MSHSML Meet 3, Event A
Study Guide

3A Systems of Linear Equations in Two (or on occasion three) Variables

Numeric and literal systems
Relation to graphical procedures
Word problems leading to such systems
Systems of inequalities used to define a region in the plane
Determinants

1 Contents

2 Determinants

The determinant of a square matrix **A**, which is denoted by either $|\mathbf{A}|$ or $\det(\mathbf{A})$, is defined by describing formulas for calculating it.

2.1 Computation of a Determinant for the 2×2 Case

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \det\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = a_1 b_2 - b_1 a_2.$$

Here is an algorithm for remembering the determinant (det) of a 2×2 matrix.

Step 1. Multiple the elements on the main diagonal going left to right to get $a_1 b_2$.

Step 2. Multiple the elements on the main diagonal going right to left to get $b_1 a_2$.

Step 3. Subtract the results of Steps 1 and 2.

$$\det\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} - \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$$

$$= \quad a_1 b_2 \quad - \quad b_1 a_2$$

2.2 Computation of a Determinant for the 3×3 Case

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \det\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$$

$$= (a_1 b_2 c_3 + b_1 c_2 a_3 + c_1 a_2 b_3) - (b_1 a_2 c_3 + a_1 c_2 b_3 + c_1 b_2 a_3).$$

Here is an algorithm for remembering the determinant (det) of a 3×3 matrix.

Step 1. Augment (add on) the 3×3 matrix with the first two columns

$$\begin{pmatrix} a_1 & b_1 & c_1 & \boldsymbol{a_1} & \boldsymbol{b_1} \\ a_2 & b_2 & c_2 & \boldsymbol{a_2} & \boldsymbol{b_2} \\ a_3 & b_3 & c_3 & \boldsymbol{a_3} & \boldsymbol{b_3} \end{pmatrix}.$$

Step 2. Multiple the elements on the three main diagonals going left to right and then add the results to get $a_1 b_2 c_3 + b_1 c_2 a_3 + c_1 a_2 b_3$.

$$\begin{pmatrix} a_1 & b_1 & c_1 & a_1 & b_1 \\ a_2 & b_2 & c_2 & a_2 & b_2 \\ a_3 & b_3 & c_3 & a_3 & b_3 \end{pmatrix}$$

$$a_1 b_2 c_3 + b_1 c_2 a_3 + c_1 a_2 b_3$$

Step 3. Repeat Step 2 but going right to left to get $b_1 a_2 c_3 + a_1 c_2 b_3 + c_1 b_2 a_3$.

$$\begin{pmatrix} a_1 & b_1 & c_1 & a_1 & b_1 \\ a_2 & b_2 & c_2 & a_2 & b_2 \\ a_3 & b_3 & c_3 & a_3 & b_3 \end{pmatrix}$$

$$b_1 a_2 c_3 + a_1 c_2 b_3 + c_1 b_2 a_3$$

Step 4. Subtract the results of Steps 2 and 3.

$$\det\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$$

$$= (a_1 b_2 c_3 + b_1 c_2 a_3 + c_1 a_2 b_3) - (b_1 a_2 c_3 + a_1 c_2 b_3 + c_1 b_2 a_3).$$

CAUTION! The method of multiplying the elements on diagonals and adding when going left to right but subtracting when going right to left DOES NOT (unfortunately) extend to finding the determinant of a 4×4 or higher.

2.3 Computation of a Determinant for the 4×4 Case

The formula below shows how to reduce the problem of finding the determinant of a 4×4 square matrix to the "simpler" problem of finding the determinant of four separate 3×3 square matrices.

$$\det\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{33} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{pmatrix}$$

$$= a_{1,1} \cdot \det\begin{pmatrix} a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,2} & a_{4,3} & a_{4,4} \end{pmatrix} - a_{1,2} \cdot \det\begin{pmatrix} a_{2,1} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,3} & a_{4,4} \end{pmatrix}$$

$$+ a_{1,3} \cdot \det\begin{pmatrix} a_{2,1} & a_{2,2} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,4} \end{pmatrix} - a_{1,4} \cdot \det\begin{pmatrix} a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \\ a_{4,1} & a_{4,2} & a_{4,3} \end{pmatrix}.$$

2.4 Computation of a Determinant in the $n \times n$ Case

Imagine drawing a line through the i^{th} row and j^{th} column of the $n \times n$ square matrix A. Below we demonstrate the case where we have drawn a line through the 4^{th} row and 2^{nd} column of the 4×4 square A.

$$A = \begin{pmatrix} a_{1,1} & \cancel{a_{1,2}} & a_{1,3} & a_{1,4} \\ a_{2,1} & \cancel{a_{2,2}} & a_{2,3} & a_{2,4} \\ a_{3,1} & \cancel{a_{3,2}} & a_{33} & a_{3,4} \\ \cancel{a_{4,1}} & \cancel{a_{4,2}} & \cancel{a_{4,3}} & \cancel{a_{4,4}} \end{pmatrix}$$

Crossing out the 4^{th} row and 2^{nd} column of the 4×4 square matrix A.

In general, what is left is a matrix of size $(n-1) \times (n-1)$. In the above example, what is left is the 3×3 matrix

$$\begin{pmatrix} a_{1,1} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,3} & a_{3,4} \end{pmatrix}.$$

The determinant of the matrix that remains after crossing out a row and column is known as a ***minor*** of the original matrix.

The minor that results from calculating the determinant of the matrix that remains after crossing out the i^{th} row and j^{th} column is referred to as the (i, j) minor and is by $M_{i,j}$.

In the above 4×4 matrix

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{33} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{pmatrix}$$

we can see that

$$M_{4,2} = \det \begin{pmatrix} a_{1,1} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,3} & a_{3,4} \end{pmatrix}.$$

If we multiple $M_{i,j}$ by $(-1)^{i+j}$ the result is known as the (i, j) ***cofactor*** of matrix A and is denoted by $A_{i,j}$.

In the above example,

$$A_{4,2} = (-1)^{4+2} M_{4,2} = (-1)^{4+2} \det \begin{pmatrix} a_{1,1} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,3} & a_{3,4} \end{pmatrix}.$$

It turns out that there is a relationship between the determinant of the $n \times n$ matrix A and its cofactors.

2.4.1 Expanding Along a Row

If $a_{i,j}$ is the element in the i^{th} row and j^{th} column of the $n \times n$ square matrix A and if $A_{i,j}$ is the (i,j) cofactor of the matrix A, then for every $i = 1,2, \dots, n$

$$\det(A) = a_{i,1}A_{i,1} + a_{i,2}A_{i,2} + \cdots + a_{i,n}A_{i,n}.$$

Using this formula to find $\det(A)$ is known as the method of “expanding along the i^{th} row. It does not matter which row you pick – you get the same result regardless of which row you “expand along”.

Which row should you pick? Again, you will get the same answer regardless of which row you to pick to “expand along”. So, unless you have some special reason for picking a particular row, it is typical to “expand along” the top (or 1st row). In this case you have

$$\det(A) = a_{1,1}A_{1,1} + a_{1,2}A_{1,2} + \cdots + a_{1,n}A_{1,n}.$$

What might be a good “special reason” for expanding along a different row other than the top row? Well, imagine the i^{th} row has several zeros in it. That is, several of the entries $a_{i,1}$, $a_{i,2}$,…, $a_{i,n}$ equal 0.

Then expanding along that row is a smart idea because if $a_{i,j} = 0$ then you will not need to find $A_{i,j}$ because you will just be multiplying it by $a_{i,j} = 0$ which makes that term drop out.

So, it can be a time saver to expand along a row with lots of zeros, if one exists.

Look back now and notice that the formula given in the previous section for finding the determinant of a 4×4 matrix is just the result of expanding that 4×4 matrix along the top row.

Can you expand along a column instead of a row? YES!

2.4.2 Expanding Along a Column

If $a_{i,j}$ is the element in the i^{th} row and j^{th} column of the $n \times n$ square matrix A and if $A_{i,j}$ is the (i,j) cofactor of the matrix A, then for every $j = 1,2, \dots, n$

$$\det(A) = a_{1,J}A_{1,J} + a_{2,J}A_{2,J} + \cdots + a_{n,j}A_{n,j}.$$

Using this formula to find $\det(A)$ is known as the method of “expanding along the j^{th} column. It does not matter which row you pick – you get the same result regardless of which row you “expand along”.

So, again it can be a time saver to pick a column with a lot of zeros.

3 Determinants: Geometric Applications

3.1 Two-Point Form of the Equation of Line

An equation of the line passing through the distinct points (x_1, y_1) and (x_2, y_2) is given by

$$\det\begin{pmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{pmatrix} = 0.$$

3.2 Distance from a Point to a Line

The shortest (*i.e*. perpendicular) distance from the line passing through the distinct points (x_1, y_1) and (x_2, y_2) to a point (x_0, y_0) not on that line is given by

$$\pm\frac{\det\begin{pmatrix} x_0 & y_0 & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{pmatrix}}{\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}}$$

where the sign $(\pm)$ is chosen to give a positive distance.

3.3 Distance from a Point to a Plane

$$\frac{Aa + Bb + Cc + D}{\sqrt{A^2 + B^2 + C^2}}.$$

3.4 Area of a Triangle in the xy-Plane

The area of a triangle with vertices $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) is given by

$$\pm\frac{1}{2}\det\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix}$$

where the sign $(\pm)$ is chosen to give a positive area.

3.5 Area of a Parallelogram

The area of a parallelogram where *any* three of the four vertices of this parallelogram are $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) is given by

$$\pm \det\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix}$$

where the sign $(\pm)$ is chosen to give a positive area.

Note 1: There are 3 different parallelograms that share these three given vertices but all three of these possible parallelograms have the same area. So, it does not matter which three of the four vertices of this parallelogram you use to find the area.

Note 2: If a parallelogram is denoted $ABCD$ and you know which three of these four vertices are given, then there is only one possible parallelogram that can be formed because the notation $ABCD$ implies the vertices follow the cyclic order $A \to B \to C \to D \to A$.

3.6 Volume of a Tetrahedron

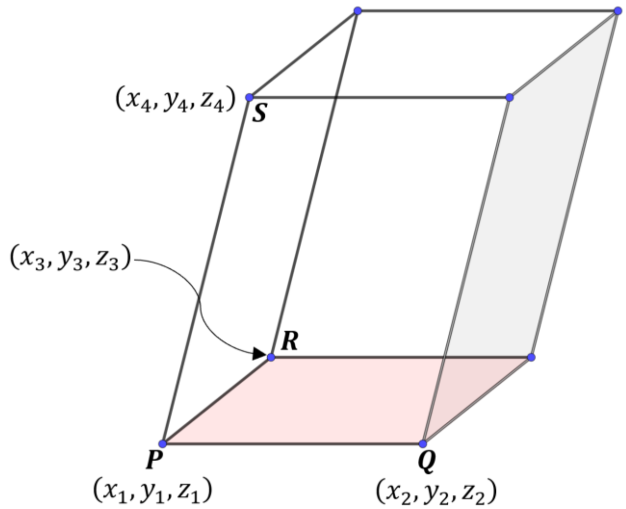
The volume of a tetrahedron with vertices (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) and (x_4, y_4, z_4) is given by

$$\pm \frac{1}{6}\det\begin{pmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{pmatrix}$$

where the sign $(\pm)$ is chosen to give a positive volume.

3.7 Volume of a Parallelepiped

A parallelepiped is a solid in which each face is a parallelogram.



If vertex P in the above figure has coordinates (x_1, y_1, z_1) and if the three vertices Q, R and S adjacent to P have coordinates (x_2, y_2, z_2), (x_3, y_3, z_3) and (x_4, y_4, z_4), then the volume of the parallelepiped is

$$\pm\det\begin{pmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{pmatrix}$$

where the sign $(\pm)$ is chosen to give a positive volume.

3.8 Test for Collinear Points in the xy-Plane

The three points (x_1, y_1), (x_2, y_2) and (x_3, y_3) are collinear if and only if

$$\det\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix} = 0.$$

Note: Remembering that

$$\pm\frac{1}{2}\det\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix}$$

is the area of the triangle with vertices $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3), we see that the test of collinearity of three points is equivalent to a check on whether the area of the triangle with these three points as vertices equals 0.

3.9 Test for Concurrent Lines

The three lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2$ and $a_3x + b_3y + c_3 = 0$ are concurrent (intersect at a single point) if and only if

$$\det\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} = 0.$$

3.10 Test for Coplanar Points in Space

The four points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) and (x_4, y_4, z_4) are coplanar if and only if

$$\det\begin{pmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{pmatrix} = 0.$$

Note: Remembering that

$$\pm\frac{1}{6}\det\begin{pmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{pmatrix}$$

is the volume of a tetrahedron with vertices (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) and (x_4, y_4, z_4), we see that the test of coplanarity of four points is equivalent to a check on whether the volume of the tetrahedron built from these four vertices equals 0.

3.11 Three-Point Form of the Equation of a Plane

An equation of the plane containing the three non-collinear points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) is given by

$$\det\begin{pmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{pmatrix} = 0.$$

3.12 Three-Point Form of the Equation of a Circle

An equation of the circle going through the three non-collinear points (x_1, y_1), (x_2, y_2) and (x_3, y_3) is given by

$$\det\begin{pmatrix} x^2+y^2 & x & y & 1 \\ x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \end{pmatrix} = 0.$$

3.13 Three-Point Form of the Equation of a Parabola

An equation of the parabola passing through the three points (x_1, y_1), (x_2, y_2) and (x_3, y_3) is given by

$$\det\begin{pmatrix} x^2 & x & y & 1 \\ x_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3 & y_3 & 1 \end{pmatrix} = 0$$

provided these three points are not collinear.

3.14 Four-Point Form of the Equation of an Ellipse

An equation of the ellipse passing through the four points (x_1, y_1), (x_2, y_2), (x_3, y_3) and (x_4, y_4) is given by

$$\det\begin{pmatrix} x^2 & y^2 & x & y & 1 \\ (x_1)^2 & (y_1)^2 & x_1 & y_1 & 1 \\ (x_2)^2 & (y_2)^2 & x_2 & y_2 & 1 \\ (x_3)^2 & (y_3)^2 & x_3 & y_3 & 1 \\ (x_4)^2 & (y_4)^2 & x_4 & y_4 & 1 \end{pmatrix} = 0$$

provided no three of these points are collinear.

