MSHSML Meet 2, Event A
Study Guide

2A Linear Equations in One Unknown

Solving numeric equations (perhaps involving a second-degree term which drops out)
Solving literal equations
Story problems leading to linear equations in one variable
Linear inequalities

1 Contents

2 Linear Inequalities

2.1 The Four Basic Rules for Inequalities

(i) An inequality still holds if you add or subtract a constant or a function.

e.g.

$(a)\quad f(x) < g(x) < h(x) \Leftrightarrow f(x) + c < g(x) + c < h(x) + c$

$(b)\quad f(x) \le g(x) < h(x) \Leftrightarrow f(x) - c \le g(x) - c < h(x) - c$

$(c)\quad f(x) < g(x) \le h(x) \Leftrightarrow f(x) + r(x) < g(x) + r(x) \le h(x) + r(x)$

$(d)\quad f(x) \le g(x) \le h(x) \Leftrightarrow f(x) - r(x) \le g(x) - r(x) < h(x) - r(x).$

(ii) An inequality still holds if you multiply or divide by a *positive* constant.

e.g.

$(a)\quad f(x) < g(x) < h(x) \Leftrightarrow c \cdot f(x) < c \cdot g(x) < c \cdot h(x), \quad c > 0$

$(b)\quad f(x) \le g(x) < h(x) \Leftrightarrow c \cdot f(x) \le c \cdot g(x) < c \cdot h(x), \quad c > 0$

$(c)\quad f(x) \le g(x) \le h(x) \Leftrightarrow \dfrac{f(x)}{c} \le \dfrac{g(x)}{c} \le \dfrac{h(x)}{c}, \quad c > 0.$

(iii) An inequality “flips” if you multiply or divide by a *negative* constant.

e.g.

$(a)\quad f(x) < g(x) < h(x) \Leftrightarrow c \cdot f(x) > c \cdot g(x) > c \cdot h(x), \quad c < 0$

$(b)\quad f(x) \le g(x) < h(x) \Leftrightarrow c \cdot f(x) \ge c \cdot g(x) > c \cdot h(x), \quad c < 0$

$(c)\quad f(x) \le g(x) \le h(x) \Leftrightarrow \dfrac{f(x)}{c} \ge \dfrac{g(x)}{c} \ge \dfrac{h(x)}{c}, \quad c < 0$

$(d)\quad f(x) < g(x) \le h(x) \Leftrightarrow \frac{f(x)}{c} > \frac{g(x)}{c} \ge \frac{h(x)}{c}, \quad c < 0.$

(iv) An inequality "flips" if you take reciprocals of terms which are *all* positive or *all* negative.

e.g.

$(a1)\quad 0 < f(x) < g(x) < h(x) \Leftrightarrow \frac{1}{f(x)} > \frac{1}{g(x)} > \frac{1}{h(x)}$ $(a2)\quad f(x) < g(x) < h(x) < 0 \Leftrightarrow \frac{1}{f(x)} > \frac{1}{g(x)} > \frac{1}{h(x)}$
$(b1)\quad 0 < f(x) \le g(x) < h(x) \Leftrightarrow \frac{1}{f(x)} \ge \frac{1}{g(x)} > \frac{1}{h(x)}$ $(b2)\quad f(x) \le g(x) < h(x) < 0 \Leftrightarrow \frac{1}{f(x)} \ge \frac{1}{g(x)} > \frac{1}{h(x)}$
$(c1)\quad 0 < f(x) \le g(x) \le h(x) \Leftrightarrow \frac{1}{f(x)} \ge \frac{1}{g(x)} \ge \frac{1}{h(x)}$ $(c2)\quad f(x) \le g(x) \le h(x) < 0 \Leftrightarrow \frac{1}{f(x)} \ge \frac{1}{g(x)} \ge \frac{1}{h(x)}$
$(d1)\quad 0 < f(x) < g(x) \le h(x) \Leftrightarrow \frac{1}{f(x)} > \frac{1}{g(x)} \ge \frac{1}{h(x)}.$ $(d2)\quad f(x) < g(x) \le h(x) < 0 \Leftrightarrow \frac{1}{f(x)} > \frac{1}{g(x)} \ge \frac{1}{h(x)}.$

1. Find all x such that $-10 < -2x + 4 \leq 6$.

Solution

	Justification
$-10 < -2x + 4 \leq 6$	
$\Leftrightarrow -10 - 4 < -2x \leq 6 - 4$	Subtracting a constant from each part
$\Leftrightarrow \dfrac{-10-4}{-2} > x \geq \dfrac{6-4}{-2}$	Dividing by a negative constant, flip all inequality signs.
$\Leftrightarrow 7 > x \geq -1$	

Equivalently, we can write this as $-1 \leq x < 7$. In set notation we can write this as $[-1,7)$.

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2.2 Breaking Up a String of Inequalities

The string of inequalities $f(x) \leq g(x) < h(x)$ is (for example) equivalent to the statement

$$f(x) \leq g(x) \textbf{ and } g(x) < h(x).$$

Breaking a string into two parts like this is necessary when you need to perform different simplification steps in one part than in the other part.

2. Find all x such that $-x - 10 < 3x + 6 \leq 4x - 2$.

Solution

$-x - 10 < 3x + 6 \leq 4x - 2$

$\Leftrightarrow$	$-x - 10 < 3x + 6$	**and**	$3x + 6 \leq 4x - 2$
$\Leftrightarrow$	$-3x - x < 6 + 10$	**and**	$-4x + 3x \leq -2 - 6$
$\Leftrightarrow$	$-4x < 16$	**and**	$-x \leq -8$
$\Leftrightarrow$	$x > \dfrac{16}{-4} = -4$	**and**	$x \geq \dfrac{-8}{-1} = 8$

So, the solution set consists of all x's such that $x > -4$ **and** $x \geq 8$. Now think about this carefully. The only values of x which are simultaneously greater than -4 and greater than or equal to 8 are just those values of x which are greater than or equal to 8. That is,

$$(x > -4 \text{ **and** } x \geq 8) \Leftrightarrow x \geq 8.$$

Using set notation our solution set is $[8, \infty)$.

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3.	Find all x such that $x - 1 < -x + 5 \leq 2x + 17$.

Solution

$x - 1 < -x + 5 \leq 2x + 17$

$\Leftrightarrow \quad x - 1 < -x + 5 \qquad$ **and** $\qquad -x + 5 \leq 2x + 17$

$\Leftrightarrow \quad 2x < 6 \qquad$ **and** $\qquad -3x \leq 12$

$\Leftrightarrow \quad x < 3 \qquad$ **and** $\qquad x \geq -4$

$\Leftrightarrow \quad x \geq -4$ **and** $x < 3$

$\Leftrightarrow \quad x \in [-4,3)$.

The symbol "∈" is read "is an element of the set". So "$x \in [-4,3)$" is read as "x is an element of the set $[-4,3)$.

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4.	How many integers n satisfy the inequalities $2n < 7n - 5 \leq 6n$? (MSHSML 2A162)

Solution

$2n < 7n - 5 \leq 6n$

$\Leftrightarrow \quad 2n < 7n - 5 \qquad$ **and** $\qquad 7n - 5 \leq 6n$

$\Leftrightarrow \quad -5n < -5 \qquad$ **and** $\qquad n \leq 5$

$\Leftrightarrow \quad n > 1 \qquad$ **and** $\qquad n \leq 5$

$\Leftrightarrow \quad n \in \{2,3,4,5\}$.

So, there are 4 integer values of n that satisfy this chain of inequalities.

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3 System of Linear Inequalities

The general theory of maximizing (or minimizing) a linear function of any number of variables which are constrained by a series of linear inequalities is considered in the field of linear programming.

However, in this section of our prep work for Test 2A it is not necessary to have any background in linear programming. The questions in MSHSML Test 2A involving a system of linear inequalities do not require any advanced techniques to solve.

5. Integers a, b and c satisfy the inequalities

$$a < 2b - 3$$
$$b < 5c - 7$$
$$c < 11a - 13.$$

What is the least possible value for a? (MSHSML 2A124)

Solution

But

$$a < 2b - 3$$

$$2b - 3 < 2(5c - 7) - 3 = 10c - 17$$

$$a < 10c - 17$$

$$a < 10c - 17 < 10(11a - 13) - 17 = 110a - 147$$

$$109a - 147 > 0$$

$$109a > 147$$

$$a > 147/109 = 1 + 38/109.$$

But we are given that a is an integer. Therefore, $a \geq 2$. But we aren't done until we know if there is at least one $(a = 2, b, c)$ solution satisfying all three inequalities. By "guess and check" we see that $(a, b, c) = (2,3,3)$ does in fact satisfy all three inequalities. So $a = 2$ is indeed the smallest possible value of a.

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6. Positive integers a, b, c and d satisfy the inequalities

$$a < b < c < d < 2016$$

and

$$2016 < 2d < 3c < 4b < 5a.$$

What is the largest possible value for $a + b + c - d$? (MSHSML 2T162)

Solution

First note that

$$a + b + c - d = (a + b) + (c - d) < (a + b) + (-1)$$

because $c < d$ implies that $c - d \leq 1$.

So, we want to maximize $(a + b)$ subject to the given constraints. To make the sum $a + b$ as large as possible we want to choose a and b to be as large as possible. But a and b are both less than c so we want c to be as large as possible. We know $c < d < 2016$. But as we noted above we want $c - d = -1$. Therefore, we should choose $c = 2014$ and $d = 2015$.

Therefore, we want to choose a and b as large as possible subject to the constraint that $0 < a < b < 2014$. That is, we want to take $a = 2012$ and $b = 2013$.

Does the choice $(a, b, c, d) = (2012, 2013, 2014, 2015)$ satisfy both sets of required inequalities?

Check! If we take $(a, b, c, d) = (2012, 2013, 2014, 2015)$ is it true that

$$a < b < c < d < 2016 \quad \text{and} \quad 2016 < 2d < 3c < 4d < 5a \ ?$$

Yes, both sets of inequalities hold for $(a, b, c, d) = (2012{,}2013{,}2014{,}2015)$. And we've already made the argument that this choice for (a, b, c, d) will maximize $a + b + c - d$.

So, the largest possible value of $a + b + c - d$ equals $2012 + 2013 + 2014 - 2015 = 4024$.

