-Simple Roots and Tangents to Polynomials

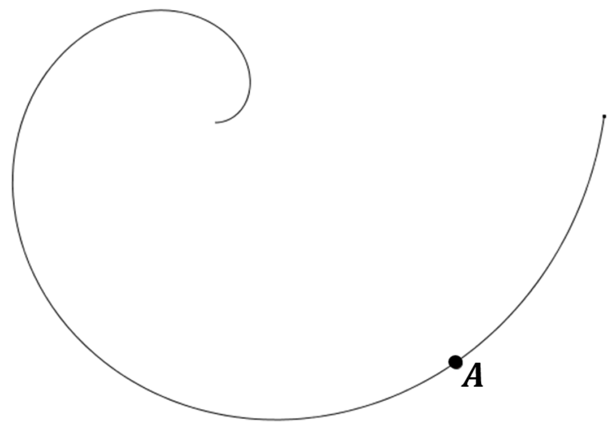
We need to take a side trip at this point and explain what we mean when we talk about a "tangent to a curve at a point". Then we will be ready to talk about how tangents are connected to the multiplicity of roots of polynomial equations.

Tangent to a Curve

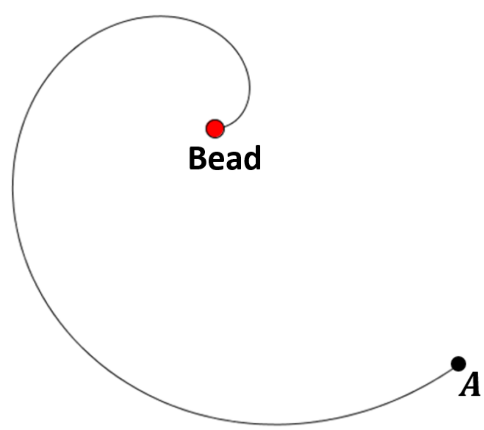
The word *tangent* is derived from the Latin word *tangens*, which means "touching."

The tangent line to a curve at a given point on that curve is a line that touches (passes through) the curve at that point and "has the same direction" as the curve at that point.

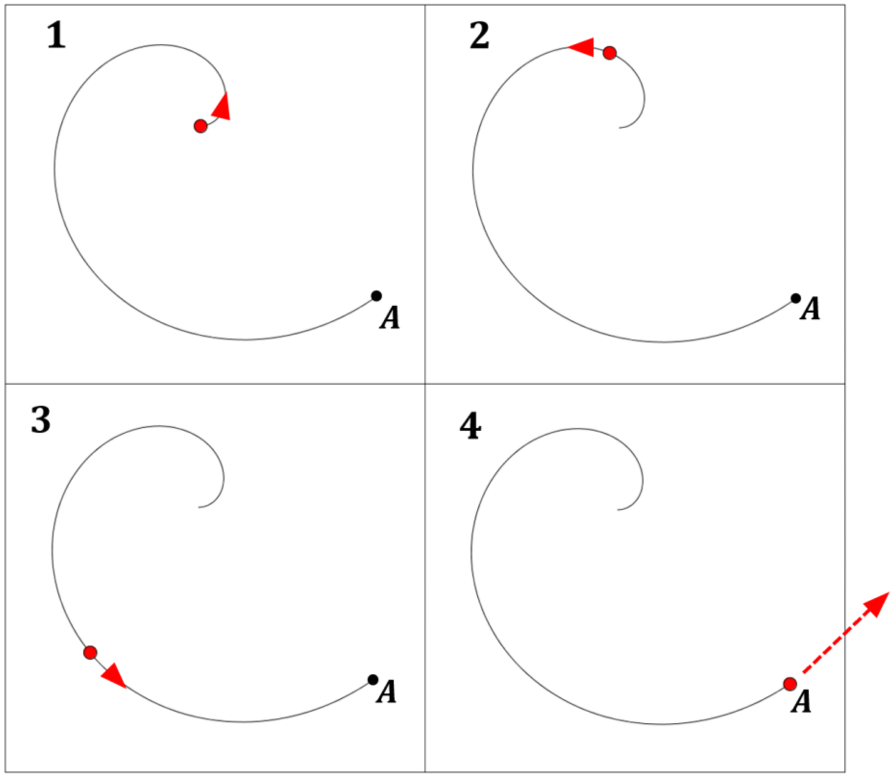
Example. Find the tangent line to the curve below at the point A.



To get a physical understanding of a tangent line, imagine this curve as a wire with a bead strung on it. Further, suppose we cut the wire at the point A



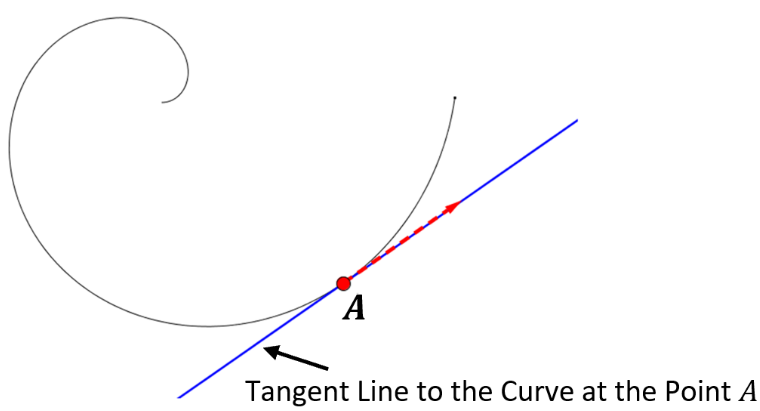
Now *give the bead a big push* and watch it travel along this wire (curve).



The bead would move from Position 1 to Position 2 to Position 3 etc.

Because the bead has nothing stopping it when it reaches the end of the wire (point A) it will continue on "in the same direction" as the curve at the point A.

The tangent line is defined as the line that coincides with this red vector (*i.e.* the direction the bead takes when it leaves the wire (curve) at point A).



In a first course in calculus you will learn that the **slope of the tangent line** of the curve $f(x)$ at the point $A = (a, f(a))$ equals $f'(a)$, where $f'(x)$ is the derivative of $f(x)$ as we discussed earlier. By the point-slope formula for a line this tells us that:

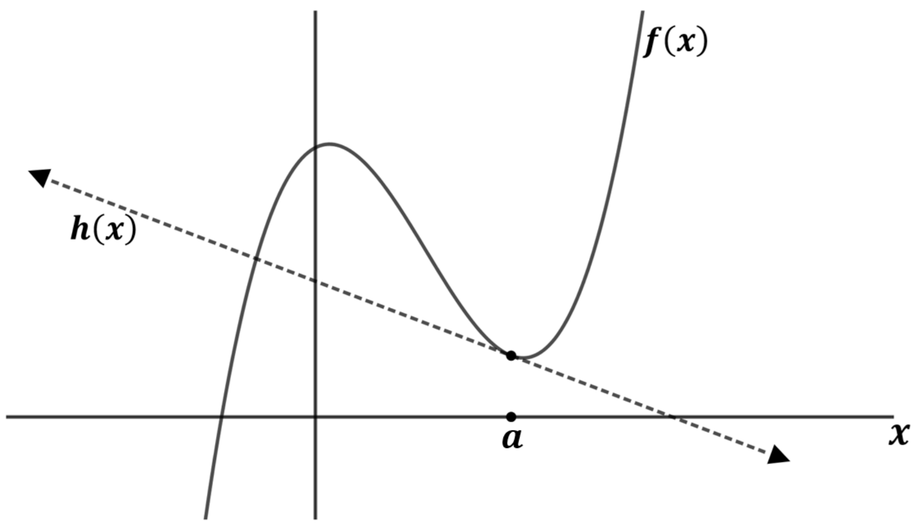
Tangent Line Formula

The formula for the tangent line $h(x)$ to the curve $f(x)$ at the point $A = (a, f(a))$ is

$$h(x) = f(a) + f'(a)(x - a).$$

Tangent Line and Non-Simple Roots Theorem

If $h(x)$ is the line tangent line to the polynomial $f(x)$ at the point $x = a$, then a is a non-simple root of the polynomial equation $g(x) = f(x) - h(x) = 0$.



If line $h(x)$ is tangent to the polynomial $f(x)$ at $x = a$, then $x = a$ is a non-simple root of the polynomial equation $g(x) = f(x) - h(x) = 0$.

Exercise 2.38 (Source: MSHSML 1T066)

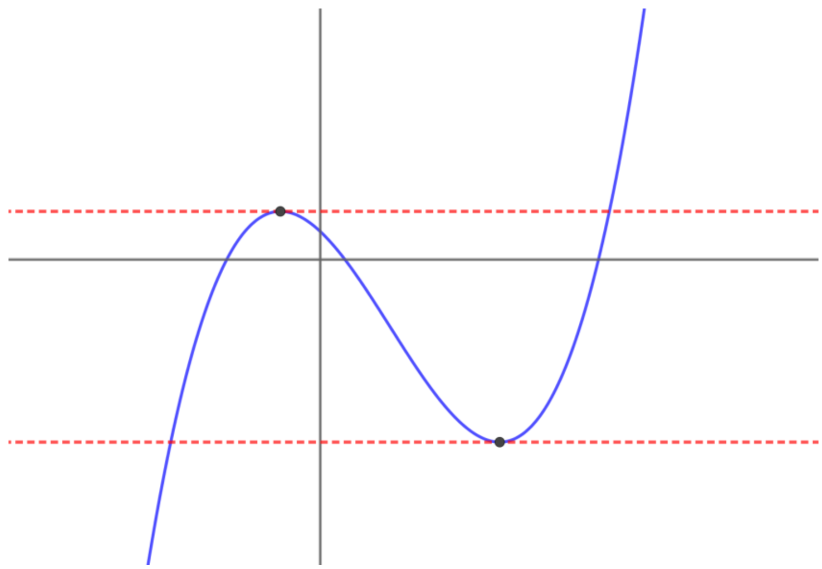
For what choices of k will the graphs of $y = k$ and $y = 2x^3 - 7x^2 - 12x + 6$ have exactly two distinct points of intersection?

Solution

The only way that the horizontal line $y = k$ can intersect the cubic function

$$f(x) = y = 2x^3 - 7x^2 - 12x + 6$$

exactly twice is that this horizontal line is <u>tangent</u> to $f(x)$ at one point.



By our result Tangent Line and Non-Simple Roots Theorem, this means that $g(x) = f(x) - k = 0$ has a double root at that value of x where $y = k$ is tangent to $f(x)$.

Therefore, by the Non-Simple Root Theorem (see page 107), $g(x) = 0$ and $g'(x) = 0$ at that value of x where $y = k$ is tangent to $f(x)$.

Now recall our discuss on how to find the derivative of a polynomial $g(x)$ (see page 106). We saw that every term ax^n in $g(x)$ was matched with a term of the form $n \cdot ax^{n-1}$ in $g'(x)$.

Derivative of a Constant Term

This means that any constant term $k = kx^0$ in $g(x)$ is matched with $0 \cdot kx^{0-1} = 0$. That is, any constant term in $g(x)$ is "zeroed out" in the corresponding $g'(x)$.

Keeping this in mind, what can we saw about the derivative of $g(x) = f(x) - k$?

The constant term k will be "zeroed out" and thus the derivative of the polynomial $g(x)$ will be same as the derivative of the polynomial $f(x)$.

That is, if $g(x) = f(x) - k$, then $g'(x) = f'(x)$.

Now we have already stated that $g'(x) = 0$ at any value of x where $y = k$ is tangent to $f(x)$. Therefore, $f'(x) = g'(x)$ will necessarily equal 0 at any value of x where $y = k$ is tangent to $f(x) = 2x^3 - 7x^2 - 12x + 6$.

That is, $f'(x) = 6x^2 - 14x - 12$ will equal 0 at any value of x where $y = k$ is tangent to $f(x)$. But we can solve $f'(x) = 0$ because it is a quadratic equation.

We note that

$$f'(x) = 6x^2 - 14x - 12 = 2(3x + 2)(x - 3).$$

Therefore, the roots of $f'(x) = 0$ are $x = -2/3$ and $x = 3$.

Now we need to back solve and find the value of k that makes $g(-2/3) = 0$ and the value of k that makes $g(3) = 0$.

$$g(-2/3) = f(-2/3) - k = 2(-2/3)^3 - 7(-2/3)^2 - 12(-2/3) + 6 - k = \frac{278}{27} - k$$

Therefore $g(-2/3) = 0$ if $k = 278/27$.

