MSHSML Meet 2, Event D
Study Guide

2D Analytic Geometry of Straight Lines and Circles

Slope, families of parallel, perpendicular, or coincident lines
Point-slope, slope-intercept, intercept, normal forms of the straight line
Intersections (solution of simultaneous systems)

1 Contents

2 Slope, Families of Parallel, Perpendicular, or Coincident Lines

Let $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, $P_3(x_3, y_3)$ and $P_4(x_4, y_4)$ be four distinct points in the xy-plane. Let $\overleftrightarrow{P_iP_j}$ denote the line containing distinct points P_i and P_j.

2.1 Slope of a Line

We define the *slope* of line $\overleftrightarrow{P_1P_2}$ containing points P_1 and P_2 by the formula

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1}$$

provided $x_1 \neq x_2$. If $x_1 = x_2$ then line $\overleftrightarrow{P_1P_2}$ is vertical and the slope is undefined.

2.2 Identifying Which Points are on a Given Line

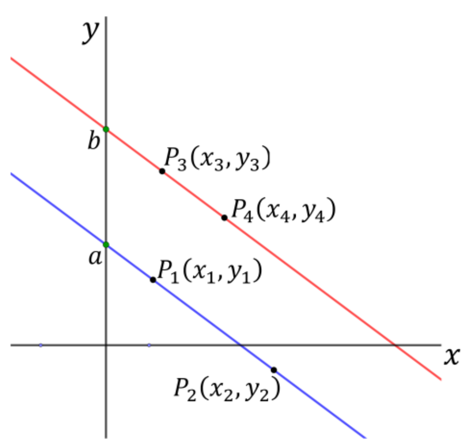
The point $P_3(x_3, y_3)$ is on $\overleftrightarrow{P_1P_2}$ if and only if

$$\frac{y_3 - y_1}{x_3 - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

with $x_1 \neq x_2$ and $x_1 \neq x_3$.

2.3 Parallel Lines

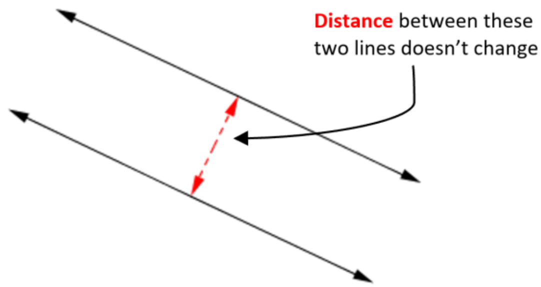
Lines $\overleftrightarrow{P_1P_2}$ and $\overleftrightarrow{P_3P_4}$ in the xy-plane are parallel if they have the same slope but different y-intercepts.



That is, if

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_4 - y_3}{x_4 - x_3}$$

with $x_1 \neq x_2$ and $x_3 \neq x_4$ but $a \neq b$.



Two lines are said to be **parallel** if the non-zero distance between the two lines never changes.

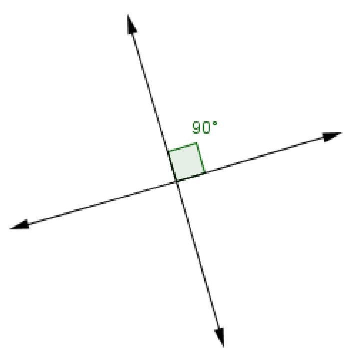
Notation: We write $l_1 \parallel l_2$ to denote that lines l_1 and l_2 are parallel.

2.4 Perpendicular Lines

Lines $\overleftrightarrow{P_1P_2}$ and $\overleftrightarrow{P_3P_4}$ in the xy-plane are perpendicular if their slopes are negative reciprocals. That is, if

$$\left(\frac{y_2 - y_1}{x_2 - x_1}\right) = -\frac{1}{\left(\frac{y_4 - y_3}{x_4 - x_3}\right)}.$$

It follows from this definition that two lines are **perpendicular** if the angle between the lines equals $90°$, that is, a right angle.



Notation: We write $l_1 \perp l_2$ to denote that lines l_1 and l_2 are perpendicular. The symbol "⊥" is spoken as "perp". So, you will hear mathematicians say l_1 "perp" l_2 as shorthand for saying the two lines are perpendicular.