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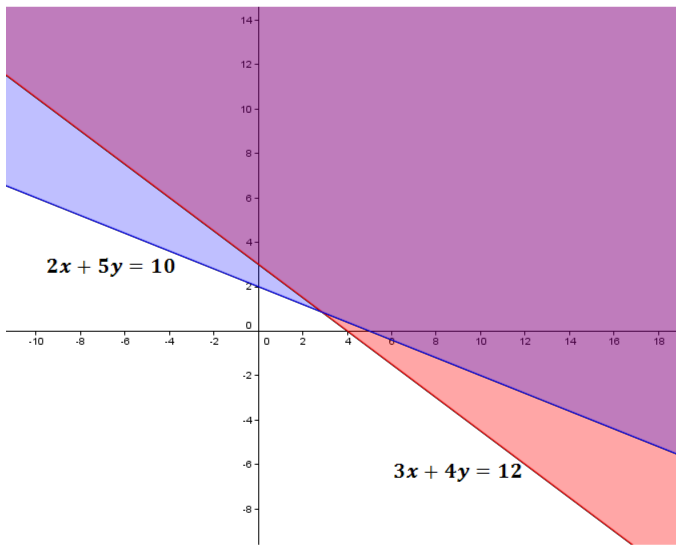
3.1 Linear Programming

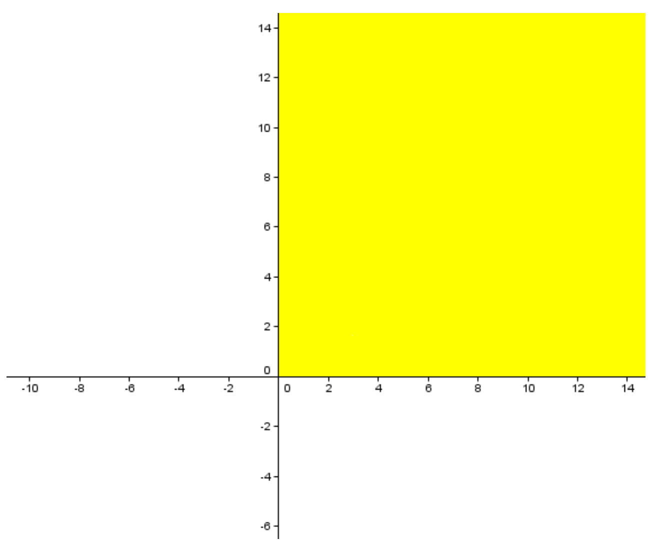
7. Suppose x and y are positive real numbers, and the point (x, y) lies on or above both of the lines having equations $2x + 5y = 10$ and $3x + 4y = 12$. What is the least possible value of $8x + 13y$? (MATHCOUNTS).

<u>Solution</u>

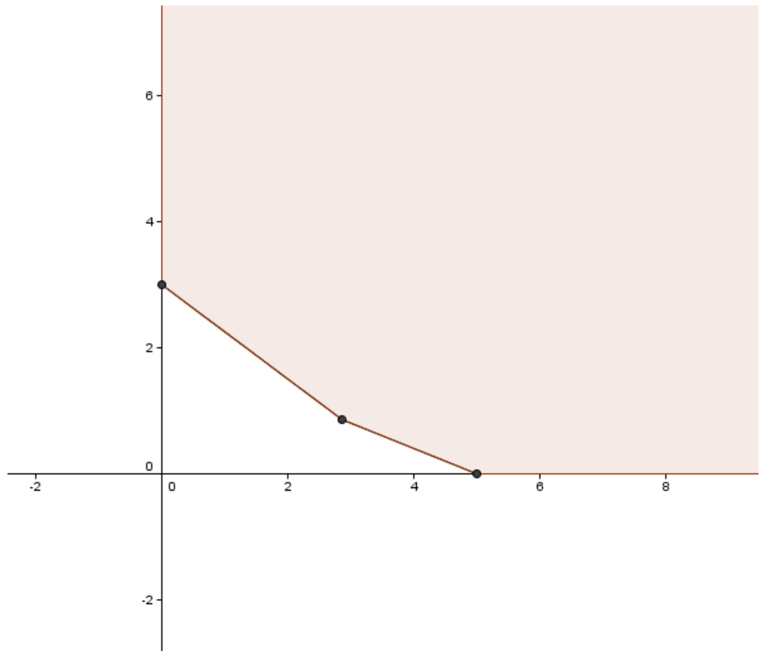
The region above the line $2x + 5y = 10$ is shown in blue and the region above the line $3x + 4y = 12$ is shown in red. Hence, the region that is above BOTH lines becomes purple (blue+red=purple).



We also are told that $x \geq 0$ and $y \geq 0$ (*i.e.* the region in yellow).



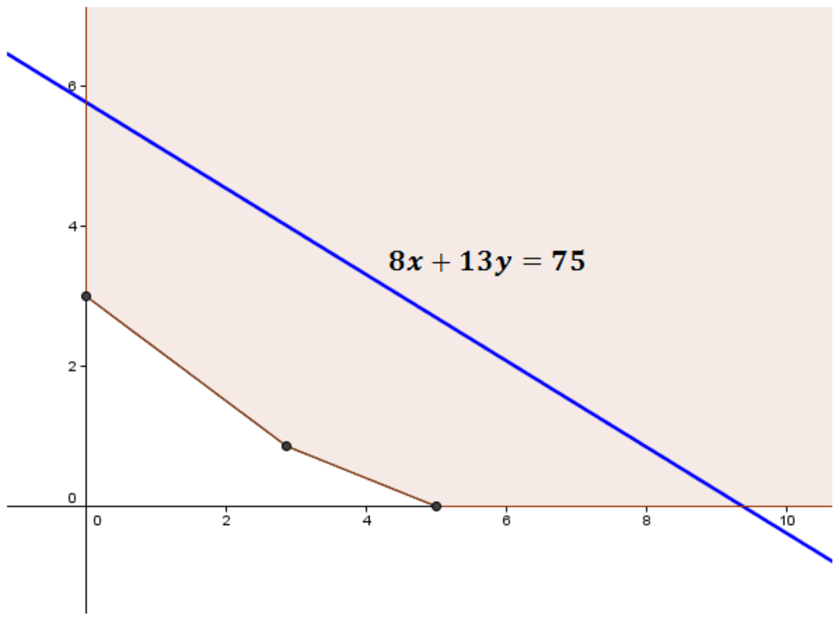
The (x, y) values that are in the purple and the yellow are highlighted below.



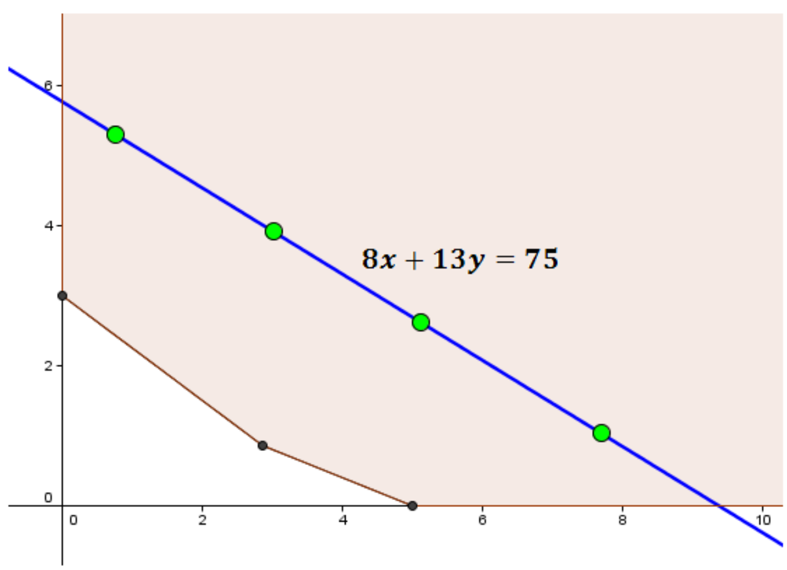
The goal of the problem is to find the smallest value of $8x + 13y$ if (x, y) has to be a point in the above shaded region. Problems of this type are known as **linear programming** problems.

Now let's play with this a bit. Could (just as an example) we find an (x, y) within this shaded region where $8x + 13y = 75$?

To see if this is possible we need to graph the line $8x + 13y = 75$. We show this line below.

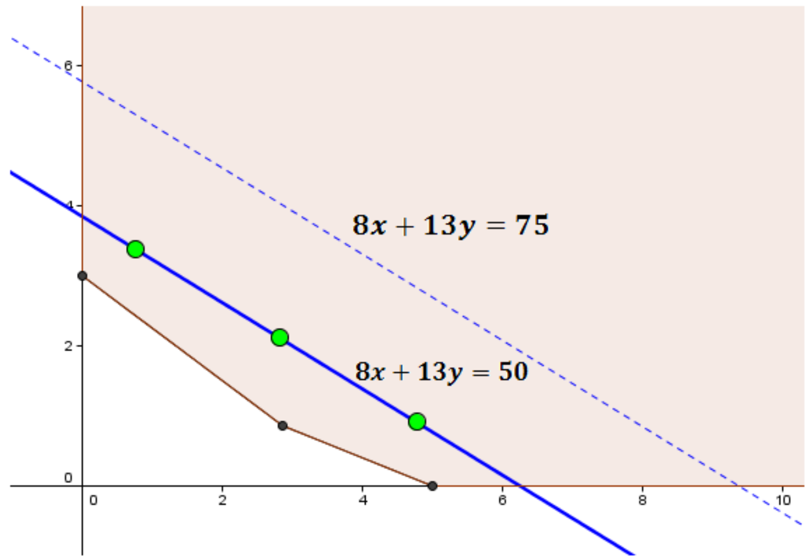


We can see that $8x + 13y$ equals 75 at each of the points marked in green (because they fall on the line where $8x + 13y = 75$) and these green points are in the highlighted region.