What value of k makes $g(3) = 0$?

$$g(3) = f(3) - k = 2(3)^3 - 7(3)^2 - 12(3) + 6 - k = -39 - k$$

Therefore $g(3) = 0$ if $k = -39$.

■

Exercise 2.39 (Source: MSHSML 1T003)

The graph of $f(x) = ax^4 + bx^3 + cx^2 + dx + e$ is symmetric about the y axis, reaches a high point of $y = 1$ when $x = 2$, and has a y intercept of -3. Find all values of x for which $f(x) = 0$. Hint: If the graph reaches a high of $y = 1$ when $x = 2$, then it is tangent to (*i.e.* has a double root with) the graph of $y = 1$ at $x = 2$.

Solution

By definition, a function $f(x)$ is symmetric with respect to the y-axis if $f(x) = f(-x)$ for all x. Now consider

$$\begin{aligned} f(x) &= ax^4 + bx^3 + cx^2 + dx + e \\ &= (ax^4 + cx^2 + e) + (bx^3 + dx) \\ &= f_E(x) + f_O(x). \end{aligned}$$

Notice that

$$f_E(-x) = a(-x)^4 + c(-x)^2 + e = ax^4 + cx^2 + e = f_E(x)$$

and

$$f_O(-x) = b(-x)^3 + d(-x) = -bx^3 - dx = -(bx^3 + dx) = -f_O(x).$$

So

$$f(-x) = f_E(-x) + f_O(-x) = f_E(x) - f_O(x).$$

Hence the requirement that $f(x) = f(-x)$ for all x tells us that

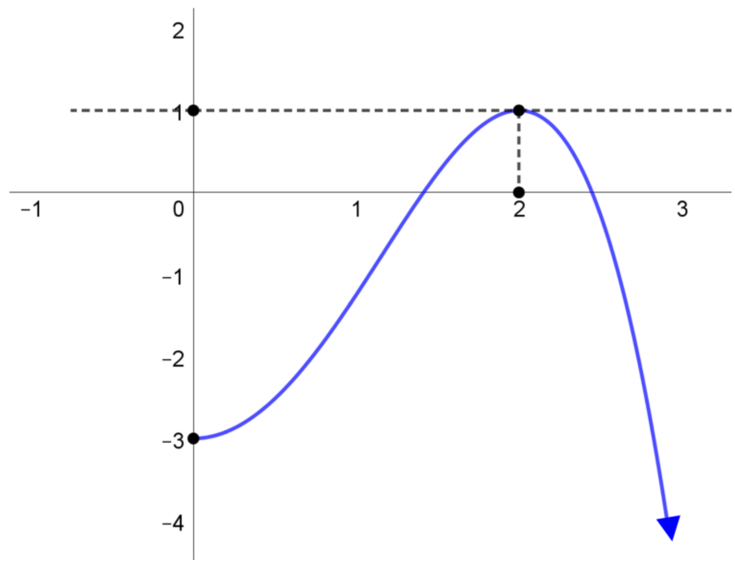
$$f_E(x) + f_O(x) = f_E(x) - f_O(x)$$

$$f_O(x) = -f_O(x)$$

$$f_0(x) = 0, \text{ for all } x.$$

That is, $f(x) = f(-x)$ for all x requires that $b = 0$ and $d = 0$.

What we have just shown is that (in general), a polynomial symmetric about the y-axis **cannot have any odd power terms**.



With the information that is given the graph of $f(x) = ax^4 + cx^2 + e$ for $x \geq 0$ will have to have this basic shape.

We know $a < 0$ because we are told that $f(x)$ reaches a max at 2. If $a > 0$ then $f(x)$ would go to ∞ as x gets larger and larger.

From the information given we know that $y = 1$ (the dotted line) is tangent to $f(x)$ at the point $x = 2$. There will be another tangent line at $y = -3$.

We can use the information that $(2,1)$ and $(0,-3)$ are on this graph to eliminate some of the unknowns.

$$-3 = a(0^4) + c(0^2) + e \Longrightarrow e = -3$$

$$1 = a(2^4) + c(2^2) - 3 = 16a + 4c - 3 \Longrightarrow 1 = 4a + c.$$

Therefore,

$$f(x) = ax^4 + cx^2 + e = ax^4 + (1 - 4a)x^2 - 3.$$

We are given that the line $y = 1$ is tangent to the polynomial $f(x)$ at $x = 2$. So it follows from the Tangent Line and Non-Simple Roots Theorem (see page 117) that 2 is a non-simple root of the polynomial equation $g(x) = f(x) - 1 = 0$. But by the Non-Simple Root Theorem (see page 107) this tells us that $g(2) = 0$ and $g'(2) = 0$.

However, the derivative of the constant term -1 in $g(x) = f(x) - 1$ “zeros out” (see page 118) so $g'(x) = f'(x) = 4ax^3 + 2(4a - 1)x$. This allows us to solve for a in the equation $g'(2) = 0$. That is,

$$0 = g'(2) = 4a(2^3) + 2(1-4a)\cdot 2 = 32a - 16a + 4 = 16a + 4.$$

Thus,

$$a = \frac{-4}{16} = \frac{-1}{4} \quad \text{and} \quad c = 1 - 4a = 1 - 4\left(\frac{-1}{4}\right) = 2.$$

Therefore,

$$f(x) = (-1/4)x^4 + 0x^3 + 2x^2 + 0x - 3 = (-1/4)x^4 + 2x^2 - 3.$$

Finally, we are in position to solve the original question, namely what are the roots of $f(x) = 0$?

$$(-1/4)x^4 + 2x^2 - 3 = 0$$

$$\Leftrightarrow x^4 - 8x^2 + 12 = 0$$

$$\Leftrightarrow (x^2 - 2)(x^2 - 6) = 0$$

$$\Leftrightarrow x = \pm\sqrt{2} \text{ or } x = \pm\sqrt{6}.$$

■

Exercise 2.40 (Source: MSHSML SD162)

Determine exactly the constant a for which the line $y = 2(x + a)$ is tangent to $y = x^2 + 4$.

Solution

Denote the x value of the point of tangency as x_0. By our formula for the tangent line $h(x)$ to the curve $f(x) = y = x^2 + 4$ at x_0 (see page **Error! Bookmark not defined.**) we have

$$h(x) = f(x_0) + f'(x_0)(x - x_0).$$

For $f(x) = x^2 + 4$ we have $f'(x) = 2x$. Therefore,

$$h(x) = (x_0^2 + 4) + (2x_0)(x - x_0).$$

But this has to agree with the given equation for the tangent line, $y = 2(x + a)$. That is,

$$(x_0^2 + 4) + (2x_0)(x - x_0) = 2(x + a)$$

$$(2x_0)x + (x_0^2 + 4 - 2x_0^2) = 2x + 2a.$$

We have previously stated (see page 15) the "Equality of Two Polynomials Theorem" which states that two polynomials are equal *for all* x if and only if their coefficients are equal term by term.

Hence, $2x_0 = 2$ and $x_0^2 + 4 - 2x_0^2 = 2a$. From this first equation we see $x_0 = 1$. Using this value for x_0 in the second equation we have $1^2 + 4 - 2(1^2) = 2a$. Solving for a, we have $2a = 3$ or $a = 3/2$.

■

Exercise 2.41 (Source: MSHSML 1D103)

For what value of k will the graphs of $y = k$ and $y = -\frac{1}{4}x^2 + \frac{3}{2}x - \frac{1}{2}$ intersect at only one point?

Solution

This problem is really an exercise in recognition. We are given a horizontal line and a vertically-oriented parabola, which will have a single intersection only at the vertex. The y-coordinate of the vertex can be found, among other methods, by calculating:

$$-\left(\frac{b^2}{4a}+c\right) = -\left(\frac{\left(\frac{3}{2}\right)^2}{4\left(-\frac{1}{4}\right)}+\frac{1}{2}\right) = -\left(-\frac{9}{4}+\frac{2}{4}\right) = \frac{7}{4}.$$

■

Exercise 2.42 (Source: MSHSML ST874)

A line is said to be tangent to a parabola if and only if the line meets the parabola in just one point. What choice of b yields a member of the family $y = x + b$ that is tangent to the graph of $y^2 - 6y + 10 = x$?

Solution

We seek a double root to the equation

$$y^2 - 6y + 10 = x = y - b$$

$$y^2 - 7y + 10 + b = 0$$

Discrim $= 49 - 4(10+b) = 9 - 4b = 0$

$$b = \frac{9}{4}$$

■

Exercise 2.43 (Source: MSHSML 1T976)

Find a negative number m so that the line $y = mx + 2$ will be tangent to the parabola $y = 4x(1 - x)$. [Note: A line is tangent to a parabola if it intersects the parabola at exactly one point.]

Solution

Method 1

By the Tangent Line Formula (see page 117), the unique tangent line to the polynomial $f(x)$ at $x = u$ is

$$h(x) = f(u) + f'(u)(x - u).$$

In our problem $f(u) = 4u(1 - u) = 4u - 4u^2$ and $f'(u) = 4 - 8u$. Therefore, the unique tangent line equals

$$\begin{aligned} h(x) &= (4u - 4u^2) + (4 - 8u)(x - u) \\ &= (4 - 8u)x + (4u - 4u^2 - 4u + 8u^2) \\ &= 4(1 - 2u)x + 4u^2. \end{aligned}$$

We were given that the tangent line has the form $y = mx + 2$. Therefore, for all x,

$$mx + 2 = 4(1 - 2u)x + 4u^2.$$

This means that these two polynomials must match coefficient by coefficient. Therefore

$$4u^2 = 2 \text{ and } 4(1 - 2u) = m.$$

It follows from the first equation that $u = \pm\sqrt{1/2}$ and using this in the second equation we find that

$$m = 4\left(1 - \frac{2}{\sqrt{2}}\right) \text{ or } m = 4\left(1 + \frac{2}{\sqrt{2}}\right).$$

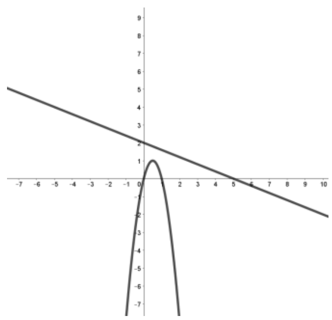
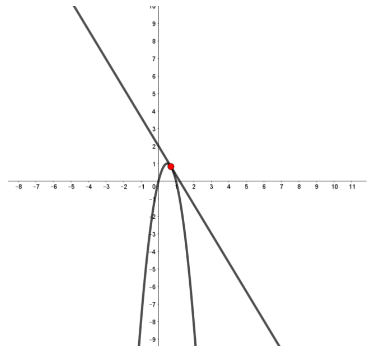
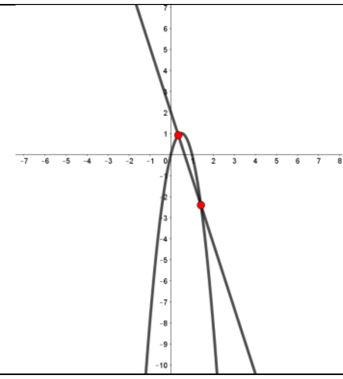
But we are told that m is negative, therefore

$$m = 4\left(1 - \frac{2}{\sqrt{2}}\right) = 4(1 - \sqrt{2}).$$

■

Method 2

Depending on the value of its slope m, the line $y = mx + 2$ will intersect the parabola $y = 4x(1 - x)$

		
0 times	1 time	2 times.

For fixed m we can determine intersection point(s) by solving the quadratic

$$4x(1-x) = mx + 2 \quad \text{or} \quad 4x^2 + (m-4)x + 2 = 0.$$

By the quadratic formula the solutions are

$$x = \frac{-(m-4) \pm \sqrt{(m-4)^2 - 32}}{8}.$$

The discriminant of this quadratic is seen to be $d = (m-4)^2 - 32$. Recall that it is the discriminant that tells us whether we will have 0 real solutions $(d < 0)$, 1 real solution that occurs twice $(d = 0)$ or 2 distinct real solutions $(d > 0)$.

It is the case of 1 real solution of multiplicity two that identifies when the line will be tangent to the parabola.

