MSHSML Meet 2, Event B
Study Guide

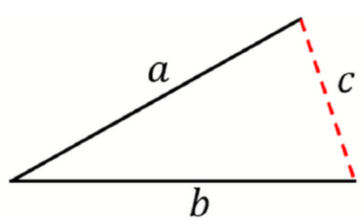
2B Triangular Figures and Solids

Medians
Angle bisectors
Altitudes
Ceva's and Stewart's Theorems
Area of a triangle (including Hero's Formula)
Triangular prisms & pyramids (including volume and surface area)

Contents

1. Triangle Basics

Criteria for separating right, acute and obtuse triangle	
Let a, b and c be the lengths of the three sides of triangle ΔABC with longest side c. Right Triangle $\Leftrightarrow$ $a^2 + b^2 = c^2$ Acute Triangle $\Leftrightarrow$ $a^2 + b^2 > c^2$ Obtuse Triangle $\Leftrightarrow$ $a^2 + b^2 < c^2$	**(1)**

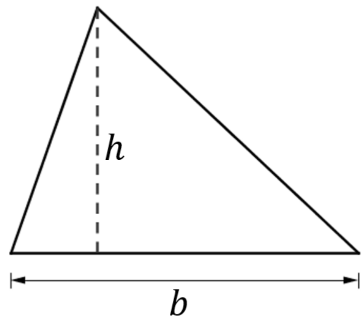
Missing Third Side of a Triangle	
If side lengths a and b are known, the missing third side length c must satisfy the inequalities $|a - b| < c < |a + b|$ 	**(2)**

Ordered Angles	
The smallest angle in a triangle is opposite the shortest side of a triangle and the largest angle in a triangle is oppoiste the longest side of a triangle.	**(3)**

2. Area of a Triangle

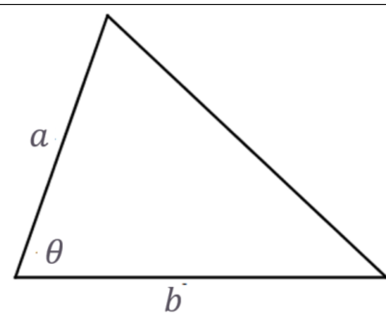
Base Height Formula

$$\text{Area} = \frac{1}{2} \cdot b \cdot h \qquad (4)$$

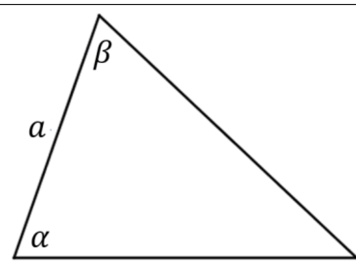


Side Angle Side Formula

$$\text{Area} = \frac{1}{2} \cdot a \cdot b \cdot \sin(\theta) \qquad (5)$$

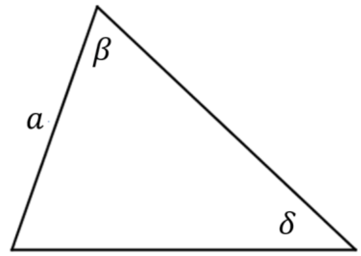


Angle Side Angle Formula



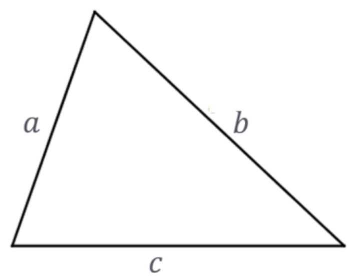
$$\text{Area} = \frac{a^2 \sin(\alpha) \sin(\beta)}{2 \sin(180° - \alpha - \beta)} \qquad (6)$$

Side Angle Angle Formula



$$\text{Area} = \frac{a^2 \sin(180° - \beta - \delta)\sin(\beta)}{2\sin(\delta)} \tag{7}$$

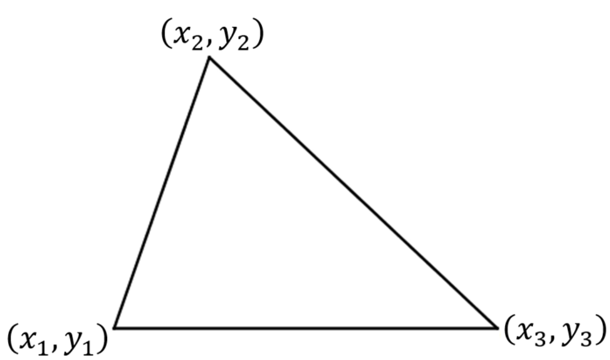
Side Side Side Formula (Heron’s or Hero’s Formula)



$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)} \tag{8}$$

where $s = (1/2)(a+b+c)$.

Vertex Coordinates Formula (Shoelace Formula)



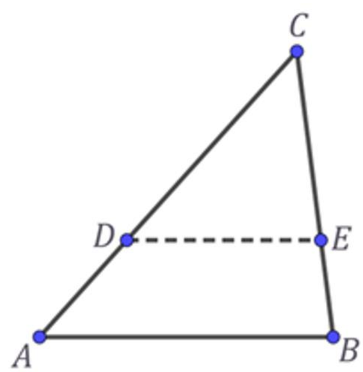
Shoelace Mnemonic

$$\text{Area} = \frac{1}{2}|x_1y_2 + x_2y_3 + x_3y_1 - y_1x_2 - y_2x_3 - y_3x_1| \tag{9}$$

Note 1: You can easily construct the formulas for Cases 3 and 4 by using the Law of Sines and the formula for Case 2. I find this easier than trying to remember the easily confused formulas for these two cases.

Note 2: There are additional formulas for finding the area of a triangle that we will consider later in this guide.

3. Triangle Proportionality Theorems

<table>
<tr><th colspan="3">Triangle Proportionality Theorems</th></tr>
<tr><td rowspan="2">
</td><td><u>Side</u> Proportionality (Side Splitter Theorem)

$\frac{AD}{CD} = \frac{BE}{CE}$ if and only if $DE \parallel AB$</td><td rowspan="2">(10)</td></tr>
<tr><td><u>Base</u> Proportionality

$\frac{DE}{AB} = \frac{DC}{AC} = \frac{EC}{BC}$ if and only if $DE \parallel AB$</td></tr>
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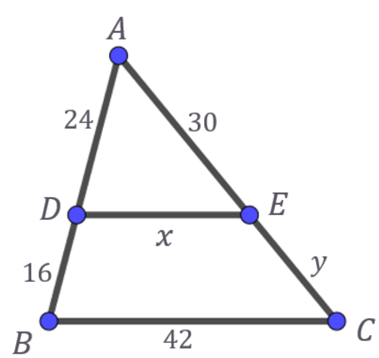
Notation: $l_1 \parallel l_2$ means lines l_1 and l_2 are parallel.

Example: Find x and y assuming $DE \parallel BC$.



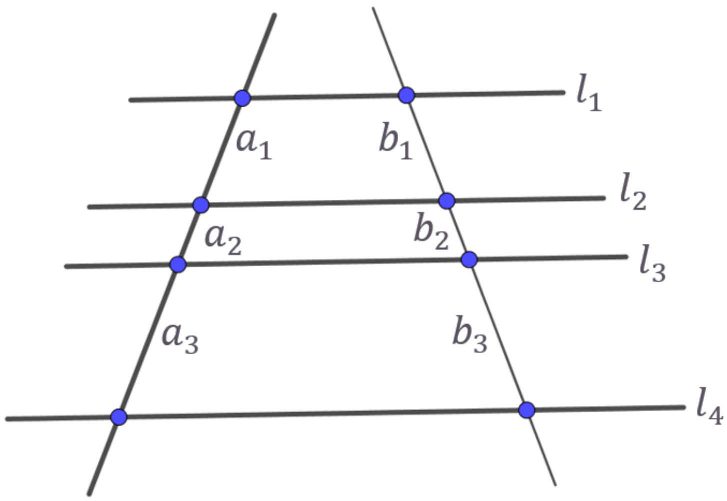
Solution

By the Side Proportionality Theorem (Side Splitter Theorem)

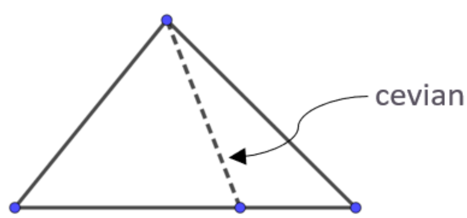
$$\frac{24}{16} = \frac{30}{y} \Longrightarrow y = \frac{30 \cdot 16}{24} = 20.$$

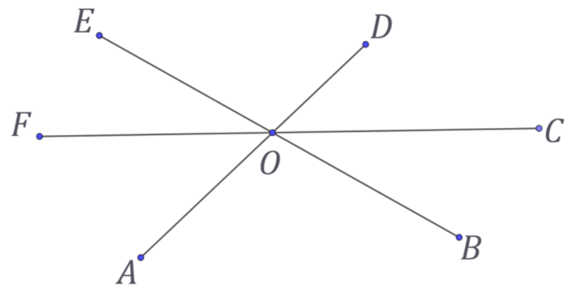
and by the Base Proportionality Theorem,

$$\frac{24}{24+16} = \frac{x}{42} \Longrightarrow x = 25.2.$$

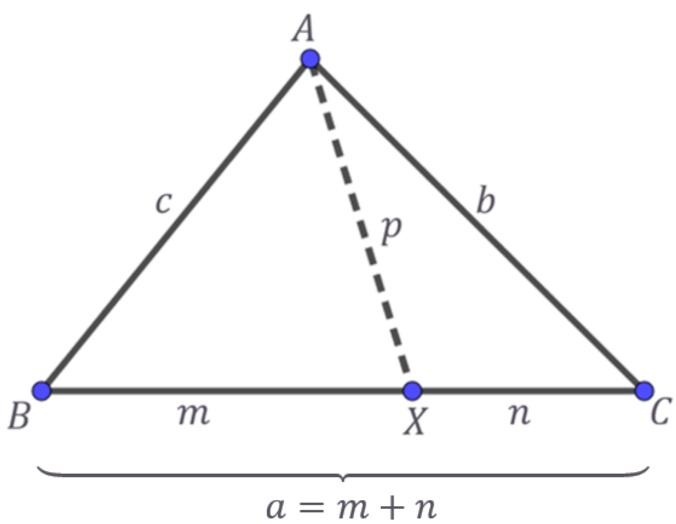
Extended Side Splitter Theorem (multiple parallel lines)	
If lines l_1, l_2, l_3 and l_4 are parallel, then $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$. 	**(11)**

4. Definitions

Cevian	
A cevian is any segment from a vertex of a triangle to a point on the opposite side. 	**(12)**

Concurrent Lines	
Three or more lines that intersect at a single point are said to be concurrent.  Lines $\overline{AD}$, $\overline{BE}$ and $\overline{CF}$ are concurrent because they all intersect at the single point O.	**(13)**

5. Stewarts's Theorem

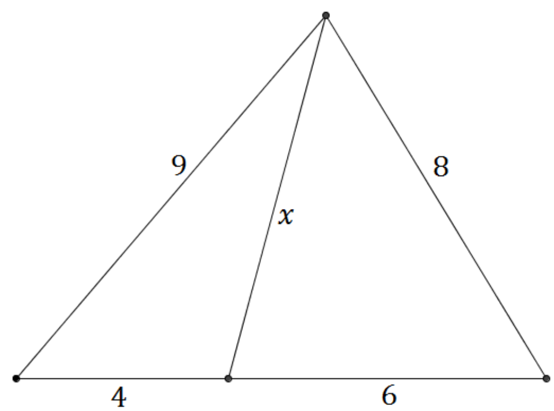
Stewart's Theorem	
$a(p^2 + mn) = b^2m + c^2n$ 	**(14)**

Special cases of Stewart's Theorem	
a) $\overline{AX}$ is a median (*i.e.* $m = n$) $$2p^2 = b^2 + c^2 - 2m^2$$	**(15)**

b)	ΔABC is an isosceles triangle with base $\overline{BC}$ (*i.e.* $b = c$) $$p^2 = b^2 - mn$$	**(16)**
c)	$\overline{AX}$ bisects $\angle ABC$ $$m = \frac{ac}{b+c}, \;\; n = \frac{ab}{b+c}$$ and $$p^2 = bc\left(\frac{(b+c)^2 - a^2}{(b+c)^2}\right) = bc - mn$$	**(17)**

Two simple examples to illustrate using Stewarts's Theorem

(*i*) Use Stewart's Theorem to show that $x^2 = 50.2$ in the diagram below.