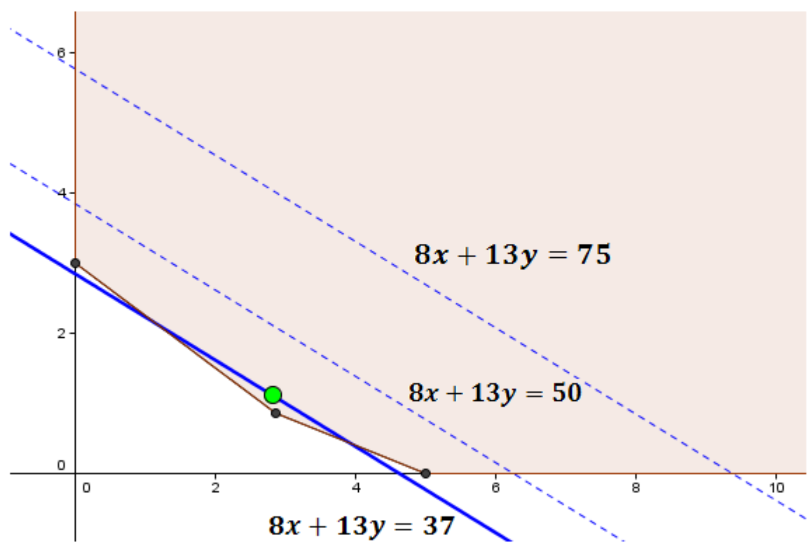
This shows that $8x + 13y$ can get at least as small as 75.

But can we make it smaller? Can we find a point(s) within the highlighted region where $8x + 13y = 50$?



The graph of the line $8x + 13y = 50$ is shown above in solid blue and we see that there are points on this line that are within the shaded region. (Clearly there are an infinite number of points on this line that are within this shaded region. The three points shown in green just show particular cases.)

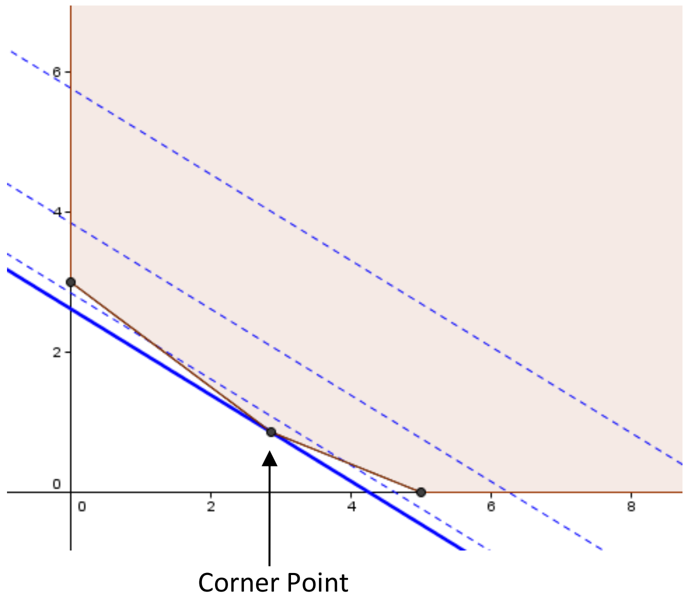
This shows that $8x + 13y$ can get at least as small as 50. Can we make it smaller yet? Can we find a point(s) within the highlighted region where $8x + 13y = 37$?



The graph of the line $8x + 13y = 37$ is shown above in solid blue and we see that there are points (e.g. the green points) on this line that are within the shaded region. (Clearly there are an infinite number of points on this line that are within this shaded region. The point in green just shows a particular case.)

This shows that $8x + 13y$ can get at least as small as 37.

By now you most likely notice that we should lower this line until it hits the corner point.



The entire of this argument has come around to showing that

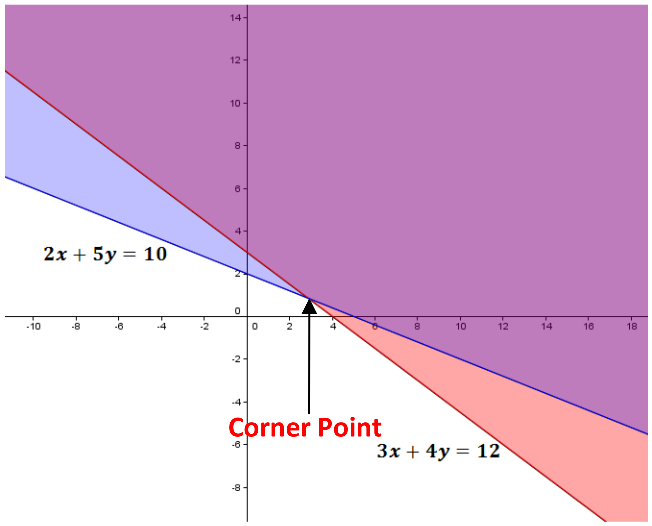
$$8x + 13y$$

will be minimized (among points in the shaded region) at this corner point. If we label the coordinates of this corner point as (x_0, y_0) then

$$8x_0 + 13y_0$$

is the minimum value we are looking for.

So how can we find this corner point? This corner point is the point where the lines $2x + 5y = 10$ and $3x + 4y = 12$ intersect (*i.e.* cross).



Finding the intersection of two lines boils down to solving two linear equations in two unknowns.

That is, we want to find x and y so that

$$2x + 5y = 10 \text{ and } 3x + 4y = 12.$$

The point of intersection is the point that lies on both lines which means this point satisfies BOTH of these equations.

Solving for y in the first equation we have

$$y = \frac{10 - 2x}{5}.$$

Substituting this into the second equation we get

$$3x + 4\left(\frac{10 - 2x}{5}\right) = 12$$

$$3x + 4\left(\frac{10 - 2x}{5}\right) = 12$$

$$\Longrightarrow 15x + 4(10 - 2x) = 60$$

$$\Longrightarrow 15x + 40 - 8x = 60$$

$$\Longrightarrow 7x = 20$$

$$\Longrightarrow x = \frac{20}{7}.$$

From here we can solve for y.

$$y = \frac{10 - 2\left(\frac{20}{7}\right)}{5} = \frac{70 - 40}{35} = \frac{30}{35} = \frac{6}{7}.$$

So the coordinates of this corner point are

$$(x_0, y_0) = \left(\frac{20}{7}, \frac{6}{7}\right).$$

Finally, we can plug these values in to our function $8x + 13y$ to get the minimum possible value of this function over all points in the shaded region.

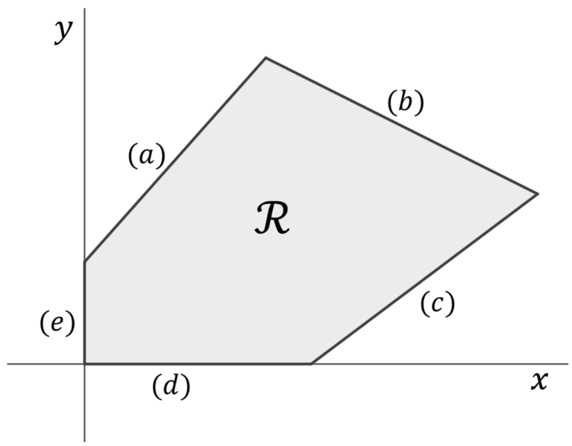
$$8x_0 + 13y_0 = 8\left(\frac{20}{7}\right) + 13\left(\frac{6}{7}\right) = \frac{238}{7} = 34.$$

That is, the smallest value of $8x + 13y$ that can be obtained by considering only the (x, y) points in the shaded region defined above is 34.

$\blacksquare$

8. The region $\mathcal{R}$ shown in the figure shown below is the closed set (*i.e.* it includes the points on the boundary lines) bounded by

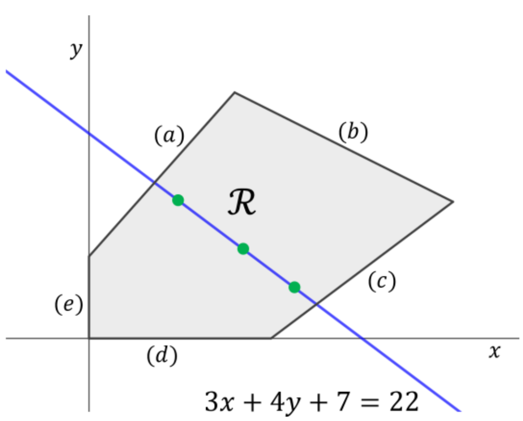
(a) $9x - 8y = -12$
(b) $3x + 6y = 35$
(c) $9x - 12y = 30$
(d) $y = 0$
(e) $x = 0$.

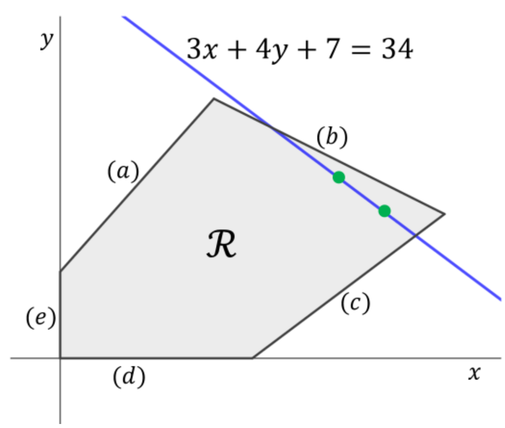


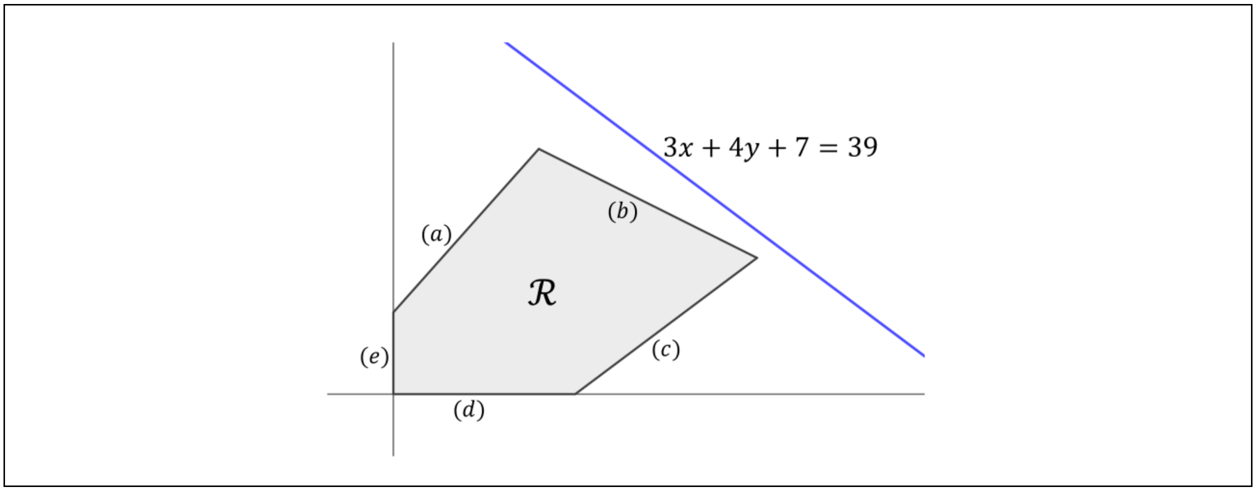
What is the maximum value of $f(x, y) = 3x + 4y + 7$ for a point (x, y) in $\mathcal{R}$? (MSHSML 4A894)