3.15 Five-Point Form of the Equation of a Conic Section (Parabola, Hyperbola, Ellipse)

An equation of a conic section (parabola, hyperbola or an ellipse) passing through the five points (x_1, y_1), (x_2, y_2),..., (x_5, y_5) is given by

$$\det\begin{pmatrix} x^2 & xy & y^2 & x & y & 1 \\ (x_1)^2 & x_1y_1 & (y_1)^2 & x_1 & y_1 & 1 \\ (x_2)^2 & x_2y_2 & (y_2)^2 & x_2 & y_2 & 1 \\ (x_3)^2 & x_3y_3 & (y_3)^2 & x_3 & y_3 & 1 \\ (x_4)^2 & x_4y_4 & (y_4)^2 & x_4 & y_4 & 1 \\ (x_5)^2 & x_5y_5 & (y_5)^2 & x_5 & y_5 & 1 \end{pmatrix} = 0$$

provided no three of these points are collinear.

4 Systems of Linear Equations

4.1 Matrix Notation

Consider the 3×3 system of linear equations in the variables x_1, x_2 and x_3:

$$\begin{aligned} a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3 &= b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + a_{2,3}x_3 &= b_2 \\ a_{3,1}x_1 + a_{3,2}x_2 + a_{3,3}x_3 &= b_b. \end{aligned}$$

We can rewrite this system in "matrix form" as

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,3} & a_{3,2} & a_{3,3} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$

The matrix

$$\mathbf{A} = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,3} & a_{3,2} & a_{3,3} \end{pmatrix}$$

is called the "coefficient matrix" of the system, the vector

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

is called the "solution vector", and the vector

$$\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

is called the "constant vector" of the system. Using this notation, we can express this system of linear equations as $\mathbf{Ax} = \mathbf{b}$.

Of course, there is nothing special about a 3×3 system. This matrix notation is applicable for any $n \times n$ linear system of equations.

1.	Express the 3×3 linear system $$\begin{aligned} 2x + 3y - z &= 1 \\ y + z &= 2 \\ -3x + 4y - 3z &= 0 \end{aligned}$$ in matrix form.

Solution

$$\begin{pmatrix} 2 & 3 & -1 \\ 0 & 1 & 1 \\ -3 & 4 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}.$$

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2.	Express the 4×4 linear system $$\begin{aligned} 2x_1 + 3x_2 - x_3 - 4x_4 &= 1 \\ x_1 - x_2 - 4x_3 - x_4 &= -2 \\ -3x_2 + x_4 &= -3 \\ 2x_1 - 3x_3 - 4x_4 &= 0 \end{aligned}$$ in matrix form.

Solution

$$\begin{pmatrix} 2 & 3 & -1 & -4 \\ 1 & -1 & -4 & -1 \\ 0 & -3 & 0 & 1 \\ 2 & 0 & -3 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ -3 \\ 0 \end{pmatrix}.$$

■

4.2 Homogeneous and Inhomogeneous Systems of Linear Equations

Suppose we write a system of linear equations in matrix form $\mathbf{Ax} = \mathbf{b}$. If the constant vector $\mathbf{b}$ consists of all 0's, *i.e.*

$$\mathbf{b} = \mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}$$

then the linear system $\mathbf{Ax} = \mathbf{0}$ is called ***homogeneous***. If the constant vector $\mathbf{b} \neq \mathbf{0}$, *i.e.* the constant vector $\mathbf{b}$ does not consist of ***all*** zeros, then the linear system $\mathbf{Ax} = \mathbf{b}$ is called ***nonhomogeneous***.

$\mathbf{Ax} = \mathbf{0}$, homogeneous

$\mathbf{Ax} = \mathbf{b}$, $\mathbf{b} \neq \mathbf{0}$, nonhomogeneous

4.3 Determinants and Solution Vectors of a Linear System

4.3.1 Number of Solutions of a Homogeneous Linear System

A homogeneous system of linear equations $\mathbf{Ax} = \mathbf{0}$ will have an infinite number of solution vectors $\mathbf{x}$ if and only if $\det(\mathbf{A}) = 0$.

If $\det(\mathbf{A}) \neq 0$ for the homogeneous system of linear equations $\mathbf{Ax} = \mathbf{0}$, then there is exactly one solution vector $\mathbf{x}$, namely the trivial solution where $\mathbf{x}$ consists of all 0's.

4.3.2 Number of Solutions of a Nonhomogeneous Linear System

A nonhomogeneous system of linear equations $\mathbf{Ax} = \mathbf{b}$, $\mathbf{b} \neq \mathbf{0}$ has a unique non-trivial solution if and only if $\det(\mathbf{A}) \neq 0$.

If $\det(\mathbf{A}) = 0$ for the nonhomogeneous system of linear equations $\mathbf{Ax} = \mathbf{b}$, $\mathbf{b} \neq \mathbf{0}$ then the system either has no solutions (called an inconsistent system) or an infinite number of solutions.

4.3.3 Cramer's Rule for Solving a System of Linear Equations

Consider a system of n linear equations for n unknowns, represented in matrix multiplication form as follows:

$$Ax = b$$

where the $n \times n$ matrix A has a nonzero determinant, and the vector $x = (x_1, \dots, x_n)^{\mathrm{T}}$ is the column vector of the variables. Then the theorem states that the system has a unique solution, whose individual values for the unknowns are given by:

$$x_i = \frac{\det(A_i)}{\det(A)} \quad i = 1,2, \dots, n$$

where A_i is the matrix formed by replacing the i^{th} column of A by the column vector b.

2×2 Case

Given

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

which in matrix format is

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Then the values of x and y can be found as follows:

$$x = \frac{\begin{vmatrix} \boldsymbol{c_1} & b_1 \\ \boldsymbol{c_2} & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{c_1b_2 - b_1c_2}{a_1b_2 - b_1a_2} \text{ and } y = \frac{\begin{vmatrix} a_1 & \boldsymbol{c_1} \\ a_2 & \boldsymbol{c_2} \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{a_1c_2 - c_1a_2}{a_1b_2 - b_1a_2}.$$

3×3 Case

Given

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

which in matrix format is

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}.$$

Then the values of x, y and z can be found as follows:

$$x = \frac{\begin{vmatrix} \mathbf{d_1} & b_1 & c_1 \\ \mathbf{d_2} & b_2 & c_2 \\ \mathbf{d_3} & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a_1 & \mathbf{d_1} & c_1 \\ a_2 & \mathbf{d_2} & c_2 \\ a_3 & \mathbf{d_3} & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \text{ and } z = \frac{\begin{vmatrix} a_1 & b_1 & \mathbf{d_1} \\ a_2 & b_2 & \mathbf{d_2} \\ a_3 & b_3 & \mathbf{d_3} \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}.$$

5 Reduced Row Echelon Form

Theorem

A linear system of equations (such as the system shown below) will have the same solution(s) as the linear system called the "reduced row echelon form" of that system (such as the system shown after Step 8 below.)

Example

System of 3 linear equations in 5 unknowns.

$$\begin{aligned} 3x_2 - 6x_3 + 6x_4 + 4x_5 &= -5 \\ 3x_1 - 7x_2 + 8x_3 - 5x_4 + 8x_5 &= 9 \\ 3x_1 - 9x_2 + 12x_3 - 9x_4 + 6x_5 &= 15 \end{aligned}$$

Step 1.

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

Switch row 1 and row 3. All leading zeros are now below non-zero leading entries.

Step 2.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

Replace row 2 with (row 2 minus row 1). This eliminates the first entry of row 2.

Step 3.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

Multiply row 2 by 3 and row 3 by 2.

Step 4.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 6 & -12 & 12 & 6 & -18 \\ 0 & 6 & -12 & 12 & 8 & -10 \end{bmatrix}$$

Replace row 3 with (row 3 minus row 2). This will eliminate the second entry of row 3.

Step 5.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 6 & -12 & 12 & 6 & -18 \\ 0 & 0 & 0 & 0 & 2 & 8 \end{bmatrix}$$

Multiply each row by the reciprocal of its first non-zero value. This will make every row start with a 1.

Step 6.

$$\begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Replace row 2 with (row 2 minus row 3).

Replace row 1 with (row 1 minus 2 times row 3).

Step 7.

$$\begin{bmatrix} 1 & -3 & 4 & -3 & 0 & -3 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Replace row 1 with (row 1 plus 3 times row 2).

Step 8.

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Reduced Row Echelon Form of the linear system given before Step 1 above.

$$x_1 - 2x_3 + 3x_4 = -24$$
$$x_2 - 2x_3 + 2x_4 = -7$$
$$x_5 = 4$$

We can rewrite this system as

$$x_1 = 2x_3 - 3x_4 - 24$$

$$x_2 = 2x_3 - 2x_4 - 7$$

$$x_5 = 4.$$

This is the solution of the system. The variables x_3 and x_4 can take any values and are thus called **free variables**. The solution is valid for any x_3 and x_4.

6 Inequalities

6.1 Algebra of Inequalities

- If you add or subtract a number (positive or negative) from both sides of an inequality you don't flip the inequality.

- If you multiply or divide both sides of an inequality by a positive number you don't flip the inequality.

- If you multiply or divide both sides of an inequality by a negative number you DO flip the inequality.

- These rules also apply to a string of inequalities. That is,

$$x < y < z \Leftrightarrow x + a < y + a < z + a \text{ for any number } a$$

$$x < y < z \Leftrightarrow x - a < y - a < z - a \text{ for any number } a$$

$$x < y < z \Leftrightarrow ax < ay < az \text{ for any number } a > 0$$

$$x < y < z \Leftrightarrow \frac{x}{a} < \frac{y}{a} < \frac{z}{a} \text{ for any number } a > 0$$

$$x < y < z \Leftrightarrow az < ay < ax \text{ for any number } a < 0$$

$$x < y < z \Leftrightarrow \frac{z}{a} < \frac{y}{a} < \frac{x}{a} \text{ for any number } a < 0$$

Note: In general, the inequality $x < y < z$ is equivalent to the inequality $z > y > x$.

- Taking reciprocals flips inequalities.

$$x < y < z \Leftrightarrow \frac{1}{z} < \frac{1}{y} < \frac{1}{x}$$

$$x > y > z \Leftrightarrow \frac{1}{x} < \frac{1}{y} < \frac{1}{z}$$

- "Zeros become Infinities" when you take reciprocals in an equality where 0 is either the lower or upper endpoint (but not both) of the inequality.

  Look carefully at the following two examples.

$$0 < \frac{1}{x} < 2 \iff \frac{1}{2} < x < \infty$$

$$-2 < \frac{1}{x} < 0 \iff -\infty < x < -\frac{1}{2}.$$

6.2 Compound (or System) of Inequalities

Conjunctions (**AND** statements)

The compound (two part) inequality (inequality statement #1) **and** (inequality statement #2) is called an inequality **conjunction**. The "and" implies an intersection (overlap) of the answers of the two inequality statements.

Note that the simple conjunction $x > a$ **and** $x < b$ simplifies to $a < x < b$.

Disjunctions (**OR** statements)

The compound (two part) inequality (inequality statement #1) **or** (inequality statement #2) is called an inequality **disjunction**. The "or" implies the union (take everything) of the answers of the two inequality statements.

Absolute value inequalities are examples of either a conjunction or a disjunction.

- $|x| < a \iff (x < a$ **and** $x > -a)$ [a **conjunction**] for any number $a \geq 0$
  Note that this conjunction simplifies to $-a < x < a$.

- $|x| > a \iff (x > a$ **or** $x < -a)$ [a **disjunction**] for any number $a \geq 0$

In summary, absolute value inequalities can be simplified into "regular" inequalities.

$$|x| < a \iff -a < x < a \text{ provided } a \geq 0$$

$$|x| > a \iff x < -a \text{ or } x > a \text{ provided } a \geq 0$$

These statements are also true of $\leq$ and $\geq$.

6.3 Selected Solved Inequality Problems

6.3.1 Solving two-sided absolute value inequalities

3.	Find the solution set for $1 \leq \|2x - 1\| \leq 5$.

<u>Solution</u>

The key to working with two-sided absolute value problems is to break the absolute value into its two cases.

$$|2x - 1| = \begin{cases} (2x - 1) & \text{if } 2x - 1 \geq 0 \\ -(2x - 1) & \text{if } 2x - 1 < 0. \end{cases}$$

Plugging this into our problem we have

$$1 \leq |2x - 1| \leq 5 = \begin{cases} 1 \leq (2x - 1) \leq 5 & \text{if } 2x - 1 \geq 0 \\ 1 \leq -(2x - 1) \leq 5 & \text{if } 2x - 1 < 0. \end{cases}$$

■

4.	Solve for x in $3(x - 3) > 4(7 - 3x)$ as a quotient of relatively prime numbers.

<u>Solution</u>

$$3x - 9 > 28 - 12x$$

$$\iff 3x + 12x > 28 + 9$$

$$\iff 15x > 37$$

$$\iff x < 37/15$$

■

5.	Solve for x if $2 < 3(4-x) < 9$.

Solution

$$2 < 3(4-x) < 9$$

$$\Leftrightarrow \frac{2}{3} < 4 - x < \frac{9}{3}$$

$$\Leftrightarrow \frac{2}{3} - 4 < -x < \frac{9}{3} - 4$$

$$\Leftrightarrow \left(\frac{2}{3} - 4\right) \cdot (-1) > x > \left(\frac{9}{3} - 4\right) \cdot (-1)$$

$$\Leftrightarrow \left(\frac{9}{3} - 4\right) \cdot (-1) < x < \left(\frac{2}{3} - 4\right) \cdot (-1)$$

$$\Leftrightarrow (-1) \cdot (-1) < x < \left(\frac{-10}{3}\right) \cdot (-1)$$

$$\Leftrightarrow 1 < x < \frac{10}{3}.$$

■

6.	Solve for x if $\lvert 4-x \rvert < 7$.

Solution

$$-7 < 4 - x < 7$$

$$\Leftrightarrow -7 - 4 < -x < 7 - 4$$

$$\Leftrightarrow (-7-4)(-1) > x > (7-4)(-1)$$

$$\Leftrightarrow (-11)(-1) > x > 3(-1)$$

$$\Leftrightarrow 11 > x > -3$$

$$\Leftrightarrow -3 < x < 11$$

■

7.	What is the lowest value of x that satisfies the inequality $|3+x| \leq 100$?

Solution

$$-100 < 3 + x < 100$$

$$\Leftrightarrow -100 - 3 < x < 100 - 4$$

$$\Leftrightarrow -103 < x < 96$$

So, the lowest value of x that satisfies this inequality is -103.