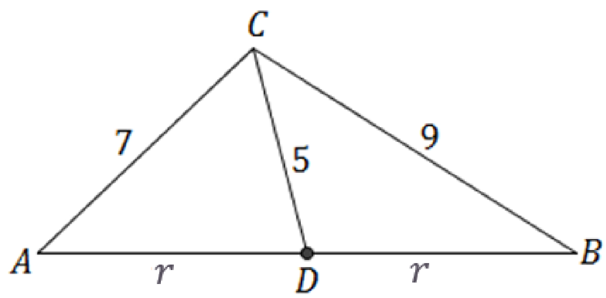
By a direct application of Stewart's Theorem we have

$$(4+6)(x^2 + 4 \cdot 6) = 9^2 \cdot 6 + 8^2 \cdot 4$$

$$10(x^2 + 24) = 486 + 256$$

$$x^2 + 24 = (486 + 256)/10 \Longrightarrow x^2 = \frac{486 + 256}{10} - 24 = 50.2$$

(*ii*) Use Stewart's Theorem to show that $r^2 = 40$ in the diagram below.



By a direct application of Stewart's Theorem we have

$$(2r)(5^2 + r \cdot r) = 7^2 r + 9^2 r$$

$$r \cdot 2 \cdot (5^2 + r^2) = r \cdot (7^2 + 9^2)$$

$$2(5^2 + r^2) = 7^2 + 9^2$$

$$5^2 + r^2 = \frac{7^2 + 9^2}{2} \Longrightarrow r^2 = \left(\frac{7^2 + 9^2}{2}\right) - 5^2 = 40$$

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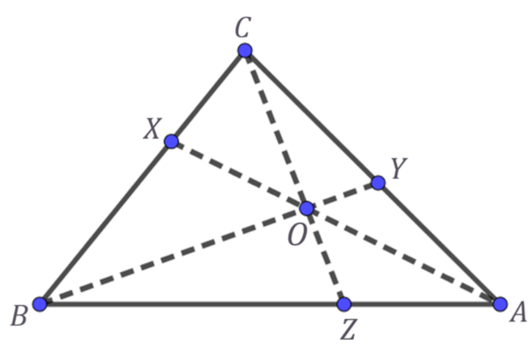
6. Ceva's Theorem

Ceva's Theorem



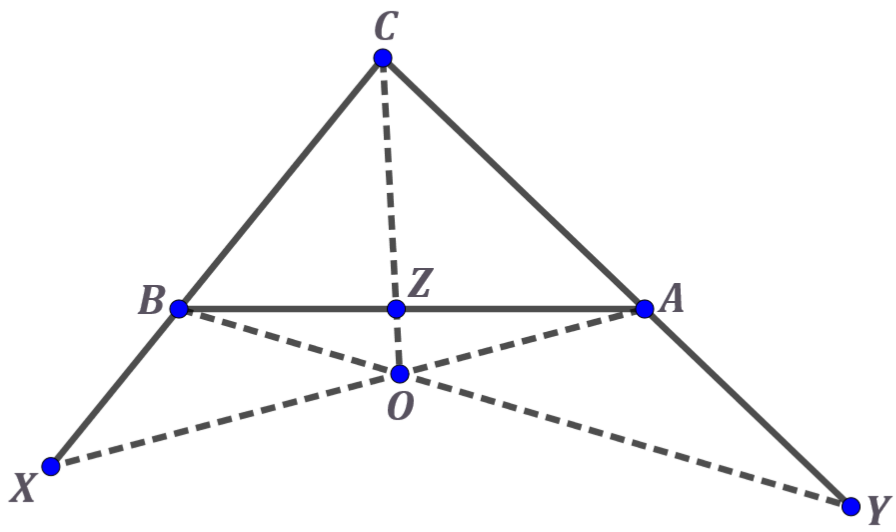
$\dfrac{BX}{XC} \cdot \dfrac{CY}{YA} \cdot \dfrac{AZ}{ZB} = 1$ if and only if cevians $\overline{AX}, \overline{BY}$ and $\overline{CZ}$ are concurrent **(18)**

and

$$\frac{OX}{AX}+\frac{OY}{BY}+\frac{OZ}{CZ}=1. \tag{19}$$

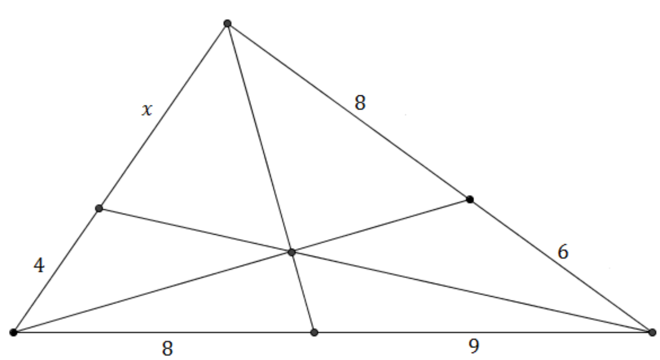
Note: Ceva's Theorem remains true even if the point of concurrency O of $\overline{AX}, \overline{BY}$ and $\overline{CZ}$ falls ***outside*** ΔABC. That is, in the figure below it is still the case that

$\frac{BX}{XC}\cdot\frac{CY}{YA}\cdot\frac{AZ}{ZB}=1$ if and only if cevians $\overline{AX}, \overline{BY}$ and $\overline{CZ}$ are concurrent.



Two simple examples to illustrate using Ceva's Theorem

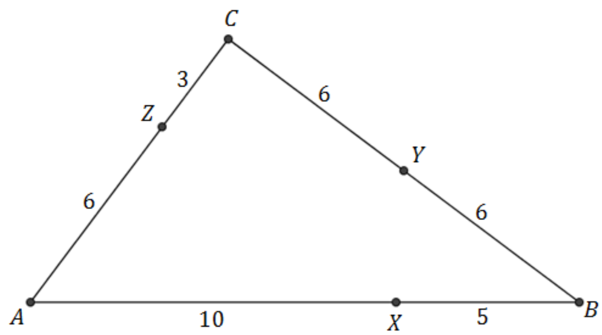
(*i*) The three cevians in the figure below appear to be concurrent (*i.e.* having a common point of intersection). Assuming that the three cevians really are concurrent, use Ceva's Theorem to show that $x=6$.



Because these three cevians are concurrent, we can apply Ceva's Theorem. Ceva's Theorem tells us that

$$\frac{8}{9} \cdot \frac{6}{8} \cdot \frac{x}{4} = 1 \implies x = \frac{4 \cdot 8 \cdot 9}{8 \cdot 6} = 6.$$

(*ii*) If you draw in the segments $\overline{AY}$, $\overline{BZ}$, and $\overline{CX}$ they appear to be concurrent, that is, they appear to have a common point of intersection. Use the converse of Ceva's Theorem to prove that they really are concurrent.



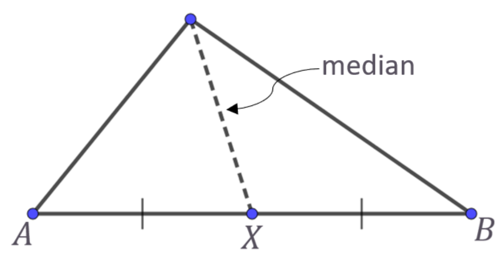
By Ceva's Theorem and its converse, these three cevians are concurrent if and only if

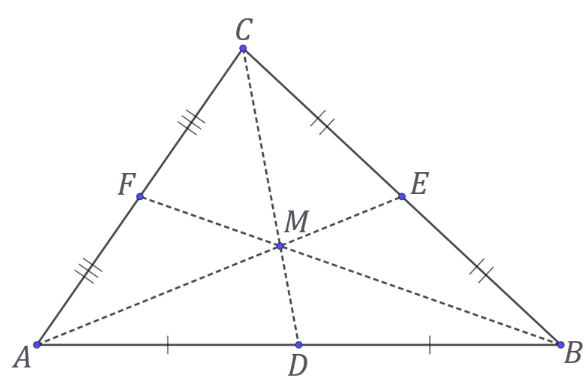
$$\frac{AX}{XB} \cdot \frac{BY}{YC} \cdot \frac{CZ}{ZA} = 1.$$

Checking this criteria in this example, we find

$$\frac{AX}{XB} \cdot \frac{BY}{YC} \cdot \frac{CZ}{ZA} = \frac{10}{5} \cdot \frac{6}{6} \cdot \frac{3}{6} = \frac{180}{180} = 1.$$

7. Medians

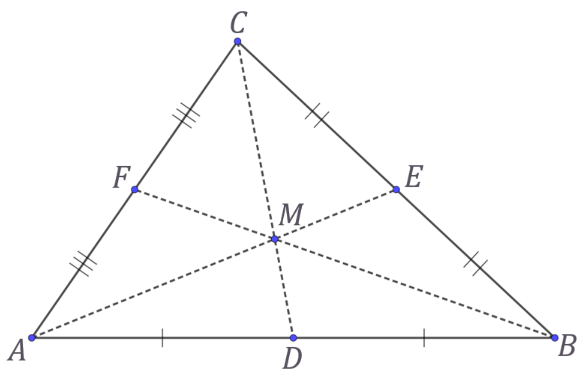
Median	
The median is that cevian which bisects the opposite side ($AX = BX$).  A triangle has three medians (one from each vertex).	**(20)**

Centroid Theorem	
The medians of a triangle are concurrent (intersect at a common point). The point of concurrence M of the medians is called the **center** or **centroid** of a triangle. The centroid is the balance point or center of gravity.  	**(21)**

Ordered Medians Theorem	
The shortest median is drawn to the longest side of a triangle and the longest median is drawn to the shortest side of a triangle. Notice how *shortest* is paired with *longest* which is counter to how angles are paired with sides.	**(22)**

Median Ratio Theorem (Trisection Property of Centroid)

The centroid M divides each median into segments whose lengths are in the ratio 2 : 1 where the longer segment is the segment from the vertex to the centroid.



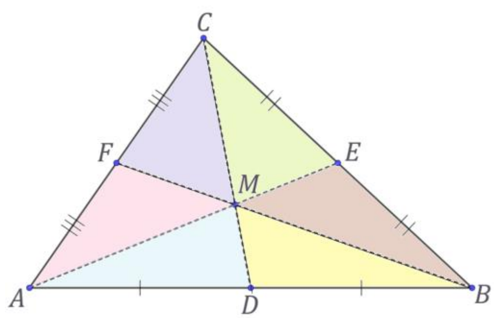
That is,

$$\frac{AM}{ME} = \frac{BM}{MF} = \frac{CM}{MD} = 2.$$

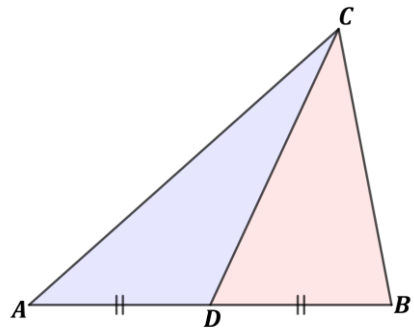
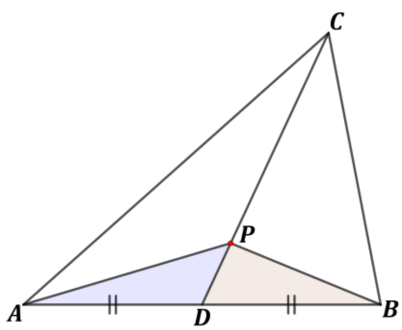
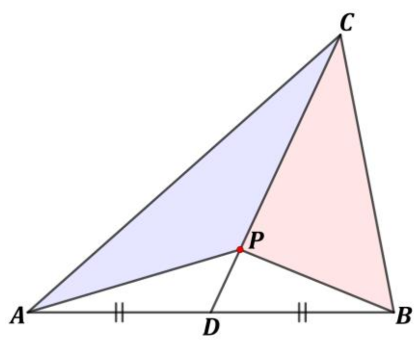
(23)

Centroid Area Theorem

The medians of ΔABC split ΔABC into 6 smaller triangles of equal area. That is,
Area $\Delta AMD =$ Area $\Delta BMD =$ Area $\Delta BME =$ Area $\Delta CME =$ Area $\Delta CMF =$
Area $\Delta AMF = (1/6)$Area ΔABC.