Solution

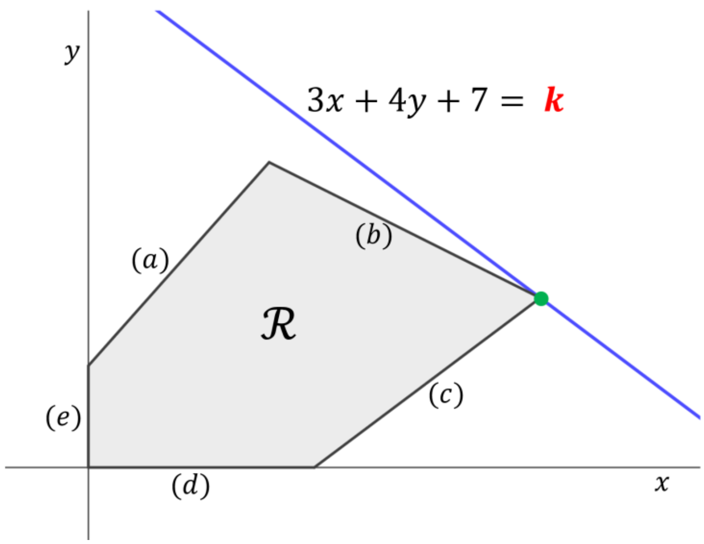
The figures below show that there are points (shown in green) in $\mathcal{R}$ where $3x + 4y + 7 = 22$ and where $3x + 4y + 7 = 34$. But there are no points in $\mathcal{R}$ where $3x + 4y + 7 = 39$.







So, the largest value of $3x + 4y + 7$ must be somewhere between 34 and 39 and can be determined by finding that value k between 34 and 39 where the line $3x + 4y + 7 = k$ goes through the intersection point of Line (b) $3x + 6y = 35$ and Line (c) $9x - 12y = 30$.



We can solve $3x + 6y = 35$ and $9x - 12y = 30$ simultaneously to locate this point of intersection.

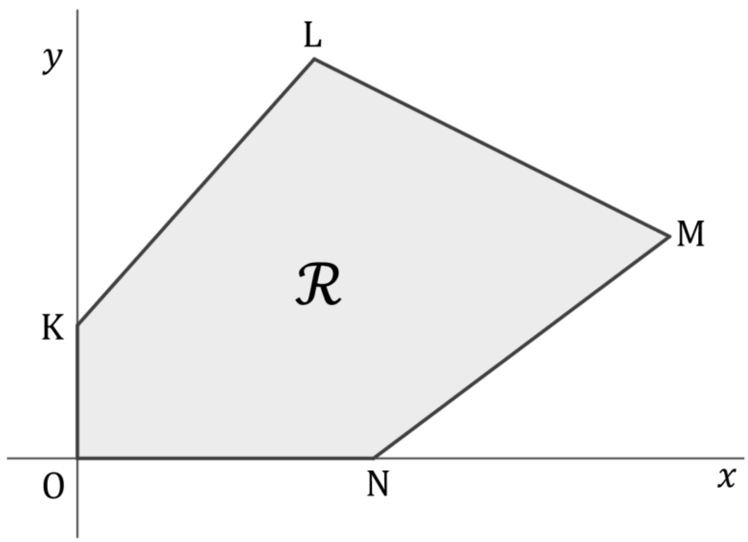
$$3(3x + 6y) - (9x - 12y) = 3(35) - 30$$
$$30y = 75$$
$$y = 5/2$$
$$3x + 6(5/2) = 35 \Longrightarrow 3x = 20 \Longrightarrow x = 20/3$$

So, this point of intersection has coordinates $(x, y) = (20/3, 5/2)$. We can solve for $\boldsymbol{k}$ by evaluating $3x + 4y + 7$ at this intersection point $(x, y) = (20/3, 5/2)$.

$$\boldsymbol{k} = 3\left(\frac{20}{3}\right) + 4\left(\frac{5}{2}\right) + 7 = 20 + 10 + 7 = 37.$$

We note that this answer is between 34 and 39 as we determined graphically that it must be. Therefore, the maximum value of $f(x, y) = 3x + 4y + 7$ for a point (x, y) in $\mathcal{R}$ is $\mathbf{37}$ and this maximum occurs at the point $(x, y) = (20/3, 5/2)$.

Can we solve this problem without all the careful graphing? **YES*!*** (Which is great because you aren't allowed a graphing calculator on this test!)



The key geometric insight is that a linear function (in our problem, the line $3x + 4y + 7$) necessarily achieves its maximum (and minimum) at one of the intersection points of the linear boundary lines of $\mathcal{R}$.

So, a purely algebraic approach to solve this problem is to find the coordinates of all the intersection points and then evaluate our objective function (*i.e.* the line $3x + 4y + 7$) at each of these intersection points to determine where the objective function is maximized or minimized.

In our problem we can find the coordinates of the remaining intersection points (called *corner points* in the field of linear programming) just as we did for the corner point $M(20/3, 5/2)$.

We will find that $L(8/3,9/2)$, $N(10/3,0)$, $O(0,0)$ and $K(0,3/2)$. We already have determined $M(20/3,5/2)$. Now we need to evaluate $f(x,y) = 3x + 4y + 7$ at each of these corner points.

$$\mathrm{L} : f\left(\frac{8}{3},\frac{9}{2}\right) = 3\left(\frac{8}{3}\right) + 4\left(\frac{9}{2}\right) + 7 = 33$$

$$\mathrm{M} : f\left(\frac{20}{3},\frac{5}{2}\right) = 3\left(\frac{20}{3}\right) + 4\left(\frac{5}{2}\right) + 7 = 37$$

$$\mathrm{N} : f\left(\frac{10}{3},0\right) = 3\left(\frac{10}{3}\right) + 4(0) + 7) = 30$$

$$\mathrm{O} : f(0,0) = 3(0) + 4(0) + 7 = 7$$

$$\mathrm{K} : f\left(0,\frac{3}{2}\right) = 3(0) + 4\left(\frac{3}{2}\right) + 7 = 13.$$

So, without any graphing we know that by evaluating the objective function $f(x,y) = 3x + 4y + 7$ over all points (x,y) in the *feasible region* (the term used in linear programming for the region $\mathcal{R}$) the minimum value of the objective function $f(x,y)$ equals 0 and this occurs at the corner point $O(0,0)$ and the maximum value of the objective function $f(x,y)$ equals 37 and this occurs at the corner point $M(20/3,5/2)$.

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4 Story Problems Leading to Linear Equations in One Variable

4.1 Steps in Solving Word Problems

(i) Draw a Diagram

Draw a diagram or graph of what the problem is saying. Try to scale it accurately so that you can get some visual clues to the solution or at least an estimate of the solution.

(ii) Assign Names to Variables

Give letter names to the unknown(s) in the problem.

(iii) Identify the Goal

Read through the problem and identify (in words and in terms of the names assigned in Step (ii)) exactly what the problem is asking you to solve for.

(iv) Identify Relationships

Make a list of the relationships (equations) between and among the knowns and the unknowns in the problem.

(v) Number of variables compared to the number of equations.

You need to identify *at least* as many relationships (equations) between the variables as the number of unknowns in the problem.

If you have more unknowns (variables) than relationships (equations) go back to the statement of the problem, read it again, and look for another relationship in the variables. *Sometimes these relationships appear to be hidden because of the wording of the problem*. So, READ CAREFULLY!

(vi) Solve for the unknown(s).

(vii) Finishing up.

Be sure that your final answer really answers what the problem is specifically asking you to find. (*e.g.* Don't give them the diameter if they are asking for the radius.)

Be sure that your final answer is completely simplified according to MSHSML standards.

Be sure your answer includes the unit of measure (feet, pages, seconds, etc.).

4.2 Recognizing Key Words

(This section is taken directly from the website "PurpleMath.Com".)

Mathematical Translation of Certain Frequently Occurring Key Words:

Addition:
- increased by
- more than
- combined, together
- total of
- sum, plus
- added to

Subtraction:
- decreased by
- minus, less
- difference between/of
- less than, fewer than
- left, left over, after

Multiplication:

of
times, multiplied by
product of
increased/decreased by a factor of
twice, triple, etc
each ("they got three each", etc)

Division:
per, a
out of
ratio of, quotient of
percent (divide by 100)
equal pieces, split
average

Equals
is, are, was, were, will be, would be
gives, yields
sold for, cost
is equivalent to, must be, yields.

5 Absolute Value Equations

5.1 Problem Type I. $|ax + b| = c$

Solve $ax + b = c$ and $ax + b = -c$ separately to find both solutions.

9.	Find all x such that $|8 + 5x| = 45$.

$$|8 + 5x| = 14 - x$$

Solution

$$8 + 3x = 43 \Leftrightarrow 3x = 35 \Leftrightarrow x = 7.$$

$$8 + 2x = -43 \Leftrightarrow 5x = -51 \Leftrightarrow x = 30.$$

■

5.2 Problem Type II. $|ax + b| + |cx + d| + \cdots = rx + s$

The method of *break points*.

10.	Find all x such that $|8 + 5x| = 14 - x$.

<u>Solution</u>

Before starting this problem, let's consider what goes wrong if we simply mimic the approach used in the previous problem and separately solve $(8 + 5x) = -(14 - x)$ and $(8 + 5x) = +(14 - x)$? The problem is that $(14 - x)$ involves the variable x and hence can be negative or positive depending on the value of x.

The method of break points is a technique that adjusts for both possibilities.

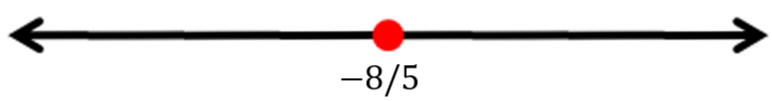
Step 1. Find the value(s) of x where the functions inside the absolute value sign(s) equal 0. We will call these the *break points*.

