MSHSML Meet 2, Event C
Study Guide

2C Trigonometry (No Calculators)

Functions of sums of angles and sums of functions of angles
Half and double angle formulas
Reduction formulas

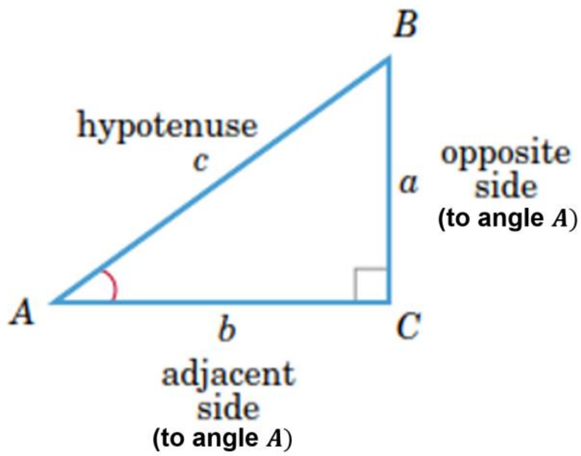
Contents

1. Two Models for Trigonometry

There are two models (leading to all equivalent results) for defining trigonometric values – the right triangle model and the unit circle model.

1.1 Right Triangle Trigonometry

For an acute angle A, we can define the trigonometric functions by looking at the ratios of the side lengths of a right triangle ABC with a right angle at C.



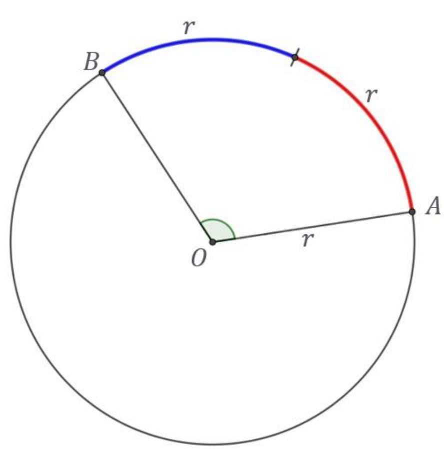
$\sin A = \dfrac{\text{opposite side}}{\text{hypotenuse}} = \dfrac{a}{c}$	$\tan A = \dfrac{\text{opposite side}}{\text{adjacent side}} = \dfrac{a}{b}$	$\sec A = \dfrac{\text{hypotenuse}}{\text{adjacent side}} = \dfrac{c}{b}$
$\cos A = \dfrac{\text{adjacent side}}{\text{hypotenuse}} = \dfrac{b}{c}$	$\cot A = \dfrac{\text{adjacent side}}{\text{opposite side}} = \dfrac{b}{a}$	$\csc A = \dfrac{\text{hypotenue}}{\text{opposite side}} = \dfrac{c}{a}$

For angles greater than 90°, apply the right-triangle definition to a reference angle and attach the appropriate $\pm$ sign.

1.2 Unit Circle Trigonometry

Radian – a unit of angle, equal to an angle at the center of a circle whose arc is equal in length to the radius.

$$\angle AOB = 2 \text{ radians}$$

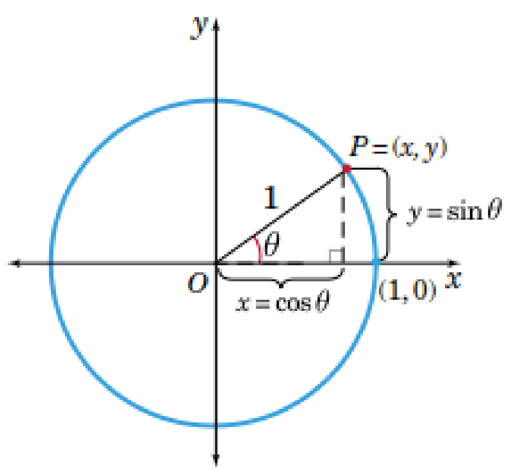


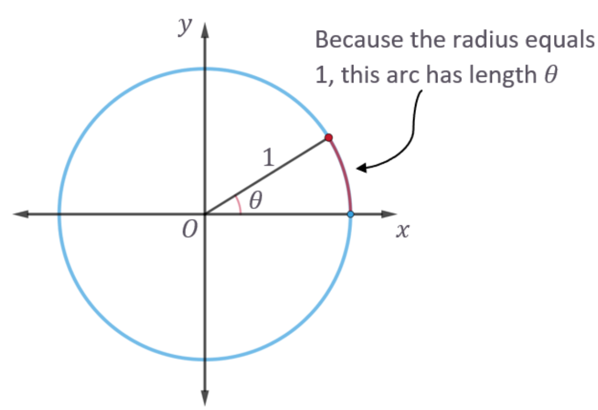
The red and blue arcs each have the same length as the radius of this circle. So, Arc $\widehat{AB}$ has the same length as 2 radii. Therefore, by the definition of radians, central angle $\angle AOB$ equals 2 radians.