■

8.	What is the lowest value of y that satisfies the inequality $|3+x| + |4+y| \leq 100$?

Solution

$$|3+x| + |4+y| \leq 100$$

$$\Leftrightarrow |4+y| \leq 100 - |3+x|$$

$$\Leftrightarrow -(100 - |3+x|) < 4 + y < (100 - |3+x|)$$

$$\Leftrightarrow -(100 - |3+x|) - 4 < y < (100 - |3+x|) - 4$$

$$\Leftrightarrow -100 + |3+x| - 4 < y < 100 - |3+x| - 4$$

$$\Leftrightarrow -104 + |3+x| < y < 96 - |3+x|$$

So, the lowest value of y that satisfies this inequality is $-104 + |3+x|$.

But x can be any value. By considering all possible values of x how low can $-104 + |3+x|$ get?

The least possible value is -104 because $|3+x| \geq 0$ and by adding the positive $|3+x|$ to -104 it makes it larger. So, the lowest possible value of $-104 + |3+x|$ is -104 and this occurs when $x = -3$ which makes $|3+x| = |3 \pm 3| = |0| = 0$.

■

9.	Write $2 \leq x \leq 4$ as an absolute value inequality.

Solution

Remember that

$$|\boldsymbol{x}-\boldsymbol{a}| \leq \boldsymbol{b} \Leftrightarrow \boldsymbol{a}-\boldsymbol{b} \leq \boldsymbol{x} \leq \boldsymbol{a}+\boldsymbol{b}.$$

Notice that a is the midpoint between $a-b$ and $a+b$. So, equating $a-b=2$ and $a+b=4$ we can immediately determine that $a=3$ because 3 is the midpoint between 2 and 4.

So $3-b=2 \Longrightarrow b=1$

∴

$$2 \leq x \leq 4 \Leftrightarrow |x-3| \leq 1.$$

■

10.	Find the solution set of the inequalities $x+2<-2x+13<4x-6$.

Solution

As we noted above, this is interpreted as:

$$x+2<-2x+13 \quad \textbf{and} \quad -2x+13<4x-6$$

This simplifies to

$x+2<-2x+13$	**and**	$-2x+13<4x-6$
$3x<11$	**and**	$-6x<-19$
$x<11/3$	**and**	$x>-19/-6=19/6$
$\frac{19}{6}<x<\frac{11}{3}$		

■

6.3.2 Solving two-sided absolute value inequalities

11.	Find the solution set for $1 \leq \lvert 2x-1 \rvert \leq 5$.

<u>Solution</u>

The key to working with two-sided absolute value problems is to break the absolute value into its two cases.

$$\lvert 2x-1 \rvert = \begin{cases} (2x-1) & \text{if } \ 2x-1 \geq 0 \\ -(2x-1) & \text{if } \ 2x-1 < 0. \end{cases}$$

Plugging this into our problem we have

$$1 \leq \lvert 2x-1 \rvert \leq 5 = \begin{cases} 1 \leq (2x-1) \leq 5 & \text{if } \ 2x-1 \geq 0 \\ 1 \leq -(2x-1) \leq 5 & \text{if } \ 2x-1 < 0. \end{cases}$$

Pay <u>VERY</u> close attention to the details of how the set

$$\{1 \leq \lvert 2x-1 \rvert \leq 5\} = \begin{cases} 1 \leq (2x-1) \leq 5 & \text{if } \ 2x-1 \geq 0 \\ 1 \leq -(2x-1) \leq 5 & \text{if } \ 2x-1 < 0 \end{cases}$$

translates to

$$\big(\{1 \leq (2x-1) \leq 5\} \textbf{ and } \{2x-1 \geq 0\}\big) \textbf{ or } \big(\{1 \leq -(2x-1) \leq 5\} \textbf{ and } \{2x-1 < 0\}\big)$$

using conjunctions (AND's) and disjunctions (OR's).

Now we just need to break this apart step by step.

	$1 \leq 2x-1 \leq 5$	**and**	$2x-1 \geq 0$
	$2 \leq 2x \leq 6$	**and**	$2x \geq 1$
	$1 \leq x \leq 3$	**and**	$x \geq \frac{1}{2}$
	$1 \leq x \leq 3$		

or			
	$1 \le -(2x-1) \le 5$	**and**	$2x - 1 < 0$
	$1 \le -2x + 1 \le 5$	**and**	$2x - 1 < 0$
	$0 \le -2x \le 4$	**and**	$2x < 1$
	$\frac{0}{-2} \ge x \ge \frac{4}{-2}$	**and**	$x < \frac{1}{2}$
	$-2 \le x \le 0$	**and**	$x < \frac{1}{2}$
	$-2 \le x \le 0.$		

So, it has all reduced to

$$\{-2 \le x \le 0\} \;\; \textbf{or} \;\; \{1 \le x \le 3\}.$$

■

6.3.3 Solving mixed absolute value and linear inequalities

12.	Find the solution set of $|2x-5| \le x + 3$.

<u>Solution</u>

We use the same idea as in the problems with two-sided absolute value inequalities. Namely, we replace $|2x-5|$ with the two possible cases.

The key to working with two-sided absolute value problems is to break the absolute value into its two cases.

$$|2x-5| = \begin{cases} (2x-5) & \text{if } \; 2x-5 \ge 0 \\ -(2x-5) & \text{if } \; 2x-5 < 0 \end{cases}$$

Plugging this into our problem we have

$$|2x-5| \leq x+3 = \begin{cases} (2x-5) \leq x+3 & \text{if } \ 2x-5 \geq 0 \\ -(2x-5) \leq x+3 & \text{if } \ 2x-5 < 0 \end{cases}$$

Just as in the previous section, the set

$$\{|2x-5| \leq x+3\} = \begin{cases} (2x-5) \leq x+3 & \text{if } \ 2x-5 \geq 0 \\ -(2x-5) \leq x+3 & \text{if } \ 2x-5 < 0 \end{cases}$$

should be interpreted as

$$\big(\{(2x-5) \leq x+3\} \textbf{ and } \{2x-5 \geq 0\}\big) \textbf{ or } \big(\{-(2x-5) \leq x+3\} \textbf{ and } \{2x-5 < 0\}\big)$$

As before, we need to break this apart step by step.

	$(2x-5) \leq x+3$	**and**	$2x-5 \geq 0$
	$x \leq 8$	**and**	$2x \geq 5$
	$x \leq 8$	**and**	$x \geq \frac{5}{2}$
	$\frac{5}{2} \leq x \leq 8$		
or			
	$-(2x-5) \leq x+3$	**and**	$2x-5 < 0$
	$-2x+5 \leq x+3$	**and**	$2x-5 < 0$
	$-3x \leq -2$	**and**	$2x < 5$
	$x \geq \frac{-2}{-3} = \frac{2}{3}$	**and**	$x < \frac{5}{2}$
	$\frac{2}{3} \leq x < \frac{5}{2}.$		

So, it has all reduced to

$$\left\{\frac{2}{3} \leq x < \frac{5}{2}\right\} \textbf{ or } \left\{\frac{5}{2} \leq x \leq 8\right\}.$$

But this simplifies to

$$\left\{\frac{2}{3} \le x \le 8\right\}.$$

■

6.3.4 Solving inequalities involving two absolute values

You can handle a problem involving two absolute values in the same way we handled the previous problems - breaking the absolute value(s) into its two cases.

13.	Find the solution set of the inequality $\lvert 2x+5\rvert + \lvert 4x+7\rvert < 30$.

<u>Solution</u>

Again, the key to working with two sided absolute value problems is to break the absolute value into its two cases.

$$|2x+5| = \begin{cases} (2x+5) & \text{if } \ 2x+5 \ge 0 \\ -(2x+5) & \text{if } \ 2x+5 < 0 \end{cases}$$

$$|4x+7| = \begin{cases} (4x+7) & \text{if } \ 4x+7 \ge 0 \\ -(4x+7) & \text{if } \ 4x+7 < 0 \end{cases}$$

Dealing with two absolute values simultaneously means we will end up with 4 cases.

Case 1. $2x+5 \ge 0$ **and** $4x+7 \ge 0$

Case 2. $2x+5 \ge 0$ **and** $4x+7 < 0$

Case 3. $2x+5 < 0$ **and** $4x+7 \ge 0$

Case 4. $2x+5 < 0$ **and** $4x+7 < 0$

(Notice that if you had a problem with 3 absolute value signs you would have $2 \times 2 \times 2 = 8$ cases to handle.)

$$|2x+5| + |4x+7| < 30 = \begin{cases} (2x+5) + (4x+7) < 30 & \text{if in Case 1} \\ (2x+5) - (4x+7) < 30 & \text{if in Case 2} \\ -(2x+5) + (4x+7) < 30 & \text{if in Case 3} \\ -(2x+5) - (4x+7) < 30 & \text{if in Case 4} \end{cases}$$

Let's start by simplifying these four cases.

Case 1. $2x+5 \geq 0$ **and** $4x+7 \geq 0$

$2x+5 \geq 0$	**and**	$4x+7 \geq 0$
$2x \geq -5$	**and**	$4x \geq -7$
$x \geq -5/2$	**and**	$x \geq -7/4$
$x \geq -7/4$		

Case 2. $2x+5 \geq 0$ **and** $4x+7 < 0$

$2x+5 \geq 0$	**and**	$4x+7 < 0$
$2x \geq -5$	**and**	$4x < -7$
$x \geq -5/2$	**and**	$x < -7/4$
$-5/2 \leq x < -7/4$		

Case 3. $2x+5 < 0$ **and** $4x+7 \geq 0$

$2x+5 < 0$	**and**	$4x+7 \geq 0$
$2x < -5$	**and**	$4x \geq -7$
$x < -5/2$	**and**	$x \geq -7/4$
$\{\ \}$ i.e. the empty set		

Case 4. $2x + 5 < 0$ **and** $4x + 7 < 0$

$2x + 5 < 0$	**and**	$4x + 7 < 0$
$2x < -5$	**and**	$4x < -7$
$x < -5/2$	**and**	$x < -7/4$
$x < -5/2$		

The problem has become

$$|2x+5| + |4x+7| < 30 = \begin{cases} (2x+5) + (4x+7) < 30 & \text{if} \quad x \geq -7/4 \\ (2x+5) - (4x+7) < 30 & \text{if} \quad -5/2 \leq x < -7/4 \\ -(2x+5) + (4x+7) < 30 & \text{if} \quad x \in \{\ \} \\ -(2x+5) - (4x+7) < 30 & \text{if} \quad x < -5/2 \end{cases}$$

which simplifies to

$$|2x+5| + |4x+7| < 30 = \begin{cases} 6x + 12 < 30 & \text{if} \quad x \geq -7/4 \\ -2x - 2 < 30 & \text{if} \quad -5/2 \leq x < -7/4 \\ 2x + 2 < 30 & \text{if} \quad x \in \{\ \} \\ -6x - 12 < 30 & \text{if} \quad x < -5/2 \end{cases}$$

which simplifies to

$$|2x+5| + |4x+7| < 30 = \begin{cases} 6x < 18 & \text{if} \quad x \geq -7/4 \\ -2x < 32 & \text{if} \quad -5/2 \leq x < -7/4 \\ 2x < 28 & \text{if} \quad x \in \{\ \} \\ -6x < 42 & \text{if} \quad x < -5/2 \end{cases}$$

which simplifies to

$$|2x+5|+|4x+7|<30=\begin{cases} x<3 & \text{if} \quad x\geq -7/4 \\ x>-16 & \text{if} \quad -5/2\leq x<-7/4 \\ x<14 & \text{if} \quad x\in\{\ \} \\ x>-7 & \text{if} \quad x<-5/2. \end{cases}$$

Just as in the previous section, the set

$$\{|2x+5|+|4x+7|<30\}=\begin{cases} x<3 & \text{if} \quad x\geq -7/4 \\ x>-16 & \text{if} \quad -5/2\leq x<-7/4 \\ x<14 & \text{if} \quad x\in\{\ \} \\ x>-7 & \text{if} \quad x<-5/2 \end{cases}$$

should be interpreted as

$$\big(\{x<3\} \textbf{ and } \{x\geq -7/4\}\big) \textbf{ or } \big(\{x>-16\} \textbf{ and } \{-5/2\leq x<-7/4\}\big)$$

$$\textbf{or } \big(\{x<14\} \textbf{ and } \{x\in\{\ \}\}\big) \textbf{ or } \big(\{x>-7\} \textbf{ and } \{x<-5/2\}\big).$$

And as before we just need to evaluate this step by step.

$$\big(x<3 \quad \textbf{and} \quad x\geq -7/4\big) \iff \frac{-7}{4}\leq x<3$$

or

$$\big(x>-16 \quad \textbf{and} \quad -5/2\leq x<-7/4\big) \iff -5/2\leq x<-7/4$$

or

$$\big(x<14 \quad \textbf{and} \quad x\in\{\ \}\big) \iff x\in\{\ \}$$

or

$$\big(x>-7 \quad \textbf{and} \quad x<-5/2\big) \iff -7<x<-5/2.$$

So, the set of x values (*i.e.* the solution set) where

$$|2x+5|+|4x+7|<30$$

equals

$$\left(\frac{-7}{4} \leq x < 3\right) \ \textbf{or} \ \left(-\frac{5}{2} \leq x < -7/4\right) \ \textbf{or} \ \left(x \in \{\ \}\right) \ \textbf{or} \ \left(-7 < x < -\frac{5}{2}\right).$$

Now remember that "**or**" translates to "x belongs to **at least one** of these four regions". Looking at these four regions we can see it includes all x in

$$-7 < x < 3.$$

■

6.3.5 Solving inequalities involving reciprocals

14.	Find the solution set of the inequality $\frac{1}{\lvert 4x+2\rvert} \geq 5$.

Solution

From the rules regarding reciprocals and inequalities, we have

$$|4x + 2| \leq \frac{1}{5}.$$

Now we are back on familiar grounds.

$$|4x + 2| \leq \frac{1}{5}$$

$$\Rightarrow -\frac{1}{5} \leq 4x + 2 \leq \frac{1}{5}$$

$$\Rightarrow -\frac{1}{5} - 2 \leq 4x \leq \frac{1}{5} - 2$$

$$\Rightarrow \frac{\left(-\frac{1}{5} - 2\right)}{4} \leq x \leq \frac{\left(\frac{1}{5} - 2\right)}{4}$$

$$\Rightarrow -\frac{3}{5} \leq x \leq -\frac{9}{20}$$

■

15.	Find the solution set of the inequality $0 < \frac{1}{\|2x-4\|} < \frac{1}{2}$.

Solution

$$0 < \frac{1}{|2x-4|} < \frac{1}{2}$$

$$\Longrightarrow 2 < |2x-4| < \infty$$

$$\Longrightarrow |2x-4| > 2$$

$$\Longrightarrow 2x-4 > 2 \text{ or } 2x-4 < -2$$

$$\Longrightarrow 2x > 6 \text{ or } 2x < 2$$

$$\Longrightarrow x > 3 \text{ or } x < 1$$

■

6.3.6 Inequality problems involving multiplying or dividing by functions of x

16.	Find the solution set of the inequality $\frac{x-2}{-4x+5} \leq -3$.

Solution

It is tempting (**but don't do it**) to multiply both sides of this inequality by $-4x+5$.