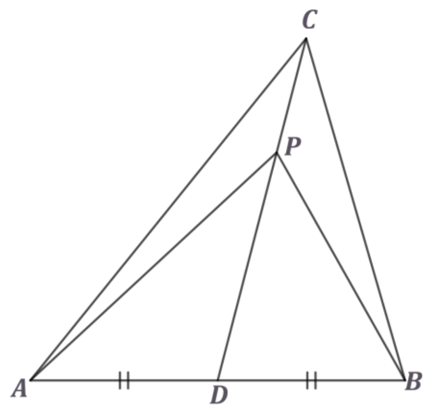


(24)

Median Area Theorems	
Medians split a triangle into two triangles of equal area. That is if $\overline{CD}$ is a median of ΔABC then ΔACD and ΔBCD have the same area.  $\text{Area}(ACD) = \text{Area}(BCD)$	**(25)**
For *any* point P on median $\overline{CD}$ of ΔABC, ΔADP and ΔBDP have the same area and also ΔAPC and ΔBPC have the same area.  $\text{Area}(ADP) = \text{Area}(BDP)$  $\text{Area}(APC) = \text{Area}(BPC)$	**(26)**

Exercise

Suppose $\overline{CD}$ is a median of triangle ABC and suppose that P is the point on $\overline{CD}$ such that $CP\!:PD = 1\!:2$. What is the ratio $\text{Area}(ACP)\!:\text{Area}(ABC)$?



Solution

By the Side Angle Side formula for the area of a triangle, we have that

$$\text{Area}(ADP) = AD \cdot DP \cdot \sin(\angle ADP)$$
$$\text{Area}(ADC) = AD \cdot DC \cdot \sin(\angle ADC).$$

But $\angle ADP \equiv \angle ADC$, so

$$\text{Area}(ADC) = AD \cdot DC \cdot \sin(\angle ADP).$$

Therefore,

$$\frac{\text{Area}(ADP)}{\text{Area}(ADC)} = \frac{AD \cdot DP \cdot \sin(\angle ADP)}{AD \cdot DC \cdot \sin(\angle ADC)} = \frac{DP}{DC} = \frac{2}{3}.$$

From the Median Area Theorem, Area(ADC) / Area$(ABC) = 1/2$. So it follows that

$$\frac{\text{Area}(ADP)}{\text{Area}(ABC)} = \left(\frac{\text{Area}(ADC)}{\text{Area}(ABC)}\right) \cdot \left(\frac{\text{Area}(ADP)}{\text{Area}(ADC)}\right) = \left(\frac{1}{2}\right)\left(\frac{2}{3}\right) = \frac{1}{3}.$$

Additionally, we know that

$$\text{Area}(ACP) = \text{Area}(ADC) - \text{Area}(ADP).$$

Therefore,

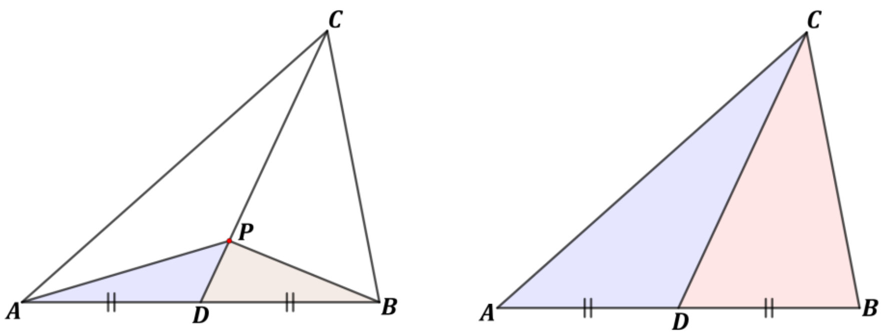
$$\frac{\text{Area}(ACP)}{\text{Area}(ABC)} = \frac{\text{Area}(ADC) - \text{Area}(ADP)}{\text{Area}(ABC)} = \left(\frac{\text{Area}(ADC)}{\text{Area}(ABC)}\right) - \left(\frac{\text{Area}(ADP)}{\text{Area}(ABC)}\right)$$
$$= \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

∎

Using the idea developed in this last exercise we can state the following general result.

Median Area Ratio Theorem

For *any* point P on median $\overline{CD}$ of ΔABC, the areas of ΔABP and ΔABC are in the same ratio as DP and DC.

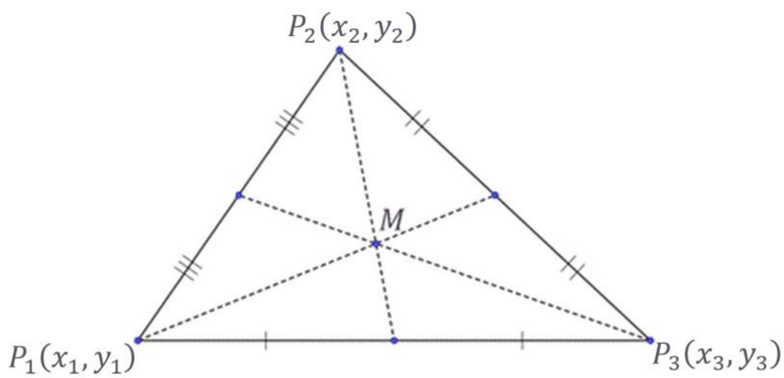


(27)

$$\text{Area}(ABP) : \text{Area}(ABC) = DP : DC.$$

Coordinates of the Centroid

For a triangle with vertices $P_1(x_1, y_1), P_2(x_2, y_2)$ and $P_3(x_3, y_3)$,



(28)

the centroid M has coordinates

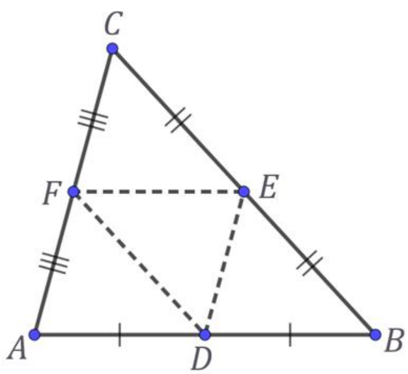
$$M = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right).$$

Medial Triangle

The triangle whose vertices are the midpoints of the sides of a given triangle is called the *medial triangle*. **(29)**

Medial Triangle Theorem

The medial triangle subdivides a triangle into four congruent sub-triangles. **(30)**



ΔDEF is the medial triangle of ΔABC

$AD = BD = EF$

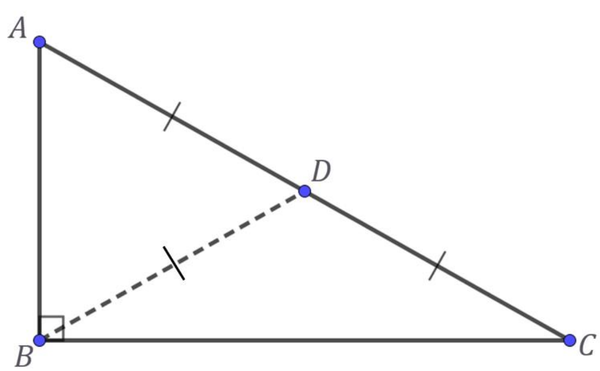
$BE = CE = DF$

$AF = CF = ED$

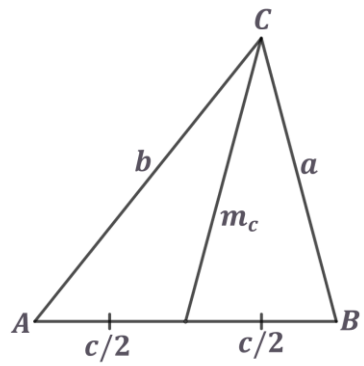
$\Delta ADF \equiv \Delta DBE \equiv \Delta FEC \equiv \Delta EFD$

Median to Hypotenuse Theorem

The median drawn from a right angle to the hypotenuse in a right triangle is half as long as the hypotenuse. That is, $BD = AD = CD$. **(31)**



Length of a Median **(Apollonuis's Formula)**



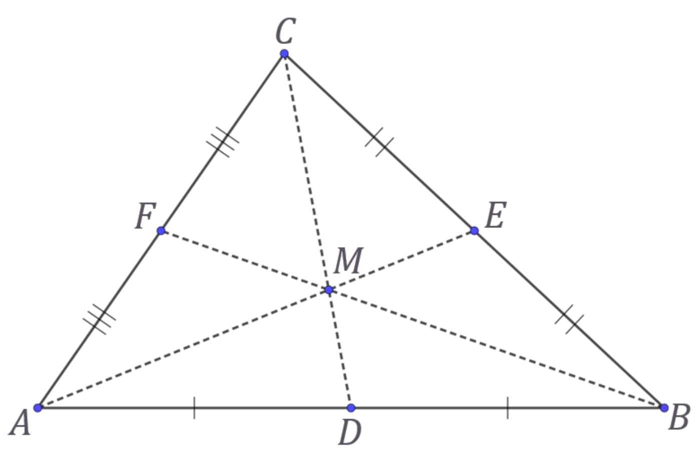
Length of Median: $m_c = \sqrt{\frac{a^2}{2} + \frac{b^2}{2} - \frac{c^2}{4}}$

(32)

Heron "like" formula for area of a triangle using the medians

The area of ΔABC with medians $m_a = AE,\ m_b = BF$ and $m_c = CD$ equals

$$\text{Area} = \frac{4}{3}\sqrt{t(t - m_a)(t - m_b)(t - m_c)}$$

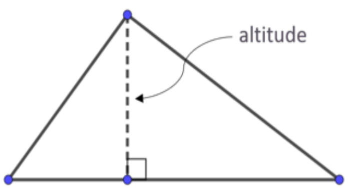
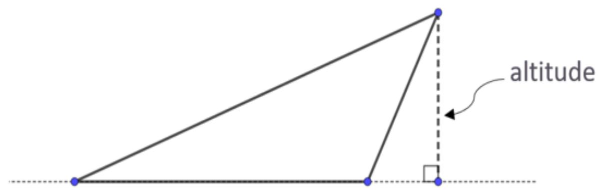


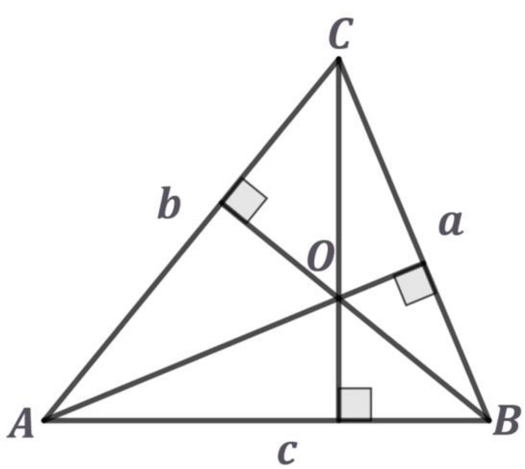
where $t = (1/2)(m_a + m_b + m_c)$.

(33)

Note: As far as I know there has never been a MSHSML problem where you needed the above formula for finding the area of a triangle using the medians. I have included for the most part in case you are using these notes for other purposes.