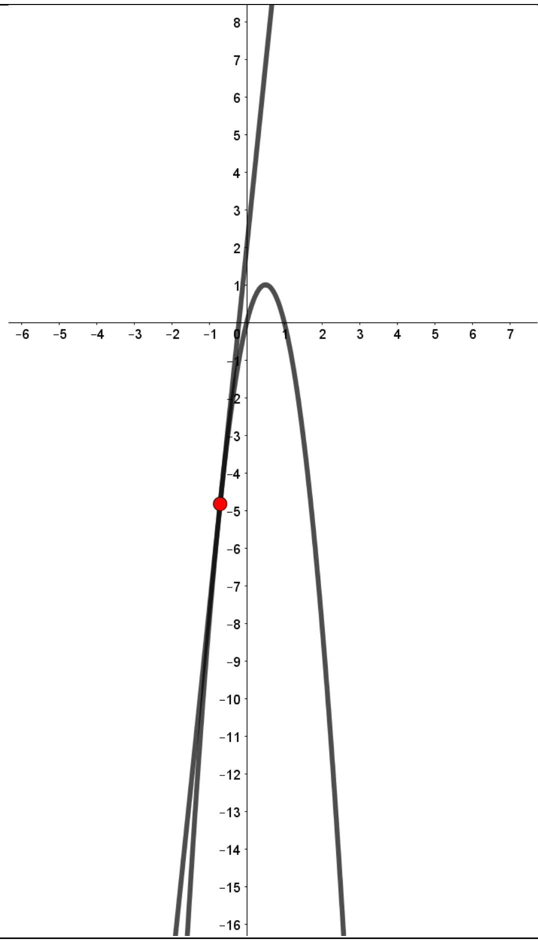
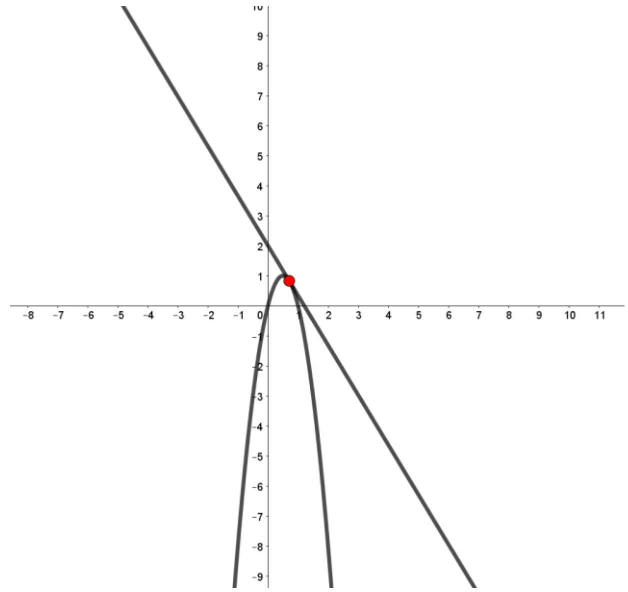
For what value(s) of m is $d = 0$?

$$(m-4)^2 - 32 = 0$$

$$(m-4)^2 = 32$$

$$m - 4 = \pm\sqrt{32}$$

$$m = 4 \pm \sqrt{32} = 4 \pm 4\sqrt{2}.$$

	
slope $= m = 4 + 4\sqrt{2} > 0$	slope $= m = 4 - 4\sqrt{2} < 0$

The problem specifies they only want the case of a negative slope so our only answer is

$$m = 4 - 4\sqrt{2} \approx -1.6569.$$

■

Exercise 2.44 (Source: MSHSML 4T846)

Two lines, having slopes of m_1 and m_2, can be drawn through the origin, tangent to the parabola $8x^2 - 37x + 8y + 32 = 0$. Find the smallest of these slopes.

Solution

We wish there to be one solution only to $8x^2+(8m-37)x+32=0$

discriminant $=(8m-37)^2-4\cdot8\cdot32 = 64m^2-592m+37^2-32^2=0$

$64m^2-592m+5\cdot69=0$

$(8m-5)(8m-69)=0$

$m=5/8$, $m=69/8$

Exercise 2.45 (Source: MSHSML SA162)

Determine exactly the constant a for which the line $y=2(x+a)$ is tangent to $y=x^2+4$.

Solution

Equate the functions to find the intersection point. This gives $2(x+a)=x^2+4 \Rightarrow x^2-2x+(4-2a)=0$. *Since these functions are tangent, the discriminant will be 0. Therefore,* $(-2)^2-4\bullet1\bullet(4-2a)=0 \Rightarrow 8a-12=0 \Rightarrow a=\frac{3}{2}$.

Exercise 2.46 (Source: MSHSML 2D154)

Determine exactly the value of c^2 so that the circle $x^2+y^2=c^2$ is tangent to the line $cx+2y=6$.

Solution

Exercise 2.47 (Source: MSHSML SD104)

A circle of radius 6, centered somewhere on the positive y-axis, is tangent to the graph of $x^2-y^2=4$ at a point P in the first quadrant. Determine exactly the slope of the line that is tangent to both curves at P.

Solution

Since $x^2 = 4 + y^2$, *we can write the equation of the circle as* $\left(4+y^2\right)+\left(y-k\right)^2 = 36$. *After rearranging terms, the quadratic equation gives us* $y = \dfrac{2k \pm \sqrt{4k^2 - 4\cdot 2\cdot\left(k^2-32\right)}}{4} = \dfrac{2k \pm 2\sqrt{64-k^2}}{4}$ *, which has a double root at the point of tangency* (*when* $k = 8$). *This means the center of the circle is at* $(0, 8)$, *with point P located at* $\left(2\sqrt{5}, 4\right)$. *The radius between these two points has slope* $\dfrac{-4}{2\sqrt{5}} = \dfrac{-2}{\sqrt{5}}$, *so the slope of the tangent line is* $\dfrac{\sqrt{5}}{2}$.

■

Exercise 2.48 (Source: MSHSML SD993)

The line $y = 2x + b$ is tangent to the circle $x^2 + y^2 = 1$. Find all possible exact values of b.

Solution

$$\left.\begin{array}{l} y = 2x+b \\ x^2+y^2=1 \end{array}\right\} \Rightarrow x^2 + (2x+b)^2 = 1 \Rightarrow 5x^2 + 4bx + (b^2-1) = 0$$

This equation has a double-root when

$$(4b)^2 - 4(5)(b^2-1) = 0$$
$$16b^2 - 20b^2 + 20 = 0$$
$$4b^2 = 20$$
$$b^2 = 5$$
$$\boxed{b = \pm\sqrt{5}}$$

■

Exercise 2.49 (Source: MSHSML 1T066)

For what choices of k will the graphs of $y = k$ and $y = 2x^3 - 7x^2 - 12x + 6$ have exactly two distinct points of intersection?

Solution

2.15 Descartes' Rule of Signs

Variation in Sign

Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ where $a_n \neq 0$. Consider the string of $+$'s and $-$'s you get if you go in order, starting at a_n and write

a "$+$" sign if a coefficient is positive

a "$-$" sign if a coefficient is negative

nothing if a coefficient is zero (*i.e.* if a term is missing).

For example, for $f(x) = -2x^3 + x + 3$ you would get the string "$- + +$". The -2 gives you the first "$-$" sign, you don't include anything for the missing x^2 term, the next coefficient is an implicit 1 and this gives you a "$+$" sign and the final coefficient is 3 and this gives you the final "$+$" sign.

As a second example, the polynomial $f(x) = 3x^5 + 2x^4 - x^3 + 5x - 4$ would generate the string "$+ + - + -$".

The number of sign variations for a given string is the number of times the sign changes (from $+$ to $-$ or from $-$ to $+$ as you go in order from one sign to the next in the string.)

For example, the number of sign variations associated with the string "$- + +$" equals 1 (which occurred as the sign switched from $-$ to $+$ with the first two signs).

As a second example, the number of sign variations associated with the string "$+ + - + -$" equals 3 (the switch from $+$ to $-$ between the second and third signs, the switch from $-$ to $+$ between the third and fourth signs and the switch from $+$ to $-$ between the fourth and fifth signs).

Descartes' Rule of Signs

Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ where all the coefficients are real and $a_n \neq 0$.

(***Positive Roots***) The number of positive roots of $f(x) = 0$ is either equal to the number of sign variations in the coefficients of $f(x)$ or else is less than this number by an even integer.

(***Negative Roots***) The number of negative roots of $f(x) = 0$ is either equal to the number of sign variations in the coefficients of $f(-x)$ or else is less than this number by an even integer.

In applying Descartes' Rule of Signs, we ignore terms with zero coefficients when determining sign variations.

Exercise 2.50

How many positive roots, negative roots and non-real complex roots does the equation $x^4 + 15x^2 + 7x - 11 = 0$ have?

Solution

Let's apply Descartes Rule of Signs to this problem where $f(x) = x^4 + 15x^2 + 7x - 11$.

Let a equal the number of positive real roots of $f(x) = 0$.

Let b equal the number of negative real roots of $f(x) = 0$.

Let c equal the number of non-real complex roots of $f(x) = 0$.

We know that $a + b + c = 4$, the degree of $f(x)$.

The polynomial $f(x) = x^4 + 15x^2 + 7x - 11$ has sign string "$+ + + -$" and this string has 1 variation in signs. By Rule 2 we know that $1 - a$ is an even, nonnegative integer. So, the *only* possible value for a is 1. In this case $1 - a = 1 - 1 = 0$ is an even, nonnegative integer.

The polynomial $f(-x) = (-x)^4 + 15(-x)^2 + 7(-x) - 11 = x^4 + 15x^2 - 7x - 11$ has sign string "$+ + - -$" and this string also has 1 variation in signs. Therefore, by Rule 2, we know that $1 - b$ is an even, nonnegative integer. It follows that $b = 1$.

We can find c, the number on non-real complex roots by remembering that $a + b + c = 4$. Therefore $c = 4 - a - b = 4 - 1 - 1 = 2$.

■

Exercise 2.51

How many positive roots, negative roots and non-real complex roots does the equation $3x^7 + 2x^5 + 4x^3 + 11x - 12 = 0$ have?

Solution

Apply Descartes' Rule of Signs to the equation $f(x) = 3x^7 + 2x^5 + 4x^3 + 11x - 12 = 0$.

Let a equal the number of positive real roots of $f(x) = 0$.

Let b equal the number of negative real roots of $f(x) = 0$.

Let c equal the number of non-real complex roots of $f(x) = 0$.

We know that $a + b + c = 7$, the degree of $f(x)$.

The polynomial $f(x) = 3x^7 + 2x^5 + 4x^3 + 11x - 12$ has sign string "$+ + + + -$" and this string has 1 variation in signs. By Rule 2 we know that $1 - a$ is an even, nonnegative integer. So, the *only* possible value for a is 1. In this case $1 - a = 1 - 1 = 0$ is an even, nonnegative integer.

The polynomial $f(-x) = 3(-x)^7 + 2(-x)^5 + 4(-x)^3 + 11(-x) - 12 = -3x^7 - 2x^5 - 4x^3 - x - 12$ has sign string "$- - - - -$" and this string also has 0 variation in signs. Therefore, by Rule 2, we know that $0 - b$ is an even, nonnegative integer. It follows that $b = 0$.

We can find c, the number on non-real complex roots by remembering that $a + b + c = 4$. Therefore $c = 7 - a - b = 7 - 1 - 0 = 6$.

■

3 Symmetry

In this chapter we will consider multivariate polynomials. Multivariate polynomials arise naturally whenever you are considering a function of the roots of a univariate (single variable) polynomial.

For example, if the univariate polynomial $f(x) = 3x^4 + 2x^2 - 2x - 1$ has roots r_1, r_2, r_3, r_4 then suppose it "just happens" that we wanted to consider the sum of all possible products of pairs of the roots. That is,

$$g(r_1, r_2, r_3, r_4) = r_1 r_2 + r_1 r_3 + r_1 r_4 + r_2 r_3 + r_2 r_4 + r_3 r_4.$$

At this point we are talking about a multivariate polynomial $g(\ )$ in the variables r_1, r_2, r_3 and r_4.

This begs the question of why anyone would "just happen" to be interested in a polynomial of the particular form of $g(\ )$ as given above.

Suppose I were to tell you that I could just look at $f(x) = 3x^4 + 2x^2 - 2x - 1$ and $g(r_1, r_2, r_3, r_4) = r_1 r_2 + r_1 r_3 + r_1 r_4 + r_2 r_3 + r_2 r_4 + r_3 r_4$ and immediately know that $g(r_1, r_2, r_3, r_4) = 2/3$ without knowing anything about the values of the individual roots r_1, r_2, r_3 and r_4? That would be pretty amazing, but it's true.