In this problem, we need to find the value of x where $(8 + 5x) = 0$.

$$8 + 5x = 0 \Longrightarrow 5x = -8 \Longrightarrow x = -8/5.$$

So, in this problem there is a single *break point*, namely $x = -8/5$.

Step 2. Draw a number line (just a quick sketch will do) and mark the break points. It does not have to be drawn too accurately.



Step 3. The break points split the number line into regions. Figure out whether the expressions inside the absolute value signs are ≥ 0 or ≤ 0 in each of these regions and make a chart.

Expressions inside absolute value signs		
$8 + 5x$	$-$	$+$
	 $-8/5$	

Step 4. Considering each region separately, remove the absolute value signs according to the definition

$$|g(x)| = \begin{cases} g(x) & \text{if } g(x) \geq 0 \\ -g(x) & \text{if } g(x) \leq 0. \end{cases}$$

In the region $x \in (-\infty, -8/5]$, $|8 + 5x| = -(8 + 5x)$ because (as delineated in the above chart), $8 + 5x$ is ≤ 0 in this region.

Therefore, provided $x \in (-\infty, -8/5]$

$$|8 + 5x| = 14 - x$$

is equivalent to

$$-(8 + 5x) = 14 - x.$$

Solving for x we find

$$-8 - 5x = 14 - x$$

$$4x = -22$$

$$x = -22/4 = -11/2.$$

Is this solution within the region we are currently considering, namely the region $(-\infty, -8/5]$? Yes. $-11/2 \in (-\infty, -8/5]$. So, this is not an extraneous solution.

Now on to the other region, $[-8/5, \infty)$.

For $x \in [-8/5, \infty)$,

$$|8 + 5x| = 14 - x$$

is equivalent to

$$+(8 + 5x) = 14 - x.$$

Solving for x we find

$$8 + 5x = 14 - x$$

$$6x = 6$$

$$x = 1.$$

Is this solution within the region we are currently considering, namely the region $[-8/5, \infty)$? Yes. $1 \in [-8/5, \infty)$. So, this is not an extraneous solution.

So, there are two values of x where $|8 + 5x| = 14 - x$. Namely at $x = -11/2$ and $x = 1$.

■

11.	Find all x such that $|3x - 1| + |2 - x| = 5 - x$.

<u>Solution</u>

Step 1. Find the value(s) of x where the functions inside the absolute value sign(s) equal 0. We will call these the *break points*.

In this problem, we need to find the value of x where $(3x - 1) = 0$ and the value of x where $(2 - x) = 0$.

$$3x - 1 = 0 \Longrightarrow 3x = 1 \Longrightarrow x = 1/3.$$

$$2 - x = 0 \Longrightarrow x = 2.$$

So there are two *break points*, namely $x = 1/3$ and $x = 2$.

Step 2. Draw a number line (just a quick sketch will do) and mark the break points. It does not have to be drawn too accurately.

Step 3. The break points split the number line into regions. Figure out whether the expressions inside the absolute value signs are ≥ 0 or ≤ 0 in each of these regions and make a chart.

Expressions inside absolute value signs			
$3x - 1$	$-$	$+$	$+$
$2 - x$	$+$	$+$	$-$

Step 4. Considering each region separately, remove the absolute value signs according to the definition

$$|g(x)| = \begin{cases} g(x) & \text{if } g(x) \geq 0 \\ -g(x) & \text{if } g(x) \leq 0. \end{cases}$$

In the region $x \in (-\infty, 1/3]$, $|3x - 1| = -(3x - 1)$ because (as delineated in the above chart), $3x - 1$ is ≤ 0 in this region. Also, $|2 - x| = +(2 - x) = 2 - x$ in this region because $2 - x$ is ≥ 0 in this region.

Therefore, provided $x \in (-\infty, 1/3]$

$$|3x - 1| + |2 - x| = 5 - x$$

Is equivalent to

$$-(3x - 1) + (2 - x) = 5 - x.$$

Now we can easily solve this equation.

$$-3x + 1 + 2 - x = 5 - x$$

$$-3x = 2$$

$$x = -2/3.$$

Is this solution within the region we are currently considering, namely the region $(-\infty, 1/3]$? Yes. $-2/3 \in (-\infty, 1/3]$. So, this is not an extraneous solution.

Now on to the next region, $[1/3, 2]$. According to our chart, $3x - 1$ and $2 - x$ are both ≥ 0 in this region. So, for $x \in [1/3, 2]$,

$$|3x - 1| + |2 - x| = 5 - x$$

Is equivalent to

$$(3x - 1) + (2 - x) = 5 - x.$$

Now we can easily solve this equation.

$$3x - 1 + 2 - x = 5 - x$$

$$3x = 4$$

$$x = 4/3.$$

Is this solution within the region we are currently considering, namely the region $[1/3, 2]$? Yes. $4/3 \in [1/3, 2]$. So, this is not an extraneous solution.

Now on to the final region, $[2, \infty)$. According to our chart, $3x - 1$ is ≥ 0 and $2 - x$ is ≤ 0 in this region. So, for $x \in [2, \infty)$,

$$|3x - 1| + |2 - x| = 5 - x$$

is equivalent to

$$(3x - 1) - (2 - x) = 5 - x.$$

Now we can easily solve this equation.

$$3x - 1 - 2 + x = 5 - x$$

$$5x = 8$$

$$x = 8/5.$$

Is this solution within the region we are currently considering, namely the region $[2,\infty)$? No. $8/5 \notin [2,\infty)$. So, this **is** an extraneous solution. It is not a solution to the original problem.

So, there are two values of x where $|3x-1|+|2-x|=5-x$. Namely at $x=-2/3$ and $x=4/3$.

$\blacksquare$

12.	Find all x such that $$\left\|\frac{2\|x\|-3}{4\|x-1\|}\right\|=2.$$

Solution

Is this really a problem of the form $|ax+b|+|cx+d|+\cdots=rx+s$?

To see that it is, we start by separating the problem into two distinct parts. In the first part we will find all values of x such that

$$\frac{2|x|-3}{4|x-1|}=2.$$

In the second part we will find all values of x such that

$$\frac{2|x|-3}{4|x-1|}=-2.$$

Our final answer is the set of all x values that satisfy *either* the first part of the second part.

Now notice that in the first part,

$$\frac{2|x|-3}{4|x-1|}=2 \Leftrightarrow 2|x|-3=8|x-1| \Leftrightarrow 2|x|-8|x-1|=3.$$

And in the second part

$$\frac{2|x|-3}{4|x-1|}=-2 \Leftrightarrow 2|x|-3=-8|x-1| \Leftrightarrow 2|x|+8|x-1|=3.$$

So, what we really have here is two separate problems, both of the form
$|ax+b|+|cx+d|+\cdots = rx+s$.

Part 1. $\dfrac{2|x|-3}{4|x-1|} = 2$ or $2|x| - 8|x-1| = 3$.

Break points: 0,1

x	$-$	$+$	$+$
$x-1$	$-$	$-$	$+$
			

Case 1 (of Part 1). $\boldsymbol{x \in (-\infty, 0]}$

$$2|x| - 3 = 8|x-1|$$
$$2(-x) - 3 = 8\big(-(x-1)\big)$$
$$-2x - 3 = -8x + 8$$
$$6x = 11$$
$$x = 11/6$$

Is this solution within the region we are currently considering, namely the region $(-\infty, 0]$? ***No***. $11/6 \notin (-\infty, 0]$. So, this is not a valid solution. It is an extraneous solution.

Case 2 (of Part 1). $\boldsymbol{x \in [0, 1]}$

$$2|x| - 3 = 8|x-1|$$
$$2(+x) - 3 = 8\big(-(x-1)\big)$$
$$2x - 3 = -8x + 8$$

$$10x = 11$$

$$x = 11/10.$$

Is this solution within the region we are currently considering, namely the region $[0,1]$? ***No***. $11/10 \notin [0,1]$. So, this is not a valid solution. It is an extraneous solution.

Case $\mathbf{3}$ (of Part $\mathbf{1}$). $\boldsymbol{x \in [1, \infty)}$

$$2|x| - 3 = 8|x - 1|$$

$$2(+x) - 3 = 8\big(+(x - 1)\big)$$

$$2x - 3 = 8x - 8$$

$$-6x = -5$$

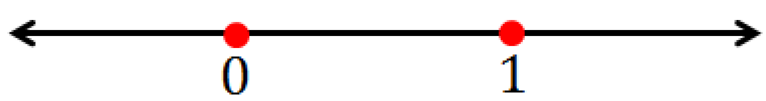
$$x = 5/6$$

Is this solution within the region we are currently considering, namely the region $[1, \infty)$? ***No***. $5/6 \notin [1, \infty)$. So this is not a valid solution. It is an extraneous solution.

So, Part 1 has no solutions. That is, there are no values of x where $2|x| - 8|x - 1| = 3$.

Part 2. $\dfrac{2|x| - 3}{4|x - 1|} = -2$ or $2|x| + 8|x - 1| = 3.$

Break points: 0,1

x	$-$	$+$	$+$
$x - 1$	$-$	$-$	$+$
			

Case $\mathbf{1}$ (of Part $\mathbf{2}$). $\boldsymbol{x \in (-\infty, 0]}$

$$2|x| - 3 = -8|x - 1|$$
$$2(-x) - 3 = -8\big(-(x - 1)\big)$$
$$-2x - 3 = 8x - 8$$
$$-10x = -5$$
$$x = 1/2$$

Is this solution within the region we are currently considering, namely the region $(-\infty, 0]$? ***No***. $1/2 \notin (-\infty, 0]$. So, this is not a valid solution. It is an extraneous solution.

Case $\mathbf{2}$ (of Part $\mathbf{2}$). $\boldsymbol{x \in [0, 1]}$

$$2|x| - 3 = -8|x - 1|$$
$$2(+x) - 3 = -8\big(-(x - 1)\big)$$
$$2x - 3 = 8x - 8$$
$$-6x = -5$$
$$x = 5/6$$

Is this solution within the region we are currently considering, namely the region $[0,1]$? ***Yes***. $5/6 \in [0,1]$. So, this is a valid solution. It is not an extraneous solution.