But when you multiply both sides of an inequality by a quantity you have to know whether that quantity is positive or negative. This determines whether we have to flip the inequality or not.

But $-4x+5$ is positive for some values of x and negative for others.

To work around this problem we have to approach this in a way similar to the way we approached absolute values. We break the problem into its two cases.

$$\frac{x-2}{-4x+5} \leq -3 \Longrightarrow \begin{cases} (x-2) \leq (-3)(-4x+5) & \text{if } (-4x+5) > 0 \\ (x-2) \geq (-3)(-4x+5) & \text{if } (-4x+5) < 0 \end{cases}$$

Note: The case $(-4x+5)=0$ is impossible because this would mean we would be dividing by 0 in

$$\frac{x-2}{-4x+5}.$$

Just as in previous sections,

$$\begin{cases}(x-2) \le (-3)(-4x+5) & \text{if } \ (-4x+5) > 0 \\ (x-2) \ge (-3)(-4x+5) & \text{if } \ (-4x+5) < 0\end{cases}$$

should be interpreted as

$$\Big(\{(x-2) \le (-3)(-4x+5)\} \textbf{ and } \{(-4x+5) > 0\}\Big)$$

or

$$\Big(\{(x-2) \ge (-3)(-4x+5)\} \textbf{ and } \{(-4x+5) < 0\}\Big)$$

which simplifies to

$$\Big(x-2 \le 12x-15 \ \textbf{ and } \ -4x+5 > 0\Big)$$

or

$$\Big(x-2 \ge 12x-15 \ \textbf{ and } \ -4x+5 < 0\Big)$$

which simplifies to

$$\Big(13 \le 11x \ \textbf{ and } \ -4x > -5\Big)$$

or

$$\Big(13 \ge 11x \ \textbf{ and } \ -4x < -5\Big)$$

which simplifies to

$$\left(x \ge \frac{13}{11} \ \textbf{ and } \ x < \frac{5}{4}\right)$$

or

$$\left(x \le \frac{13}{11} \textbf{ and } x > \frac{5}{4}\right)$$

which simplifies to

$$\left(\frac{13}{11} \le x < \frac{5}{4}\right) \textbf{ or } \Big(x \in \{\ \}\Big)$$

Note: There cannot be any values of x such that

$$x \le \frac{13}{11} \text{ and } x > \frac{5}{4}$$

because

$$\frac{5}{4} = \frac{55}{44} > \frac{52}{44} = \frac{13}{11}.$$

Therefore, our final answer is

$$\frac{13}{11} \le x < \frac{5}{4}.$$

■

6.4 Geometry and Inequalities

Theorem: The sum of the lengths of any two sides of a triangle must be greater than the third side.

17.	If the sides of a triangle have measures of $3x + 4, 6x - 1$ and $8x + 2$, find all possible values of x.

Solution

$$(3x + 4) + (6x - 1) > (8x + 2)$$

and

$$(3x + 4) + (8x + 2) > (6x - 1)$$

and

$$(6x - 1) + (8x + 2) > (3x + 4)$$

which simplifies to

$$9x + 3 > 8x + 2$$

and

$$11x + 6 > 6x - 1$$

and

$$14x + 1 > 3x + 4$$

which simplifies to

$$x > -1$$

and

$$5x > -7$$

and

$$11x > 3$$

which simplifies to

$$(x > -1) \ \textbf{and} \ \left(x > -\frac{7}{5}\right) \ \textbf{and} \ \left(x > \frac{3}{11}\right).$$

The values of x which satisfy all three of these inequalities are the values of x such that

$$x > \frac{3}{11}.$$

■

6.5 Miscellaneous

18.	For how many positive integral values of a is it true that $x = 2$ is the only positive integer solution of the system of inequalities $$\begin{Bmatrix} 2x > 3x - 3 \\ 3x - a > -6 \end{Bmatrix}?$$

Solution

$2x > 3x - 3$	**and**	$3x - a > -6$
$-x > -3$	**and**	$3x > -6 + a$
$x < 3$	**and**	$x > (-6 + a)/3$

$$\boxed{\Longrightarrow \quad \frac{-6+a}{3} < x < 3}$$

For $x = 2$ to be the only integer in the interval

$$\frac{-6+a}{3} < x < 3$$

it must be true that

$$1 \leq \frac{-6+a}{3} < 2.$$

Using our standard approach for isolating a we have

$$1 \leq \frac{-6+a}{3} < 2$$

$$\Longrightarrow \quad 3 \leq -6+a < 6$$

$$\Longrightarrow \quad 9 \leq a < 12$$

So the integral values of a that satisfy this inequality are $a = 9, a = 10$ and $a = 11$. So there are 3 positive integral values of a that make $x = 2$ the only positive integer solution of the original pair of inequalities.

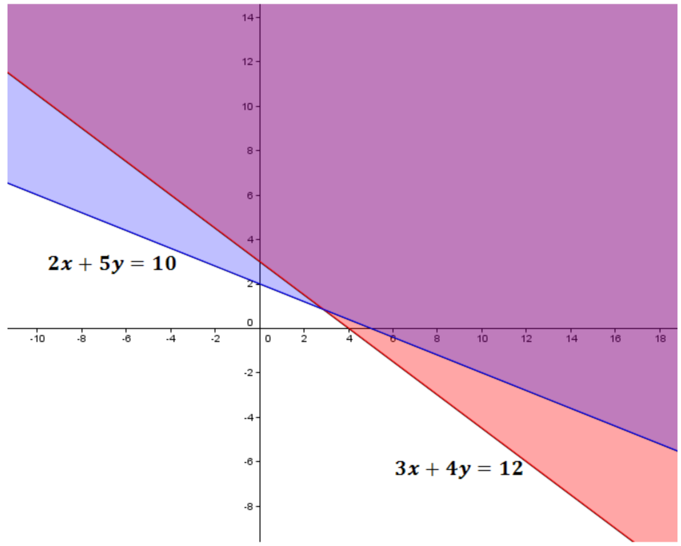
■

6.6 Linear Programming Problems

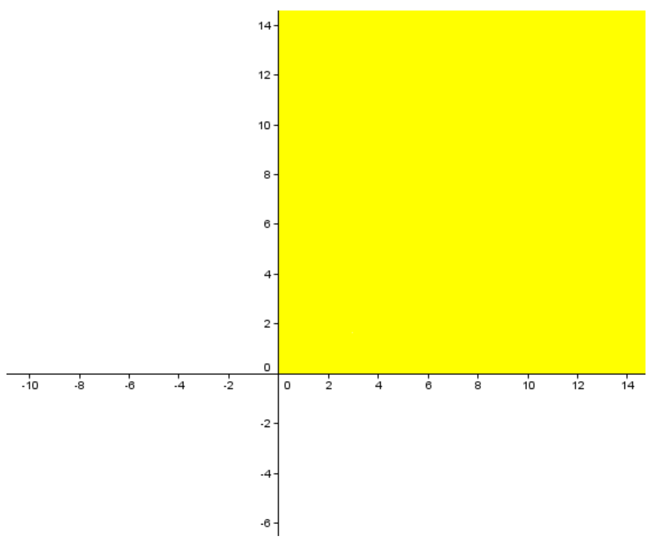
19.	Both x and y are positive real numbers, and the point (x, y) lies on or above both of the lines having equations $2x + 5y = 10$ and $3x + 4y = 12$. What is the least possible value of $8x + 13y$? (Source: Mathcounts).

Solution

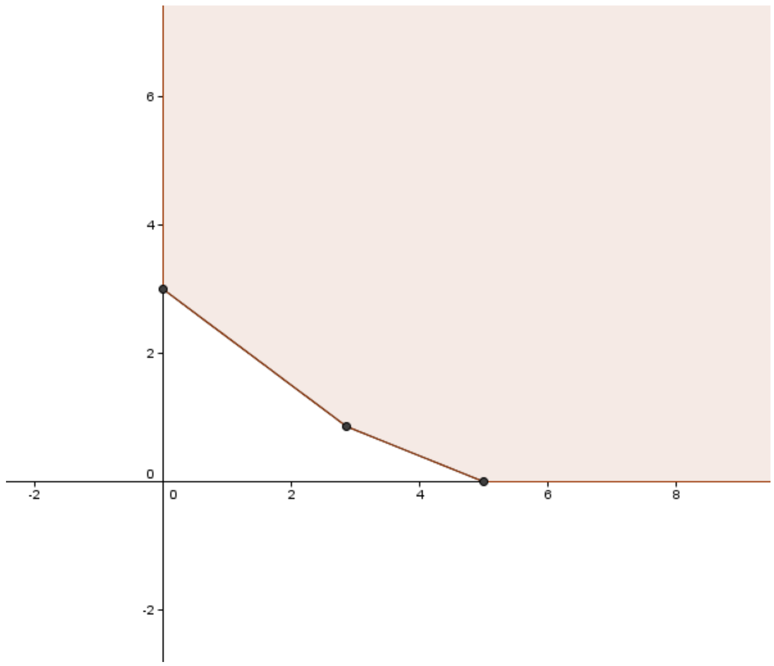
The region above the line $2x + 5y = 10$ is shown in blue and the region above the line $3x + 4y = 12$ is shown in red. Hence, the region that is above BOTH lines becomes purple (blue+red=purple).



We also are told that $x \geq 0$ and $y \geq 0$ (*i.e.* the region in yellow).



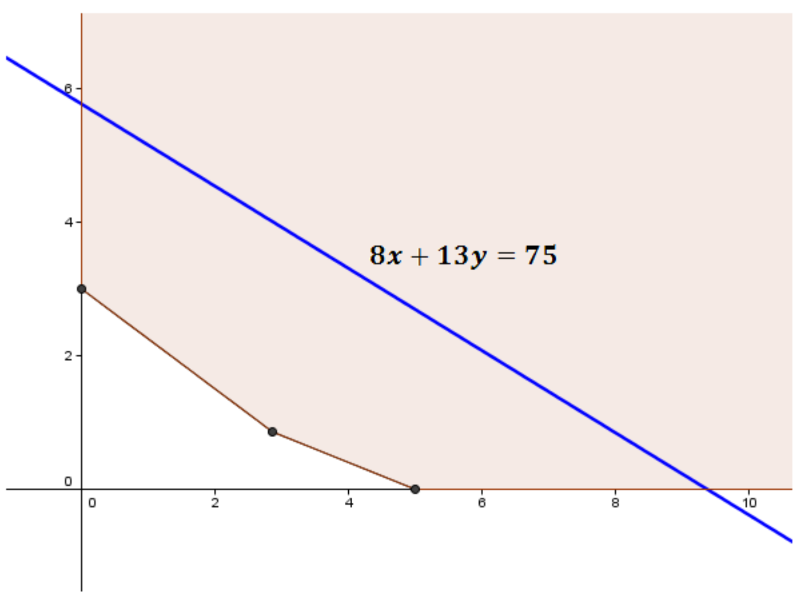
The (x, y) values that are in the purple and the yellow are highlighted below.



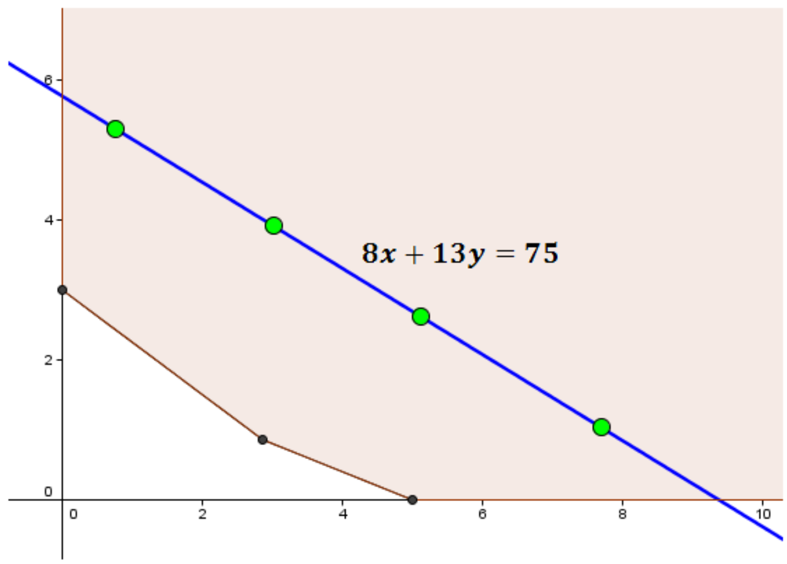
The goal of the problem is to find the smallest value of $8x + 13y$ if (x, y) has to be a point in the above shaded region. Problems of this type are known as **linear programming** problems.