But the magic is just getting going. I could also tell you that $h(r_1, r_2, r_3, r_4) = r_1 r_2 r_3 r_4 = 1/3$ and $q(r_1, r_2, r_3, r_4) = r_1 + r_2 + r_3 + r_4 = 0$, among other things.

A insightful question at this point is whether you can take all this information about these multivariate polynomials $g(\ ), h(\ )$ and $q(\ )$ built on the roots of $f(x) = 3x^4 + 2x^2 - 2x - 1$ and reverse engineer that information to figure out the individual values of the roots r_1, r_2, r_3 and r_4.

3.1 Terms

We will begin by defining the necessary terms.

Monomial in n variables

An expression of the form $c \cdot x_1^{k_1} \cdot x_2^{k_2} \cdots x_n^{k_n}$ with each $k_j \in \{0,1,2,3,\dots\}$ is called a *monomial* in the n variables $x_1, x_2, \dots, x_n$. The constant c can be any real number and is called the coefficient of that monomial.

e.g. $2 \cdot x_1^3 \cdot x_2^2$ and $(-\pi) \cdot x_1 \cdot x_3 \cdot x_4^3$ are both monomials

Polynomial in n variables

A *polynomial* in n variables $x_1, x_2, \dots, x_n$ is a finite sum of monomials.

e.g. $x_1x_2 - 2x_2x_3 + x_1x_3$ and $x_1^2 + x_2^2 + x_3x_4$ and $x_1 - x_1x_2 + 5$ are all polynomials

Symmetric Polynomial

A polynomial $f(x_1, x_2, \dots, x_n)$ is called *symmetric* if for every ordering $y_1, y_2, \dots, y_n$ of the variables $x_1, x_2, \dots, x_n$ we have $f(y_1, y_2, \dots, y_n) = f(x_1, x_2, \dots, x_n)$.

e.g. $f(x, y, z) = xy + xz + yz$ is symmetric in the variables x, y, z because

$$\begin{aligned} f(x,y,z) &= xy + xz + yz \qquad & f(y,z,x) &= yz + yx + zx \\ f(x,z,y) &= xz + xy + zy \qquad & f(z,x,y) &= zx + zy + xy \\ f(y,x,z) &= yx + yz + xz \qquad & f(z,y,x) &= zy + zx + yx \end{aligned}$$

are all equal.

Elementary Symmetric Polynomials

The symmetric polynomials $s_1, s_2, \dots, s_n$, as defined below are called the *elementary symmetric polynomials* in the variables $x_1, x_2, \dots, x_n$.

$$\begin{aligned} s_1 &= x_1 + x_2 + \cdots + x_n \\ s_2 &= x_1x_2 + x_1x_3 + \cdots + x_1x_n + x_2x_3 + \cdots \end{aligned}$$

$$
\begin{aligned}
s_3 &= x_1x_2x_3 + x_1x_2x_4 + \cdots + x_2x_3x_4 + \cdots \\
&\vdots \\
s_n &= x_1x_2 \cdots x_n
\end{aligned}
$$

To be clear, s_3 is, for example, the sum over *all possible* monomials in three variables of the form $x_ix_jx_k$ where i, j and k are distinct.

And s_k is the sum over *all possible* monomials of k variables the form $x_{j_1}x_{j_2} \cdots x_{j_k}$ where $j_1, j_2, \dots, j_k$ are all distinct.

That is, s_k is the sum over all possible monomials involving k of the n variables $x_1, x_2, \dots, x_n$ each raised to the first power and each with coefficient $c = 1$.

3.2 Viète's Theorem

Let $r_1, r_2, \dots, r_n$ be the roots of the polynomial equation

$$p(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = 0.$$

Then the following holds:

$$
\begin{aligned}
s_1 = r_1 + r_2 + \cdots + r_n &= -\frac{a_{n-1}}{a_n} \\
s_2 = r_1r_2 + r_1r_3 + \cdots + r_1r_n + r_2r_3 + \cdots &= +\frac{a_{n-2}}{a_n} \\
s_3 = r_1r_2r_3 + r_1r_2r_4 + \cdots + r_2r_3r_4 + \cdots &= -\frac{a_{n-3}}{a_n} \\
&\vdots \\
s_n = r_1r_2 \dots r_n &= (-1)^n \frac{a_0}{a_n}.
\end{aligned}
$$

Collectively, these results are known as Viète's Theorem. They tell us how the elementary symmetric polynomials $s_1, s_2, \dots, s_n$ can be determined from the coefficients of the polynomial

$$p(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$$

with roots $r_1, r_2, \dots, r_n$.

Exercise 3.1

Find all solutions to the equation $x^3 - 5x^2 - 16x + 80 = 0$ given that the sum of two of its roots is zero.

Solution

Let the roots be a, b and c and assume that $a + b = 0$. From Viète's theorem,

$$\text{sum of all three roots} = -a_2/a_3$$

where a_j is the coefficient of x^j. Therefore, from Viète's theorem,

$$a + b + c = -a_2/a_3 = -(-5)/1 = 5.$$

And as $a + b = 0$ we can deduce that $c = 5$. Because $a + b = 0$ we can write $b = -a$. We can again turn to Viète's theorem to find the remaining roots. From Viète's theorem, we have

$$\text{product of all three roots} = -a_0/a_3 = -80/1 = -80.$$

Therefore

$$-80 = abc = (a)(-a)(5) = 5a^2$$

which tells us that

$$a^2 = 16 \Longrightarrow a = \pm 4.$$

So, the three roots are $4, -4$ and 5.

■

Exercise 3.2

Find all solutions to the equation $x^3 - 12x^2 + 39x - 28 = 0$ given that the roots are in *arithmetic progression*.

Solution

Because the roots are in arithmetic progression, we can express them as $a - d, a, a + d$ for some a and d. From Viète's theorem,

$$\text{sum of all three roots} = -a_2/a_3 = -(-12)/1 = 12.$$

Therefore, from Viète's theorem,

$$12 = (a - d) + a + (a + d) = 3a \Longrightarrow a = 4.$$

Now again using Viète's theorem, we have

$$\text{product of all three roots} = -a_0/a_3 = -(-28)/1 = 28.$$

Therefore

$$28 = (a-d)(a)(a+d) = (4-d)(4)(4+d) = 4(16-d^2).$$

Hence,

$$16 - d^2 = 7 \Longrightarrow d^2 = 9 \Longrightarrow d = \pm 3.$$

So, the three roots are $4-3, 4, 4+3$. That is, $1, 4$ and 7.

■

Exercise 3.3 (Source: Golden Algebra, N.P. Bali)

Find all solutions to the equation $x^4 + 5x^3 - 30x^2 - 40x + 64 = 0$ given that the roots are in *geometric progression*.

Solution

Because the four roots are in geometric progression, we can express them as $\alpha = A, \beta = Ad, \gamma = Ad^2$ and $\delta = Ad^3$ for some A and d. From Viète's theorem,

$$\text{product of all four roots} = a_0/a_4 = 64/1 = \alpha\beta\gamma\delta = A^4 d^6.$$

Therefore,

$$\alpha\delta = \beta\gamma = A^2 d^3 = \pm\sqrt{64} = \pm 8$$

which means we have two cases to consider.

Case 1. $\boldsymbol{\alpha\delta = \beta\gamma = 8}$

By the factor theorem,

$$x^4 + 5x^3 - 30x^2 - 40x + 64 = c(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$$

and because the coefficient of x^4 on the left equals 1 we know $c = 1$. So,

$$\begin{aligned}
x^4 + 5x^3 - 30x^2 - 40x + 64 &= \big((x-\alpha)(x-\delta)\big)\big((x-\beta)(x-\gamma)\big) \\
&= (x^2 - (\alpha+\delta)x + \alpha\delta)(x^2 - (\beta+\gamma)x + \beta\gamma) \\
&= (x^2 - (\alpha+\delta)x + 8)(x^2 - (\beta+\gamma)x + 8) \\
&= (x^2 - px + 8)(x^2 - qx + 8) \\
&= x^4 - (p+q)x^3 + (16 + pq)x^2 - 8(p+q)x + 64
\end{aligned}$$

where $p = \alpha + \delta$ and $q = \beta + \gamma$. Matching coefficients of like terms we have

$$-(p+q)x^3 = 5x^3 \Longrightarrow p+q = -5$$
$$-8(p+q)x = -40x \Longrightarrow p+q = 5.$$

This is impossible, so Case 1 is impossible.

Case 2. $\boldsymbol{\alpha\delta = \beta\gamma = -8}$

Using some of the work from Case 1 we have

$$\begin{aligned} x^4 + 5x^3 - 30x^2 - 40x + 64 &= \big((x-\alpha)(x-\delta)\big)\big((x-\beta)(x-\gamma)\big) \\ &= (x^2 - (\alpha+\delta)x + \alpha\delta)(x^2 - (\beta+\gamma)x + \beta\gamma) \\ &= (x^2 - (\alpha+\delta)x - 8)(x^2 - (\beta+\gamma)x - 8) \\ &= (x^2 - px - 8)(x^2 - qx - 8) \\ &= x^4 - (p+q)x^3 + (-16+pq)x^2 + 8(p+q)x + 64 \end{aligned}$$

where $p = \alpha + \delta$ and $q = \beta + \gamma$. Matching coefficients of like terms we have

$$-(p+q)x^3 = 5x^3 \Longrightarrow p+q = -5$$
$$8(p+q)x = -40x \Longrightarrow p+q = -5$$
$$(-16+pq)x^2 = -30x^2 \Longrightarrow pq = -14.$$

From $p + q = -5$ we have $q = -5 - p$. Therefore

$$-14 = pq = p(-5-p) = -5p - p^2$$

$$p^2 + 5p - 14 = 0$$

$$(p+7)(p-2) = 0 \Longrightarrow p = -7, 2$$

which leads to $(p, q) = (-7,2)$ or $(2,-7)$. Whichever of these we take we are left with

$$x^4 + 5x^3 - 30x^2 - 40x + 64 = (x^2 - px - 8)(x^2 - qx - 8)$$

$$= (x^2 + 7x - 8)(x^2 - 2x - 8)$$

$$= (x+8)(x-1)(x-4)(x+2).$$

So, the four roots are $1, -2, 4, -8$. Notice that these roots do form a geometric progress:

$$1, -2, 4, -8 = A, Ad, Ad^2, Ad^3 = 1, 1(-2), 1(-2)^2, 1(-2)^3.$$

■

Exercise 3.4 (Source: MSHSML 1T951)

The equation $x^3 + bx^2 + 9x + c = 0$ has three positive roots. One of them is 4; the other two are equal. Find c.

Solution

Let the roots be $4, r$ and r. By Viète's theorem we know that

$$4r^2 = -c \text{ and } 4r + 4r + r^2 = 9.$$

Therefore,

$$0 = r^2 + 8r - 9 = (r + 9)(r - 1) \Rightarrow r = -9, r = 1.$$

But we are told that all three roots are positive so $r = 1$. Therefore $c = -4r^2 = -4(1^2) = -4$.

■

Exercise 3.5 (Source: Competition Algebra, Xing Zhou)

Find the sum of all integers m such that the equation $x^2 - mx + m + 1 = 0$ has two positive integer roots.

Solution

Let the two positive integer roots of $x^2 - mx + (m + 1) = 0$ be a and b and without loss of generality take $b \geq a$.

Then by Viète's Theorem,

$$a + b = -(-m)/1 = m$$

$$a \cdot b = m + 1$$

Therefore, $a + b = ab - 1$.

$$ab - a - b = 1$$

$$(a - 1)(b - 1) - 1 = 1$$

$$(a - 1)(b - 1) = 2.$$

The only two positive integers a and b (with $b \geq a$) that satisfy this relation are $a - 1 = 1$ and $b - 1 = 2$. That is $a = 2$ and $b = 3$.

Therefore, $m = a + b = 5$ is the only possible value for m.

■

Exercise 3.6 (Source: Challenge and Thrill of Pre-College Mathematics, Krishnamurthy)

Find the value(s) of c for which the roots a and b of the equation $t^2 - 8t + c = 0$ satisfy the condition that $a^2 + b^2 = 4$ and then find a and b.