Case $\mathbf{3}$ (of Part $\mathbf{2}$). $\boldsymbol{x \in [1, \infty)}$

$$2|x| - 3 = -8|x - 1|$$
$$2(+x) - 3 = -8\big(+(x - 1)\big)$$
$$2x - 3 = -8x + 8$$
$$10x = 11$$
$$x = 11/10$$

Is this solution within the region we are currently considering, namely the region $[1, \infty)$? ***Yes***. $11/10 \in [1, \infty)$. So, this is a valid solution. It is not an extraneous solution.

So, Part 2 has two solutions. That is, there are two values of x where $2|x| + 8|x - 1| = 3$. Namely, $x = 5/6$ and $x = 11/10$.

Combining the solutions from parts 1 and 2 we see that $x = 5/6$ and $x = 11/10$ are the only two solutions of

$$\left|\frac{2|x|-3}{4|x-1|}\right| = 2.$$

■

5.3 Problem Type III. Nested Absolute Values. $\left|ax+\left|bx+|cx+d|\right|\right| = rx+s$

Work from the inside and go out.

13.	Find the sum of all solutions of $x = \left|2x - |60-2x|\right|$.

<u>Solution</u>

First, let's simplify $\left|2x-|60-2x|\right|$, working from the inside and going out.

$$|2x-|60-2x|| = \begin{cases} |2x-(+1)(60-2x)| & \text{if } 60-2x \geq 0 \\ |2x-(-1)(60-2x)| & \text{if } 60-2x \leq 0 \end{cases}$$

$$= \begin{cases} |4x-60| & \text{if } x \leq 30 \\ 60 & \text{if } x \geq 30 \end{cases}$$

$$= \begin{cases} 4x-60 & \text{if } x \leq 30 \text{ and } 4x-60 \geq 0 \\ -(4x-60) & \text{if } x \leq 30 \text{ and } 4x-60 \leq 0 \\ 60 & \text{if } x \geq 30 \end{cases}$$

$$= \begin{cases} 4x-60 & \text{if } x \leq 30 \text{ and } x \geq 15 \\ 60-4x & \text{if } x \leq 30 \text{ and } x \leq 15 \\ 60 & \text{if } x \geq 30. \end{cases}$$

$$= \begin{cases} 4x-60 & \text{if } 15 \leq x \leq 30 \\ 60-4x & \text{if } x \leq 15 \\ 60 & \text{if } x \geq 30. \end{cases}$$

So, the problem reduces to finding all x in $[15,30]$ where $x = 4x-60$, all x in $(-\infty, 30]$ where $x = 60-4x$ and all x in $[30, \infty)$ where $x = 60$.

$$x = 4x-60 \Longrightarrow -3x = -60 \Longrightarrow x = 20$$

and 20 is in the required interval $[15,30]$ so this is a valid solution.

$$x = 60 - 4x \Longrightarrow 5x = 60 \Longrightarrow x = 12$$

and 12 is in the required interval $(-\infty, 15]$ so this is a valid solution.

$$x = 60 \Longrightarrow x = 60$$

and 60 is in the required interval $[30, \infty)$ so this is a valid solution.

So, the three solutions are $x = 12, x = 20$ and $x = 60$ is in the required interval $[30, \infty)$. So this is a valid solution.

Therefore, the sum equals of all solutions equals $12 + 20 + 60 = 92$.

■

14.	Find all x such that $\big|3x - |x+4|\big| = 3x + 2$.

Solution

Solution

First, let's simplify $\big|3x - |x+4|\big|$, working from the inside and going out.

$$|3x - |x+4|| = \begin{cases} |3x - (+1)(x+4)| & \text{if } x + 4 \geq 0 \\ |3x - (-1)(x+4)| & \text{if } x + 4 \leq 0 \end{cases}$$

$$= \begin{cases} |2x - 4| & \text{if } x + 4 \geq 0 \\ |4x + 4| & \text{if } x + 4 \leq 0 \end{cases}$$

$$= \begin{cases} 2x - 4 & \text{if } x + 4 \geq 0 \text{ and } 2x - 4 \geq 0 \\ -(2x - 4) & \text{if } x + 4 \geq 0 \text{ and } 2x - 4 \leq 0 \\ 4x + 4 & \text{if } x + 4 \leq 0 \text{ and } 2x - 4 \geq 0 \\ -(4x + 4) & \text{if } x + 4 \leq 0 \text{ and } 2x - 4 \leq 0 \end{cases}$$

$$= \begin{cases} 2x-4 & \text{if } x \geq -4 \text{ and } x \geq 2 \\ -(2x-4) & \text{if } x \geq -4 \text{ and } x \leq 2 \\ 4x+4 & \text{if } x \leq -4 \text{ and } x \geq 2 \\ -(4x+4) & \text{if } x \leq -4 \text{ and } x \leq 2 \end{cases}$$

$$= \begin{cases} 2x-4 & \text{if } x \geq 2 \\ -(2x-4) & \text{if } -4 \leq x \leq 2 \\ 4x+4 & \text{impossible} \\ -(4x+4) & \text{if } x \leq -4. \end{cases}$$

Now we can start solving for x.

$$3x + 2 = 2x - 4 \Leftrightarrow x = -6$$

But -6 is ***not*** in the required interval $[2, \infty]$ so this is ***not*** a valid solution.

$$3x + 2 = -(2x - 4) \Leftrightarrow 5x = 2 \Leftrightarrow x = 2/5$$

And $2/5$ is in the required interval $[-4,2]$ so this is a valid solution.

$$3x + 2 = -(4x + 4) \Leftrightarrow 7x = -6 \Leftrightarrow x = -6/7$$

But $-6/7$ is ***not*** in the required interval $(-\infty, -4]$ so this is ***not*** a valid solution.

Therefore, $x = 2/5$ is the only value of x such that $\big|3x - |x+4|\big| = 3x + 2$.

■

6 Absolute Value Inequalities

6.1 Two Key Results

The two keys to working with absolute value inequalities are

(i) $\quad |f(x)| < a \Leftrightarrow -a < f(x) < a$, for all $a \geq 0$

and

(ii) $\quad |f(x)| < a \Leftrightarrow f(x) < -a$ or $f(x) > a$, for all $a \geq 0$.

15. Find the solution set for $|x+5|<2$.

Solution

$|x+5|<2$

$\Leftrightarrow -2<x+5<2$

$\Leftrightarrow -2-5<x<2-5$

$\Leftrightarrow -7<x<-3$

In set notation, this is $(-7,-3)$.

■

16. Find the solution set for $|x+5|>2$.

Solution

$|x+5|>2$

$\Leftrightarrow x+5<-2$ or $x+5>2$

$\Leftrightarrow x<-2-5$ or $x>2-5$

$\Leftrightarrow x<-7$ or $x>-3$

In set notation, this is $(-\infty,-7)\cup(-3,\infty)$.

■

We can handle a two-sided absolute value inequality by breaking it into two separate problems and then find the intersection of the two solutions.

17. Find the solution set of $1 \leq |2x-1| \leq 5$.

Solution

$$|2x-1| \leq 5 \textbf{ and } |2x-1| \geq 1$$

$$-5 \leq 2x-1 \leq 5$$
$$-4 \leq 2x \leq 6$$
$$-2 \leq x \leq 3$$

and

$$|2x-1| \geq 1$$
$$2x-1 \leq -1 \textbf{ or } 2x-1 \geq 1$$
$$x \leq 0 \textbf{ or } x \geq 2.$$

Now we need to use the distributive rule for sets, namely, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. Recall that "∩" means "intersection" which means "and". Also recall that "∪" means "union" which means "or".

$$(-2 \leq x \leq 3) \textbf{ and } (x \leq 0 \textbf{ or } x \geq 2)$$

$$(-2 \leq x \leq 3 \textbf{ and } x \leq 0) \textbf{ or } (-2 \leq x \leq 3 \textbf{ and } x \geq 2)$$

$$(-2 \leq x \leq 0) \textbf{ or } (2 \leq x \leq 3).$$

In set notation, this is $[-2,0] \cup [2,3]$.

■

Problem Type 4: Mixed absolute value and linear inequalities

Find the solution set of $|2x - 5| \leq x + 3$.

Problem Type 5: Inequalities involving two absolute value functions

Find the solution set of $|2x + 5| + |4x + 7| < 30$.

For $a \leq b$,

$$\left|x - \left(\frac{a+b}{2}\right)\right| < \frac{b-a}{2} \iff x > a \text{ and } x < b$$

$$\left|x - \left(\frac{a+b}{2}\right)\right| > \frac{b-a}{2} \iff x < a \text{ or } x > b$$

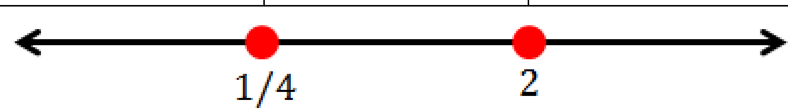
Problem 8.	Find the solution set to $|4x - 1| + 2x \geq 1 + |2 - x|$.

<u>Answer</u>

$$x \in \color{red}{(-\infty, -2]} \cup \color{red}{[4/7, \infty)}.$$

<u>Proof</u>

Break Points : $4x - 1 = 0 \Longrightarrow x = \color{red}{\mathbf{\frac{1}{4}}}$, $2 - x = 0 \Longrightarrow x = \color{red}{\mathbf{2}}$.