Now let's play with this a bit. Could (just as an example) we find an (x, y) within this shaded region where $8x + 13y = 75$?

To see if this is possible, we need to graph the line $8x + 13y = 75$. We show this line below.

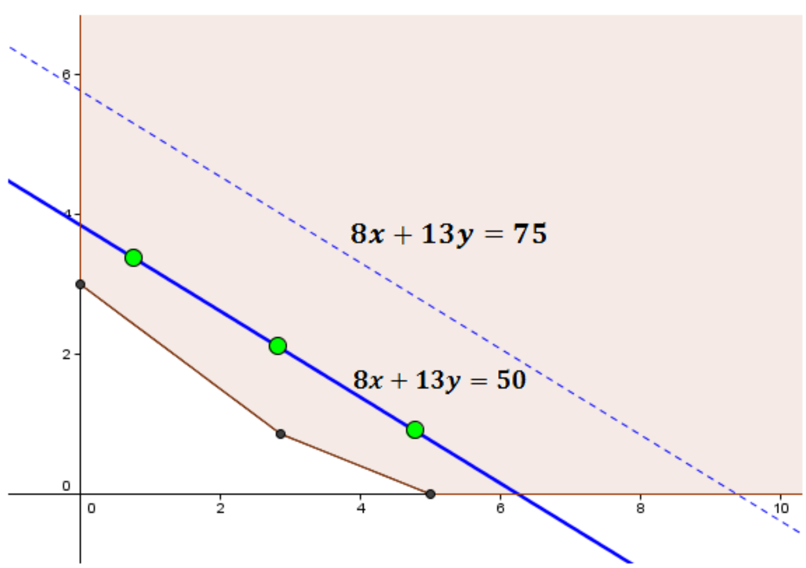


We can see that $8x + 13y$ equals 75 at each of the points marked in green (because they fall on the line where $8x + 13y = 75$) and these green points are in the highlighted region.



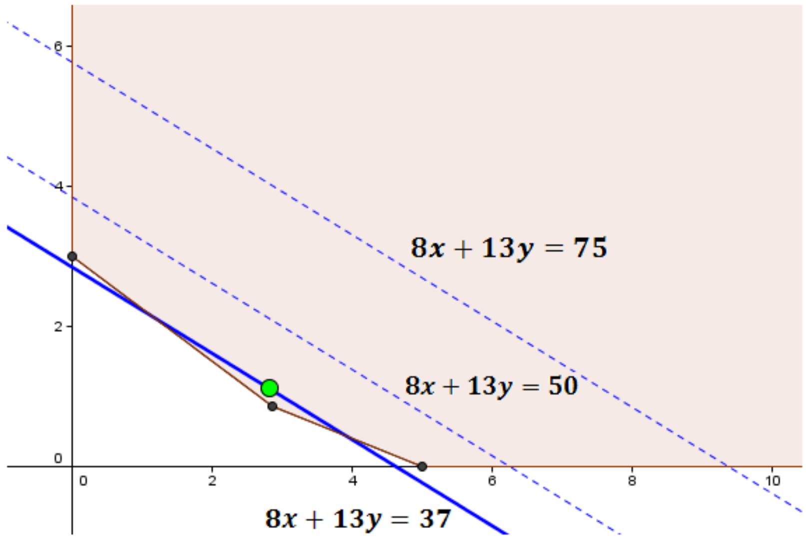
This shows that $8x + 13y$ can get at least as small as 75.

But can we make it smaller? Can we find a point(s) within the highlighted region where $8x + 13y = 50$?



The graph of the line $8x + 13y = 50$ is shown above in solid blue and we see that there are points on this line that are within the shaded region. (Clearly there are an infinite number of points on this line that are within this shaded region. The three points shown in green just show particular cases.)

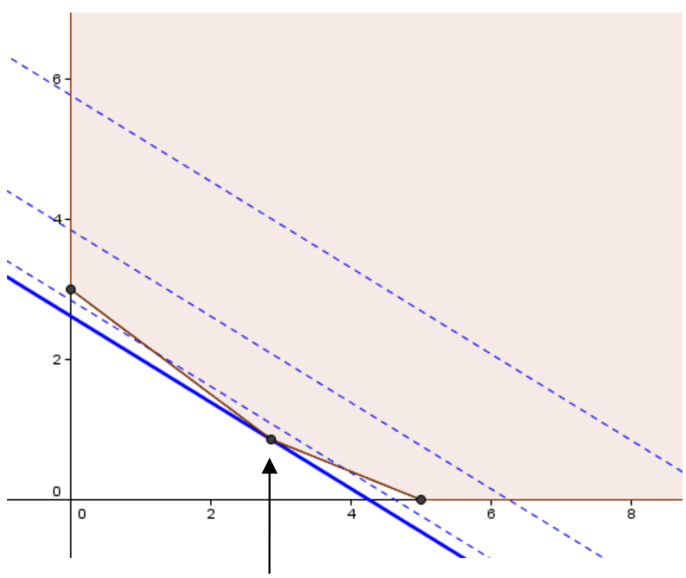
This shows that $8x + 13y$ can get at least as small as 50. Can we make it smaller yet? Can we find a point(s) within the highlighted region where $8x + 13y = 37$?



The graph of the line $8x + 13y = 37$ is shown above in solid blue and we see that there are points (e.g. the green points) on this line that are within the shaded region. (Clearly there are an infinite number of points on this line that are within this shaded region. The point in green just shows a particular case.)

This shows that $8x + 13y$ can get at least as small as 37.

By now you most likely notice that we should lower this line until it hits the corner point.



Corner Point

The entire of this argument has come around to showing that

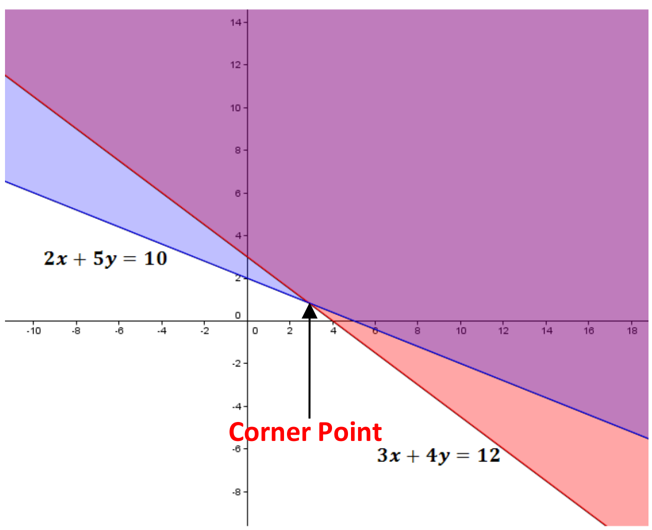
$$8x + 13y$$

will be minimized (among points in the shaded region) at this corner point. If we label the coordinates of this corner point as (x_0, y_0) then

$$8x_0 + 13y_0$$

is the minimum value we are looking for.

So how can we find this corner point? This corner point is the point where the lines $2x + 5y = 10$ and $3x + 4y = 12$ intersect (*i.e.* cross).



Finding the intersection of two lines boils down to solving two linear equations in two unknowns.

That is, we want to find x and y so that

$$2x + 5y = 10 \text{ and } 3x + 4y = 12.$$

The point of intersection is the point that lies on both lines which means this point satisfies BOTH of these equations.

Solving for y in the first equation we have

$$y = \frac{10 - 2x}{5}.$$

Substituting this into the second equation we get

$$3x + 4\left(\frac{10 - 2x}{5}\right) = 12$$

$$3x + 4\left(\frac{10 - 2x}{5}\right) = 12$$

$$\Longrightarrow 15x + 4(10 - 2x) = 60$$

$$\Longrightarrow\ 15x + 40 - 8x = 60$$

$$\Longrightarrow\ 7x = 20$$

$$\Longrightarrow x = \frac{20}{7}.$$

From here we can solve for y.

$$y = \frac{10 - 2\left(\frac{20}{7}\right)}{5} = \frac{70 - 40}{35} = \frac{30}{35} = \frac{6}{7}.$$

So, the coordinates of this corner point are

$$(x_0, y_0) = \left(\frac{20}{7}, \frac{6}{7}\right).$$

Finally, we can plug these values in to our function $8x + 13y$ to get the minimum possible value of this function over all points in the shaded region.

$$8x_0 + 13y_0 = 8\left(\frac{20}{7}\right) + 13\left(\frac{6}{7}\right) = \frac{238}{7} = 34.$$

That is, the smallest value of $8x + 13y$ that can be obtained by considering only the (x, y) points in the shaded region defined above is 34.

■

20.	Find all x such that $$\left\lvert\frac{2\lvert x\rvert-3}{4\lvert x-1\rvert}\right\rvert = 2.$$		

Solution

Situation 1. $\dfrac{2|x|-3}{4|x-1|} = 2,\quad$ *i.e.* $\;2|x|-3 = 8|x-1|$

Break points: 0,1

x	$-$	$+$	$+$
$x-1$	$-$	$-$	$+$
			

Case 1. $x \in (-\infty, 0]$

$$2|x| - 3 = 8|x-1|$$

$$2(-x) - 3 = 8\big(-(x-1)\big)$$

$$-2x - 3 = -8x + 8$$

$$6x = 11$$

$$x = 11/6$$

Is this solution within the region we are currently considering, namely the region $(-\infty, 0]$? No. $11/6 \notin (-\infty, 0]$. So, this is not a valid solution. It is an extraneous solution.

Case 2. $x \in [0, 1]$

$$2|x| - 3 = 8|x-1|$$

$$2(+x) - 3 = 8\big(-(x-1)\big)$$

$$2x - 3 = -8x + 8$$

$$10x = 11$$

$$x = 11/10$$

Is this solution within the region we are currently considering, namely the region $[0,1]$? No. $11/10 \notin [0,1]$. So, this is not a valid solution. It is an extraneous solution.

Case 3. $\boldsymbol{x \in [1, \infty)}$

$$2|x| - 3 = 8|x - 1|$$

$$2(+x) - 3 = 8\big(+(x - 1)\big)$$

$$2x - 3 = 8x - 8$$

$$-6x = -5$$

$$x = 5/6$$

Is this solution within the region we are currently considering, namely the region $[1, \infty)$? No. $5/6 \notin [1, \infty)$. So this is not a valid solution. It is an extraneous solution.

So, there are no values of x where $2|x| - 3 = 8|x - 1|$.

Situation 2. $\dfrac{2|x| - 3}{4|x - 1|} = -2$, *i.e.* $2|x| - 3 = -8|x - 1|$

Break points: 0,1

x	$-$	$+$	$+$
$x - 1$	$-$	$-$	$+$
			

Case 1. $\boldsymbol{x \in (-\infty, 0]}$

$$2|x| - 3 = -8|x - 1|$$

$$2(-x) - 3 = -8\big(-(x - 1)\big)$$

$$-2x - 3 = 8x - 8$$

$$-10x = -5$$

$$x = 1/2$$

Is this solution within the region we are currently considering, namely the region $(-\infty, 0]$? No. $1/2 \notin (-\infty, 0]$. So this is not a valid solution. It is an extraneous solution.

Case 2. $x \in [0,1]$

$$2|x| - 3 = -8|x - 1|$$

$$2(+x) - 3 = -8\big(-(x - 1)\big)$$

$$2x - 3 = 8x - 8$$

$$-6x = -5$$

$$x = 5/6$$

Is this solution within the region we are currently considering, namely the region $[0,1]$? Yes. $5/6 \in [0,1]$. So this is a valid solution. It is not an extraneous solution.

Case 3. $x \in [1, \infty)$

$$2|x| - 3 = -8|x - 1|$$

$$2(+x) - 3 = -8\big(+(x - 1)\big)$$

$$2x - 3 = -8x + 8$$

$$10x = 11$$

$$x = 11/10$$

Is this solution within the region we are currently considering, namely the region $[1, \infty)$? Yes. $11/10 \in [1, \infty)$. So this is a valid solution. It is not an extraneous solution.

So, there are two values of x where $2|x| - 3 = -8|x - 1|$.

∎

21.	Find all x such that $\lvert 4x-1\rvert+2x=1+\lvert 2-x\rvert$.

Solution

Break Points : $4x-1=0 \Longrightarrow x=\mathbf{\frac{1}{4}}$, $2-x=0 \Longrightarrow x=\mathbf{2}$.

$4x-1$	$-$	$+$	$+$
$2-x$	$+$	$+$	$-$
			

Case 1. $\boldsymbol{x \in \left(-\infty, \frac{1}{4}\right]}$

$$|4x-1|+2x=1+|2-x|$$

$$-(4x-1)+2x=1+\big(+(2-x)\big)$$

$$-4x+1+2x=1+2-x$$

$$-x=2$$

$$x=-2$$

Is this solution within the region we are currently considering, namely the region $\left(-\infty, \frac{1}{4}\right]$? Yes. $-2 \in \left(-\infty, \frac{1}{4}\right]$. So this is a valid solution. It is not an extraneous solution.

Case 2. $\boldsymbol{x \in [1/4, 2]}$

$$|4x-1|+2x=1+|2-x|$$

$$+(4x-1)+2x=1+\big(+(2-x)\big)$$

$$4x-1+2x=1+2-x$$

$$7x=4$$

$$x=4/7$$

Is this solution within the region we are currently considering, namely the region $[1/4, 2]$? Yes. $4/7 \in [1/4, 2]$. So, this is a valid solution. It is not an extraneous solution.