8. Altitudes

Altitude	
The perpendicular from a vertex to the line containing the opposite side of a triangle. A triangle has three altitudes (one from each vertex) 	**(34)**

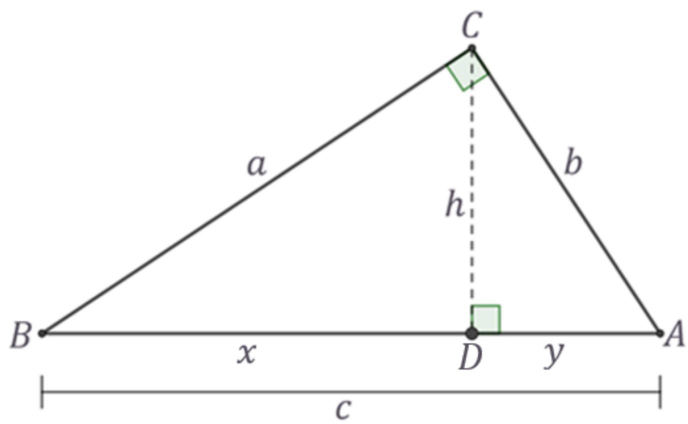
Orthocenter	
The three altitudes of a triangle intersect at a common point O called its *orthocenter*. 	**(35)**

Ordered Altitudes Theorem	
The shortest altitude is drawn to the longest side of a triangle and the longest altitude is drawn to the smallest side of a triangle. Notice how *shortest* is paired with *longest* which is counter to how angles are paired with sides.	**(36)**

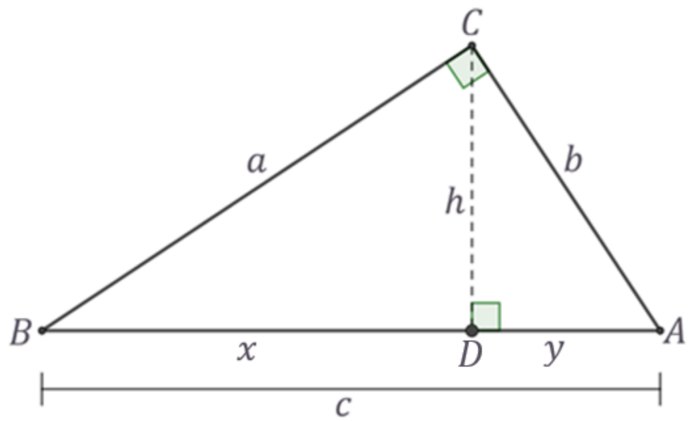
Orthocenter Theorem	
The altitudes of a triangle are concurrent. The point of concurrence is called the orthocenter of that triangle.	**(37)**

Orthic Triangle	
The triangle whose vertices are the feet of the altitudes of a given triangle.	**(38)**

Orthic Triangle Theorem	
The orthic triangle has the minimum perimeter among all triangles whose vertices are on the three sides.	**(39)**

Altitude to Hypotenuse Theorem (Three Similar Triangles)	
 $\Delta ACB \sim \Delta ADC \sim \Delta CDB$	**(40)**

Geometric Means Theorem



$$h^2 = x \cdot y \qquad ab = ch$$

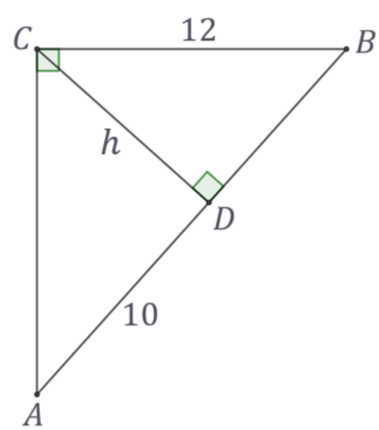
$$a^2 = x \cdot c \qquad \frac{1}{h^2} = \frac{1}{a^2} + \frac{1}{b^2}$$

$$b^2 = y \cdot c$$

(41)

A simple example to illustrate using the Geometric Means Theorem

Find h.



Let $x = \overline{BD}$. Then by the Geometric Means Theorem above we have

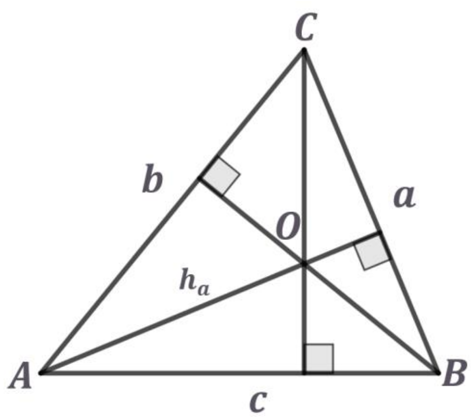
$$h^2 = 10x \ \text{ and } \ 12^2 = x(10 + x).$$

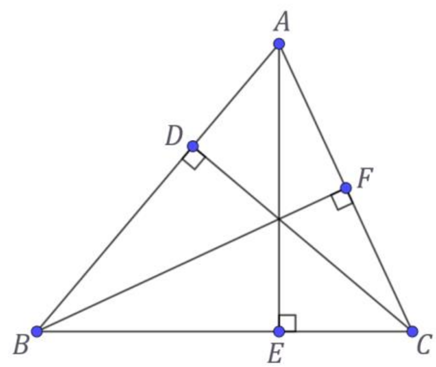
$$\begin{aligned} 12^2 = x(10 + x) &\Longrightarrow x^2 + 10x - 144 = 0 \\ &\Longrightarrow (x - 8)(x + 18) = 0 \end{aligned}$$

$$\Longrightarrow x = 8 \text{ or } x = -18.$$

But x cannot be negative so x must equal 8. Therefore, $h^2 = 10x = 80$ and $h = 4\sqrt{5}$.

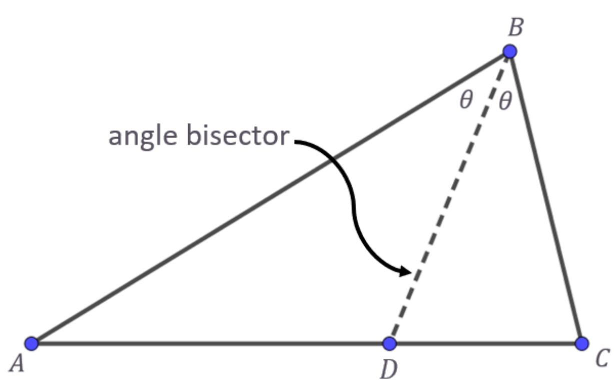
■

<table>
<tr><th colspan="2">Length of an Altitude</th></tr>
<tr><td>The length h_a of the altitude drawn from A to the opposite side can be expressed in two ways.
$$h_a = c\sin(B) = b\sin(C)$$

</td><td>(42)</td></tr>
</table>

Hero "based" formula for area of a triangle using the altitudes	
The area of ΔABC with altitudes $h_a = AE,\ h_b = BF$ and $h_c = CD$ equals $$\text{Area} = \frac{1}{\sqrt{\left(\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}\right)\left(-\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}\right)\left(\frac{1}{h_a}-\frac{1}{h_b}+\frac{1}{h_c}\right)\left(\frac{1}{h_a}+\frac{1}{h_b}-\frac{1}{h_c}\right)}}$$  	**(43)**

Note: As far as I know there has never been a MSHSML problem where you needed the above formula for finding the area of a triangle using the altitudes. I have included for the most part in case you are using these notes for other purposes.

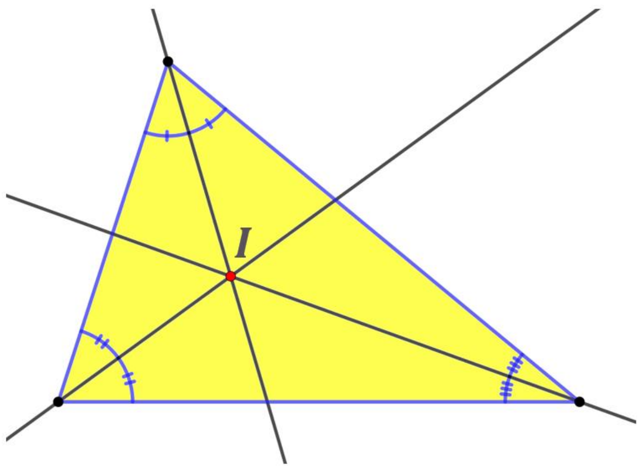
9. Angle Bisectors

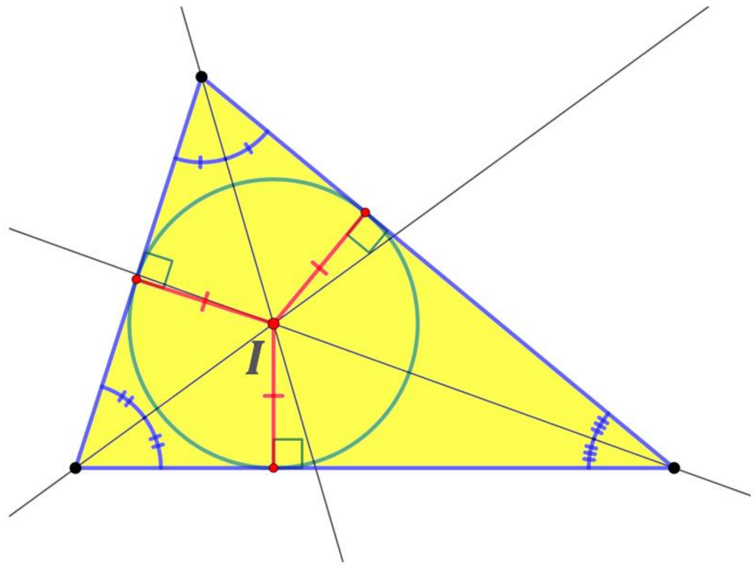
Angle Bisector	
An angle bisector is a cevian which bisects the angle it is drawn from $(\angle ABD = \angle DBC)$.  	**(44)**

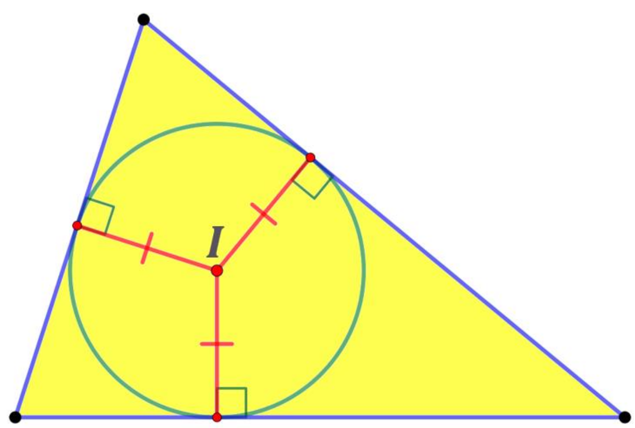
A triangle has three angle bisectors (one from each vertex).	

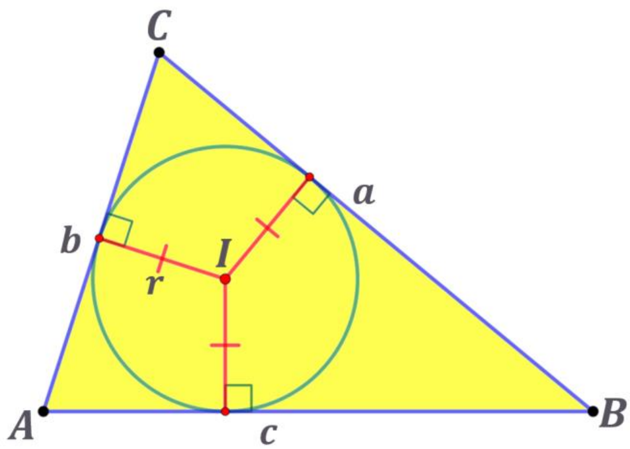
Ordered Angle Bisectors Theorem	
The shortest angle bisector is drawn to the longest side of a triangle and the longest angle bisector is drawn to the shortest side of a triangle. Notice how *shortest* is paired with *longest* which is counter to how angles are paired with sides.	**(45)**

Equal Angle Bisectors Implies Equal Angles Theorem	
Two angle bisectors are equal in length if and only if the angles they are drawn from are equal.	**(46)**

Incenter	
The incenter I is the point of intersection of the angle bisectors of each side of a triangle. 	**(47)**

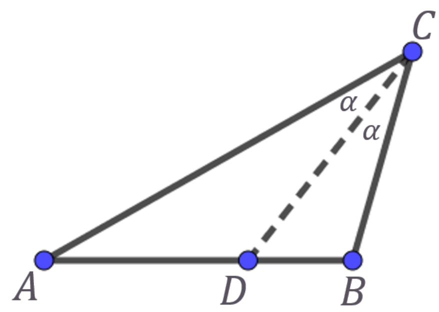
Incircle Theorem	
Every triangle has a unique inscribed circle whose center is the incenter I, the point of intersection of the angle bisectors of each side of a triangle.  Notice that the points of tangency (where the incircle is tangent to the triangle) are ***not*** necessarily the same as the points where the angle bisectors intersect the sides of the triangle.	**(48)**

Incenter Equidistant from all Three Sides	
The incenter I is the center of the incircle, the circle that is tangent to all three sides of this triangle. It follows that the incenter is equidistant from all three sides. 	**(49)**

Inradius of the Incircle	
Let a, b and c be the lengths of the three sides of ΔABC. The radius r of the unique inscribed circle of ΔABC (called the inradius) is given by $$r = \frac{2 \cdot \text{Area}(\Delta ABC)}{\text{Perimeter}(\Delta ABC)}.$$ 	**(50)**

Area of a Triangle and the Inradius	
We can solve for $\text{Area}(\Delta ABC)$ in the above formula for the inradius r of the incircle of ΔABC to get another formula for the area of a triangle. $$\text{Area}(\Delta ABC) = \frac{r \cdot \text{Perimeter}(\Delta ABC)}{2}$$	**(51)**

Angle Bisector Theorem



If $\overline{CD}$ bisects $\angle ACB$ in ΔACB, then

$$\frac{AC}{BC} = \frac{AD}{BD}. \tag{52}$$

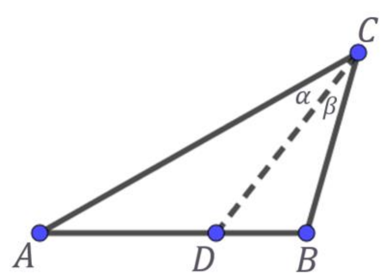
In this case we can solve for AD and BD in terms of the sides of ΔABC to get

$$AD = \frac{CA \cdot AB}{AC + BC} \tag{53}$$

$$BD = \frac{AB \cdot BC}{AC + BC}. \tag{54}$$

Converse of the Angle Bisector Theorem

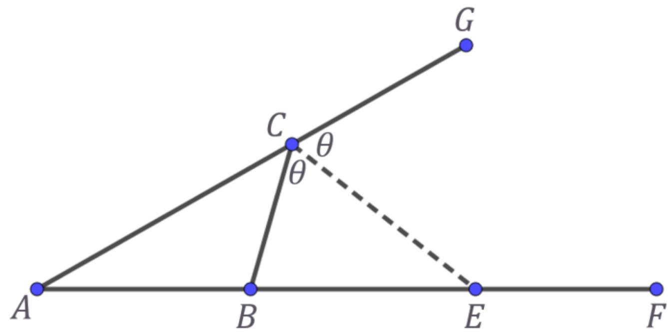
Suppose cevian $\overline{CD}$ of ΔABC divides $\angle C$ into $\alpha = \angle ACD$ and $\beta = \angle DCB$.