$4x - 1$	$-$	$+$	$+$
$2 - x$	$+$	$+$	$-$
			

Case 1. $\color{red}{x \in \left(-\infty, \frac{1}{4}\right]}$

$$|4x - 1| + 2x \geq 1 + |2 - x|$$

$$-(4x-1)+2x \geq 1+\left(+(2-x)\right)$$
$$-4x+1+2x \geq 1+2-x$$
$$-x \geq 2$$
$$x \leq -2$$

So what we are looking for are those $x \in (-\infty, 1/4]$ **and** $(-\infty, -2]$. That is,

$$x \in (-\infty, 1/4] \cap (-\infty, -2] = (-\infty, -2].$$

Case 2. $x \in [1/4, 2]$

$$|4x-1|+2x \geq 1+|2-x|$$
$$+(4x-1)+2x \geq 1+\left(+(2-x)\right)$$
$$4x-1+2x \geq 1+2-x$$
$$7x \geq 4$$
$$x \geq 4/7$$

So what we are looking for are those $x \in [1/4, 2]$ **and** $[4/7, \infty)$. That is,

$$x \in [1/4, 2] \cap [4/7, \infty) = [4/7, 2].$$

Case 3. $x \in [2, \infty)$

$$|4x-1|+2x \geq 1+|2-x|$$
$$+(4x-1)+2x \geq 1+\left(-(2-x)\right)$$
$$4x-1+2x \geq 1-2+x$$
$$5x \geq 0$$
$$x \geq 0$$

So what we are looking for are those $x \in [2, \infty)$ **and** $[0, \infty)$. That is,

$$x \in [2, \infty) \cap [0, \infty) = [2, \infty).$$

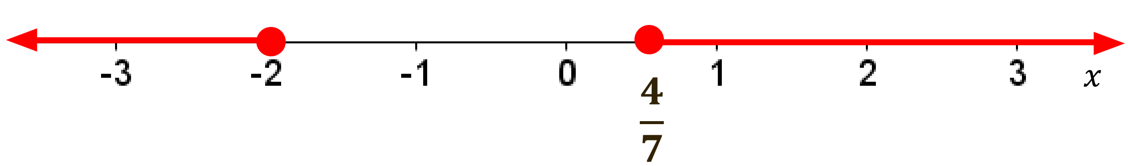
So the values of x where $|4x-1|+2x < 6-|2-x|$ are

$$x \in (-\infty, -2] \text{ or } [4/7, 2] \text{ or } [2, \infty).$$

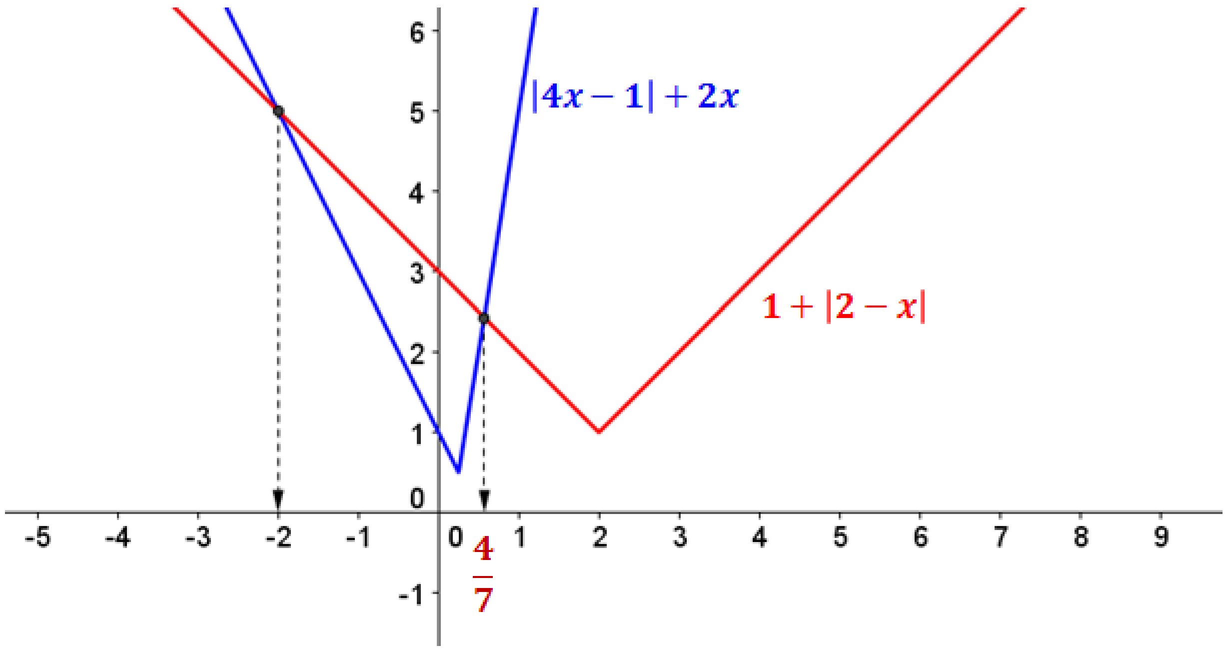
That is,

$$x \in (-\infty, -2] \cup [4/7, 2] \cup [2, \infty)$$
$$x \in (-\infty, -2] \cup [4/7, \infty).$$

In graph form, the answer is



To help you visualize what is going on, I used the computer program "Geogebra" to graph the problem. You can see from the graph that $|4x-1|+2x$ (in blue) is greater than or equal to (*i.e.* above or touching) $1 + |2 - x|$ (in red) when $-\infty < x \leq -2$ and also when $4/7 \leq x < \infty$ (as we just showed algebraically).



Example 3 Solve $|3-4x| \geq 9$.

Solution We have $|3-4x| \geq 9$.

$\Rightarrow \quad 3 - 4x \leq -9$ or $3 - 4x \geq 9 \quad$ (Since $|x| \geq a \Rightarrow x \leq -a$ or $x \geq a$)

$\Rightarrow \quad -4x \leq -12$ or $-4x \geq 6$

$\Rightarrow \quad x \geq 3 \quad$ or $\quad x \leq \dfrac{-3}{2} \quad$ (Dividing both sides by -4)

$\Rightarrow \quad x \in (-\infty, \dfrac{-3}{2}] \cup [3, \infty)$

Example 4 Solve $1 \leq |x - 2| \leq 3$.

Solution We have $1 \leq |x - 2| \leq 3$

$\Rightarrow \quad |x-2| \geq 1 \quad \text{and} \quad |x-2| \leq 3$

$\Rightarrow \quad (x - 2 \leq -1 \text{ or } x - 2 \geq 1) \quad \text{and} \quad (-3 \leq x - 2 \leq 3)$

$\Rightarrow \quad (x \leq 1 \text{ or } x \geq 3) \quad \text{and} \quad (-1 \leq x \leq 5)$

$\Rightarrow \quad x \in (-\infty, 1] \cup [3, \infty) \quad \text{and} \quad x \in [-1, 5]$

Combining the solutions of two inequalities, we have

$$x \in [-1, 1] \cup [3, 5]$$

Example 5 Solve for x, $|x+1| + |x| > 3$.

Solution On LHS of the given inequality, we have two terms both containing modulus. By equating the expression within the modulus to zero, we get $x = -1, 0$ as critical points. These critical points divide the real line in three parts as $(-\infty, -1)$, $[-1, 0)$, $[0, \infty)$.

Case I When $-\infty < x < -1$

$$|x+1| + |x| > 3 \Rightarrow -x - 1 - x > 3 \Rightarrow x < -2.$$

Case II When $-1 \leq x < 0$,

$$|x+1| + |x| > 3 \Rightarrow x + 1 - x > 3 \Rightarrow 1 > 3 \qquad \text{(not possible)}$$

Case III When $0 \leq x < \infty$,

$$|x+1| + |x| > 3 \Rightarrow x + 1 + x > 3 \Rightarrow x > 1.$$

Combining the results of cases (I), (II) and (III), we get

$$x \in (-\infty, -2) \cup (1, \infty)$$

Example 6 Solve for x, $\frac{|x+3|+x}{x+2} > 1$

Solution We have $\frac{|x+3|+x}{x+2} > 1$

$$\Rightarrow \quad \frac{|x+3|+x}{x+2} - 1 > 0$$

$$\Rightarrow \quad \frac{|x+3|-2}{x+2} > 0$$

Now two cases arise:

Case I When $x + 3 \geq 0$, i.e., $x \geq -3$. Then

$$\frac{|x+3|-2}{x+2} > 0 \Rightarrow \frac{x+3-2}{x+2} > 0$$

$$\Rightarrow \quad \frac{x+1}{x+2} > 0$$

$$\Rightarrow \quad \{(x+1) > 0 \text{ and } x+2 > 0\} \quad \text{or} \quad \{x+1 < 0 \text{ and } x+2 < 0\}$$

$$\Rightarrow \quad \{x > -1 \text{ and } x > -2\} \quad \text{or} \quad \{x < -1 \text{ and } x < -2\}$$

$$\Rightarrow \quad x > -1 \quad \text{or} \quad x < -2$$

$$\Rightarrow \quad x \in (-1, \infty) \quad \text{or} \quad x \in (-\infty, -2)$$

$$\Rightarrow \quad x \in (-3, -2) \cup (-1, \infty) \quad [\text{Since } x \geq -3] \qquad \ldots (1)$$

Case II When $x + 3 < 0$, i.e., $x < -3$

$$\frac{|x+3|-2}{x+2} > 0 \quad \Rightarrow \quad \frac{-x-3-2}{x+2} > 0$$

$$\Rightarrow \quad \frac{-(x+5)}{x+2} > 0 \quad \Rightarrow \quad \frac{x+5}{x+2} < 0$$

$\Rightarrow$ $(x + 5 < 0 \text{ and } x + 2 > 0)$ or $(x + 5 > 0 \text{ and } x + 2 < 0)$

$\Rightarrow$ $(x < -5 \text{ and } x > -2)$ or $(x > -5 \text{ and } x < -2)$

it is not possible.

$\Rightarrow$ $x \in (-5, -2)$ ... (2)

Combining (I) and (II), the required solution is

$$x \in (-5, -2) \cup (-1, \infty)$$

7 Absolute Value as a Square Root

$$\sqrt{x^2} = |x|$$

8 Quadratic Equations where x^2 Term Drops Out

9 Rational Function Equations

Find the set of all x values such that

$$\frac{5}{2x-3} = \frac{3}{x+5}.$$

10 Rational Function Inequalities

If a nor b equals 0 then

$$a < b \Leftrightarrow \frac{1}{b} < \frac{1}{a}.$$

CUT POINTS

If $r(x) = P(x)/Q(x)$ is a rational function, the **cut points** of $r(x)$ are the values of x at which either $P(x) = 0$ or $Q(x) = 0$.

SOLVING POLYNOMIAL INEQUALITIES

1. **Move All Terms to One Side.** Rewrite the inequality so that all nonzero terms appear on one side of the inequality symbol.
2. **Factor the Polynomial.** Factor the polynomial into irreducible factors, and find the **real zeros** of the polynomial.
3. **Find the Intervals.** List the intervals determined by the real zeros.
4. **Make a Table or Diagram.** Use test values to make a table or diagram of the signs of each factor in each interval. In the last row of the table determine the sign of the polynomial on that interval.
5. **Solve.** Determine the solutions of the inequality from the last row of the table. Check whether the **endpoints** of these intervals satisfy the inequality. (This may happen if the inequality involves $\leq$ or $\geq$.)