Solution

We know from Viète's Theorem that $a + b = -(-8)$ and $ab = c$. Furthermore

$$4 = a^2 + b^2 = (a+b)^2 - 2ab = 8^2 - 2c \Longrightarrow c = 30.$$

Now we can use the quadratic formula to find a and b in the equation $t^2 - 8t + 30$ or we can solve the system of equations $ab = 30$ and $a + b = 8$ for a and b. Using the quadratic formula approach, we see that the roots a and b are

$$\frac{-(-8) \pm \sqrt{(-8)^2 - 4(30)}}{2} = \frac{8 \pm 2\sqrt{-14}}{2} = 4 \pm i\sqrt{14}.$$

■

Exercise 3.7 (Source: MSHSML 2T002)

Suppose $x^3 + Bx^2 + Cx + D = 0$ has roots of $\tan(\alpha)$, $\tan(\beta)$, and $\tan(\gamma)$. Express $\tan(\alpha + \beta + \gamma)$ in terms of B, C, and D.

Solution

From Viète's formulas,

$$\begin{aligned} D &= -\tan(\alpha)\cdot\tan(\beta)\cdot\tan(\gamma) \\ C &= \tan(\alpha)\cdot\tan(\beta) + \tan(\alpha)\cdot\tan(\gamma) + \tan(\beta)\tan(\gamma) \\ B &= -(\tan(\alpha) + \tan(\beta) + \tan(\gamma)). \end{aligned}$$

$$\begin{aligned} \tan[(\alpha+\beta)+\gamma] &= \frac{\tan(\alpha+\beta) + \tan(\gamma)}{1 - \tan(\alpha+\beta)\tan(\gamma)} \\ &= \frac{\left(\frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}\right) + \tan(\gamma)}{1 - \left(\frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}\right)\tan(\gamma)} \cdot \left(\frac{1-\tan(\alpha)\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}\right) \end{aligned}$$

$$= \frac{\tan(\alpha) + \tan(\beta) + \tan(\gamma)\big(1 - \tan(\alpha)\tan(\beta)\big)}{1 - \tan(\alpha)\tan(\beta) - \big(\tan(\alpha) + \tan(\beta)\big)\tan(\gamma)}$$

$$= \frac{-B + D}{1 - C}.$$

■

Exercise 3.8 (Source: MSHSML S7073)

Let r and s be roots of $f(x) = ax^2 + x + a = 0$. Express $\frac{1}{r^2} + \frac{1}{s^2}$ in terms of a.

Solution

If $f(x)$ has roots r and s then $g(x) = (-1)^2 f(\sqrt{x}) f(-\sqrt{x})$ has roots r^2 and s^2.

If $g(x)$ has roots r^2 and s^2 then $h(x) = x^2 g\left(\frac{1}{x}\right)$ has roots $\frac{1}{r^2}$ and $\frac{1}{s^2}$.

So,

$$h(x) = x^2 g\left(\frac{1}{x}\right) = x^2 f\left(\frac{1}{\sqrt{x}}\right) f\left(-\frac{1}{\sqrt{x}}\right)$$

$$= x^2\left(a \cdot \left(\frac{1}{\sqrt{x}}\right)^2 + \left(\frac{1}{\sqrt{x}}\right) + a\right)\left(a \cdot \left(\frac{-1}{\sqrt{x}}\right)^2 + \left(\frac{-1}{\sqrt{x}}\right) + a\right)$$

$$= a^2x^2 + (2a^2 - 1)x + a^2$$

Therefore, by Viète's formulas,

$$\frac{1}{r^2} + \frac{1}{s^2} = -\left(\frac{2a^2 - 1}{a^2}\right) = \frac{1 - 2a^2}{a^2}.$$

Method 2.

The roots of $ax^2 + x + a = 0$ are r and s. Therefore, $rs = a/a$ and $r + s = -1/a$.

$$\frac{1}{r^2} + \frac{1}{s^2} = \frac{r^2 + s^2}{(rs)^2} = \frac{(r + s)^2 - 2rs}{(rs)^2} = \frac{(-1/a)^2 - 2}{1^2} = \frac{1}{a^2} - 2 = \frac{1 - 2a^2}{a^2}.$$

■

Exercise 3.9

Let a and b be the roots of $f(x) = x^2 + 2x - 3 = 0$. Find the value of $(1/a) + (1/b)$.

Solution

Method 1.

We know from Viète's theorem that $s_1 = a + b = -2$ and $s_2 = ab = -3$. Furthermore,

$$\frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab} = \frac{-2}{-3} = \frac{2}{3}.$$

Method 2.

Define $g(x) = x^2 f\left(\frac{1}{x}\right)$ for all $x \neq 0$. We can see by inspection that $x = 0$ is not a root of $f(x)$ so

$$g(x) = 0 \Leftrightarrow f\left(\frac{1}{x}\right) = 0 \Leftrightarrow \frac{1}{x} = a \text{ or } \frac{1}{x} = b \Leftrightarrow x = \frac{1}{a} \text{ or } x = \frac{1}{b}.$$

But

$$g(x) = x^2 f\left(\frac{1}{x}\right) = x^2\left(\left(\frac{1}{x}\right)^2 + 2\left(\frac{1}{x}\right) - 3\right) = x^2\left(\frac{1 + 2x - 3x^2}{x^2}\right) = -3x^2 + 2x + 1$$

and by Viète's Theorem, the sum of the roots of $g(x)$, *i.e.* $\frac{1}{a} + \frac{1}{b}$, will equal $-\left(\frac{2}{-3}\right) = 2/3$ which agrees with our previous answer.

■

3.3 Symmetric Polynomials

Fundamental Theorem of Symmetric Polynomials

Why are the elementary symmetric polynomials special? The answer follows from the *Fundamental Theorem on Symmetric Polynomials*:

Every symmetric polynomial in n variables $r_1, r_2, \dots, r_n$ can be represented in a unique way as a polynomial in the elementary symmetric polynomials $s_1, s_2, \dots, s_n$.

Note: The number of variables in the symmetric polynomial has to be specified or clear from context to make sense of this notation. For example, if $n = 2$ then $s_1 = r_1 + r_2$ and $s_2 = r_1 r_2$. But if $n = 3$ then $s_1 = r_1 + r_2 + r_3$ and $s_2 = r_1 r_2 + r_1 r_3 + r_2 r_3$.

Here are three examples illustrating how a symmetric polynomial in r_1, r_2, r_3 ... can be rewritten in terms of $s_1, s_2, s_3, \dots$.

Again, what the fundamental theorem of symmetric polynomials tells us is that it is ***always*** possible to rewrite a symmetric polynomial in $r_1, r_2, \dots$ in terms of the elementary symmetric polynomials $s_1, s_2, s_3, \dots$.

$n = 2$	$(r_1 - r_2)^2 = (r_1 + r_2)^2 - 4r_1 r_2 = s_1^2 - 4s_2$
$n = 3$	$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{r_2 r_3 + r_1 r_3 + r_1 r_2}{r_1 r_2 r_3} = \frac{s_2}{s_3}$
$n = 3$	$r_1^2 + r_2^2 + r_3^2 = (r_1 + r_2 + r_3)^2 - 2(r_1 r_2 + r_1 r_3 + r_2 r_3) = s_1^2 - 2s_2$

If you are keeping a sharp eye out for details you might notice that the second example is problematic in as much as

$$g(r_1, r_2, r_3) = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$$

is not a symmetric polynomial.

However, it *is* a symmetric rational function because $g(r_1, r_2, r_3) = g(r_1, r_3, r_2) = g(r_2, r_1, r_3) = g(r_2, r_3, r_1) = g(r_3, r_1, r_2) = g(r_3, r_2, r_1)$.

And note we *were* able to rewrite $g(\ )$ as the ratio of two symmetric polynomials and hence apply the fundamental theorem of symmetric polynomials to the numerator and denominator separately.

Every symmetric rational function can be written uniquely as the ratio of two relatively prime symmetric polynomials.

Hence, by applying the Fundamental Theorem of Symmetric Polynomials separately to the numerator and denominator symmetric polynomials we can extend the fundamental theorem to symmetric rational functions.

Fundamental Theorem on Symmetric Rational Functions:

Every symmetric rational function in n variables $r_1, r_2, \dots, r_n$ can be represented in a unique way as a rational function in the elementary symmetric polynomials $s_1, s_2, \dots, s_n$.

Exercise 3.10

Let a, b and c be the three roots of $f(x) = 7x^3 - x^2 + x - 5 = 0$. Find the *exact* value of $ab^2c^2 + a^2bc^2 + a^2b^2c$.

Solution

You *could* start by checking for a rational root (there aren't any) but the big clue you be looking for is that $ab^2c^2 + a^2bc^2 + a^2b^2c$ is symmetric in the variables a, b, and c.

That means that $ab^2c^2 + a^2bc^2 + a^2b^2c$ can be rewritten in terms of the elementary symmetric polynomials and the exact value of each elementary symmetric polynomial can be read off from the coefficients of $7x^3 - x^2 + x - 5$.

The key step is to rewrite $ab^2c^2 + a^2bc^2 + a^2b^2c$ in terms of the elementary symmetric polynomials

$$s_1 = a + b + c$$

$$s_2 = ab + ac + bc$$

$$s_3 = abc.$$

Did you see that each term has a factor of abc? So, simplify the expression by factoring abc out.

$$ab^2c^2 + a^2bc^2 + a^2b^2c = abc(bc + ac + ab).$$

Aha! That's just $s_3 \cdot s_2$. Now remember what Viète's Theorem tells you about the values of the elementary symmetric polynomials.

If a, b and c are the three roots of $C_3x^3 + C_2x^2 + C_1x + C_0 = 0$, then

$$s_1 = -\frac{C_2}{C_3},\ s_2 = \frac{C_1}{C_3},\ s_3 = -\frac{C_0}{C_3}.$$

In this problem our polynomial is $7x^3 - x^2 + x - 5$. So $C_3 = 7, C_2 = -1, C_1 - 1, C_0 = -5$.

Therefore,

$$ab^2c^2 + a^2bc^2 + a^2b^2c = abc(bc + ac + ab)$$

$$= s_3 \cdot s_2 = \left(-\frac{C_0}{C_3}\right)\left(\frac{C_1}{C_3}\right)$$

$$= \left(-\frac{-5}{7}\right)\left(\frac{1}{7}\right) = \frac{5}{49}.$$

■

Exercise 3.11

Let a, b and c be the three roots of $f(x) = 7x^3 - x^2 + x - 5 = 0$. Find the *exact* value of

$$g(a,b,c) = \frac{a}{b} + \frac{a}{c} + \frac{b}{a} + \frac{b}{c} + \frac{c}{a} + \frac{c}{b}.$$

Solution

Is $g(a,b,c)$ symmetric in the variables a, b, and c? That is, does $g(a,b,c) = g(a,c,b) = g(b,a,c) = g(b,c,a) = g(c,a,b) = g(c,b,a)$? Yes.

What does that tell us? The fundamental theorem of symmetric rational functions assures us that there is a rational function of the elementary symmetric polynomials, call it $h(\ )$, such that

$$g(a,b,c) = h(s_1, s_2, s_3)$$

where $s_1 = a + b + c,\ s_2 = ab + ac + bc,\ s_3 = abc$.

A good place to start is to rewrite $g(a,b,c)$ with a common denominator.

$$\frac{c}{b} + \frac{c}{a} + \frac{b}{c} + \frac{a}{c} + \frac{b}{a} + \frac{a}{b} = \frac{ac^2 + bc^2 + ab^2 + a^2b + b^2c + a^2c}{abc}$$

The numerator looks like it must contain the product $(a + b + c)(ab + ac + bc)$.

$$(a + b + c)(ab + ac + bc) = a^2b + a^2c + abc + ab^2 + abc + b^2c + abc + a^2c + bc^2$$

$$= (ac^2 + bc^2 + ab^2 + a^2b + b^2c + a^2c) + 3abc.$$

Now we can see that

$$ac^2 + bc^2 + ab^2 + a^2b + b^2c + a^2c = (a + b + c)(ab + ac + bc) - 3abc.$$

So,

$$\begin{aligned} g(a,b,c) &= \frac{a}{b} + \frac{a}{c} + \frac{b}{a} + \frac{b}{c} + \frac{c}{a} + \frac{c}{b} \\ &= \frac{ac^2 + bc^2 + ab^2 + a^2b + b^2c + a^2c}{abc} \\ &= \frac{(a + b + c)(ab + ac + bc) - 3abc}{abc} \\ &= \frac{s_1 \cdot s_2 - 3s_3}{s_3}. \end{aligned}$$

Recalling that

$$s_1 = -\frac{C_2}{C_3},\ s_2 = \frac{C_1}{C_3},\ s_3 = -\frac{C_0}{C_3}$$

for the general cubic $C_3x^3 + C_2x^2 + C_1x + C_0$ we have for our polynomial $7x^3 - x^2 + x - 5$ that

$$s_1 = -\frac{(-1)}{7},\ s_2 = \frac{(-1)}{7},\ s_3 = -\frac{(-5)}{7}.$$

In conclusion, if a, b and c are the three roots of $f(x) = 7x^3 - x^2 + x - 5 = 0$ then the *exact* value of

$$\begin{aligned} g(a,b,c) &= \frac{a}{b} + \frac{a}{c} + \frac{b}{a} + \frac{b}{c} + \frac{c}{a} + \frac{c}{b} \\ &= \frac{s_1 \cdot s_2 - 3s_3}{s_3} = \frac{\left(\frac{1}{7}\right)\left(\frac{-1}{7}\right) - 3\left(\frac{5}{7}\right)}{\frac{5}{7}} \\ &= -\frac{106}{35}. \end{aligned}$$

■

3.3.1 Gauss's Algorithm

In the previous two exercises we were able to find, without much trouble, a way to express the given symmetric polynomial or symmetric rational function of the roots of $f(x) = 0$ as either a polynomial or rational function of the elementary symmetric polynomials.