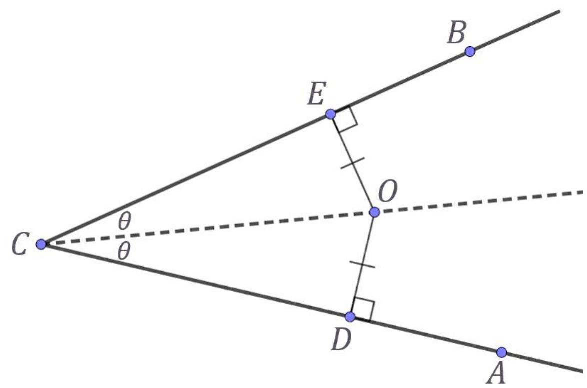


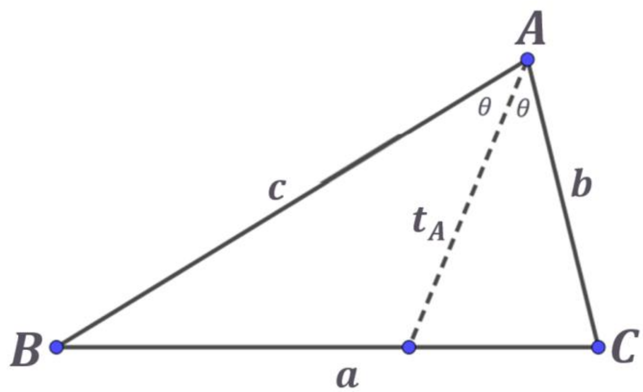
If

$$\frac{AC}{BC} = \frac{AD}{BD} \tag{55}$$

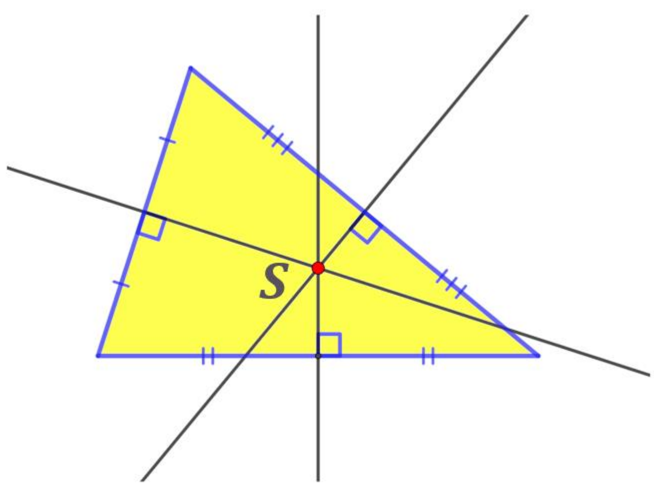
then $\alpha = \beta$. That is, $\overline{CD}$ bisects angle $\angle C$.

Exterior Angle Bisector Theorem	
 If $\overline{CE}$ bisects exterior angle $\angle BCG$ to ΔBCA, then $$\frac{AC}{BC} = \frac{AE}{BE}.$$	**(56)**

Angle Bisector Equidistance Theorem	
Every point O on the angle bisector of $\angle ACB$ is equi- (perpendicular or shortest) distance from sides $\overline{AC}$ and $\overline{BC}$. That is, $OD = OE$. 	**(57)**

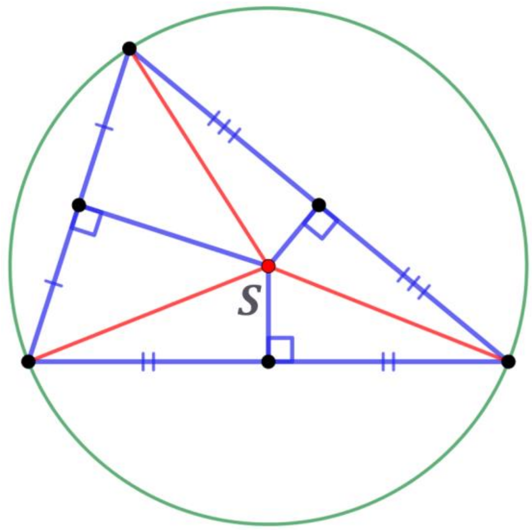
Length of an Angle Bisector	
 Length of Angle Bisector: $t_A = \sqrt{bc\left(1 - \frac{a^2}{b^2 + c^2}\right)}$	**(58)**

10. Perpendicular Bisectors

Circumcenter	
The circumcenter S is the point of intersection of the perpendicular bisectors of each side of a triangle. 	**(59)**

Circumcircle Theorem

Every triangle can be circumscribed by a unique circle whose center is the circumcenter S, the unique point of concurrence of the perpendicular bisectors of the three sides.

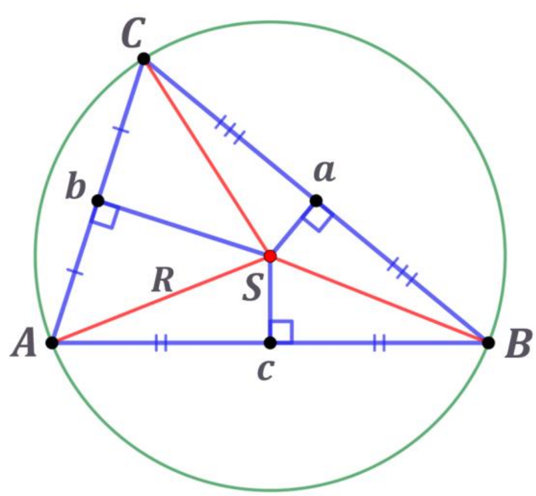


(60)

Circumradius of the Circumcircle

Let a, b and c be the lengths of the three sides of ΔABC. The radius R of the unique circumscribed circle of ΔABC (called the circumradius) is given by

$$R = \frac{abc}{4\,\text{Area}(\Delta ABC)}.$$

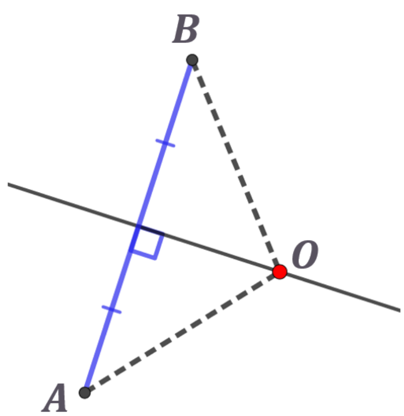


(61)

Area of a Triangle and the Circumradius	
We can solve for Area(ΔABC) in the above formula for the circumradius R of the circumcircle of ΔABC to get another formula for the area of a triangle. $$\text{Area}(\Delta ABC) = \frac{abc}{4R}.$$	**(62)**

Perpendicular Bisector Equidistance Theorem

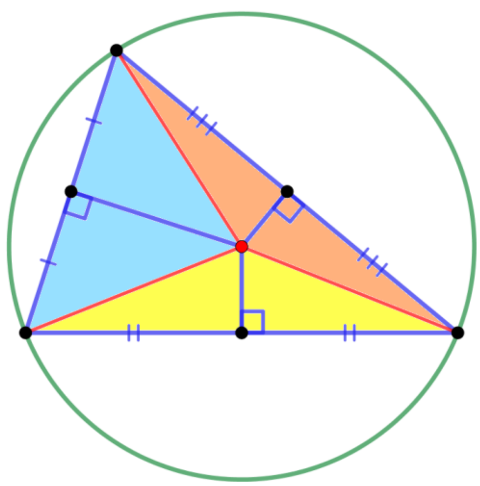
Every point O on the perpendicular bisector of segment $\overline{AB}$ is equi- (perpendicular or shortest) distance from the endpoints A and B of that segment That is, $OA = OB$.



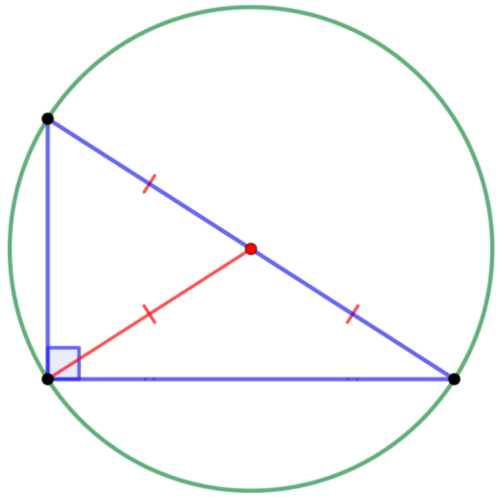
(63)

Perpendicular Bisectors Create Congruent Triangles

The same colored triangles are congruent by the SSS (side, side, side) criteria.



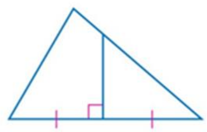
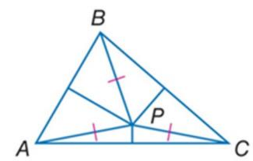
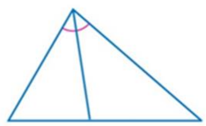
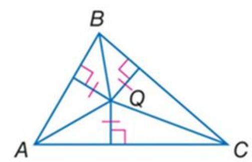
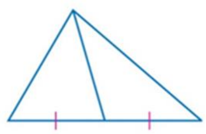
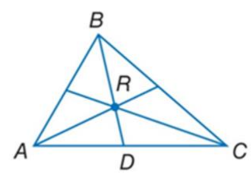
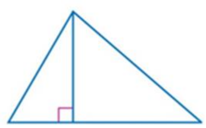
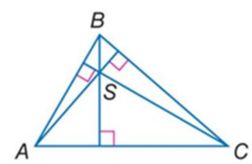
(64)

Circumcenter of a Right Triangle	
The circumcenter of a right triangle occurs at the midpoint of the hypotenuse. It follows that the midpoint of the hypotenuse is equidistant from all three vertices. 	**(65)**

11. Summary

Contrasting Ordered Angles and Ordered Side Lengths	
The smallest angle in a triangle is opposite the shortest side of a triangle. The largest (longest) cevian (median, angle bisector, altitude) is drawn to the shortest side of a triangle. Take careful note of the difference in these two results because they are easy to get confused.	**(66)**

Special Segments and Points in Triangles

Name	Example	Point of Concurrency	Special Property	Example
perpendicular bisector		circumcenter	The circumcenter P of $\triangle ABC$ is equidistant from each vertex.	
angle bisector		incenter	The incenter Q of $\triangle ABC$ is equidistant from each side of the triangle.	
median		centroid	The centroid R of $\triangle ABC$ is two thirds of the distance from each vertex to the midpoint of the opposite side.	
altitude		orthocenter	The lines containing the altitudes of $\triangle ABC$ are concurrent at the orthocenter S.	