$90° = \frac{\pi}{2}$ radians	$180° = \pi$ radians	$360° = 2\pi$ radians

x degrees $= \left(\frac{\pi}{180} \cdot x\right)$ radians	x radians $= \left(\frac{180}{\pi} \cdot x\right)$ degrees

Any angle θ defines a point $P = (x, y)$ on the unit circle (circle with radius 1, centered at the origin). The x coordinate is defined to be $\cos(\theta)$ and the y coordinate is defined to be $\sin(\theta)$.





$\sin\theta = y$	$\tan\theta = \frac{y}{x}$	$\sec\theta = \frac{1}{x}$
$\cos\theta = x$	$\cot\theta = \frac{x}{y}$	$\csc\theta = \frac{1}{y}$

Note that $\tan(\theta) = y/x$ equals the slope of the line $\overline{OP}$.

Because θ and $\theta + 2k\pi$ define the same point on the unit circle, all trigonometric functions are periodic with a period of 2π (sin, cos, sec, csc), or π (tan, cot).

1.3 Equivalence of the Two Defining Models

Figure A is the starting point for the right triangle definition of the trigonometric values and Figure B is the starting point for the unit circle definition of the trigonometric values.

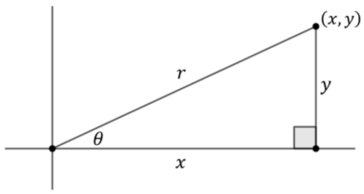
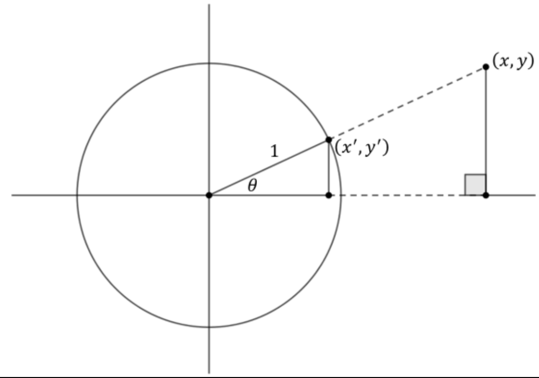
	

Figure A	Figure B

By the right triangle definition and Figure A, we have

$$\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{y}{r}.$$

By the unit circle definition and Figure B, we have

$$\sin\theta = y'.$$

However, the two triangles in Figure B are similar. Hence,

$$\frac{y}{y'} = \frac{x}{x'} = \frac{r}{1}.$$

Therefore,

$$y = ry' \Longrightarrow y' = \frac{y}{r}.$$

The equivalence of the other trig values follows similarly.

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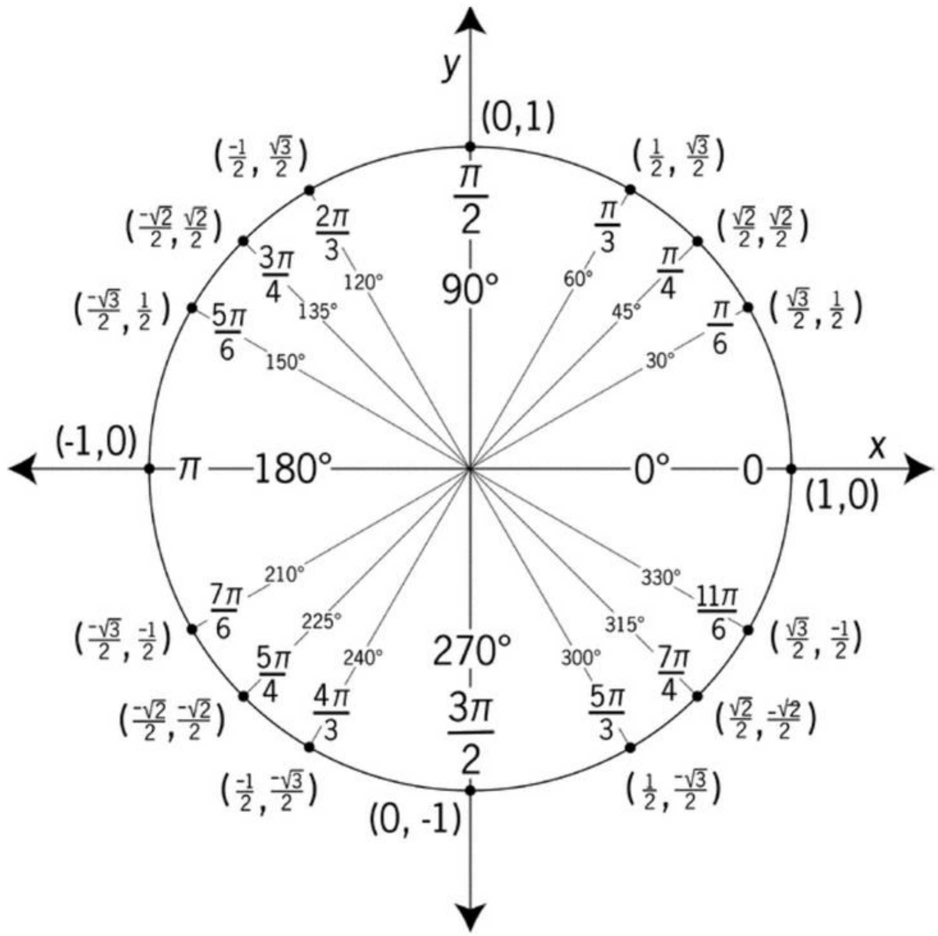
2. **Special Trig Values**