In the first exercise we found

$$ab^2c^2 + a^2bc^2 + a^2b^2c = s_2 \cdot s_3$$

and in the second exercise we found

$$\frac{a}{b} + \frac{a}{c} + \frac{b}{a} + \frac{b}{c} + \frac{c}{a} + \frac{c}{b} = \frac{s_1 \cdot s_2 - 3s_3}{s_3}.$$

But trying to find an expression just involving the elementary symmetric polynomials can get problematic very quickly.

Can you see how to express

$$-abc^2 + c^2 - ab^2c - a^2bc + bc + ac + b^2 + ab + a^2$$

in terms of s_1, s_2 and s_3 in the case of three variables?

How about expressing

$$\begin{gathered} abc^2d^2 + ab^2cd^2 + a^2bcd^2 - d^2 + ab^2c^2d + a^2bc^2d + a^2b^2cd + bcd + acd - 2cd + abd \\ - 2bd - 2ad - c^2 + abc - 2bc - 2ac - b^2 - 2ab - a^2 \end{gathered}$$

in terms of s_1, s_2, s_3 and s_4 in the case of four variables?

The answers are $s_1^2 - s_1s_3 - s_2$ (in three variables) and $s_3 - s_1^2 + s_2s_4$ (in four variables) respectively. But you might end up using up the entire time of the test trying to find these answers.

We will give two algorithms to address this difficulty. The first is due to the German mathematician Carl Friedrich Gauss (1777-1855). The second algorithm, which we will cover in the next section is due to the English mathematician Isaac Newton (1642-1726). It is the opinion of many that they are the two most influential mathematicians of all time.

But before we can explain Gauss's algorithm, we need to define the term **lexicographic ordering** of monomials. It is most common to see the term in print shorten to just **lex order**.

Lex Order

For univariate polynomials it is straight forward to rank monomials by their degree. Clearly x^5 has a higher degree than x^2. But how do we extend this to multivariate monomials? Does x^4y^2 have a higher degree than x^3y^4?

So, using total degree as our metric, x^3y^4 (total degree 7) has a higher degree than x^4y^2 (total degree 6).

Another way of ranking multivariate monomials is by lex order.

We will define the monomial $k_1x^{a_1}y^{b_1}z^{c_1}$ to have a **higher lex order** than the monomial $k_2x^{a_2}y^{b_2}z^{c_2}$ if

$$a_1 > a_2$$

or

$$a_1 = a_2 \text{ and } b_1 > b_2$$

or

$$a_1 = a_2 \text{ and } b_1 = b_2 \text{ and } c_1 > c_2.$$

Here are a few examples illustrating lex ordering of monomials $k_1x^{a_1}y^{b_1}z^{c_1}$ and $k_2x^{a_2}y^{b_2}z^{c_2}$.

$2x^2yz^3$ has a higher lex order than $5xy^2z^5$ because $a_1 > a_2$.

$5xy^3z^5$ has a lex order than $3xy^2z^3$ because $a_1 = a_2$ and $b_1 > b_2$.

$-2x^3y^2z^3$ has a lex order than $4x^3y^2z^2$ because $a_1 = a_2$, $b_1 = b_2$ and $c_1 > c_2$.

Notice that the coefficients of these monomials plays no role in deciding the lex ordering.

Now that we have defined lexicographic (lex) ordering we are ready to present Gauss's Algorithm for expressing any symmetric polynomial in terms of the elementary symmetric polynomials. We will illustrate the algorithm by showing how to express

$$f(r_1, r_2, r_3) = r_1^2 r_2^2 + r_1^2 r_3^2 + r_2^2 r_3^2$$

as a function of the elementary symmetric polynomials

$$s_1 = r_1 + r_2 + r_3$$

$$s_2 = r_1 r_2 + r_1 r_3 + r_2 r_3$$

$$s_3 = r_1 r_2 r_3.$$

Gauss's Algorithm

Step 1. Find the term in the symmetric polynomial $f(r_1, r_2, \dots, r_n)$ with the highest lex order degree.

The term in

$$f(r_1, r_2, r_3) = r_1^2 r_2^2 + r_1^2 r_3^2 + r_2^2 r_3^2$$

with the highest lex order term is $r_1^2 r_2^2$.

Step 2. If the highest order term in Step 1 is $k r_1^a r_2^b r_3^c$ then construct the new polynomial

$$g_1(r_1, r_2, r_3) = f(r_1, r_2, r_3) - k s_1^{a-b} s_2^{b-c} s_3^c.$$

The highest lex order term is $r_1^2 r_2^2 = r_1^a r_2^b r_3^c$ with $k = 1, a = b = 2$ and $c = 0$. Therefore,

$$\begin{aligned} g_1(r_1, r_2, r_3) &= f(r_1, r_2, r_3) - k s_1^{a-b} s_2^{b-c} s_3^c \\ &= (r_1^2 r_2^2 + r_1^2 r_3^2 + r_2^2 r_3^2) - s_1^0 s_2^{2-0} s_3^0 \\ &= (r_1^2 r_2^2 + r_1^2 r_3^2 + r_2^2 r_3^2) - s_2^2 \\ &= (r_1^2 r_2^2 + r_1^2 r_3^2 + r_2^2 r_3^2) - (r_1 r_2 + r_1 r_3 + r_2 r_3)^2 \end{aligned}$$

$$
\begin{aligned}
&= (r_1^2r_2^2 + r_1^2r_3^2 + r_2^2r_3^2) \\
&\quad -(r_1^2r_2^2 + r_1^2r_3^2 + r_2^2r_3^2 + 2r_1^2r_2r_3 + 2r_1r_2^2r_3 + 2r_1r_2r_3^2) \\
&= -2r_1^2r_2r_3 - 2r_1r_2^2r_3 - 2r_1r_2r_3^2.
\end{aligned}
$$

Step 3. Repeat Steps 1 and 2 but on the new polynomial formed in Step 2. Continue in this loop until the new polynomial formed in Step 2 equals 0.

The highest order term in $g_1(r_1, r_2, r_3) = -2r_1^2r_2r_3 - 2r_1r_2^2r_3 - 2r_1r_2r_3^2$ is

$$-2r_1^2\, r_2\, r_3 = -2r_1^a\, r_2^b\, r_3^c \text{ with } k = -2, a = 2, b = 1 \text{ and } c = 1.$$

Therefore,

$$
\begin{aligned}
g_2(r_1, r_2, r_3) &= g_1(r_1, r_2, r_3) - ks_1^{a-b}s_2^{b-c}s_3^c \\
&= (-2r_1^2r_2r_3 - 2r_1r_2^2r_3 - 2r_1r_2r_3^2) - (-2)s_1^{2-1}s_2^{1-1}s_3^1 \\
&= (-2r_1^2r_2r_3 - 2r_1r_2^2r_3 - 2r_1r_2r_3^2) - (-2)s_1^1s_2^0s_3^1 \\
&= (-2r_1^2r_2r_3 - 2r_1r_2^2r_3 - 2r_1r_2r_3^2) + 2s_1s_3 \\
&= (-2r_1^2r_2r_3 - 2r_1r_2^2r_3 - 2r_1r_2r_3^2) \\
&\qquad + 2(r_1 + r_2 + r_3)(r_1r_2r_3) \\
&= 0.
\end{aligned}
$$

Our newly formed polynomial is equal to 0 so we can move on to Step 4.

Step 4. Work backwards (from the bottom up) in your steps to find $f(r_1, r_2, r_3)$ in terms of the elementary symmetric polynomials s_1, s_2, s_3.

$$
\begin{aligned}
0 &= g_2(r_1, r_2, r_3) \\
&= g_1(r_1, r_2, r_3) + 2s_1s_3 \\
&= \big(f(r_1, r_2, r_3) - s_2^2\big) + 2s_1s_3.
\end{aligned}
$$

∴

$$f(r_1, r_2, r_3) = r_1^2r_2^2 + r_1^2r_3^2 + r_2^2r_3^2 = s_2^2 - 2s_1s_2.$$

■

Exercise 3.12

Use Gauss's algorithm to express the symmetric polynomial $f(r_1, r_2, r_3) = r_1^3 + r_2^3 + r_3^3$ in terms of elementary symmetric polynomials.

Solution

Step 1. Find the term in the symmetric polynomial $f(r_1, r_2, \dots, r_n)$ with the highest lex order degree.

The term in $f(r_1, r_2, r_3) = r_1^3 + r_2^3 + r_3^3$ with the highest lex order is r_1^3

Step 2. If the highest order term in Step 1 is $kr_1^a r_2^b r_3^c$ then construct the new polynomial $$g_1(r_1, r_2, r_3) = f(r_1, r_2, r_3) - ks_1^{a-b} s_2^{b-c} s_3^c.$$

The highest lex order term is $r_1^3 = kr_1^a r_2^b r_3^c$ with $k = 1$, $a = 3, b = 0$ and $c = 0$. Therefore,

$$\begin{aligned}
g_1(r_1, r_2, r_3) &= f(r_1, r_2, r_3) - ks_1^{a-b} s_2^{b-c} s_3^c \\
&= (r_1^3 + r_2^3 + r_3^3) - s_1^3 s_2^{0-0} s_3^0 \\
&= (r_1^3 + r_2^3 + r_3^3) - s_1^3 \\
&= (r_1^3 + r_2^3 + r_3^3) - (r_1 + r_2 + r_3)^3 \\
&= (r_1^3 + r_2^3 + r_3^3) \\
&\qquad -(r_1^3 + r_2^3 + r_3^3 + 3r_1^2 r_2 + 3r_1^2 r_3 + 3r_1 r_2^2 + 3r_2^2 r_3 \\
&\qquad\qquad + 3r_1 r_3^2 + 3r_2 r_3^2 + 6r_1 r_2 r_3) \\
&= -3r_1^2 r_2 - 3r_1^2 r_3 - 3r_1 r_2^2 - 3r_2^2 r_3 - 3r_1 r_3^2 - 3r_2 r_3^2 \\
&\qquad - 6r_1 r_2 r_3
\end{aligned}$$

Step 3. Repeat Steps 1 and 2 but on the new polynomial formed in Step 2. Continue in this loop until the new polynomial formed in Step 2 equals 0.