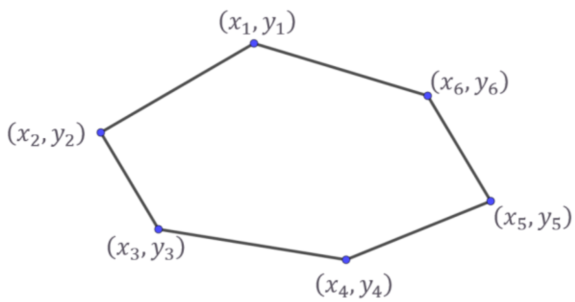
(67)

12. Polygon Area Formula

Area of a Convex Polygon (the Shoelace Algorithm)

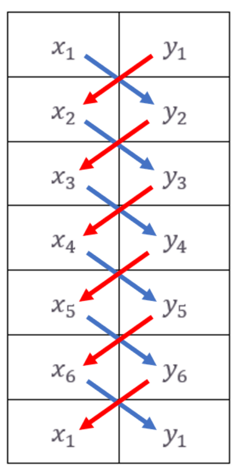
The area of the polygon with vertex coordinates $(x_1, y_1), (x_2, y_2), \dots, (x_6, y_6)$ equals

$$\text{Area } = \left|\frac{(x_1y_2 + x_2y_3 + \cdots + x_6y_1) - (y_1x_2 + y_2x_3 + \cdots + y_6x_1)}{2}\right|.$$



A mnemonic device for remembering the numerator of this formula is to put the coordinates in rows with an x coordinate column and a y coordinate column as you go around the polygon in a clockwise direction and then drawing in the “shoelaces” (the arrows) as shown in the diagram below.

(68)



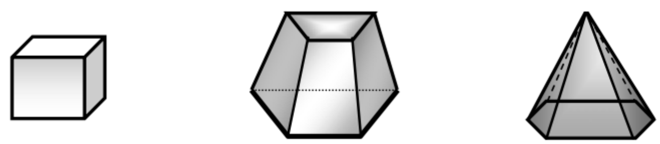
The algorithm is to add the product of numbers tied together with blue shoelaces and then to subtract the product of numbers tied together with red shoelaces.

Notice that the end with the same vertex you start the “shoelace” algorithm with.

13. Pyramids and Prisms (Surface Area and Volume Formulas)

13.1 Polyhedron

A **polyhedron** is a closed solid whose faces are polygons.



In this study guide we will consider two classes of polyhedron – prisms and pyramids.

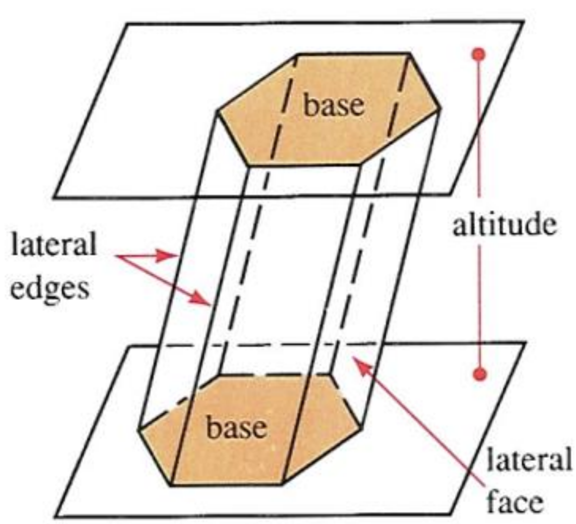
13.2 Prisms

A **prism** is a polyhedron with two congruent faces in parallel planes which are connected with parallelograms.

The top and bottom shaded faces of a prism are called its **bases**. Bases of a prism lie in parallel planes and are congruent polygons.

The faces of a prism that are not bases are called **lateral faces**.

The line segments where the lateral faces intersect each other are called **lateral edges**. Notice that lateral edges are necessarily parallel to each other.



A line segment joining the two base planes of a prism that is perpendicular to both bases is called an **altitude** of that prism.

The **lateral faces** of a prism are **parallelograms**. If they are rectangles (a parallelogram where all four angles are right angles), the prism is a **right prism**. Otherwise, the prism is an **oblique prism**.

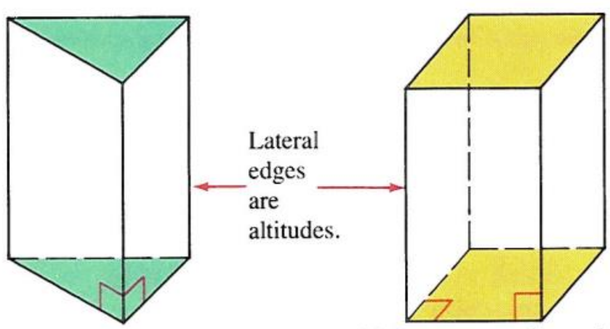
In a right prism, each lateral edge is an altitude.

The length of an altitude is the **height**, h, of a prism.

The **lateral area** (L.A.) of a prism is the sum of the areas of its lateral faces.

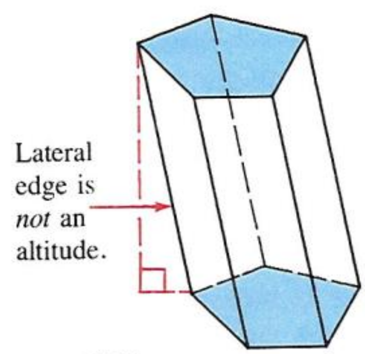
The **total area** (T.A.) of a prism is the sum of its lateral area and the area of its two bases.

The diagrams below show that a prism is also classified by the shape of its bases.

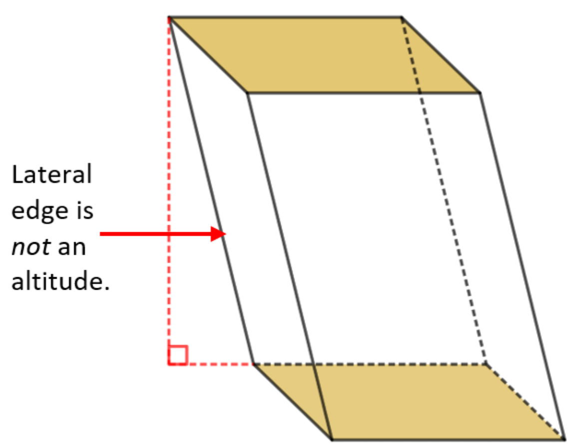


Right triangular prism

Right rectangular prism
(Rectangular solid)



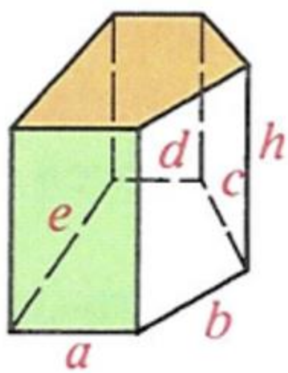
Oblique pentagonal prism



Oblique parallelogram solid
(a parallelepiped)

Lateral Surface Area of a Right Prism with Base Perimeter p and Height h	
Lateral Area $= ph$.	**(69)**

Proof

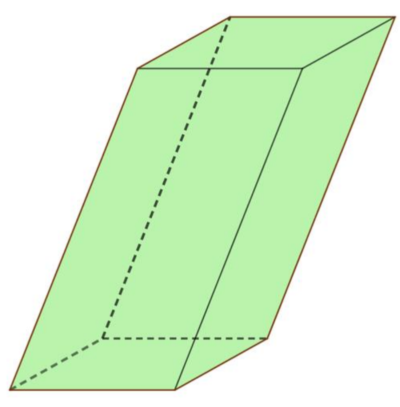


$$\begin{aligned} \text{L.A.} &= ah + bh + ch + dh + eh \\ &= (a + b + c + d + e)h \\ &= \text{perimeter} \cdot h \\ &= ph \end{aligned}$$

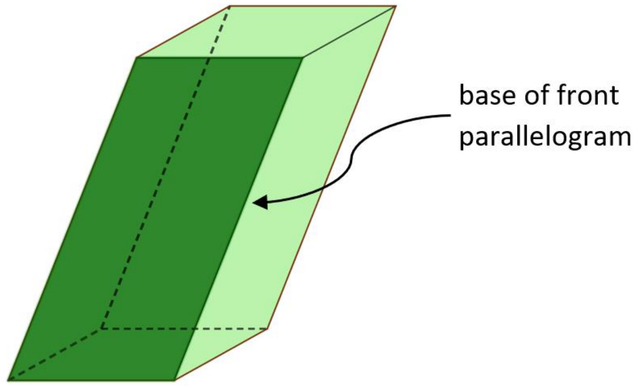
Note again that this result assumes the prism is a right prism.

13.2.1 Lateral Surface Area of a General Prism (including Oblique Prisms)

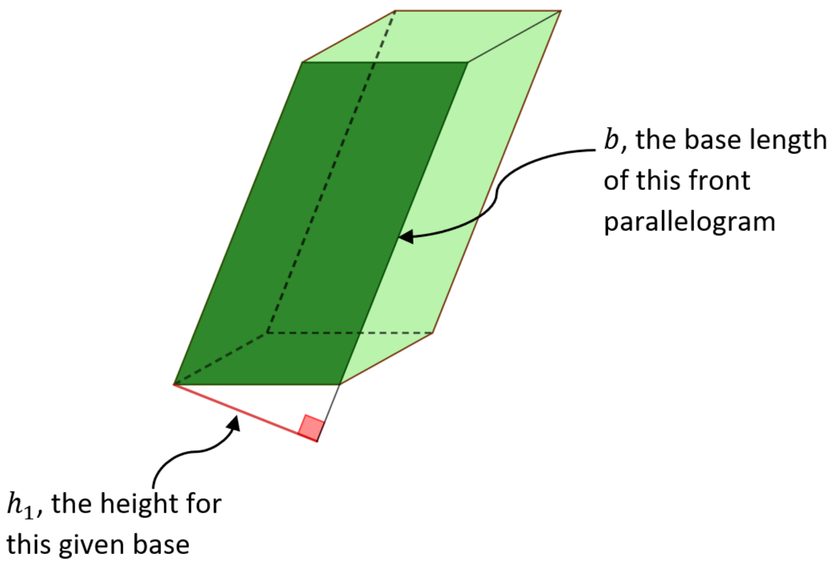
Every face of (every) prism is a parallelogram. So, to find the lateral area of a prism we will need to add the area of each face (parallelogram).



Recall that the area of a parallelogram is (base × height) where height is the perpendicular distance between the base of the parallelogram and the side parallel to the base. In the prism shown above let's start by finding the area of its front face, shown in dark green in the diagram below. Let's take our base edge to be the edge shown. Let b represent the length of this base edge (which is a lateral edge of the prism).



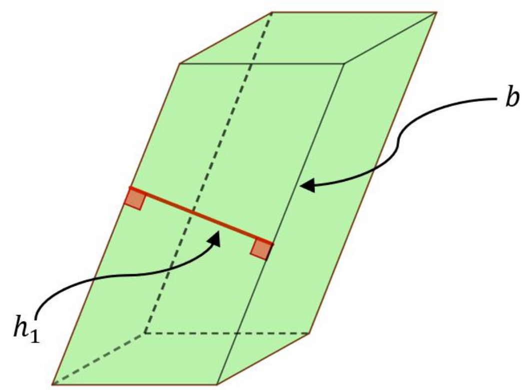
For this base edge the associated height is shown in the diagram below.



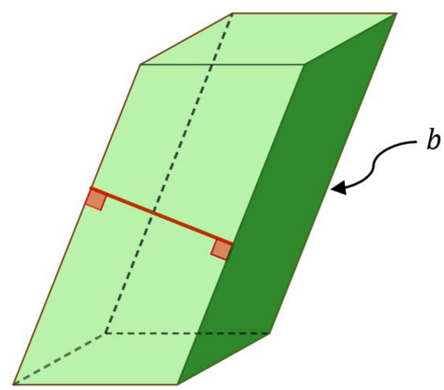
Let h_1 be the length of this height. So, the area of this font face is $h_1 b$.

I've drawn this example prism so it looks almost regular (because it was easier to draw it that way). But I do not want our argument here to be limited to just regular prisms. You can imagine that some faces are wider than others. Varying the width of a face will not change the length of the base edge (a lateral edge of the parallelogram) as we move on to the next face. However, the height will vary if the next face is wider or narrower than the previous.