SPECIAL TRIGONOMETRIC VALUES

θ (○)	θ (rad)	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
0°	0	$0=\frac{\sqrt{0}}{2}$	1	0	undefined	1	undefined
30°	$\frac{\pi}{6}$	$\frac{1}{2}=\frac{\sqrt{1}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$
90°	$\frac{\pi}{2}$	$1=\frac{\sqrt{4}}{2}$	0	undefined	1	undefined	0
120°	$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	-2	$-\frac{\sqrt{3}}{3}$
135°	$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	-1	$\sqrt{2}$	$-\sqrt{2}$	-1
150°	$\frac{5\pi}{6}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3}$	2	$-\frac{2\sqrt{3}}{3}$	$-\sqrt{3}$
180°	π	0	-1	0	undefined	-1	undefined
210°	$\frac{7\pi}{6}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	-2	$-\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
225°	$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	1	$-\sqrt{2}$	$-\sqrt{2}$	1
240°	$\frac{4\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\sqrt{3}$	$-\frac{2\sqrt{3}}{3}$	-2	$\frac{\sqrt{3}}{3}$
270°	$\frac{3\pi}{2}$	-1	0	undefined	-1	undefined	0
300°	$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\sqrt{3}$	$-\frac{2\sqrt{3}}{3}$	2	$-\frac{\sqrt{3}}{3}$
315°	$\frac{7\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	-1	$-\sqrt{2}$	$\sqrt{2}$	-1
330°	$\frac{11\pi}{6}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3}$	-2	$\frac{2\sqrt{3}}{3}$	$-\sqrt{3}$
360° = 0°	$2\pi=0$	0	1	0	undefined	1	undefined

3. Trigonometric Identities

	$\sin x$	$\cos x$	$\tan x$	$\cot x$	$\sec x$	$\csc x$
$\sin x$		$\pm\sqrt{1-\cos^2 x}$	$\frac{\tan x}{\pm\sqrt{1+\tan^2 x}}$	$\frac{1}{\pm\sqrt{1+\cot^2 x}}$	$\frac{\pm\sqrt{\sec^2 x-1}}{\sec x}$	$\frac{1}{\csc x}$
$\cos x$	$\pm\sqrt{1-\sin^2 x}$		$\frac{1}{\pm\sqrt{1+\tan^2 x}}$	$\frac{\cot x}{\pm\sqrt{1+\cot^2 x}}$	$\frac{1}{\sec x}$	$\frac{\pm\sqrt{\csc^2 x-1}}{\csc x}$
$\tan x$	$\frac{\sin x}{\pm\sqrt{1-\sin^2 x}}$	$\frac{\pm\sqrt{1-\cos^2 x}}{\cos x}$		$\frac{1}{\cot x}$	$\pm\sqrt{\sec^2 x-1}$	$\frac{1}{\pm\sqrt{\csc^2 x-1}}$
$\cot x$	$\frac{\pm\sqrt{1-\sin^2 x}}{\sin x}$	$\frac{\cos x}{\pm\sqrt{1-\cos^2 x}}$	$\frac{1}{\tan x}$		$\frac{1}{\pm\sqrt{\sec^2 x-1}}$	$\pm\sqrt{\csc^2 x-1}$
$\sec x$	$\frac{1}{\pm\sqrt{1-\sin^2 x}}$	$\frac{1}{\cos x}$	$\pm\sqrt{\tan^2 x+1}$	$\frac{\pm\sqrt{\cot^2 x+1}}{\cot x}$		$\frac{\csc x}{\pm\sqrt{\csc^2 x-1}}$
$\csc x$	$\frac{1}{\sin x}$	$\frac{1}{\pm\sqrt{1-\cos^2 x}}$	$\frac{\pm\sqrt{1+\tan^2 x}}{\tan x}$	$\pm\sqrt{1+\cot^2 x}$	$\frac{\sec x}{\pm\sqrt{\sec^2 x-1}}$	