3 Point-slope, slope-intercept, intercept, normal forms of the straight line

3.1 Standard Form of a Line

The standard form for a line with slope $-A/B$ and intercepts $-C/A$ and $-C/B$ is

$$Ax + By + C = 0.$$

In many books the standard form title is used when the line is expressed by

$$Ax + By = C.$$

In this case the slope is still $-A/B$ but the intercepts would be C/A and C/B.

3.2 Point Slope Form of a Line

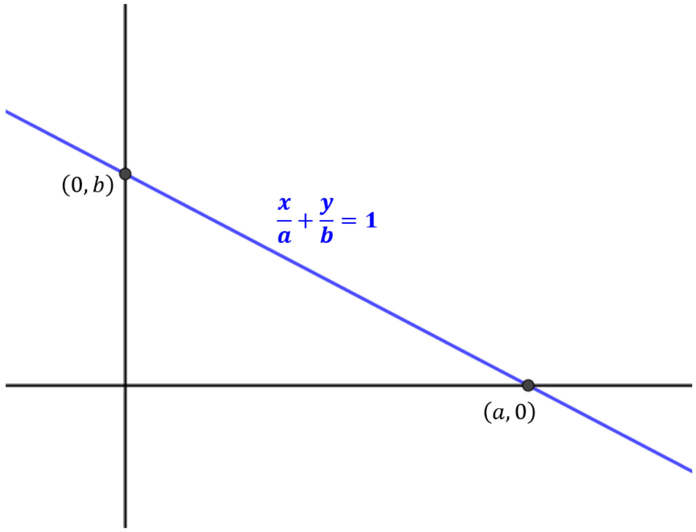
The equation of the line with slope m that includes the point (x_0, y_0) is given by

$$y - y_0 = m(x - x_0) \text{ or equivalently } y = y_0 + m(x - x_0).$$

3.3 Two Intercepts Form of a Line

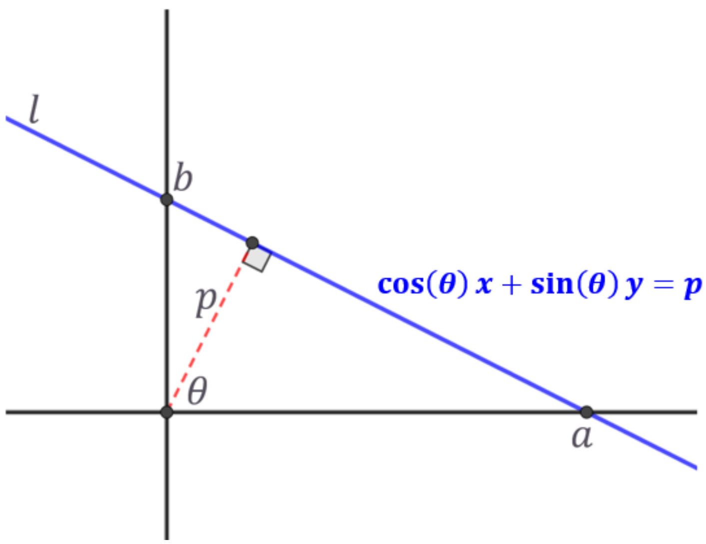
The x-y intercept form of a line with x-intercept $(a, 0)$ and y-intercept $(0, b)$ is

$$\frac{x}{a} + \frac{y}{b} = 1.$$



3.4 Normal Form of a Line

The normal form for line l is $\cos(\theta)\,x+\sin(\theta)\,y=p$



where p is the perpendicular (shortest) distance from l to the origin and θ is the angle between this perpendicular and the positive x-axis. We note that in this model

$$\cos(\theta)=\frac{b}{\sqrt{a^2+b^2}},\quad \sin(\theta)=\frac{b}{\sqrt{a^2+b^2}},\ \text{and}\ \ p=\frac{ab}{\sqrt{a^2+b^2}}.$$