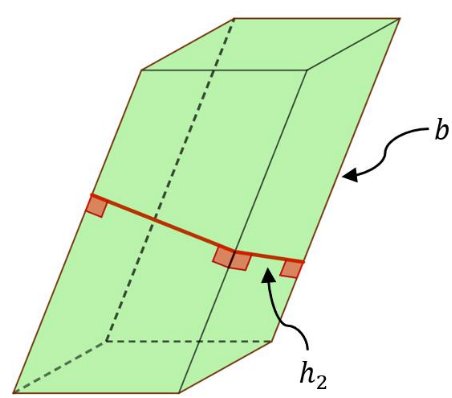
For reasons that will become clear in a few steps allow me to move height h_1 so it is drawn on the prism instead off one end. We also can see that because this height is perpendicular to the bottom base of a parallelogram then it also has to be perpendicular to the top base of this parallelogram because the top and bottom base lines are parallel.



How let's move on to the next face as shown in the diagram below. We again note that the base length b does not vary (*i.e.* all lateral edges of a prism are congruent).



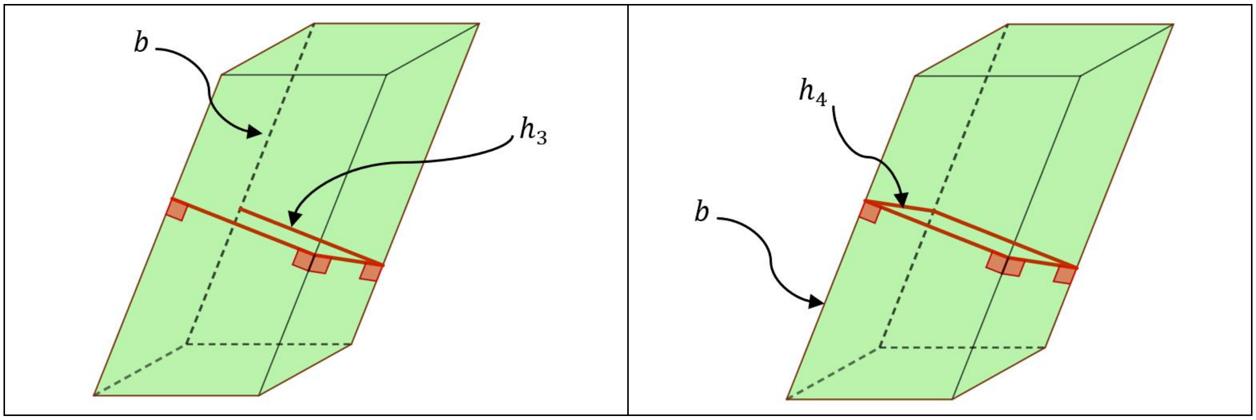
Now draw in the associated height for the base of this parallelogram and position it so it is attached to the previous height segment. Let h_2 be the length of this associated height.



So, this face (parallelogram) has area h_2b.

It is important to notice that this second height is in the same plane that contains the first height. To be formal, we appeal to Euclid's Elements, Box XI, Proposition 8, which tells us that "if of two parallel straight lines one is perpendicular to a plane, the other is so also."

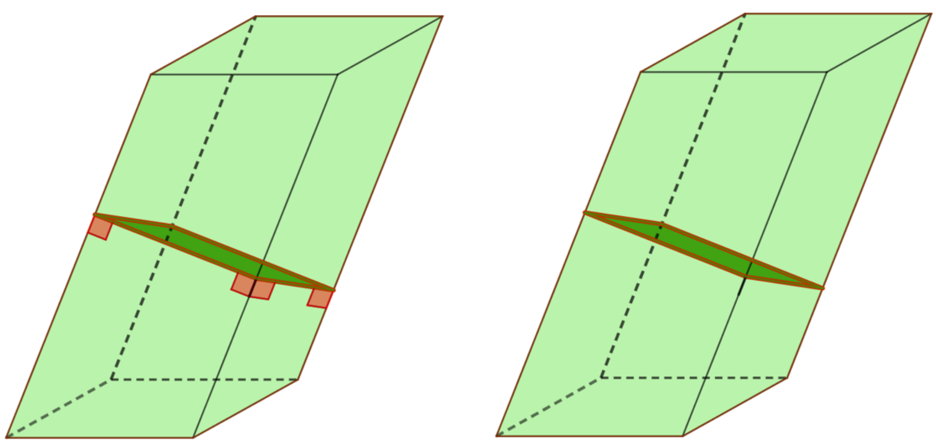
We can continue around the prism in this way and draw the heights of all edges in the plane that contains the first height.



In this way we can see that the lateral surface area of a prism will equal

$$b(h_1 + h_2 + \cdots + h_n).$$

I've drawn the diagram so it appears that the last height will exactly match up with the first height and form a polygon. Is this necessarily the case? Yes – because all of the heights are in the same plane (remember that each successive height is the same plane as the previous height). So, the polygon formed is the region formed when a plane perpendicular to all faces cuts the prism. We highlight this region in the diagrams below.



The shaded polygon above is called a *right section*.

Formally, a right section of a prism is a section formed by a plane which is perpendicular to all the lateral edges of that prism.

So, our above sum $h_1 + h_2 + \cdots + h_n$ is the perimeter of a right section of a prism. So, for general prisms (not just right prisms or just regular prisms),

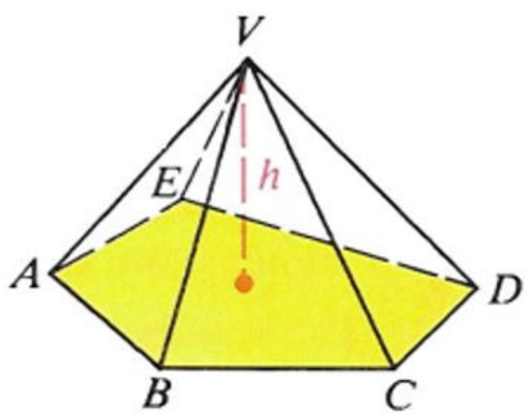
$$\text{Surface Area of a Prism} = bp$$

where b is the length of a base edge and p is the **perimeter of a right section** of that prism.

13.3 Pyramids

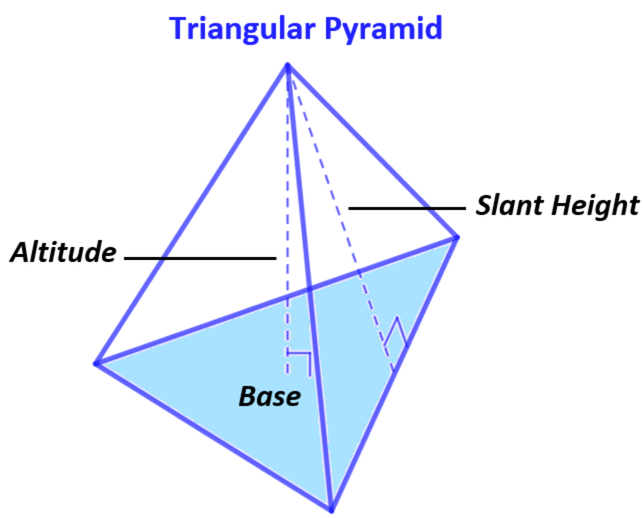
A pyramid is a polyhedron with a polygonal base and triangular faces that meet at a common point (called the vertex) not in the plane containing the base.

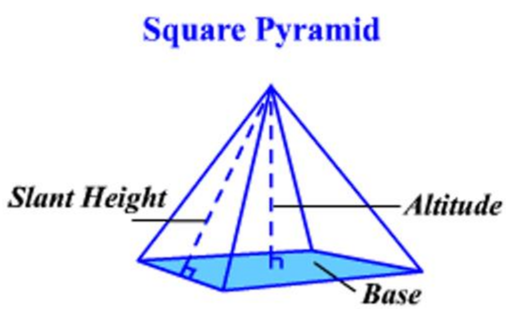
The diagram below shows **pyramid** $V - ABCDE$. Point V is the **vertex** of the pyramid and **base** $ABCDE$. The segment from the vertex perpendicular to the plane containing the base is the **altitude** and its length is the height, h, of the pyramid.

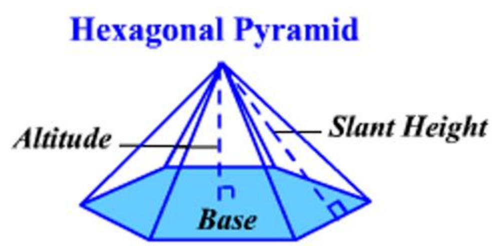


The five triangular faces with V in common, such as ΔVAB, are **lateral faces**. These faces intersect in segments called **lateral edges**.

Like prisms, pyramids are classified according to the shape of their base.

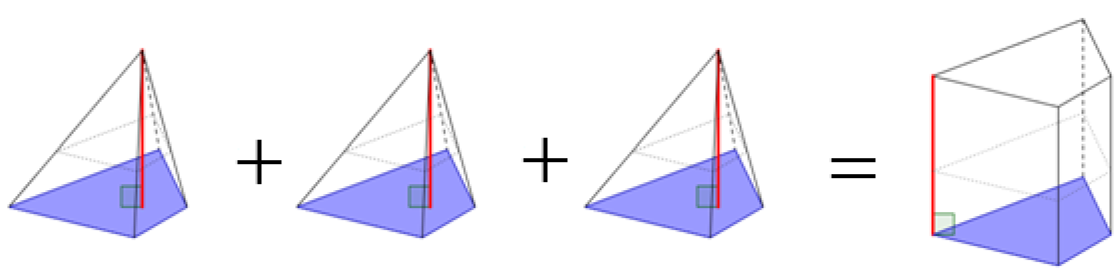




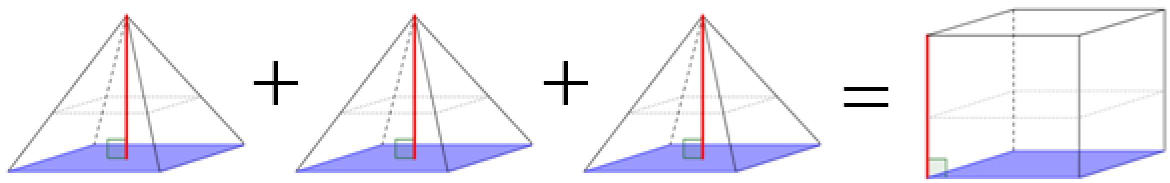


13.4 Volume of Prisms and Pyramids

There is an interesting connection between the volume of a prism and a pyramid with the same height and congruent bases.



and



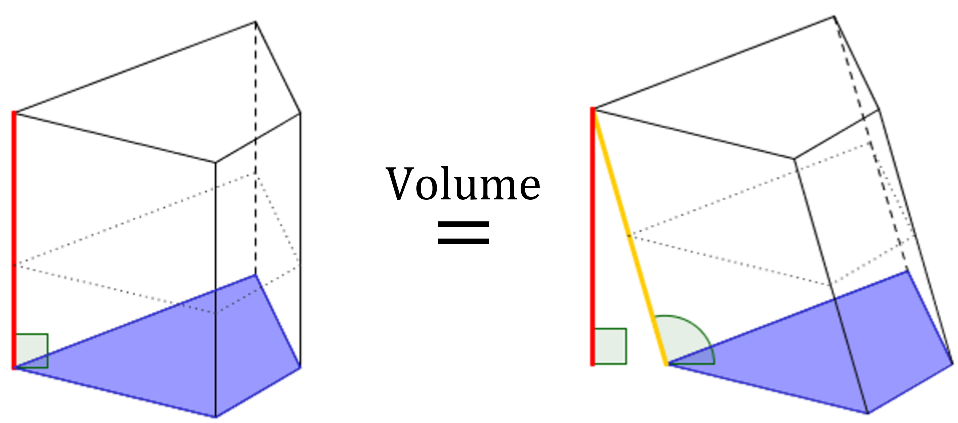
In particular, we have the formulas for the volume of a pyramid and prism.