SOLVING RATIONAL INEQUALITIES

1. **Move All Terms to One Side.** Rewrite the inequality so that all nonzero terms appear on one side of the inequality symbol. Bring all quotients to a common denominator.
2. **Factor Numerator and Denominator.** Factor the numerator and denominator into irreducible factors, and then find the **cut points**.
3. **Find the Intervals.** List the intervals determined by the cut points.
4. **Make a Table or Diagram.** Use test values to make a table or diagram of the signs of each factor in each interval. In the last row of the table determine the sign of the rational function on that interval.
5. **Solve.** Determine the solution of the inequality from the last row of the table. Check whether the **endpoints** of these intervals satisfy the inequality. (This may happen if the inequality involves $\leq$ or $\geq$.)

Find the set of all x values such that

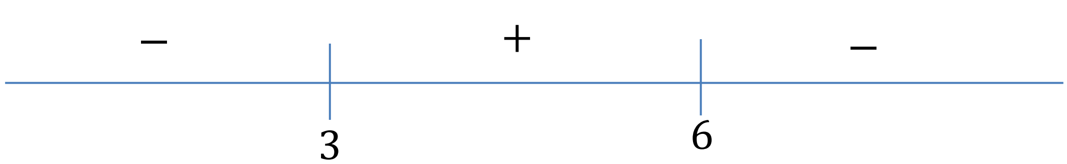
$$\frac{x+3}{x-3} < 3.$$

$$\frac{x+3}{x-3} - \frac{3(x-3)}{x-3} < 0$$

$$\frac{x+3-3x+9}{x-3} < 0$$

$$\frac{-2x+12}{x-3} < 0$$

$$\frac{-2(x-6)}{x-3} < 0$$



$$(-\infty, 3) \cup (6, \infty)$$

Find the set of all x values such that

$$\frac{1}{x-2} \geq \frac{2}{x+3}.$$

EXAMPLE 3 ■ Solving a Rational Inequality

Solve the inequality

$$\frac{1-2x}{x^2-2x-3} \geq 1$$

SOLUTION We follow the above guidelines.

Move all terms to one side. We move all terms to the left-hand side of the inequality.

$$\frac{1-2x}{x^2-2x-3} - 1 \geq 0 \qquad \text{Move terms to LHS}$$

$$\frac{(1-2x)-(x^2-2x-3)}{x^2-2x-3} \geq 0 \qquad \text{Common denominator}$$

$$\frac{4-x^2}{x^2-2x-3} \geq 0 \qquad \text{Simplify}$$

The left-hand side of the inequality is a rational function.

Example 2 Solve $\frac{x-2}{x+5} > 2$

Solution We have $\frac{x-2}{x+5} > 2$

$\Rightarrow \quad \frac{x-2}{x+5} - 2 > 0$ [Subtracting 2 from each side]

$\Rightarrow \quad \frac{-(x+12)}{x+5} > 0$

$\Rightarrow \quad \frac{x+12}{x+5} < 0$ (Multiplying both sides by – 1)

$\Rightarrow \quad x + 12 > 0$ and $x + 5 < 0$ [Since $\frac{a}{b} < 0 \Rightarrow a$ and b are of opposite signs]

or

$x + 12 < 0$ and $x + 5 > 0$

$\Rightarrow \quad x > -12$ and $x < -5$

or

$x < -12$ and $x > -5$ (Not possible)

Therefore, $-12 < x < -5$, i.e. $x \in (-12, -5)$

Example 7 Solve the following system of inequalities :

$$\frac{x}{2x+1} \geq \frac{1}{4}, \ \frac{6x}{4x-1} < \frac{1}{2}$$

Solution From the first inequality, we have $\frac{x}{2x+1} - \frac{1}{4} \geq 0$

$\Rightarrow \quad \frac{2x-1}{2x+1} \geq 0$

$\Rightarrow \quad (2x - 1 \geq 0 \text{ and } 2x + 1 > 0) \text{ or } (2x - 1 \leq 0 \text{ and } 2x + 1 < 0)$ [Since $2x + 1 \neq 0$)

$\Rightarrow \quad (x \geq \frac{1}{2} \text{ and } x > -\frac{1}{2}) \text{ or } (x \leq \frac{1}{2} \text{ and } x < -\frac{1}{2})$

$\Rightarrow \quad x \geq \frac{1}{2} \text{ or } x < -\frac{1}{2}$

$\Rightarrow \quad x \in (-\infty, -\frac{1}{2}) \cup [\frac{1}{2}, \infty)$ ... (1)

From the second inequality, we have $\frac{6x}{4x-1} - \frac{1}{2} < 0$

$\Rightarrow \quad \frac{8x+1}{4x-1} < 0$

$\Rightarrow \quad (8x + 1 < 0 \text{ and } 4x - 1 > 0) \quad \text{or} \quad (8x + 1 > 0 \text{ and } 4x - 1 < 0)$

$\Rightarrow \quad (x < -\frac{1}{8} \text{ and } x > \frac{1}{4}) \quad \text{or} \quad (x > -\frac{1}{8} \text{ and } x < \frac{1}{4})$

$\Rightarrow \quad x \in (-\frac{1}{8}, \frac{1}{4})$ (Since the first is not possible) ... (2)

Note that the common solution of (1) and (2) is null set. Hence, the given system of inequalities has no solution.

18.	Find all values of x such that $\frac{1}{3x+5} > 2$.

Solution

Would it be correct to say

$$\frac{1}{3x+5} > 2 \Leftrightarrow 3x+5 < \frac{1}{2} \ ?$$

No. Why not?

■

11 Conjunctions and Disjunctions

11.1 Definitions of Conjunction and Disjunction

11.1.1 Conjunction

The compound (two part) inequality

(inequality statement #1) **and** (inequality statement #2)

is called an inequality **conjunction**. The "and" implies an intersection (overlap) of the answers of the two inequality statements.

Note that the simple conjunction $x > a$ and $x < b$ simplifies $a < x < b$.

11.1.2 Disjunction

The compound (two part) inequality

(inequality statement #1) **or** (inequality statement #2)

is called an inequality **disjunction**. The "or" implies the union (take everything) of the answers of the two inequality statements.

11.2 Absolute Value Inequalities, Conjunctions and Disjunctions

For all real numbers a and b, $b > 0$, the following statements are true.

1. If $|a| < b$, then $a < b$ and $a > -b$ (a conjunction). This conjunction simplifies to $-b < a < b$.

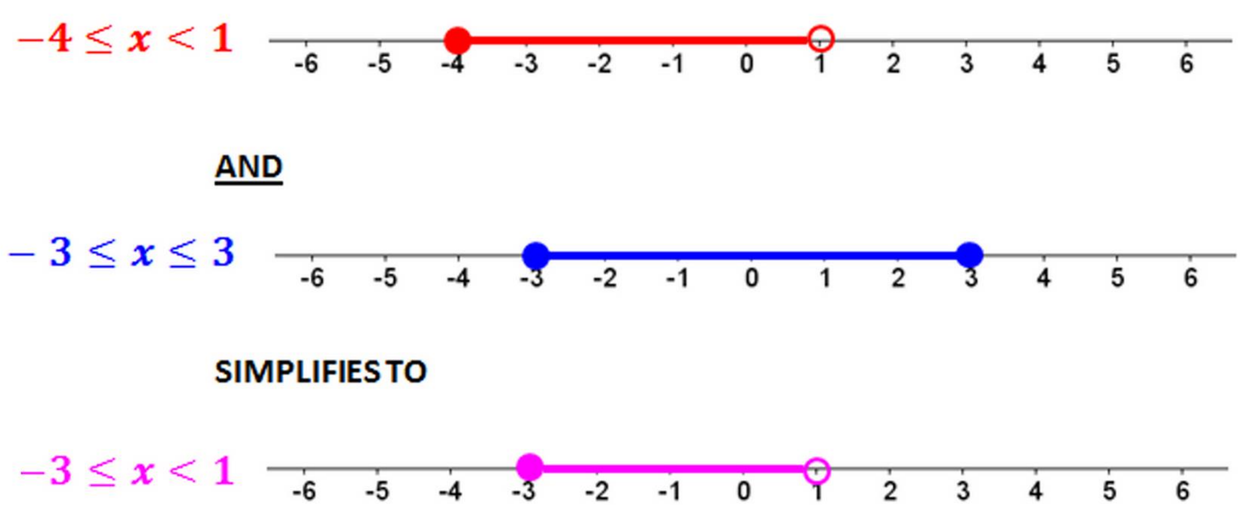
2. If $|a| > b$, then $a > b$ or $a < -b$ (a disjunction).

   These statements are also true of $\leq$ and $\geq$.

11.3 Simplifying Conjunctions and Disjunctions

Problem 1. Simplify the conjunction: $-4 \leq x < 1$ **and** $-3 \leq x \leq 3$.

Solution



In set notation

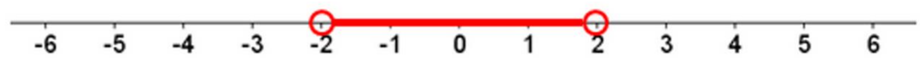
$$-4 \leq x < 1 \textbf{ and } -3 \leq x \leq 3 \equiv [-3,1)$$

■

Problem 2. Simplify the disjunction: $-2 < x < 2$ **or** $-3 \leq x \leq 0$.

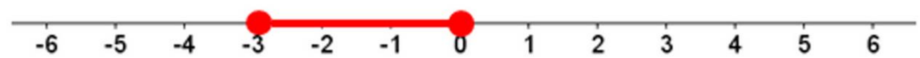
Solution

$-2 < x < 2$



OR

$-3 \leq x \leq 0$



SIMPLIFIES TO

$-3 \leq x < 2$

In set notation

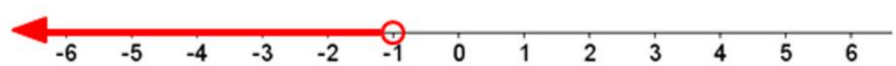
$$-2 < x < 2 \ \textbf{ or } \ -3 \leq x \leq 0 \equiv [-3,2).$$

■

Problem 3. Simplify the disjunction: $-\infty < x < -1$ **or** $-1 < x < \infty$.

Solution

$-\infty < x < -1$



OR

$-1 < x < \infty$

SIMPLIFIES TO

In set notation

$$-\infty < x < -1 \ \textbf{ or } \ -1 < x < \infty \equiv (-\infty,-1) \cup (-1,\infty).$$

■