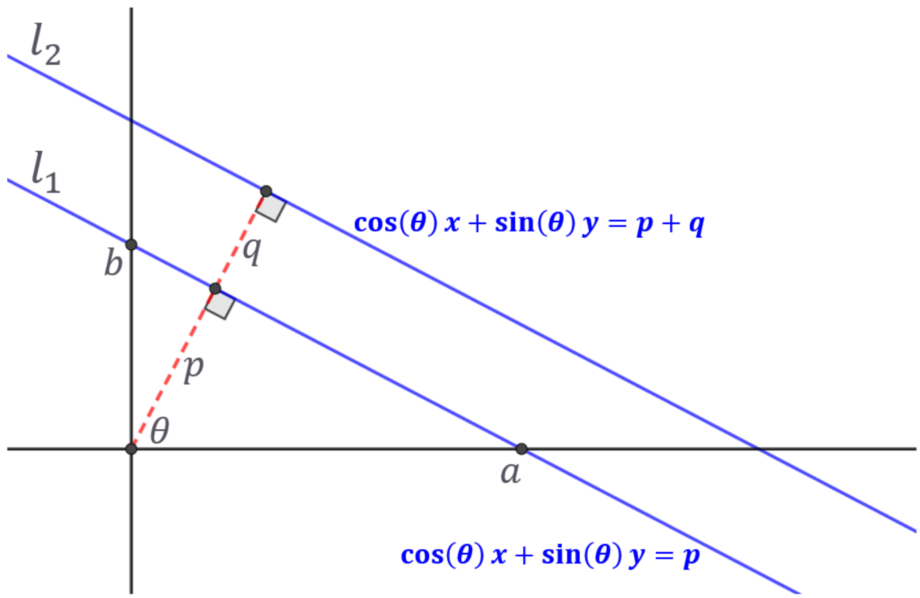
Therefore, the normal form for line l can also be written as

$$\left(\frac{b}{\sqrt{a^2+b^2}}\right)x+\left(\frac{a}{\sqrt{a^2+b^2}}\right)y=\frac{ab}{\sqrt{a^2+b^2}}$$

where a and b are respectively the x and y intercepts of line l.

3.4.1 Relationship of the Normal Forms for Two Parallel Lines

If q is the distance between two parallel lines l_1 and l_2 and if the normal form for l_1 is $\cos(\theta)\,x+\sin(\theta)\,y=p$, then the normal form for l_2 would be $\cos(\theta)\,x+\sin(\theta)\,y=p+q$.



4 Distance Results

Distance Between Two Points	
The distance d between the points (x_1, y_1) and (x_2, y_2) $$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}.$$	**(1)**

Distance from a Line to a Point Not on that Line

Let $P = (x_0, y_0)$ be a point in the same plane as the line $ax + by + c = 0$ but not on this line. Then the perpendicular (shortest) distance d from point P to this line equals $$d = \frac{\lvert ax_0 + by_0 + c \rvert}{\sqrt{a^2 + b^2}}.$$	**(2)**

Closest Point on a Line to a Given Point Not on that Line	
The point on this line which is closest to the point P at (x_0, y_0) not on this line has coordinates $$x = \frac{b(bx_0 - ay_0) - ac}{a^2 + b^2} \quad \text{and} \quad y = \frac{a(-bx_0 + ay_0) - bc}{a^2 + b^2}.$$	**(3)**

Distance Between Two Parallel Lines	
The distance d between the parallel lines $Ax + By + C_1 = 0$ and $Ax + By + C_2 = 0$ equals $$d = \frac{\lvert C_2 - C_1 \rvert}{\sqrt{A^2 + B^2}}.$$	**(4)**

5 Applications of Analytic Geometry

Midpoint of a Line Segment	
The midpoint of a line segment connecting the points (x_1, y_1) and (x_2, y_2) has the coordinates $$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$	**(5)**

Collinear Points	
The three points at $A(x_1, y_1), B(x_x, y_2)$ and $C(x_3, y_3)$ are collinear (on the same line) if $$\frac{1}{2}\lvert x_1y_2 + x_2y_3 + x_3y_1 - x_2y_1 - x_3y_2 - x_1y_3 \rvert = 0.$$ That is, if the area of the triangle these three points determine has area equal to 0.	**(6)**

Equation of a Circle	
The equation of the circle with center at (a, b) and radius r is given by $$(x - a)^2 + (y - b)^2 = r^2.$$	**(7)**

Centroid of a Triangle	
The centroid, the point of concurrency (intersection) of the three medians of a triangle with vertices at (x_1, y_1), (x_2, y_2) and (x_3, y_3) has coordinates $$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right).$$	**(8)**