The highest lex order term in $g_1(r_1, r_2, r_3) = -3r_1^2r_2 - 3r_1^2r_3 - 3r_1r_2^2 - 3r_2^2r_3 - 3r_1r_3^2 - 3r_2r_3^2 - 6r_1r_2r_3$ is

$$-3r_1^2r_2 = -3r_1^a\, r_2^b\, r_3^c \text{ with } k = -3, a = 2, b = 1 \text{ and } c = 0.$$

Therefore,

$$\begin{aligned}
g_2(r_1, r_2, r_3) &= g_1(r_1, r_2, r_3) - ks_1^{a-b}s_2^{b-c}s_3^c \\
&= (-3r_1^2r_2 - 3r_1^2r_3 - 3r_1r_2^2 - 3r_2^2r_3 - 3r_1r_3^2 - 3r_2r_3^2 \\
&\qquad - 6r_1r_2r_3) - (-3)s_1^{2-1}s_2^{1-0}s_3^0 \\
&= (-3r_1^2r_2 - 3r_1^2r_3 - 3r_1r_2^2 - 3r_2^2r_3 - 3r_1r_3^2 - 3r_2r_3^2 \\
&\qquad - 6r_1r_2r_3) - (-3)s_1^1s_2^1s_3^0 \\
&= (-3r_1^2r_2 - 3r_1^2r_3 - 3r_1r_2^2 - 3r_2^2r_3 - 3r_1r_3^2 - 3r_2r_3^2 \\
&\qquad - 6r_1r_2r_3) + 3s_1s_2 \\
&= (-3r_1^2r_2 - 3r_1^2r_3 - 3r_1r_2^2 - 3r_2^2r_3 - 3r_1r_3^2 - 3r_2r_3^2 \\
&\qquad - 6r_1r_2r_3) \\
&\qquad + 3(r_1 + r_2 + r_3)(r_1r_2 + r_1r_3 + r_2r_3) \\
&= (-3r_1^2r_2 - 3r_1^2r_3 - 3r_1r_2^2 - 3r_2^2r_3 - 3r_1r_3^2 - 3r_2r_3^2 \\
&\qquad - 6r_1r_2r_3) \\
&\qquad +3(r_1^2r_2 + r_1r_2^2 + r_1r_2r_3 + r_1^2r_3 + r_1r_2r_3 + r_1r_3^2 \\
&\qquad +r_1r_2r_3 + r_2^2r_3 + r_2r_3^2) \\
&= 3r_1r_2r_3
\end{aligned}$$

Iterating.

The highest lex order term in $g_2(r_1, r_2, r_3) = 3r_1r_2r_3$ is

$$3r_1r_2r_3 = 3r_1^a\, r_2^b\, r_3^c \text{ with } k = 3, a = 1, b = 1 \text{ and } c = 1.$$

Therefore,

$$\begin{aligned}
g_3(r_1, r_2, r_3) &= g_2(r_1, r_2, r_3) - ks_1^{a-b}s_2^{b-c}s_3^c \\
&= 3r_1r_2r_3 - 3s_1^{1-1}s_2^{1-1}s_3^1
\end{aligned}$$

$$= \quad 3r_1r_2r_3 - 3s_1^0s_2^0s_3^1$$

$$= \quad 3r_1r_2r_3 - 3s_3$$

$$= \quad 3r_1r_2r_3 - 3r_1r_2r_3$$

$$= \quad 0.$$

Our newly formed polynomial is equal to 0 so we can move on to Step 4.

Step 4. Work backwards (from the bottom up) in your steps to find $f(r_1, r_2, r_3)$ in terms of the elementary symmetric polynomials s_1, s_2, s_3.

$$\begin{aligned} 0 &= g_3(r_1, r_2, r_3) \\ &= g_2(r_1, r_2, r_3) - 3s_3 \\ &= g_1(r_1, r_2, r_3) + 3s_1s_2 - 3s_3 \\ &= f(r_1, r_2, r_3) - s_1^3 + 3s_1s_2 - 3s_3. \end{aligned}$$

∴

$$f(r_1, r_2, r_3) = r_1^3 + r_2^3 + r_3^3 = s_1^3 - 3s_1s_2 + 3s_3.$$

■

Exercise 3.13

Use Gauss's algorithm to express the symmetric polynomial $f(a, b, c) = a^3b^2 + a^3c^2 + b^3a^2 + b^3c^2 + c^3a^2 + c^3b^2$ in terms of elementary symmetric polynomials.

Solution

Step 1. Find the term in the symmetric polynomial $f(r_1, r_2, \ldots, r_n)$ with the highest lex order degree.

The term in $f(a, b, c) = a^3b^2 + a^3c^2 + b^3a^2 + b^3c^2 + c^3a^2 + c^3b^2$ with the highest lex order is a^3b^2.

Step 2. If the highest order term in Step 1 is $kr_1^a r_2^b r_3^c$ then construct the new polynomial

$$g_1(r_1, r_2, r_3) = f(r_1, r_2, r_3) - ks_1^{a-b} s_2^{b-c} s_3^c.$$

The highest lex order term is $a^3b^2 = ka^{\alpha_1}b^{\alpha_2}c^{\alpha_3}$ with $k = 1,\ \alpha_1 = 3, \alpha_2 = 2$ and $\alpha_3 = 0$. Therefore,

$$\begin{aligned} g_1(a,b,c) &= f(a,b,c) - ks_1^{\alpha_1-\alpha_2} s_2^{\alpha_2-\alpha_3} s_3^{\alpha_3} \\ &= (a^3b^2 + a^3c^2 + b^3a^2 + b^3c^2 + c^3a^2 + c^3b^2) - s_1^{3-2}s_2^{2-0}s_3^0 \\ &= (a^3b^2 + a^3c^2 + b^3a^2 + b^3c^2 + c^3a^2 + c^3b^2) - (a+b+c)(ab+ac+bc)^2 \\ &= -2abc^3 - 5ab^2c^2 - 5a^2bc^2 - 2ab^3c - 5a^2b^2c - 2a^3bc \end{aligned}$$

Step 3. Repeat Steps 1 and 2 but on the new polynomial formed in Step 2. Continue in this loop until the new polynomial formed in Step 2 equals 0.

The highest lex order term in

$$g_1(a,b,c) = -2abc^3 - 5ab^2c^2 - 5a^2bc^2 - 2ab^3c - 5a^2b^2c - 2a^3bc$$

is

$$-2a^3bc = -2a^{\alpha_1}b^{\alpha_2}c^{\alpha_3} \text{ with } k = -2, \alpha_1 = 3, \alpha_2 = 1 \text{ and } \alpha_3 = 1.$$

Therefore,

$$\begin{aligned} &g_2(a,b,c) \\ &\quad = g_1(a,b,c) - ks_1^{\alpha_1-\alpha_2} s_2^{\alpha_2-\alpha_3} s_3^{\alpha_3} \\ &\quad = (-2abc^3 - 5ab^2c^2 - 5a^2bc^2 - 2ab^3c - 5a^2b^2c - 2a^3bc) - (-2)s_1^{3-1}s_2^{1-1}s_3^1 \\ &\quad = (-2abc^3 - 5ab^2c^2 - 5a^2bc^2 - 2ab^3c - 5a^2b^2c - 2a^3bc) + 2s_1^2s_3 \\ &\quad = (-2abc^3 - 5ab^2c^2 - 5a^2bc^2 - 2ab^3c - 5a^2b^2c - 2a^3bc) + 2(a+b+c)^2(abc) \\ &\quad = -ab^2c^2 - a^2bc^2 - a^2b^2c \end{aligned}$$

Iterating.

The highest lex order term in

$$g_2(a,b,c) = -ab^2c^2 - a^2bc^2 - a^2b^2c$$

is

$$-a^2b^2c = -a^{\alpha_1}b^{\alpha_2}c^{\alpha_3} \text{ with } k = -1, \alpha_1 = 2, \alpha_2 = 2 \text{ and } \alpha_3 = 1.$$

Therefore,

$$\begin{aligned} g_3(a,b,c) &= g_2(a,b,c) - ks_1^{2-2}s_2^{2-1}s_3^1 \\ &= g_2(a,b,c) - (-1)s_2s_3 \\ &= g_2(a,b,c) + (ab+ac+bc)(abc) \\ &= (-ab^2c^2 - a^2bc^2 - a^2b^2c) + (a^2b^2c + a^2bc^2 + ab^2c^2) \\ &= 0. \end{aligned}$$

Our newly formed polynomial is equal to 0 so we can move on to Step 4.

Step 4. Work backwards (from the bottom up) in your steps to find $f(r_1, r_2, r_3)$ in terms of the elementary symmetric polynomials s_1, s_2, s_3.

$$\begin{aligned} 0 &= g_3(a,b,c) \\ &= g_2(a,b,c) + s_2s_3 \\ &= g_1(a,b,c) + 2s_1^2s_3 + s_2s_3 \\ &= f(a,b,c) - s_1s_2^2 + 2s_1^2s_3 + s_2s_3. \end{aligned}$$

∴

$$f(a,b,c) = a^3b^2 + a^3c^2 + b^3a^2 + b^3c^2 + c^3a^2 + c^3b^2 = s_1s_2^2 - 2s_1^2s_3 - s_2s_3.$$

■

Exercise 3.14 (Source: 2019 AMC 10A Problem 24)

Let p, q, and r be the distinct roots of the polynomial equation $x^3 - 22x^2 + 80x - 67 = 0$. It is given that there exist real numbers A, B, and C such that

$$\frac{1}{s^3 - 22s^2 + 80s - 67} = \frac{A}{s-p} + \frac{B}{s-q} + \frac{C}{s-r}$$

for all $s \notin \{p, q, r\}$. What is $\frac{1}{A} + \frac{1}{B} + \frac{1}{C}$?

Solution

We are given that p, q and r are the roots of $x^3 - 22x^2 + 80x - 67 = 0$. Thus,

$$x^3 - 22x^2 + 80x - 67 = (x-p)(x-q)(x-r) \text{ for all } x.$$

Therefore,

$$\frac{(x-p)(x-q)(x-r)}{x^3 - 22x^2 + 80x - 67} = 1 \text{ for all } x.$$

We are given that

$$\frac{1}{s^3 - 22s^2 + 80s - 67} = \frac{A}{s-p} + \frac{B}{s-q} + \frac{C}{s-r}$$

is an identity (true for all s). Therefore, if we multiply both sides by of this identity by $(s-p)(s-q)(s-r)$ it will remain an identity. That is, for all s,

$$\frac{(s-p)(s-q)(s-r)}{s^3 - 22s^2 + 80s - 67} = \left(\frac{A}{s-p} + \frac{B}{s-q} + \frac{C}{s-r}\right)(s-p)(s-q)(s-r).$$

But we already know that the left-hand side equals 1 for all s. Simplifying the right-hand side we have, for all s, that

$$1 = A(s-q)(s-r) + B(s-p)(s-r) + C(s-p)(s-q).$$

for all s (*i.e.* this is an identity). Therefore, this identity holds when $s = p$. If we plug in $s = p$ this identity reduces to

$$1 = A(p-q)(p-r) \Longrightarrow \frac{1}{A} = (p-q)(p-r).$$

In the same way we can plug in $s = q$ and $s = r$ into this identity to find that

$$\frac{1}{B} = (q-p)(q-r) \text{ and } \frac{1}{C} = (r-p)(r-q).$$

Therefore,

$$\frac{1}{A} + \frac{1}{B} + \frac{1}{C} = (p-q)(p-r) + (q-p)(q-r) + (r-p)(r-q)$$

$$= (p^2 + q^2 + r^2) - (pq + qr + pr)$$

where p, q and r are the roots of $x^3 - 22x^2 + 80x - 67 = 0$. Notice that

$$f(p, q, r) = (p^2 + q^2 + r^2) - (pq + qr + pr)$$

is a symmetric function of p, q and r. This means that we can solve for this function in terms of the coefficients of $x^3 - 22x^2 + 80x - 67$. I think you might be able to see what the solution is by inspection. Remember from Viète's theorem that

$$s_1 = p + q + r = -(-22) = 22$$

$$s_2 = pq + qr + pr = 80$$

and

$$s_3 = pqr = -(-67) = 67.$$

But as practice using Gauss's Algorithm let's go through the algorithm to find the solution.

Gauss's Algorithm

Step 1. Find the term in the symmetric polynomial $f(p, q, r)$ with the highest lex order degree.

The term in

$$f(p, q, r) = (p^2 + q^2 + r^2) - (pq + qr + pr)$$

with the highest lex order term is p^2.