Reciprocal Identities		
$\csc x = \frac{1}{\sin x}$	$\sec x = \frac{1}{\cos x}$	$\cot x = \frac{1}{\tan x}$
$\tan x = \frac{\sin x}{\cos x}$	$\cot x = \frac{\cos x}{\sin x}$	

Pythagorean Identities	
$\sin^2 x + \cos^2 x = 1$	$\tan^2 x + 1 = \sec^2 x$
$1 + \cot^2 x = \csc^2 x$	

Addition and Subtraction Identities	
$\sin(x+y) = \sin x \cos y + \cos x \sin y$	$\sin(x-y) = \sin x \cos y - \cos x \sin y$
$\cos(x+y) = \cos x \cos y - \sin x \sin y$	$\cos(x-y) = \cos x \cos y + \sin x \sin y$
$\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$	$\tan(x+y) = \dfrac{\sin x \cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y}$
$\tan(x-y) = \dfrac{\tan x - \tan y}{1 + \tan x \tan y}$	$\tan(x-y) = \dfrac{\sin x \cos y - \cos x \sin y}{\cos x \cos y + \sin x \sin y}$

Periodicity Identities (for integer k)		
$\sin(\theta) = \sin(\theta + 2k\pi)$	$\cos(\theta) = \cos(\theta + 2k\pi)$	$\tan(\theta) = \tan(\theta + k\pi)$
$\cot(\theta) = \cot(\theta + k\pi)$	$\sec(\theta) = \sec(\theta + 2k\pi)$	$\csc(\theta) = \csc(\theta + 2k\pi)$

Shift Identities

Shift by $\frac{\pi}{2}$	**Shift by π**	**Shift by $\frac{3\pi}{2}$**	**Shift by 2π**
$\sin\left(\theta + \frac{\pi}{2}\right) = +\cos\theta$	$\sin(\theta + \pi) = -\sin\theta$	$\sin\left(\theta + \frac{3\pi}{2}\right) = -\cos\theta$	$\sin(\theta + 2\pi) = \sin\theta$
$\cos\left(\theta + \frac{\pi}{2}\right) = -\sin\theta$	$\cos(\theta + \pi) = -\cos\theta$	$\cos\left(\theta + \frac{3\pi}{2}\right) = +\sin\theta$	$\cos(\theta + 2\pi) = \cos\theta$
$\tan\left(\theta + \frac{\pi}{2}\right) = -\cot\theta$	$\tan(\theta + \pi) = +\tan\theta$	$\tan\left(\theta + \frac{3\pi}{2}\right) = -\cot\theta$	$\tan(\theta + 2\pi) = +\tan\theta$
$\cot\left(\theta + \frac{\pi}{2}\right) = -\tan\theta$	$\cot(\theta + \pi) = +\cot\theta$	$\cot\left(\theta + \frac{3\pi}{2}\right) = -\tan\theta$	$\cot(\theta + 2\pi) = +\cot\theta$
$\sec\left(\theta + \frac{\pi}{2}\right) = -\csc\theta$	$\sec(\theta + \pi) = -\sec\theta$	$\sec\left(\theta + \frac{3\pi}{2}\right) = +\csc\theta$	$\sec(\theta + 2\pi) = \sec\theta$
$\csc\left(\theta + \frac{\pi}{2}\right) = +\sec\theta$	$\csc(\theta + \pi) = -\csc\theta$	$\csc\left(\theta + \frac{3\pi}{2}\right) = -\sec\theta$	$\csc(\theta + 2\pi) = \csc\theta$

More Shift Identities			
Shift by $\left(-\frac{\pi}{2}\right)$	**Shift by $(-\pi)$**	**Shift by $\left(-\frac{3\pi}{2}\right)$**	**Shift by (-2π)**
$\sin\left(\theta-\frac{\pi}{2}\right)=-\cos\theta$	$\sin(\theta-\pi)=-\sin\theta$	$\sin\left(\theta-\frac{3\pi}{2}\right)=+\cos\theta$	$\sin(\theta-2\pi)=\sin\theta$
$\cos\left(\theta-\frac{\pi}{2}\right)=+\sin\theta$	$\cos(\theta-\pi)=-\cos\theta$	$\cos\left(\theta-\frac{3\pi}{2}\right)=-\sin\theta$	$\cos(\theta-2\pi)=\cos\theta$
$\tan\left(\theta-\frac{\pi}{2}\right)=-\cot\theta$	$\tan(\theta-\pi)=+\tan\theta$	$\tan\left(\theta-\frac{3\pi}{2}\right)=-\cot\theta$	$\tan(\theta-2\pi)=+\tan\theta$
$\cot\left(\theta-\frac{\pi}{2}\right)=-\tan\theta$	$\cot(\theta-\pi)=+\cot\theta$	$\cot\left(\theta-\frac{3\pi}{2}\right)=-\tan\theta$	$\cot(\theta-2\pi)=+\cot\theta$
$\sec\left(\theta-\frac{\pi}{2}\right)=+\csc\theta$	$\sec(\theta-\pi)=-\sec\theta$	$\sec\left(\theta-\frac{3\pi}{2}\right)=-\csc\theta$	$\sec(\theta-2\pi)=\sec\theta$
$\csc\left(\theta-\frac{\pi}{2}\right)=-\sec\theta$	$\csc(\theta-\pi)=-\csc\theta$	$\csc\left(\theta-\frac{3\pi}{2}\right)=+\sec\theta$	$\csc(\theta-2\pi)=\csc\theta$