Angle between Two Lines	
If ϕ is an angle, measured counterclockwise, between two lines, then $$\tan(\phi) = \frac{m_2 - m_1}{1 + m_1 m_2}$$ where m_2 is the slope of the terminal side and m_1 is the slope of the initial side.	**(9)**

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Perpendicular Lines	
If m_1 and m_2 are the slopes of two perpendicular lines then $$m_1 = -\frac{1}{m_2}.$$	**(10)**

Parallel Lines	
If m_1 and m_2 are the slopes of two parallel lines then $$m_1 = m_2.$$	**(11)**

Midpoints Theorem	
In any triangle the line segment joining the midpoints of two sides is parallel to, and one-half as long as, the third side.	**(12)**

6 Determinants

6.1 Determinant (det) of a 2×2 matrix

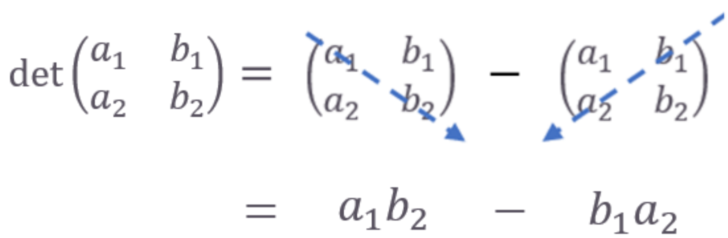
$$\det\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = a_1 b_2 - b_1 a_2.$$

Here is an algorithm for remembering the determinant (det) of a 2×2 matrix.

Step 1. Multiple the elements on the main diagonal going left to right to get $a_1 b_2$.

Step 2. Multiple the elements on the main diagonal going right to left to get $b_1 a_2$.

Step 3. Subtract the results of Steps 1 and 2.



6.2 Determinant (det) of a 3×3 matrix

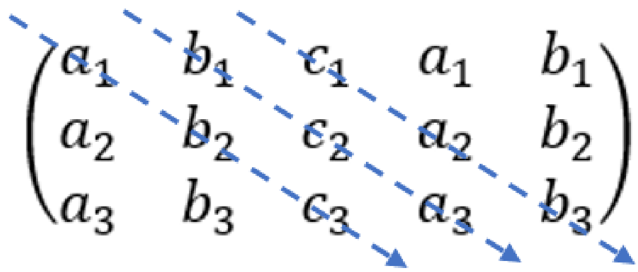
$$\det\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} = (a_1 b_2 c_3 + b_1 c_2 a_3 + c_1 a_2 b_3) - (b_1 a_2 c_3 + a_1 c_2 b_3 + c_1 b_2 a_3).$$

Here is an algorithm for remembering the determinant (det) of a 3×3 matrix.

Step 1. Augment (add on) the 3×3 matrix with the first two columns

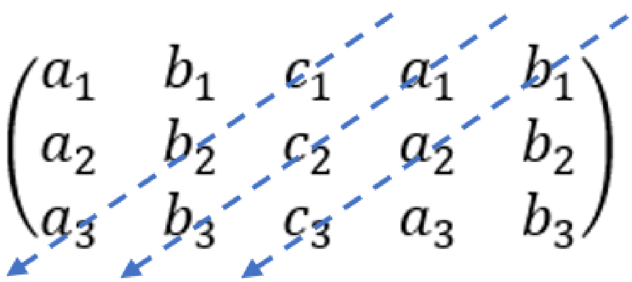
$$\begin{pmatrix} a_1 & b_1 & c_1 & \boldsymbol{a_1} & \boldsymbol{b_1} \\ a_2 & b_2 & c_2 & \boldsymbol{a_2} & \boldsymbol{b_2} \\ a_3 & b_3 & c_3 & \boldsymbol{a_3} & \boldsymbol{b_3} \end{pmatrix}.$$

Step 2. Multiple the elements on the three main diagonals going left to right and then add the results to get $a_1 b_2 c_3 + b_1 c_2 a_3 + c_1 a_2 b_3$.



$$a_1 b_2 c_3 + b_1 c_2 a_3 + c_1 a_2 b_3$$

Step 3. Repeat Step 2 but going right to left to get $b_1 a_2 c_3 + a_1 c_2 b_3 + c_1 b_2 a_3$.



$$b_1 a_2 c_3 + a_1 c_2 b_3 + c_1 b_2 a_3$$

Step 4. Subtract the results of Steps 2 and 3.

$$\det\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} = (a_1 b_2 c_3 + b_1 c_2 a_3 + c_1 a_2 b_3) - (b_1 a_2 c_3 + a_1 c_2 b_3 + c_1 b_2 a_3).$$

CAUTION! The method of multiplying the elements on diagonals and adding when going left to right but subtracting when going right to left DOES NOT (unfortunately) extend to finding the determinant of a 4×4 or higher.