Case 3. $\boldsymbol{x \in [2, \infty)}$

$$|4x - 1| + 2x = 1 + |2 - x|$$

$$+(4x - 1) + 2x = 1 + \left(-(2 - x)\right)$$

$$4x - 1 + 2x = 1 - 2 + x$$

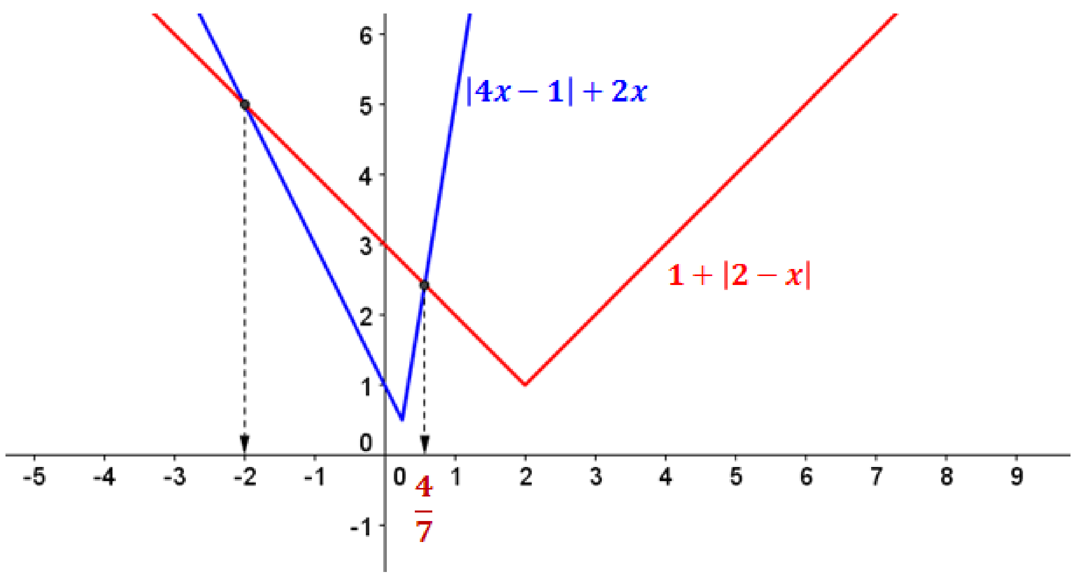
$$5x = 0$$

$$x = 0$$

Is this solution within the region we are currently considering, namely the region $[2, \infty)$? No. $0 \notin [2, \infty)$. So, this is not a valid solution. It is an extraneous solution.

So, there are two values of x where $|4x - 1| + 2x = 1 + |2 - x|$. Namely at $x = -2$ and at $x = 4/7$.

A graph will help you to visualize what is going on. You can see from the graph that $|4x - 1| + 2x$ and $1 + |2 - x|$ intersect (*i.e.* are equal) at two places, namely, $x = -2$ and $x = 4/7$ (as we just showed algebraically).

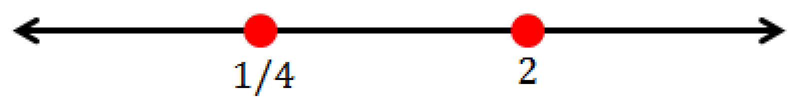


■

22.	Find the solution set to $|4x-1|+2x \geq 1+|2-x|$.

Solution

Break Points : $4x-1=0 \Longrightarrow x=\frac{1}{4}$, $2-x=0 \Longrightarrow x=2$.

$4x-1$	$-$	$+$	$+$
$2-x$	$+$	$+$	$-$
			

Case 1. $x \in \left(-\infty, \frac{1}{4}\right]$

$$|4x-1|+2x \geq 1+|2-x|$$

$$-(4x-1)+2x \geq 1+\big(+(2-x)\big)$$

$$-4x+1+2x \geq 1+2-x$$

$$-x \geq 2$$

$$x \leq -2.$$

So, what we are looking for are those $x \in (-\infty, 1/4]$ **and** $(-\infty,-2]$. That is,

$$x \in (-\infty, 1/4] \cap (-\infty,-2] = (-\infty, -2].$$

Case 2. $x \in [1/4, 2]$

$$|4x-1|+2x \geq 1+|2-x|$$

$$+(4x-1)+2x \geq 1+\big(+(2-x)\big)$$

$$4x-1+2x \geq 1+2-x$$

$$7x \geq 4$$

$$x \geq 4/7.$$

So, what we are looking for are those $x \in [1/4,2]$ **and** $[4/7, \infty)$. That is,

$$x \in [1/4,2] \cap [4/7, \infty) = [\mathbf{4/7}\,, \mathbf{2}].$$

Case 3. $\boldsymbol{x \in [2, \infty)}$

$$|4x - 1| + 2x \geq 1 + |2 - x|$$

$$+(4x - 1) + 2x \geq 1 + \big(-(2 - x)\big)$$

$$4x - 1 + 2x \geq 1 - 2 + x$$

$$5x \geq 0$$

$$x \geq 0$$

So, what we are looking for are those $x \in [2, \infty)$ **and** $[0, \infty)$. That is,

$$x \in [2, \infty) \cap [0, \infty) = [\mathbf{2}, \infty).$$

So, the values of x where $|4x - 1| + 2x < 6 - |2 - x|$ are

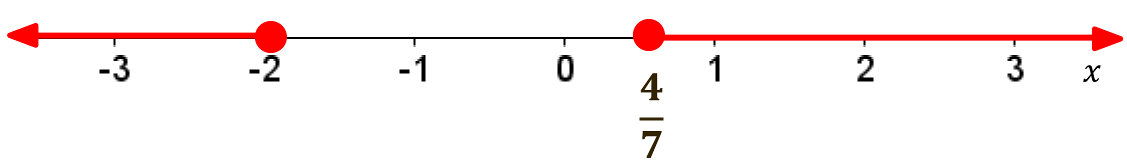
$$x \in (-\infty, -\mathbf{2}] \ \textbf{ or } [\mathbf{4/7}\,, \mathbf{2}] \textbf{ or } [\mathbf{2}, \infty).$$

That is,

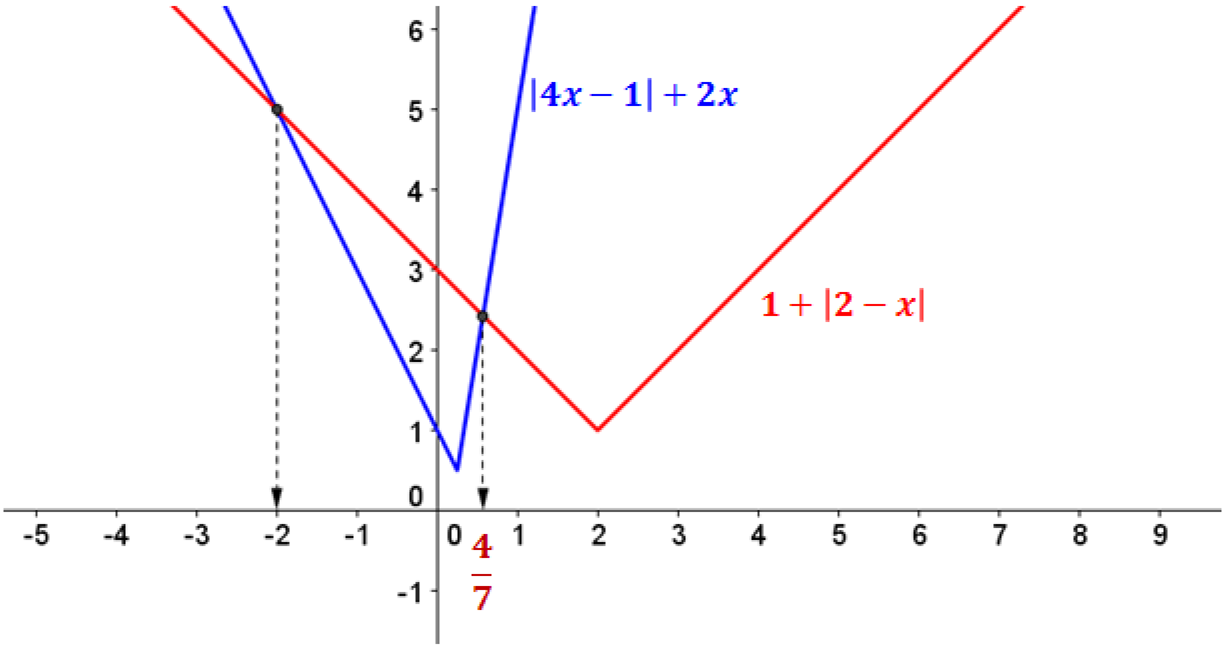
$$x \in (-\infty, -\mathbf{2}] \cup [\mathbf{4/7}\,, \mathbf{2}] \cup [\mathbf{2}, \infty)$$

$$x \in (-\infty, -\mathbf{2}] \cup [\mathbf{4/7}, \infty).$$

In graph form, the answer is



You can see from the graph that $|4x - 1| + 2x$ (in blue) is greater than or equal to (*i.e.* above or touching) $1 + |2 - x|$ (in red) when $-\infty < x \leq -2$ and also when $4/7 \leq x < \infty$ (as we just showed algebraically).



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7 Extra Solved Problems

23. The hour hand and the minute hand coincide sometime between 3 o'clock and 4 o'clock. Express the exact time as a rational number of minutes after 3 o'clock. (Source: MSHSML 3A054)

Solution

$$\text{In } m \text{ minutes, the } \begin{cases} \text{hr. hand moves } \dfrac{360}{60(12)} m \text{ deg} \\ \text{min hand moves } \dfrac{360}{60} m \text{ deg} \end{cases}$$

$$m \text{ min after 3, the } \begin{cases} \text{hr hand is at } 90 + \dfrac{m}{2} \text{ deg} \\ \text{min hand is at } 6m \text{ deg} \end{cases}$$

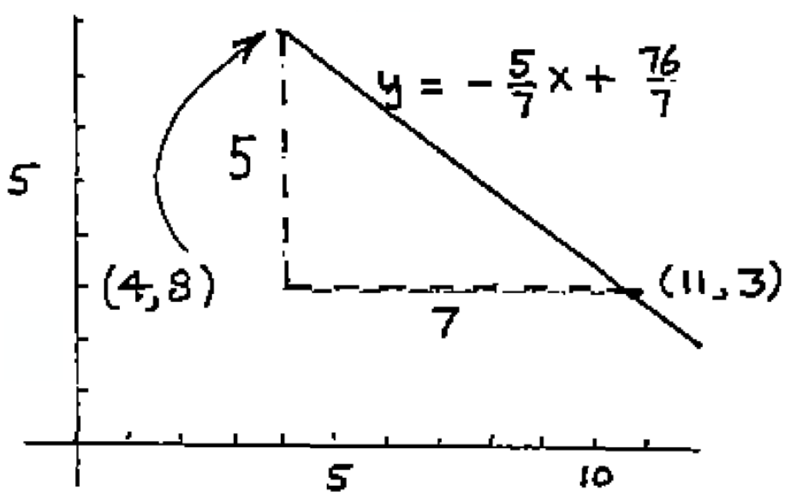
$$90 + \frac{m}{2} = 6m$$

$$180 + m = 12m\ ; \quad m = \frac{180}{11} \text{ min after 3,}$$

■

24. The graph of the equation $5x + 7y = 76$ passes through $(11,3)$. Find a second lattice point (i.e. a point having integer coordinates) in the first quadrant that lies on the same line. (Source: MSHSML 3A023)

Solution



The slope of $5x + 7y = 76$ equals $-5/7$. So, if you go left 7 units and up 5 units from any point on this line you will end up at another point on this line. And if you start at a lattice point and go left an integer number of units and go up an integer number of units, you will necessarily end up at another lattice point (which is also on this line).

Note: In MSHSML Meet 1, Event D we saw similar problems about lattice points on a line when they don't give you an initial lattice point to work from. Then you have to use number theory.

■

25. An auto is twice as old as its tires were when the auto was as old as the tires are now. When the tires are as old as the auto is, the combined ages of the tires and the auto will be two years and three months. How old (in months) are the tires now? (Source: MSHSML 3A014)

Solution

	AGES	
	auto	tires
time past (when)	a	t
now	$2t$	a
time future	$2t+(2t-a)$	$2t$

$$\left.\begin{array}{l} 2t-a=a-t \\ [4t-a]+2t=27 \end{array}\right\} \Rightarrow \begin{array}{rcl} 3t-2a &=& 0 \\ 6t-a &=& 27 \\ \hline 3a &=& 27 \end{array}$$

$$a=9$$

■

26. Find the <u>sum</u> of all positive integers a and b where $a > b$ and the determinant

$$\begin{vmatrix} 1 & 0 & a \\ 0 & 1 & a \\ b & b & 1 \end{vmatrix} = -13.$$

(Source: MSHSML 3A164)

Solution

$$\text{The determinant of} \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & a \\ b & b & 1 \end{bmatrix} = -13 \text{ when}$$

$1(1 \bullet 1 - a \bullet b) - 0(0 \bullet 1 - a \bullet b) + a(0 \bullet b - b \bullet 1) = -13 \Rightarrow 1 - 2ab = -13 \Rightarrow ab = 7.$ *Since 7 is prime and $a > b$, $a = 7$ and $b = 1$, so $a + b$ is 8.*

■

27. Determine exactly <u>all</u> values of the real number c for which the determinant of $$\begin{bmatrix} 1 & 2 & 3 \\ 4 & c & 6 \\ 7 & 8 & 9 \end{bmatrix}$$ Is positive. (Source: MSHSML 3A153)

Solution

$$\text{The determinant of} \begin{bmatrix} 1 & 2 & 3 \\ 4 & c & 6 \\ 7 & 8 & 9 \end{bmatrix} \text{ is positive when}$$

$$1(9c - 48) - 2(4 \bullet 9 - 7 \bullet 6) + 3(32 - 7c) > 0 \Rightarrow 60 - 12c > 0 \Rightarrow c < 5.$$

■

28. $A = \begin{bmatrix} x & y \\ 5 & 8 \end{bmatrix}$ and $B = \begin{bmatrix} x & y \\ 2 & 3 \end{bmatrix}$. If $\det A = 10$ and $\det B = 36$, determine x and y exactly. (Source: MSHSML 3A143)

Solution

det A = 8x – 5y = 10. Likewise, det B = 3x – 2y = 36. We now have a system of two linear equations: $\begin{cases} 8x-5y=10 \\ 3x-2y=36 \end{cases}$*. Multiply the first equation by –2 and the second by 5 to get* $\begin{cases} -16x+10y=-20 \\ 15x-10y=180 \end{cases}$*. Add the two equations to obtain x = –160; substitution yields y = –258.*

■

29. It is known that the system of equations $\begin{cases} Ax+y+z=1 \\ x+By+z=1 \\ x+y+Cz=1 \end{cases}$ has no solution. Determine exactly the value of $ABC-(A+B+C)$. (Source: MSHSML 3A124)

Solution

For the system to be inconsistent, the determinant of $\begin{bmatrix} A & 1 & 1 \\ 1 & B & 1 \\ 1 & 1 & C \end{bmatrix}$ *must equal zero. Using the cofactors of row 1:* $A\cdot\begin{vmatrix} B & 1 \\ 1 & C \end{vmatrix}-1\cdot\begin{vmatrix} 1 & 1 \\ 1 & C \end{vmatrix}+1\cdot\begin{vmatrix} 1 & B \\ 1 & 1 \end{vmatrix}=A(BC-1)-(C-1)+(1-B)$

$= ABC - A - C + 1 + 1 - B = 0$, *so* $ABC - A - B - C = ABC - (A + B + C) = -2$.