Step 2. If the highest order term in Step 1 is $kp^a q^b r^c$ then construct the new polynomial $$g_1(p, q, r) = f(p, q, r) - ks_1^{a-b} s_2^{b-c} s_3^c.$$

The highest order term is $p^2 = kp^a q^b r^c$ with $a = 2, b = 0, c = 0$ and $k = 1$. Therefore,

$$g_1(p, q, r) \quad = \quad f(p, q, r) - s_1^{2-0} s_2^{0-0} s_3^0$$

$$
\begin{aligned}
&= f(p,q,r) - s_1^2 \\
&= \left((p^2+q^2+r^2)-(pq+qr+pr)\right)-(p+q+r)^2 \\
&= \begin{array}{l}\left((p^2+q^2+r^2)-(pq+qr+pr)\right) \\ \quad -(p^2+q^2+r^2+2pq+2pr+2qr)\end{array} \\
&= -3pq-3pr-3qr
\end{aligned}
$$

The highest order term is $p^2 = kp^aq^br^c$ with $a=2$, $b=0, c=0$ and $k=1$. Therefore,

$$
\begin{aligned}
g_1(p,q,r) &= f(p,q,r) - s_1^{2-0}s_2^{0-0}s_3^0 \\
&= f(p,q,r) - s_1^2 \\
&= \left((p^2+q^2+r^2)-(pq+qr+pr)\right)-(p+q+r)^2 \\
&= \begin{array}{l}\left((p^2+q^2+r^2)-(pq+qr+pr)\right) \\ \quad -(p^2+q^2+r^2+2pq+2pr+2qr)\end{array}
\end{aligned}
$$

Step 3.	Repeat Steps 1 and 2 but on the new polynomial formed in Step 2. Continue in this loop until the new polynomial formed in Step 2 equals 0.

The highest order term in

$$g_1(p,q,r) = -3pq-3pr-3qr$$

is

$$-3pq = kp^aq^br^c \text{ with } a=1, b=1, c=0 \text{ and } k=-3.$$

Therefore,

$$
\begin{aligned}
g_2(p,q,r) &= g_1(p,q,r) - ks_1^{a-b}s_2^{b-c}s_3^c \\
&= g_1(p,q,r) - (-3)s_1^{1-1}s_2^{1-0}s_3^0 \\
&= g_1(p,q,r) + 3s_2 \\
&= (-3pq-3pr-3qr) + 3(pq+pr+qr)
\end{aligned}
$$

$$= \quad 0.$$

Our newly formed polynomial is equal to 0 so the algorithm stops.

What have we found out by using this algorithm?

$$g_1(p,q,r) = f(p,q,r) - s_1^2$$

and

$$g_2(p,q,r) = g_1(p,q,r) + 3s_2 = 0.$$

Let's combine these two results.

$$\begin{aligned} 0 &= g_1(p,q,r) + 3s_2 \\ &= (f(p,q,r) - s_1^2) + 3s_2. \end{aligned}$$

Solving for $f(p,q,r)$ we see that

$$(p^2 + q^2 + r^2) - (pq + qr + pr) = f(p,q,r) = s_1^2 - 3s_2$$

where

$$\begin{aligned} s_1 &= p + q + r = -(-22) = 22 \\ s_2 &= pq + qr + pr = 80 \\ s_3 &= pqr = -(-67) = 67. \end{aligned}$$

Therefore,

$$\begin{aligned} (p^2 + q^2 + r^2) - (pq + qr + pr) &= s_1^2 - 3s_2 \\ &= (22)^2 - 3(80) \\ &= 244. \end{aligned}$$

■

3.3.2 Newton's Sums Algorithm

The Newton's algorithm is a recursive algorithm for expressing symmetric polynomials of the form $r_1^k + r_2^k + \cdots + r_n^k$ directly in terms of the coefficients of the polynomial equation $f(x) = 0$ with roots $r_1, r_2, \dots, r_n$.

Define

$$PS_1 = r_1 + r_2 + r_3 + \cdots + r_n$$
$$PS_2 = r_1^2 + r_2^2 + r_3^2 + \cdots + r_n^2$$
$$PS_3 = r_1^3 + r_2^3 + r_3^3 + \cdots + r_n^3$$
$$\vdots$$

We call $PS_k = r_1^k + r_2^k + r_3^k + \cdots + r_n^k$ the k^{th} **power sum** in n variables. It is clearly a symmetric polynomial. Therefore, by the fundamental theorem of symmetric polynomials it is possible for every k to express PS_k in terms of the elementary symmetric polynomials.

Looking back at Exercise 3.12 you can see how we used Gauss's algorithm to express the 3rd power sum in three variables in terms of the elementary symmetric polynomials s_1, s_2, s_3 in three variables.

$$r_1^3 + r_2^3 + r_3^3 = s_1^3 - 3s_1s_2 + 3s_3$$

In principle we can use Gauss's algorithm to find the k^{th} power sum in n variables for all k and all n. However, for k and/or n larger than 3 Gauss's algorithm is too time consuming. Newton's Sums Algorithm is a better alternative.

Understand that Newton's Sums Algorithm is a better alternative than Gauss's algorithm but is a very limited use alternative. Newton's Sums Algorithm only can be used for expressing the k^{th} power sums in terms of the symmetric polynomials while Gauss's algorithm can be used for any symmetric polynomial.

The following recursive results are called ***Newton's Sum Identities***. The essence of Newton's Sums Algorithm is just to iteratively use Newton's Identities to build up to the power sum you are interested in. We will illustrate this through the exercises.

Newton's Sum Identities

$$PS_1 = s_1 \tag{EqNS-1}$$

$$PS_k = \sum_{i=1}^{k-1} (-1)^{k-1+i}\, s_{k-i} PS_i + (-1)^{k-1} k s_k, \quad 2 \leq k \leq n \tag{EqNS-2}$$

$$PS_k = \sum_{i=k-n}^{k-1} (-1)^{k-1+i}\, s_{k-i} PS_i, \quad k > n \tag{EqNS-3}$$

$$s_k = \frac{1}{k} \cdot \sum_{i=1}^{k} (-1)^{i-1}\, s_{k-i} PS_i, \;\; 1 \leq k \leq n \text{ and where } \; s_0 = 1 \tag{EqNS-4}$$

Exercise 3.15

Repeat Exercise 3.12 but this time use Newton's algorithm instead of Gauss's algorithm to express the symmetric polynomial $f(r_1, r_2, r_3) = r_1^3 + r_2^3 + r_3^3$ in terms of elementary symmetric polynomials.

Solution

We begin with

$$PS_1 = s_1.$$

From Newton's identity (EqNS-2)

$$PS_k = \sum_{i=1}^{k-1} (-1)^{k-1+i}\, s_{k-i} PS_i + (-1)^{k-1} k s_k, \quad 2 \leq k \leq n$$

we find

$$\begin{aligned} PS_2 &= (-1)^{2-1+1} s_{2-1} PS_1 + (-1)^{2-1} \cdot 2 s_2 \\ &= s_1^2 - 2 s_2 \end{aligned}$$

and

$$PS_3 = \big((-1)^3 s_2 PS_1 + (-1)^4 s_1 PS_2\big) + (-1)^{3-1} \cdot 3 s_3$$

$$= -s_2 s_1 + s_1(s_1^2 - 2s_2) + 3s_3$$
$$= s_1^2 - 3s_1 s_2 + 3s_3.$$

■

Exercise 3.16

Let a and b be the roots of $f(x) = x^2 + 2x - 3 = 0$. Find the value of $a^4 + b^4$ using Newton's Sums Algorithm.

Solution

The problem is asking for PS_4 with $n = 2$ variables.

Viète's Theorem tells us that for the general quadratic $f(x) = a_2 x^2 + a_1 x + a_0$,

$$s_1 = -a_1/a_2 \ \text{ and } \ s_2 = a_0/a_2$$

So, in this problem we know $s_1 = -2$ and $s_2 = -3$. We know $PS_1 = s_1 = -2$ and we can use (EqNS-2) to find PS_2 but we need to use (EqNS-3) to find PS_3 and PS_4. From (EqNS-1) we have

$$\begin{aligned} PS_2 &= (-1)^{2-1+1} s_{2-1} PS_1 + (-1)^{2-1} 2s_2 \\ &= s_1^2 - 2s_2 \\ &= (-2)^2 - 2(-3) \\ &= 10. \end{aligned}$$

And from (EqNS-3) we have

$$\begin{aligned} PS_3 &= (-1)^{3-1+1} s_{3-1} PS_1 + (-1)^{3-1+2} s_{3-2} PS_2 \\ &= -s_1 s_2 + s_1(10) \\ &= -(-2)(-3) + (-2)(10) \\ &= -26 \end{aligned}$$

and

$$\begin{aligned} PS_4 &= (-1)^{4-1+2} s_{4-2} PS_2 + (-1)^{4-1+3} s_{4-3} PS_3 \\ &= -s_2 PS_2 + s_1 PS_3 \\ &= -(-3)(10) + (-2)(-26) \\ &= 82. \end{aligned}$$

■

Exercise 3.17 (Source: MSHSML ST861)

Given that

$$\begin{aligned} a+b+c &= 1 \\ a^2+b^2+c^2 &= 2 \\ a^3+b^3+c^3 &= 3. \end{aligned}$$

Find $a^4+b^4+c^4$.

Solution

We have $n=3$ variables: a, b and c. We are given the information that $PS_1=1$, $PS_2=2, PS_3=3$ and the problem is asking for the value of PS_4. From (EqNS-3) we have

$$\begin{aligned} PS_4 &= (-1)^{4-1+1}s_{4-1}PS_1+(-1)^{4-1+2}s_{4-2}PS_2+(-1)^{4-1+3}s_{4-3}PS_3 \\ &= s_3PS_1-s_2PS_2+s_1PS_3 \\ &= s_3-2s_2+3s_1 \end{aligned}$$

We now can use (EqNS-4) to find s_1, s_2 and s_3. Clearly, $s_1=PS_1=1$.

$$\begin{aligned} s_2 &= \frac{1}{2}\left((-1)^{1-1}s_{2-1}PS_1+(-1)^{2-1}s_{2-2}PS_2\right) \\ &= \frac{1}{2}\left(s_1PS_1-s_0PS_2\right) \\ &= \frac{1}{2}\left(1(1)-1(2)\right) \\ &= -\frac{1}{2} \end{aligned}$$

$$\begin{aligned} s_3 &= \frac{1}{3}\left((-1)^{(1-1)}s_{3-1}PS_1+(-1)^{(2-1)}s_{3-2}PS_2+(-1)^{(3-1)}s_{3-3}PS_3\right) \\ &= \frac{1}{3}\left(s_2PS_1-s_1PS_2+s_0PS_3\right) \\ &= \frac{1}{3}\left(\left(-\frac{1}{2}\right)(1)-1(2)+1(3)\right) \end{aligned}$$

$$= \frac{1}{6}.$$

Therefore,

$$\begin{aligned} PS_4 &= s_3 - 2s_2 + 3s_1 \\ &= \left(\frac{1}{6}\right) - 2\left(-\frac{1}{2}\right) + 3(1) \\ &= \frac{25}{6}. \end{aligned}$$

■

Exercise 3.18 (Source: MSHSML 1T056)

Given: $x + y + z = 0$; $x^2 + y^2 + z^2 = 36$; $x^3 + y^3 + z^3 = 105$. One of the unknowns, say x, (by symmetry, it could be any one of them) is real; the other two complex. Find x.

Solution

Let $f(t)$ be a polynomial such that $f(t) = 0$ has roots x, y and z. Without loss of generality we can assume $f(t)$ is monic (*i.e.* the leading coefficient equals 1).

Then $f(t) = (t - x)(t - y)(t - z) = t^3 + a_2t^2 + a_1t + a_0$ and by Viète's formulas

$$x + y + z = -a_2$$

$$xy + xy + yz = a_1$$

$$xyz = -a_0.$$

Define

$$PS_1 = x + y + z = 0$$

$$PS_2 = x^2 + y^2 + z^2 = 36$$

$$PS_3 = x^3 + y^3 + z^3 = 105.$$

Then

$$PS_1 + a_2 = 0$$
$$PS_2 + a_2 PS_1 + 2a_1 = 0$$
$$PS_3 + a_2 PS_2 + a_1 PS_1 + 3a_0 = 0.$$

or

$$0 + a_2 = 0$$
$$36 + a_2(0) + 2a_1 = 0$$
$$105 + a_2(36) + a_1(0) + 3a_0 = 0.$$

Therefore, $a_2 = 0$, $a_1 = -18$, and $a_0 = -105/3 = -35$. Hence,

$$t^3 + a_2 t^2 + a_1 t + a_0 = t^3 - 18t - 35 = 0$$

has roots x, y and z. The candidates for rational roots are $\{\pm 1, \pm 5, \pm 7, \pm 35\}$. Only 5 turns out to be a rational root.

5	1	0	-18	-35
		5	25	35
	1	5	7	0

The quotient is $q(x) = x^2 + 5x + 7$ and the discriminant of $q(x) = 5^2 - 4(1)(7) = -3 < 0$ which verifies that the remaining two roots are nonreal.

Note: There is a typo in the official MSHSML solutions where they state that $a_0 = 35$ at one point.

■