Reflection Identities			
θ over 0 **(even, odd identities)**	**θ over $\frac{\pi}{4}$** **(cofunction identities)**	**θ over $\frac{\pi}{2}$**	**θ over $\frac{3\pi}{4}$**
$\sin(-\theta)=-\sin\theta$	$\sin\left(\frac{\pi}{2}-\theta\right)=+\cos\theta$	$\sin(\pi-\theta)=+\sin\theta$	$\sin\left(\frac{3\pi}{2}-\theta\right)=-\cos\theta$
$\cos(-\theta)=+\cos\theta$	$\cos\left(\frac{\pi}{2}-\theta\right)=+\sin\theta$	$\cos(\pi-\theta)=-\cos\theta$	$\cos\left(\frac{3\pi}{2}-\theta\right)=-\sin\theta$
$\tan(-\theta)=-\tan\theta$	$\tan\left(\frac{\pi}{2}-\theta\right)=+\cot\theta$	$\tan(\pi-\theta)=-\tan\theta$	$\tan\left(\frac{3\pi}{2}-\theta\right)=+\cot\theta$
$\cot(-\theta)=-\cot\theta$	$\cot\left(\frac{\pi}{2}-\theta\right)=+\tan\theta$	$\cot(\pi-\theta)=-\cot\theta$	$\cot\left(\frac{3\pi}{2}-\theta\right)=+\tan\theta$
$\sec(-\theta)=+\sec\theta$	$\sec\left(\frac{\pi}{2}-\theta\right)=+\csc\theta$	$\sec(\pi-\theta)=-\sec\theta$	$\sec\left(\frac{3\pi}{2}-\theta\right)=-\csc\theta$
$\csc(-\theta)=-\csc\theta$	$\csc\left(\frac{\pi}{2}-\theta\right)=+\sec\theta$	$\csc(\pi-\theta)=\csc\theta$	$\csc\left(\frac{3\pi}{2}-\theta\right)=-\sec\theta$

Double-Angle Formulas	
$\sin 2x = 2\sin x\cos x$	$\cos 2x = \cos^2 x - \sin^2 x$ $= 1 - 2\sin^2 x$ $= 2\cos^2 x - 1$
$\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$	

Half-Angle Formulas	
$\sin\dfrac{x}{2} = \pm\sqrt{\dfrac{1-\cos x}{2}}$	$\cos\dfrac{x}{2} = \pm\sqrt{\dfrac{1+\cos x}{2}}$
The choice of the $+$ or – sign depends on the quadrant in which $x/2$ lies.	
$\tan\dfrac{x}{2} = \dfrac{1-\cos x}{\sin x} = \dfrac{\sin x}{1+\cos x}$	

Triple-Angle Identities	
$\sin 3x = 3\sin x - 4\sin^3 x$	$\cos 3x = 4\cos^3 x - 3\cos x$
$\tan 3x = \dfrac{3\tan x - \tan^3 x}{1-3\tan^2 x}$	

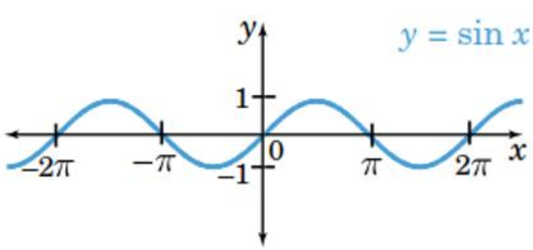
Formulas for Lowering Powers	
$\sin^2 x = \dfrac{1-\cos 2x}{2}$	$\cos^2 x = \dfrac{1+\cos 2x}{2}$
$\tan^2 x = \dfrac{1-\cos 2x}{1+\cos 2x}$	

Sum to Product Identities
$\sin(x) + \sin(y) = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$
$\sin(x) - \sin(y) = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$
$\cos(x) + \cos(y) = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$
$\cos(x) - \cos(y) = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$