■

30. The vertices of ΔAOB are $A=(a,b)$, $O=(0,0)$, and $B=(c,d)$. The values of a,b,c, and d are all positive integers and they are all different. If the area of ΔAOB is 13, what is the smallest possible of value of the sum $a+b+c+d$? (Source: 3T171)

Solution

The area of ΔAOB is the determinant $\frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ a & b & 1 \\ c & d & 1 \end{vmatrix}$, making $ad-bc=26$. To minimize the sum choose bc as small as possible, making ad as small as possible. Let $b=1$ and $c=2$, then $ad=28$. Because 1 and 2 have already been assigned, the smallest a can be is 4, making $d=7$. Therefore, the minimum value of $a+b+c+d=1+2+4+7=14$.

■

Test 3A, Linear Equations

2014-15 Meet 3, Individual Event A

4. Determine exactly the necessary values of a and b in the linear equation $ax + by = -6$ so that the line passes through the intersection point of the lines $x + y = -4$ and $2x + 3y = -10$, and is also parallel to the line $-8x + 2y = 16$.

Solution

First find the point of intersection between the lines $x + y = -4$ and $2x + 3y = -10$. Using either elimination or substitution, that point is $(-2, -2)$. Because the line $ax + by = -6$ passes through this point, substitute -2 for x and y to obtain the equation $-2a - 2b = -6$, i.e. $a + b = 3$. Also, $ax + by = -6$ is parallel to $-8x + 2y = 16$, so their slopes must be the same. This means $\frac{-a}{b} = 4 \Rightarrow a = -4b$. Substituting, $-4b + b = 3$, so $b = -1$ and $a = 4$.

■

2013-14 Meet 3, Individual Event A

2. Determine exactly the values of R and T such that the lines described by the system $\begin{cases} Rx - 4y = 7 \\ 3x + Ty = 11 \end{cases}$ intersect at the point $(-5, 2)$.

Solution

Substitute $x = -5$ and $y = 2$ into each equation: $Rx - 4y = 7 \Rightarrow -5R - 4(2) = 7 \Rightarrow R = -3$;

$$3x + Ty = 11 \Rightarrow 3(-5) + 2T = 11 \Rightarrow T = 13.$$

■

2011-12 Meet 3, Individual Event A

4. Of all the points (x, y) with integer coordinates for which $\begin{cases} 2x + 5y = 111 \\ 3x - 2y \ge 11 \end{cases}$, there is one such point for which $x + y$ is a minimum. Give the coordinates of this point.

Solution

First note that $3x-2y \geq 11$ *has its solutions occurring "below" the line. Moving from right to left along 2x + 5y = 111 (starting from y = 0), the first point with integer coordinates is (53, 1), with x + y = 54. Using LCM concepts, the next point to the left with integer coordinates is (48, 3), with x + y = 51. So we want to get as close to 2x + 5y = 111 as possible. 2x + 5y = 111 and 3x – 2y = 11 intersect at* $\left(14\tfrac{11}{19}, 16\tfrac{7}{19}\right)$, *so the minimum x + y occurs at (18, 15).*

■

2010-11 Meet 3, Individual Event A

2. If the graph of $ax+by=8$ passes through $\left(3, \dfrac{1}{b}\right)$ and $\left(-\dfrac{1}{a}, -5\right)$, then it will also pass through the point (15, k). Determine exactly the value of k.

Solution

Substituting the coordinates of the first point: $ax+by=8 \Rightarrow 3a+1=8 \Rightarrow a=\frac{7}{3}$.

Substituting the coordinates of the second point: $ax+by=8 \Rightarrow -1-5b=8 \Rightarrow b=-\frac{9}{5}$.

Substituting (15, k) into $\frac{7}{3}x-\frac{9}{5}y=8$ *yields* $35-\frac{9}{5}k=8 \Rightarrow -\frac{9}{5}k=-27 \Rightarrow k=15$.

■

4. Two distinct positive integers are chosen such that their difference, sum, and product (in that order) are in the ratio 1 : 7 : 36 . Calculate the product of the two numbers.
[adapted from MAA: The Contest Problem Book II]

Solution

Let (m, n) represent the two numbers, with m > n, and let d = their difference. We can then use the given ratio to create the system $\begin{cases} m-n=d \\ m+n=7d \end{cases}$ *, which can be solved quickly by elimination to discover that (m, n) = (4d, 3d). The product mn is therefore* $12d^2$ *, but is also 36d by the given ratio. Setting* $12d^2=36d$ *yields d = 3 as the only viable solution, so the product is 36(3) = 108.*

■

2009-10 Event 3A

4. Consider all possible ordered triples of integers (x, y, z) which solve the system $\begin{cases} x + 3y - z = 1 \\ 3x - y - 2z = -1 \end{cases}$. There are two specific single-digit positive integers, n and d, such that all of the x-values in those ordered triples can be written in the form $mn + d$, where m is any integer. Compute the values of n and d.

Solution

Multiply the top equation by –2 and add: $x - 7y = -3 \;\Rightarrow\; y = \dfrac{x+3}{7}$.

Then multiply the bottom equation by 3 and add: $10x - 7z = -2 \;\Rightarrow\; z = \dfrac{10x+2}{7}$.

We need x + 3 and 10x + 2 to both be multiples of 7. This occurs when x = 4, and every 7 units thereafter. So y and z will be integers when x is of the form 7m + 4.

■

2006-07 Meet 3 Team Event

5. Solve the following system for x, y, and z.

$$\begin{aligned} 2x + 3y - 2z &= -4 \\ -3x + 9y - 5z &= 4 \\ 5x + 6y + 3z &= 8 \end{aligned}$$

Solution

5. $$\begin{bmatrix} 2 & 3 & -2 & -4 \\ -9 & 0 & 1 & 16 \\ 1 & 0 & 7 & 16 \end{bmatrix} \sim \begin{bmatrix} -16 & 3 & 0 & 28 \\ -9 & 0 & 1 & 16 \\ 64 & 0 & 0 & -96 \end{bmatrix}$$

$$\sim \begin{bmatrix} 0 & 3 & 0 & 4 \\ 0 & 0 & 1 & 5/2 \\ 1 & 0 & 0 & -3/2 \end{bmatrix} \qquad \begin{aligned} y &= 4/3 \\ z &= 5/2 \\ x &= -3/2 \end{aligned}$$

(If you don't follow the matrix notation, ask your coach to explain it.)

■

2005-06 Meet 3 Team Event

1. Express in the form $(az+b,\ cz+d,\ z)$ all solutions to the system $\begin{array}{l} 2x-3y-3z=4 \\ 3x-5y+2z=-3 \end{array}$

Solution

1. $\begin{bmatrix} 2 & -3 & -3 & 4 \\ 3 & -5 & 2 & -3 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & -13 & 18 \\ 1 & -2 & 5 & -7 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 0 & -21 & 29 \\ 0 & 1 & -13 & 18 \end{bmatrix} \quad \begin{array}{l} x = 29 + 21z \\ y = 18 + 13z \end{array}$

■

2004-05 Event 3A

3. There are many solutions to the set of three equations below. Express y and z in terms of x so that every real value of x gives a solution to these equations:

$$\begin{array}{c} 2x-y+z=8 \\ 3x-y+2z=11 \\ 5x-y+4z=17 \end{array}$$

Solution

$$\begin{bmatrix} 2 & -1 & 1 & 8 \\ 3 & -1 & 2 & 11 \\ 5 & -1 & 4 & 17 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 1 & 8 \\ 1 & 0 & 1 & 3 \\ 3 & 0 & 3 & 9 \end{bmatrix} \sim \begin{bmatrix} -1 & +1 & 0 & -5 \\ 1 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} y = -5 + x \\ z = 3 - x \end{array}$$

■

2003-04 Event 3A

4. There are many solutions to the set of three equations below. Express y and z in terms of x so that every real value of x gives a solution to these equations:

$$\begin{array}{c} 2x+y-z=-2 \\ 5x+2y+z=3 \\ 4x+y+5z=12 \end{array}$$

Solution

$$\begin{bmatrix} 2 & 1 & -1 & -2 \\ 5 & 2 & 1 & 3 \\ 4 & 1 & 5 & 12 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & -1 & 2 \\ 7 & 3 & 0 & 1 \\ 14 & 6 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1/3 & 0 & 1 & 7/3 \\ 7/3 & 1 & 0 & 1/3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(Have your coach explain the notation above.)

$$z = \frac{7}{3} - \frac{1}{3}x$$

$$y = \frac{1}{3} - \frac{7}{3}x$$

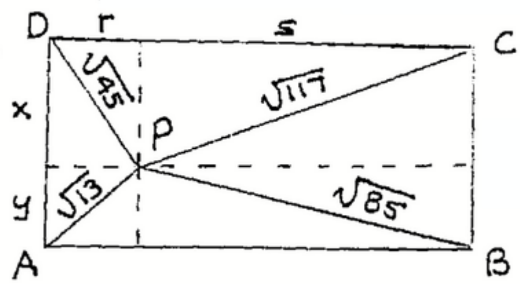
■

2004-05 Meet 3 Team Event

2. In Team Event 2, we found that if a point P is placed inside a rectangle $ABCD$ so that $AP = \sqrt{13}$, $CP = \sqrt{117}$, and $DP = \sqrt{45}$, then $BP = \sqrt{85}$. If the sides of this rectangle are integers, what must be its perimeter?

Solution

2.



With notation as in the drawing

$$AD = x + y, \quad CD = r + s$$

$$\begin{aligned} x^2 + r^2 &= 45 \\ x^2 + s^2 &= 117 \\ y^2 + r^2 &= 13 \\ y^2 + s^2 &= 85 \end{aligned}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 45 \\ 1 & 0 & 0 & 1 & 117 \\ 0 & 1 & 1 & 0 & 13 \\ 0 & 1 & 0 & 1 & 85 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 & 45 \\ 0 & 0 & -1 & 1 & 72 \\ 0 & 1 & 1 & 0 & 13 \\ 0 & 0 & -1 & 1 & 72 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 & 117 \\ 0 & 0 & 1 & -1 & -72 \\ 0 & 1 & 0 & 1 & 85 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} x^2 &= 117 - s^2 \\ y^2 &= 85 - s^2 \\ r^2 &= -72 + s^2 \end{aligned}$$

$$x^2 - y^2 = (x-y)(x+y) = 117 - 85 = 32$$

$x-y$	$x+y$	x	y
1	32	$33/2$	$31/2$
2	16	9	7
4	8	6	2

$x-y$ and $x+y$ are integer factors of 32. The table shows the possibilities

Only $x = 6$, $y = 2$ works. Then $r = 3$, $s = 9$

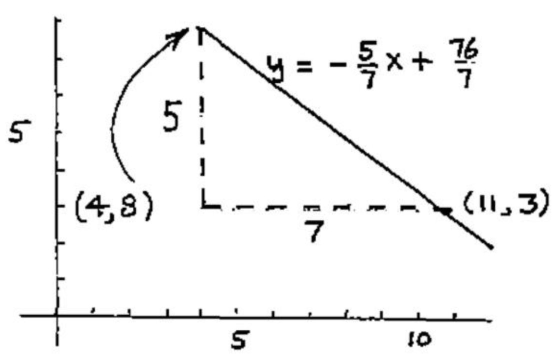
Perimeter $= 2(6+2) + 2(3+9) = 40$

■

2002-03 Event 3A

3. The graph of the equation $5x + 7y = 76$ passes through (11,3). Find a second lattice point (i.e. a point having integer coordinates) in the first quadrant that lies on the same line.

Solution



(4,8)

■

2001-02 Event 3A

The first three questions all refer to the system of equations

$$(a)\quad 2x-3y+5z=0$$
$$(b)\quad 3x-4y+7z=1$$
$$(c)\quad 4x-5y+4z=-3$$

Equations of the form $ax+by+cz=d$ **have graphs that are planes in three space. Lattice points in space are points (k,l,m) having integers for coordinates.**

1. Find a lattice point, not the origin, that lies on the plane determined by (a).

Solution

1. $x = \frac{3}{2}y - \frac{5}{2}z.$ The set of all solutions is given by
$$y\left(\tfrac{3}{2}, 1, 0\right) + z\left(-\tfrac{5}{2}, 0, 1\right)$$
Any integer choices of y and z that make x an integer will suffice; for example,
$y=3, z=1$ gives $(2,3,1)$; $y=2, z=2$ gives $(-2,2,2)$

■

2. Find two distinct lattice points that lie on the plane determined by (a) and the plane determined by (b).

Solution

$$\begin{bmatrix} 2 & -3 & 5 & 0 \\ 3 & -4 & 7 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & -3 & 5 & 0 \\ 1 & -1 & 2 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & -1 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & -1 & 2 \end{bmatrix}$$

$x = 3 - z$, $y = 2 + z$. Solutions are all of the form $(3, 2, 0) + z(-1, 1, 1)$.

Choose z to be any integer.