6.3 Applications of Determinants in Analytic Geometry

6.3.1 Concurrent Lines

The three lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2$ and $a_3x + b_3y + c_3 = 0$ are concurrent (intersect at a single point) if the 3×3 determinant of coefficients equals 0. That is, if

$$\det\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} = 0.$$

6.3.2 Finding the Equation of the Plane Containing Three Points

The equation of the plane containing the three non-collinear points $P(x_1, y_1, z_1), Q(x_2, y_2, z_2)$ and $R(x_3, y_3, z_3)$ can be expressed by

$$\det\begin{pmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{pmatrix} = 0.$$

7 Finding the Equation of the Circle Passing Through Three Points

Let $P(x_1, y_1), Q(x_2, y_2)$ and $R(x_3, y_3)$ be the three given points. Let $C(a, b)$ and r be the (unknown) center point and radius of the circle that passes through these three points.

The square of the distance from C to P equals the square of the distance from C to Q and the square of the distance from C to R. That is $|PC|^2 = |PQ|^2 = |PR|^2 = r^2$. This gives us three equations from which we can solve for the three unknowns, namely, a, b and r.

$$(x_1 - a)^2 + (y_1 - b)^2 = (x_2 - a)^2 + (y_2 - b)^2 = (x_3 - a)^2 + (y_3 - b)^2.$$

Expanding these out we see

$$x_1^2 - 2ax_1 + a^2 + y_1^2 - 2by_1 + b^2 = x_2^2 - 2ax_2 + a^2 + y_2^2 - 2by_2 + b^2$$

and

$$x_2^2 - 2ax_2 + a^2 + y_2^2 - 2by_2 + b^2 = x_3^2 - 2ax_3 + a^2 + y_3^2 - 2by_3 + b^2.$$

We can see that the a^2 and b^2 cancel out in both equations which allows us to solve the resulting system of two linear equations in two (a and b) unknowns.

Once you find a and b we can substitute these values into $|PC|^2 = x_1^2 - 2ax_1 + a^2 + y_1^2 - 2by_1 + b^2$ and solve for r in the equation $|PC|^2 = r^2$.

Example

Find the equation for the circle that passes through the three points $(2,1), (0,5)$ and $(-1,2)$.

Solution

Following in the notation used above we have

$$(x_1 - a)^2 + (y_1 - b)^2 = (x_2 - a)^2 + (y_2 - b)^2 = (x_3 - a)^2 + (y_3 - b)^2$$

or

$$(2 - a)^2 + (1 - b)^2 = (0 - a)^2 + (5 - b)^2 = (-1 - a)^2 + (2 - b)^2.$$

On simplifying we have

$$5 - 4a - 2b = 25 - 10b = 5 + 2a - 4b.$$

From Equation 1 = Equation 2 and Equation 2 = Equation 3, we have

$$5 - 4a - 2b = 25 - 10b \quad \text{and} \quad 25 - 10b = 5 + 2a - 4b.$$

On simplifying these two equations we have

$$a - 2b = -5 \quad \text{and } a + 3b = 10.$$

Solving simultaneously for a and b we find that $a = 1$ and $b = 3$.