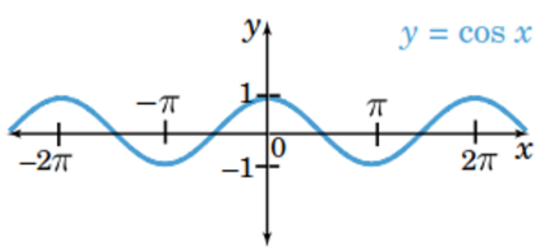
Product to Sum Identities
$\sin(x)\cos(y) = \frac{1}{2}\left(\sin(x+y) + \sin(x-y)\right)$
$\cos(x)\sin(y) = \frac{1}{2}\left(\sin(x+y) - \sin(x-y)\right)$
$\cos(x)\cos(y) = \frac{1}{2}\left(\cos(x+y) + \cos(x-y)\right)$
$\sin(x)\sin(y) = \frac{1}{2}\left(\cos(x-y) - \cos(x+y)\right)$

4. Graphs of the Trig Functions

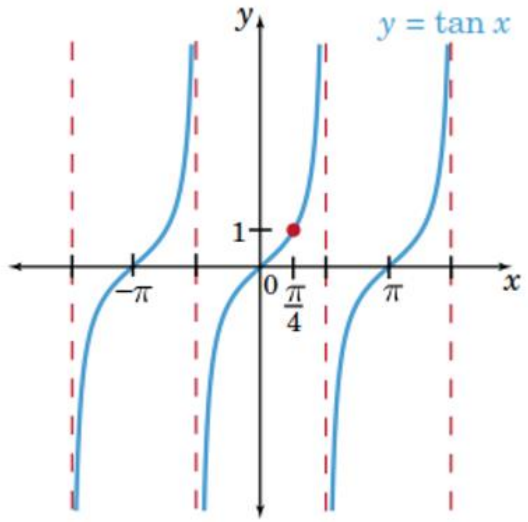
SINE



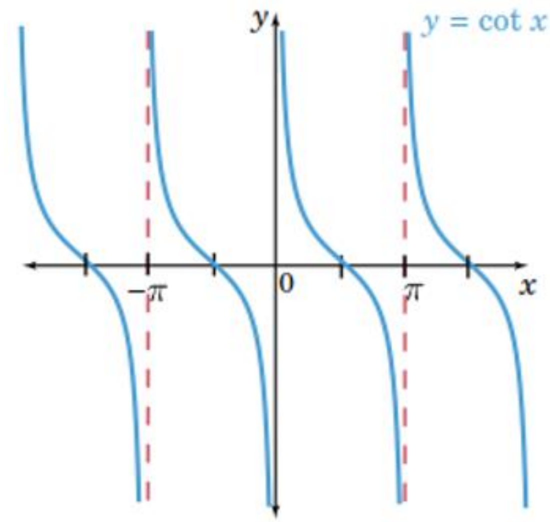
COSINE



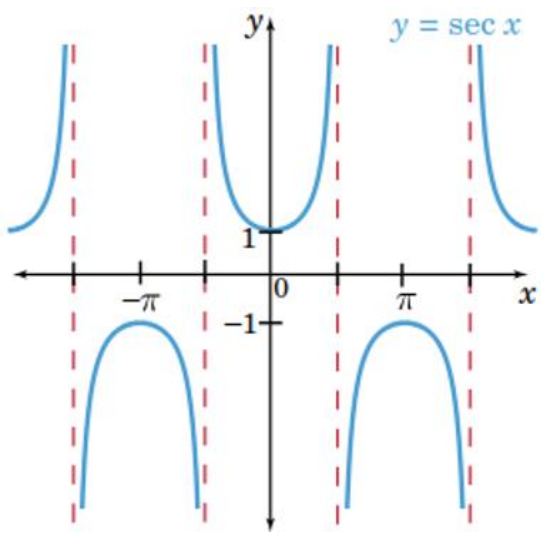
TANGENT



COTANGENT



SECANT



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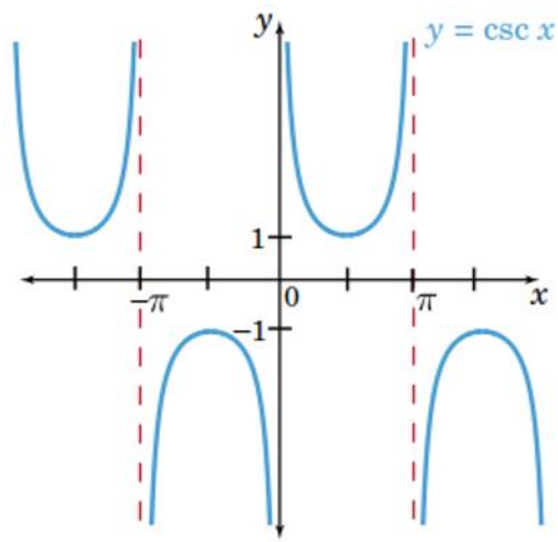
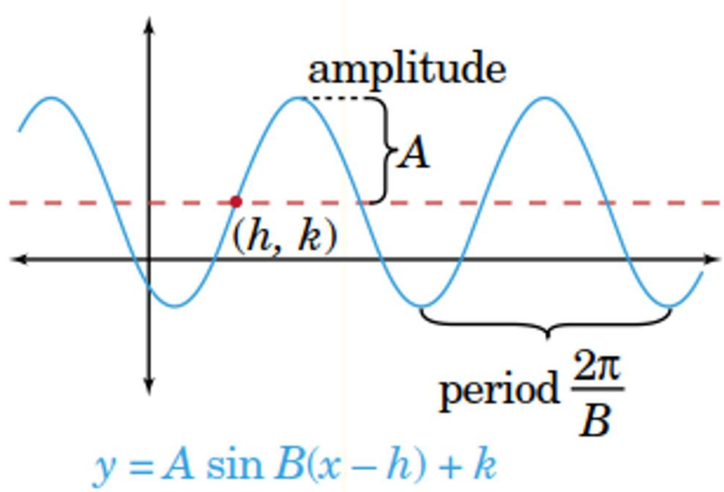


5. Graphing $y = A\sin\big(B(x-h)\big) + k$ and $y = A\cos(B(x-h)) + k$

$\lvert A\rvert$ is the **amplitude.**	
k is the is the **average value:** halfway between the maximum and the minimum value of the function.	
$\frac{2\pi}{B}$ is the **period.** There are B cycles in every interval of length 2π; so $\frac{B}{2\pi}$ is the **frequency**.	