Volume of Pyramids and Prisms	
$$\text{Volume}_{\text{Pyramid}} = \frac{1}{3}(\text{Base Area})(\text{Height})$$ $$\text{Volume}_{\text{Prism}} = (\text{Base Area})(\text{Height}).$$	**(70)**

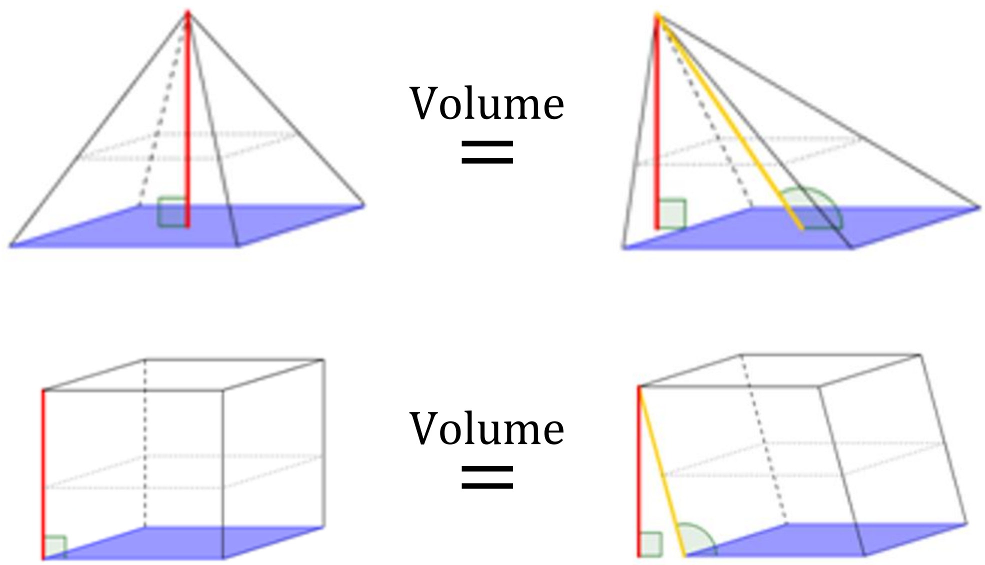
13.5 Cavalieri's Principle

Two solids with the same height and equal cross-sectional area at all values of that height will have equal volumes.

It follows from Cavalieri's Principle the following right prism and oblique prism will have equal volumes because they have the same cross sections at each value of their common height.



Other examples of the same principle are

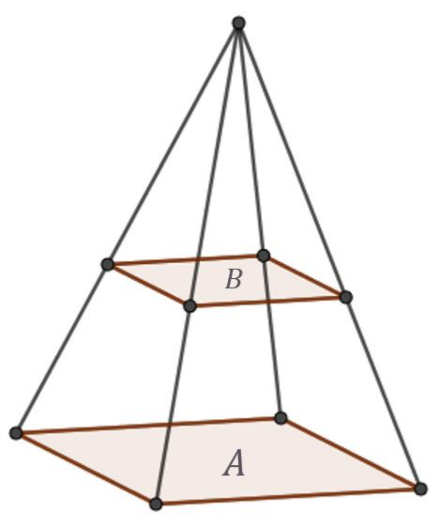


The objects on the left and right have the same volume in each of these cases because they have the same height and the same cross-sectional area at each value of that height.

13.6 Cross Sections

Cross sections of prisms are identical (congruent).

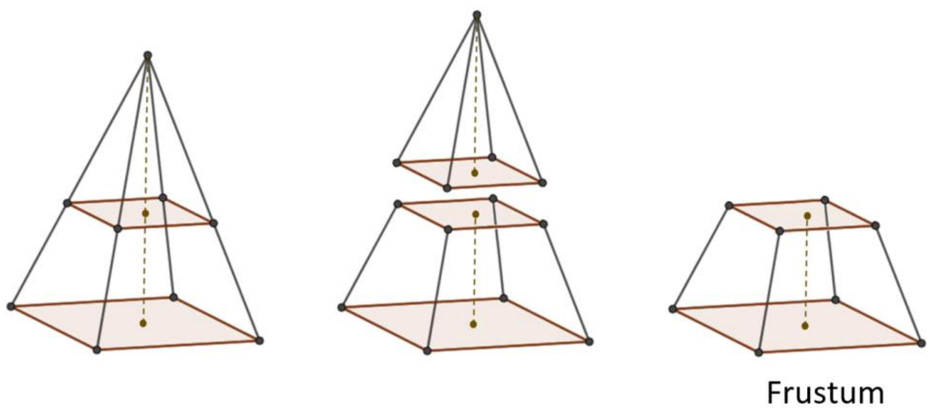
However, cross sections of pyramids get smaller and smaller but remain similar (corresponding angles are equal and the ratio of corresponding sides are equal.)

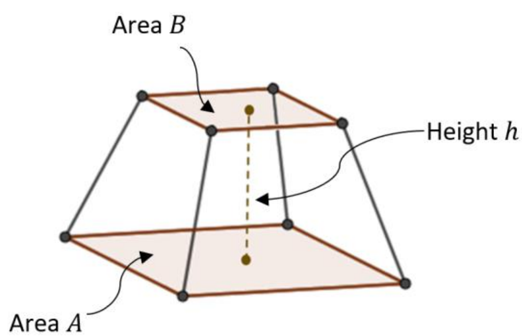


Cross sections A and B of a pyramid are similar objects (all corresponding angles are equal and the ratios of corresponding sides are equal).

13.7 Frustums

If you remove the top of a pyramid the part that remains is called a **frustum** of that pyramid.





Volume of Frustum	
$$\text{Volume}_{\text{Frustum}} = \frac{1}{3}h\left(A + B + \sqrt{AB}\right)$$	**(71)**

<u>Proof</u>

If we let y be the height of the original pyramid (before the top is removed), then this original pyramid has volume $(1/3)Ay$. In this case the pyramid that is removed has volume $(1/3)B(y-h)$. Therefore,

$$\text{Volume of Frustum} = \frac{1}{3}Ay - \frac{1}{3}B(y-h).$$

The original pyramid and the top pyramid are similar solids. (Note: Similarity is covered in depth in Test 5B.) We know from the properties of similar objects that the ratio of two dimensional measures (in particular base area) is the square of the ratio of one dimensional measures (in particular height). So we can say, just based on similarity of the two pyramids, that

$$\frac{\text{base area of top pyramid}}{\text{base area of full pyramid}} = \left(\frac{\text{height of top pyramid}}{\text{height of full pyramid}}\right)^2.$$

That is,

$$\frac{B}{A} = \left(\frac{y-h}{y}\right)^2.$$

Solving for y we find, after simplification, that

$$y = h\left(\frac{A+\sqrt{AB}}{A-B}\right).$$

Substituting this value of y into our initial result for the area of a frustum we have

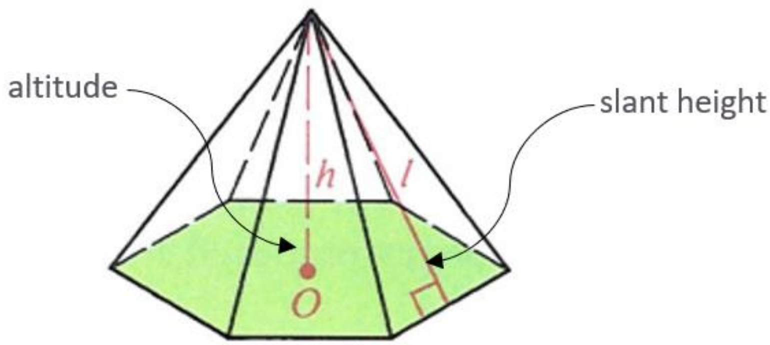
$$\text{Area Frustum} = \frac{1}{3}A\left(h\left(\frac{A+\sqrt{AB}}{A-B}\right)\right) - \frac{1}{3}B\left(\left(h\left(\frac{A+\sqrt{AB}}{A-B}\right)\right) - h\right).$$

After several steps of simplification this reduces to

$$\text{Area Frustum} = \frac{1}{3}h\left(A+B+\sqrt{AB}\right).$$

■

13.8 Regular Pyramid



Regular hexagonal pyramid

A regular pyramid is a pyramid with the following properties:

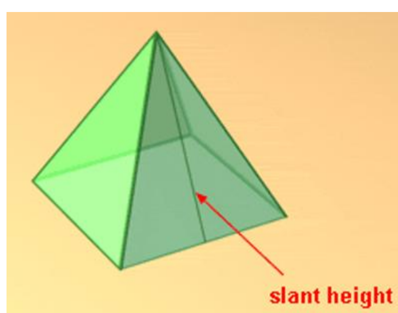
(1) The base is a regular polygon.

(2) All lateral edges are congruent.

(3) All lateral faces are congruent isosceles triangles. The height of a lateral face is called the **slant height** of the pyramid. It is denoted by l.

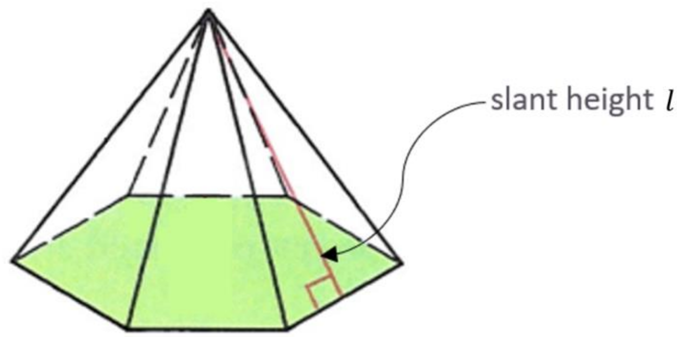
(4) The **altitude** h extends from the vertex to the base and is **perpendicular** to the base. The altitude of a regular pyramid meets the base at its center, O.

Here is another picture illustrating the slant height of a regular pyramid.



The **slant height** is the length of this line segment which goes from the vertex (or apex) of the pyramid to edge of the base and is perpendicular to the base.

The total surface area of the sides (excluding the base) of a pyramid is called the **lateral surface** area of the pyramid.

<table>
<tr><th colspan="2">Lateral Surface Area of a Regular Pyramid with base perimeter p</th></tr>
<tr><td>
Lateral Surface Area $= \frac{pl}{2}$.</td><td>(72)</td></tr>
</table>

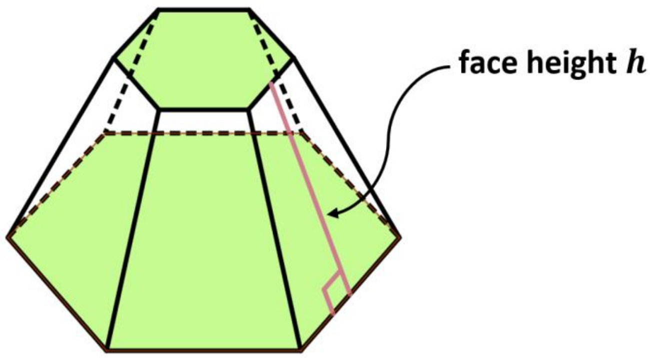
Proof

Each of the n lateral faces is an isosceles triangle with height l and because this is a regular pyramid all lateral faces are congruent. If we denote the common base width of each of these n triangles by b then the area of each of these triangles is just $(1/2)bl$. Therefore

$$\text{pyramid lateral surface area} = n \cdot \left(\frac{1}{2}bl\right) = \frac{(nb)l}{2} = \frac{pl}{2}$$

because the perimeter p equals nb.

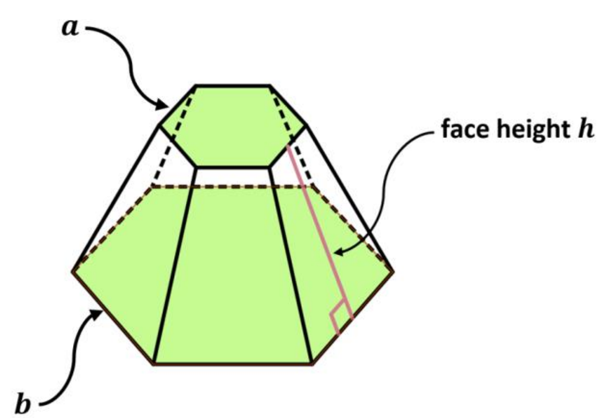
■

<table>
<tr><td colspan="2">Lateral Surface Area of a Regular Frustum of a Regular Pyramid</td></tr>
<tr><td>A frustum of a regular pyramid where each face has height h, with base perimeter p_1 and top perimeter p_2.


$$\text{Lateral Surface Area} = \left(\frac{p_1 + p_2}{2}\right) h.$$</td><td>(73)</td></tr>
</table>

<u>Proof</u>

Each of the n congruent faces of this frustum is a trapezoid with height h. Suppose we denote the common base width of each trapezoid by b and the common top width by a.



We will learn in the upcoming study notes on polygons (Test 3B) that the area of such a trapezoid equals

$$\left(\frac{a+b}{2}\right) h.$$

Therefore,

$$\text{frustum lateral surface area} = n \cdot \left(\frac{a+b}{2}\right)h = \left(\frac{na+nb}{2}\right)h = \left(\frac{p_1+p_2}{2}\right)h.$$

■