$$r^2 = (x_1 - a)^2 + (y_1 - b)^2 = (2 - 1)^2 + (1 - 3)^2 = 5.$$

Therefore, $(x - a)^2 + (y - b)^2 = r^2$, the equation for our circle becomes

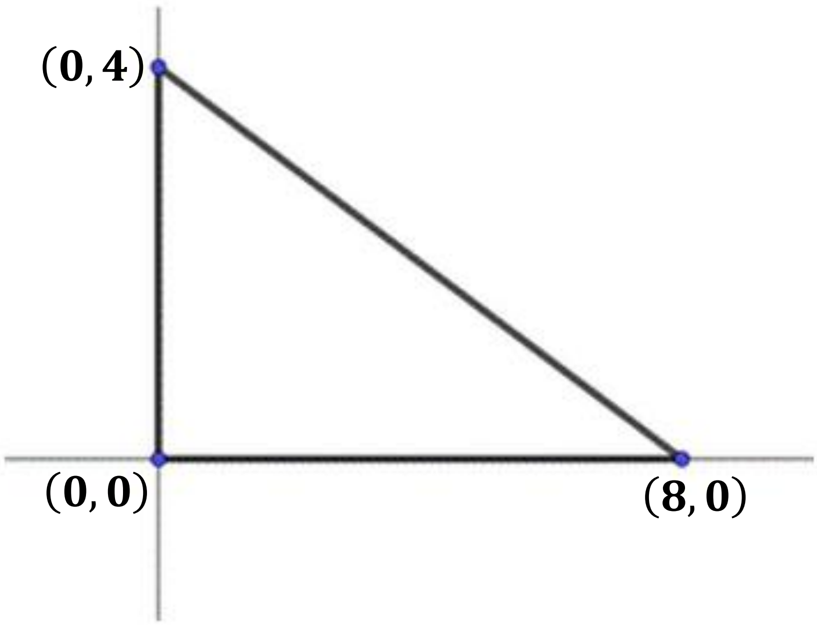
$$(x - 1)^2 + (y - 3)^2 = 5.$$

Example (MSHSML 2D041)

Write the equation of the circle that circumscribed ΔABC, given $A(0,0), B(8,0)$ and $C(0,6)$.

<u>Solution</u>

First find the center of the circle by finding where the perpendicular bisectors intersect.



Midpoint between $(0,0)$ and $(0,6)$ is $\left(\frac{0+0}{2}, \frac{6+0}{2}\right) = (0,3)$

Midpoint between $(0,0)$ and $(8,0)$ is $\left(\frac{0+8}{2}, \frac{0+0}{2}\right) = (4,0)$

Midpoint between $(0,6)$ and $(8,0)$ is $\left(\frac{0+8}{2}, \frac{6+0}{2}\right) = (4,3)$.

Slope of the line going through $(0,0)$ and $(0,6)$ is $\frac{6-0}{0-0}$. (i.e. vertical line)

Slope of the line going through $(0,0)$ and $(8,0)$ is $\frac{0-0}{8-0} = 0$. (i.e. horizontal line)

Slope of the line going through $(0,6)$ and $(8,0)$ is $\frac{0-6}{8-0} = \frac{-3}{4}$.

Slope of the perpendicular bisector to side with vertices $(0,0)$ and $(0,6)$ is 0 (i.e. horizontal line)

Slope of the perpendicular bisector to side with vertices $(0,0)$ and $(8,0)$ is undefined (i.e. vertical line)

Slope of the perpendicular bisector to side with vertices $(0,6)$ and $(8,0)$ is $-\left(\frac{-4}{3}\right) = \frac{4}{3}$.

Equation for the perpendicular bisector going through the midpoint $(0,3)$ and with a slope of 0 is the horizontal line is $y = 3$.

Equation for the perpendicular bisector going through the midpoint $(4,0)$ and with an undefined slope (*i.e.* is a vertical line) is $x = 4$.

Equation for the perpendicular bisector going through the midpoint $(4,3)$ and with a slope of $\frac{4}{3}$ is the line is $y = 3 + \left(\frac{4}{3}\right)(x - 4)$.

Now find where these three lines intersect.

$$3 + \left(\frac{4}{3}\right)(x - 4) = 3 \Longrightarrow x = 4.$$

Therefore, the center of the circle has coordinates $(x, y) = (4,3)$.

We can find the radius of the circle by finding the distance from the center point to any of the known three points on the circle. Consider $(0,0)$.

$$\text{radius} = \sqrt{(4-0)^2 + (3-0)^2} = \sqrt{16+9} = 5.$$

Therefore, the equation of the circle is

$$(x - 4)^2 + (y - 3)^2 = 5^2.$$

Write the equation of the circle that circumscribed ΔABC, given $A(3,2), B(1,4)$ and $C(5,4)$.

Answer: Center is $(3,4)$ and the radius is 2. $(x-3)^2+(y-4)^2=2^2$.

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