h is the **phase shift,** or how far the beginning of the cycle is from the y-axis.	

The basic shape of the function will stay the same. The sine curve will start at (h, k) as though it were the origin and go up if A is positive (down if A is negative). A cosine curve will start at (h, k) at the crest if A is positive (trough if A is negative).

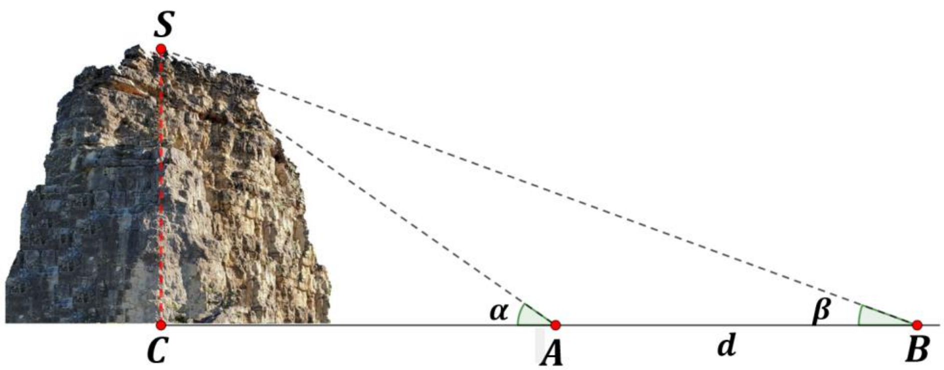


6. Morrie's Law

7. Extra Problems

Problem 1.

The Winona Rock Climbing Club wanted to determine the height of Sugar Loaf Rock. The angle of elevation angle at point A was determined to be $\alpha = 35°$ but point C was inaccessible so the distance AC could not be measured. Because this was not information to determine the height $h = CS$ of Sugar Loaf the club went $d = 112$ feet further to point B and found the angle of elevation there to be $\beta = 20°$.



Find $h = CS$, the height of Sugar Loaf Rock, assuming ΔBCS is a right triangle.

Solution

We have $AC = h/\tan\alpha$ and $BC = h/\tan\beta$. Therefore,

$$d = BC - AC = h\left(\frac{1}{\tan\beta} - \frac{1}{\tan\alpha}\right)$$

Hence,

$$h = \frac{d \cdot \tan\alpha \cdot \tan\beta}{\tan\alpha - \tan\beta}.$$

Evaluating h with the given data we can determine that

$$h = CS = \frac{(112)(0.700)(0.364)}{0.700 - 0.364} \approx 85 \text{ feet.}$$

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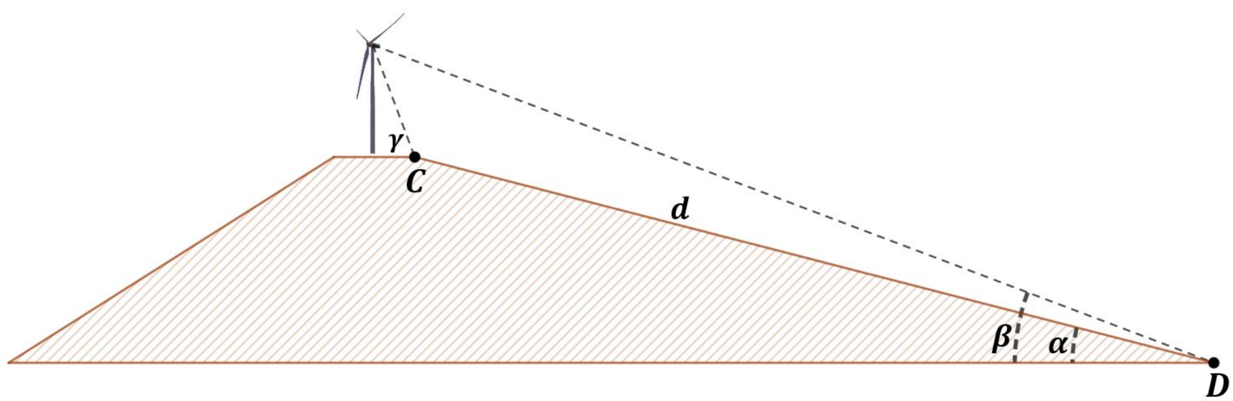
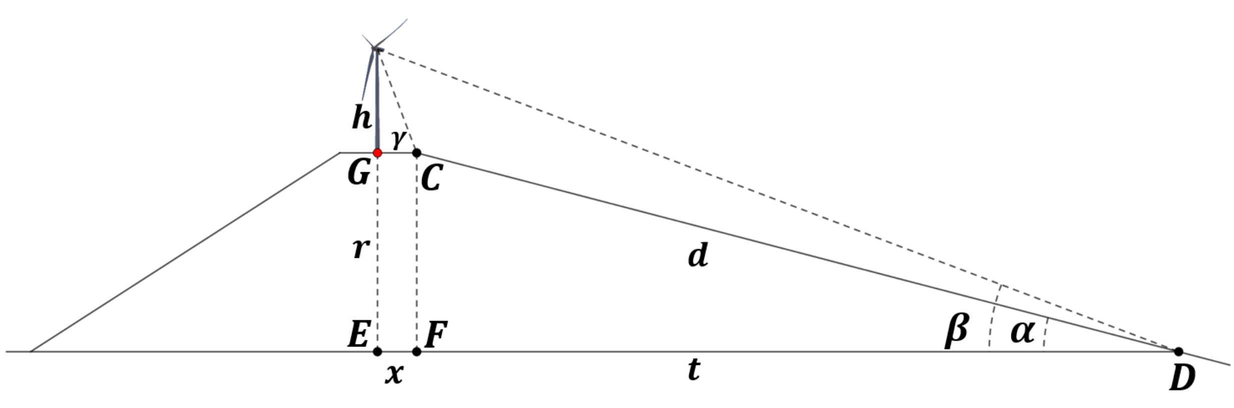
Problem 2.

A wind turbine was built at the top of a bluff to capture the maximum amount of wind. The region around the turbine was fenced off making its base point inaccessible. At point C where the bluff starts to slope down, the angle of elevation of the turbine is $\gamma = 70°$. At point D, which is $d = 2036$ feet further down the slope from point C, the angle of elevation is $\beta = 21°$. The angle of inclination on this side of the bluff is $\alpha = 15°$. How tall is the wind turbine?



Solution

Imagine dropping perpendiculars from points C and G (the bottom of the turbine) to the horizontal line containing D (as shown in the diagram below). Additional points and distances are also labeled.



Then

$$\tan(\beta) = \frac{h + r}{x + t} \Longrightarrow (x + t)\tan(\beta) = (h + r)$$

$$\Longrightarrow x = \frac{h + r - t\tan(\beta)}{\tan(\beta)}.$$

We can also see that $h = x\tan(\gamma)$. Substituting for x in the above expression yields

$$h = x\tan(\gamma) = \left(\frac{h + r - t\tan(\beta)}{\tan(\beta)}\right)\tan(\gamma).$$

But $r = d\sin(\alpha)$ and $t = d\cos(\alpha)$. Hence

$$h = \left(\frac{h + d\sin(\alpha) - d\cos(\alpha)\tan(\beta)}{\tan(\beta)}\right)\tan(\gamma)$$

$$\Longrightarrow h\tan(\beta) - h\tan(\gamma) = d\sin(\alpha)\tan(\gamma) - d\cos(\alpha)\tan(\beta)\tan(\gamma)$$

$$\Longrightarrow h = \frac{d\big(\sin(\alpha)\tan(\gamma) - \cos(\alpha)\tan(\beta)\tan(\gamma)\big)}{\tan(\beta) - \tan(\gamma)}.$$

Inputting the given data shows that the wind turbine is

$$h = \frac{2036 \cdot \big(\sin(15°)\tan(70°) - \cos(15°)\tan(21°)\tan(70°)\big)}{\tan(21°) - \tan(70°)} \text{ feet}$$

$$\approx 265 \text{ feet tall.}$$

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