Examples:

$z = 0$ gives $(3, 2, 0)$

$z = 1$ gives $(2, 3, 1)$

■

3. Find the unique lattice point that lies on all three of the planes determined by (a), (b), and (c).

Solution

Standard solution of 3 indep linear equations in 3 unknowns give $(2, 3, 1)$

■

Test 3A, Concurrent Lines

2017-18 Meet 3, Individual Event A

2. If the following three lines intersect at a single point, what is the value of $b-a$?

$$2x+y=1$$
$$3x-y=4$$
$$ax+by=7$$

Solution

Adding the first two equations gives 5x = 5, or x = 1, making y = -1. Substituting these values into the third equation, gives a – b = 7. Therefore, b – a = -7.

Test 3A Gaussian Elimination

31.	3T051 Express in the form $(az+b, cz+d, z)$ all solutions to the system $2x-3y-3z-=4$ $3x-5y+2z=-3.$

Solution

$$\begin{pmatrix} 2 & -3 & -3 & 4 \\ 3 & -5 & 2 & -3 \end{pmatrix}$$

$R1 \to R1 \times 3, \quad R2 \to R2 \times 2$

$$\begin{pmatrix} 6 & -9 & -9 & 12 \\ 6 & -10 & 4 & -6 \end{pmatrix}$$

$R2 \to R2 - R1$

$$\begin{pmatrix} 6 & -9 & -9 & 12 \\ 0 & -1 & 13 & -18 \end{pmatrix}$$

$R1 \to R1 - (R2 \times 9)$

$$\begin{pmatrix} 6 & 0 & -126 & 174 \\ 0 & -1 & 13 & -18 \end{pmatrix}$$

$R1 \to R1/6$

$$\begin{pmatrix} 1 & 0 & -21 & 29 \\ 0 & -1 & 13 & -18 \end{pmatrix}$$

$R2 \rightarrow (-1) \times R2$

$$\begin{pmatrix} 1 & 0 & -21 & 29 \\ 0 & 1 & -13 & 18 \end{pmatrix}$$

$x - 21z = 29$ $y - 13z = 18$	or	$x = 29 + 21z$ $y = 18 + 13z$

$$\mathbf{(29 + 21z, 18 + 13z, z)}$$

■

32.	3A043 There are many solutions to the set of three equations below. Express y and z in terms of x so that every real value of x gives a solution to these equations. $2x - y + z = 8$ $3x - y + 2z = 11$ $5x - y + 4z = 17$

<u>Solution</u>

$$\begin{pmatrix} 2 & -1 & 1 & 8 \\ 3 & -1 & 2 & 11 \\ 5 & -1 & 4 & 17 \end{pmatrix}$$

$R1 \rightarrow 3 \times R1, R2 \rightarrow 2 \times R2$

$$\begin{pmatrix} 6 & -3 & 3 & 24 \\ 6 & -2 & 4 & 22 \\ 5 & -1 & 4 & 17 \end{pmatrix}$$

$R2 \rightarrow R2 - R1$

$$\begin{pmatrix} 6 & -3 & 3 & 24 \\ 0 & 1 & 1 & -2 \\ 5 & -1 & 4 & 17 \end{pmatrix}$$

$R1 \rightarrow 5 \times R1, R3 \rightarrow 6 \times R3$

$$\begin{pmatrix} 30 & -15 & 15 & 120 \\ 0 & 1 & 1 & -2 \\ 30 & -6 & 24 & 102 \end{pmatrix}$$

$R3 \to R3 - R1$

$$\begin{pmatrix} 30 & -15 & 15 & 120 \\ 0 & 1 & 1 & -2 \\ 0 & 9 & 9 & -18 \end{pmatrix}$$

$R3 \to R3 - R2 \times 9$

$$\begin{pmatrix} 30 & -15 & 15 & 120 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$R1 \to R1 + R2 \times 15$

$$\begin{pmatrix} 30 & 0 & 30 & 90 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$R1 \to R1/30$

$$\begin{pmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$x + z = 3$ $y + z = -2$	$z = 3 - x$ $y = -2 - z$	$z = 3 - x$ $y = -2 - (3 - x)$	$z = 3 - x$ $y = -5 + x$

$$\boldsymbol{y = -5 + x}$$
$$\boldsymbol{z = 3 - x}$$

■

33.	3T003 Find two functions, f and g, so that for any integer m, $(f(m), g(m), m)$ is a lattice point that lies on the three planes determined by $6x + 7y - 9z = 16$ $2x + 3y - 5z = 4$ $5x + 2y + 4z = 21.$

<u>Solution</u>

$$\begin{pmatrix} 6 & 7 & -9 & 16 \\ 2 & 3 & -5 & 4 \\ 5 & 2 & 4 & 21 \end{pmatrix}$$

$R2 \to 3 \times R2$

$$\begin{pmatrix} 6 & 7 & -9 & 16 \\ 6 & 9 & -15 & 12 \\ 5 & 2 & 4 & 21 \end{pmatrix}$$

$R2 \to R2 - R1$

$$\begin{pmatrix} 6 & 7 & -9 & 16 \\ 0 & 2 & -6 & -4 \\ 5 & 2 & 4 & 21 \end{pmatrix}$$

$R1 \to R1 \times 5, R3 \to 6 \times R3$

$$\begin{pmatrix} 30 & 35 & -45 & 80 \\ 0 & 2 & -6 & -4 \\ 30 & 12 & 24 & 126 \end{pmatrix}$$

$R3 \to R3 - R1$

$$\begin{pmatrix} 30 & 35 & -45 & 80 \\ 0 & 2 & -6 & -4 \\ 0 & -23 & 69 & 46 \end{pmatrix}$$

$R2 \to 23 \times R2, R3 \to 2 \times R3$

$$\begin{pmatrix} 30 & 35 & -45 & 80 \\ 0 & 46 & -138 & -92 \\ 0 & -46 & 138 & 92 \end{pmatrix}$$

$R3 \to R3 + R2$

$$\begin{pmatrix} 30 & 35 & -45 & 80 \\ 0 & 46 & -138 & -92 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$R1 \to R1/5,\ R2 \to R2/46$

$$\begin{pmatrix} 6 & 7 & -9 & 16 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$R1 \to R1 - 7 \times R2$

$$\begin{pmatrix} 6 & 0 & 12 & 30 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$R1 \to R1/6$

$$\begin{pmatrix} 1 & 0 & 2 & 5 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$x + 2z = 5$ $y - 3z = -2$	$x = 5 - 2z$ $y = -2 + 3z$	$\boldsymbol{x(m) = 5 - 2m}$ $\boldsymbol{y(m) = -2 + 3m}$

■

34.	3A034 There are many solutions to the set of three equations below. Express y and z in terms of x so that every real value of x gives a solution to these equations. $2x + y - z = -2$ $5x + 2y + z = 3$ $4x + y + 5z = 12.$

<u>Solution</u>

$$\begin{pmatrix} 2 & 1 & -1 & -2 \\ 5 & 2 & 1 & 3 \\ 4 & 1 & 5 & 12 \end{pmatrix}$$

Move $C1$ to go between $C3$ and $C4$.

$$\begin{pmatrix} 1 & -1 & 2 & -2 \\ 2 & 1 & 5 & 3 \\ 1 & 5 & 4 & 12 \end{pmatrix}$$

$R2 \to R2 - (2 \times R1)$

$$\begin{pmatrix} 1 & -1 & 2 & -2 \\ 0 & 3 & 1 & 7 \\ 1 & 5 & 4 & 12 \end{pmatrix}$$

$R3 \to R3 - R1$

$$\begin{pmatrix} 1 & -1 & 2 & -2 \\ 0 & 3 & 1 & 7 \\ 0 & 6 & 2 & 14 \end{pmatrix}$$

$R3 \to R3 - 2 \times R2$

$$\begin{pmatrix} 1 & -1 & 2 & -2 \\ 0 & 3 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$R1 \to 3 \times R1 + R2$

$$\begin{pmatrix} 3 & 0 & 7 & 1 \\ 0 & 3 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$3y + 7x = 1$ $3z + x = 7$	$\boldsymbol{y = \frac{1}{3} - \frac{7}{3}x}$ $\boldsymbol{z = \frac{7}{3} - \frac{1}{3}x}$

■

35.	3A092 Solve the system $$2x - 3y + 6 = 0$$ $$2x + y - 10 = 0$$

Solution

Step 1. Rewrite the system in the “standard” form

$$2x - 3y = -6$$

$$2x + y = 10.$$

$$\begin{pmatrix} 2 & -3 & -6 \\ 2 & 1 & 10 \end{pmatrix}$$

$R2 \to R2 - R1$

$$\begin{pmatrix} 2 & -3 & -6 \\ 0 & 4 & 16 \end{pmatrix}$$

$R2 \to R2/4$

$$\begin{pmatrix} 2 & -3 & -6 \\ 0 & 1 & 4 \end{pmatrix}$$

$R1 \to R1 + (3 \times R2)$

$$\begin{pmatrix} 2 & 0 & 6 \\ 0 & 1 & 4 \end{pmatrix}$$

$R1 \to R1/2$

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \end{pmatrix}$$

$$\boxed{\boldsymbol{x = 3, \ \ y = 4}}$$

■

36.	3A093 Solve the system $$p + q + 2r = 1$$ $$2p - q = 3$$ $$-p + 4r = 1$$

<u>Solution</u>

$$\begin{pmatrix} 1 & 1 & 2 & 1 \\ 2 & -1 & 0 & 3 \\ -1 & 0 & 4 & 1 \end{pmatrix}$$

$R2 \to R2 - 2 \times R1$

$$\begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & -3 & -4 & 1 \\ -1 & 0 & 4 & 1 \end{pmatrix}$$

$R3 \to R3 + R1$

$$\begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & -3 & -4 & 1 \\ 0 & 1 & 6 & 2 \end{pmatrix}$$

$R3 \to 3 \times R3 + R2$

$$\begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & -3 & -4 & 1 \\ 0 & 0 & 14 & 7 \end{pmatrix}$$

$R3 \to R3/7$

$$\begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & -3 & -4 & 1 \\ 0 & 0 & 2 & 1 \end{pmatrix}$$

$R2 \rightarrow 2 \times R3 + R2$

$$\begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & -3 & 0 & 3 \\ 0 & 0 & 2 & 1 \end{pmatrix}$$

$R1 \rightarrow 3 \times R1 + R2$

$$\begin{pmatrix} 3 & 0 & 6 & 6 \\ 0 & -3 & 0 & 3 \\ 0 & 0 & 2 & 1 \end{pmatrix}$$

$R1 \rightarrow 3 \times R3 - R1$

$$\begin{pmatrix} -3 & 0 & 0 & -3 \\ 0 & -3 & 0 & 3 \\ 0 & 0 & 2 & 1 \end{pmatrix}$$

$$-3p = -3, \quad -3q = 3, \quad 2r = 1$$

$$\boxed{\boldsymbol{p = 1\,,\ q = -1,\ r = 1/2}}$$

■

37.	3A011 Find a lattice point, not the origin, that lies on the plane determined by $2x - 3y + 5z = 0.$ (Remember that a lattice points in space are points have integers for coordinates.)

Solution

Any integer choices of y and z that makes x an integer will suffice. For example, $(y, z) = (1,1)$.

Then

$$2x - 3(1) + 5(1) = 0$$

$$2x + 2 = 0$$

$$x = -1.$$

So $(x, y, z) = (-1,1,1)$ is a lattice point on the plane determined by $2x - 3y + 5z = 0$.

∎

38.	3A012 Find two distinct lattice points that lie on the plane determined by the planes. $2x - 3y + 5z = 0$ $3x - 4y + 7z = 1.$

<u>Solution</u>

$$\begin{pmatrix} 2 & -3 & 5 & 0 \\ 3 & -4 & 7 & 1 \end{pmatrix} \sim \begin{pmatrix} 6 & -9 & 15 & 0 \\ 6 & -8 & 14 & 2 \end{pmatrix} \sim \begin{pmatrix} 6 & -9 & 15 & 0 \\ 0 & 1 & -1 & 2 \end{pmatrix} \sim \begin{pmatrix} 6 & 0 & 6 & 18 \\ 0 & 1 & -1 & 2 \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & -1 & 2 \end{pmatrix}$$

$$x + z = 3$$
$$y - z = 2$$

or

$$x = 3 - z$$
$$y = 2 + z$$

Any integer choice of z that makes x and y integers will suffice. For example, $z = 1$ and $z = 2$.

For $z = 1$, $x = 3 - 1 = 2$ and $y = 2 + 1 = 3$. Therefore, $(x, y, z) = (2,3,1)$ is a lattice point on the line determined by these two planes.

For $z = 2$, $x = 3 - 2 = 1$ and $y = 2 + 2 = 4$. Therefore, $(x, y, z) = (1,4,2)$ is a lattice point on the line determined by these two planes.

∎

39.	3A013 Find a lattice point, not the origin, that lies on the line determined by $2x - 3y + 5z = 0$ $3x - 4y + 7z = 1$ $4x - 5y + 4z = -3.$

Solution

Instead of starting from the very beginning I can take advantage of the fact that this problem has the same first two equations as the previous problem. So I can make the substitutions

$$x = 3 - z$$
$$y = 2 + z$$

Into the third plane.

$$4(3 - z) - 5(2 + z) + 4z = -3$$

$$12 - 4z - 10 - 5z + 4z = -3$$

$$2 - 5z = -3$$

$$-5z = -5$$

$$z = 1.$$

Now plug $z = 1$ into the

$$x = 3 - z$$

$$y = 2 + z$$

to get

$$\boxed{(x, y, z) = (2, 3, 1).}$$

■

40.	3A003 Find $a + b$ given that $3a - 5b + c = 4$ $2a + 4b - c = 3$ $7a - 11b + 2c = 9$

Solution

$$\begin{pmatrix} 3 & -5 & 1 & 4 \\ 2 & 4 & -1 & 3 \\ 7 & -11 & 2 & 9 \end{pmatrix} \sim \begin{pmatrix} 6 & -10 & 2 & 8 \\ 6 & 12 & -3 & 9 \\ 7 & -11 & 2 & 9 \end{pmatrix} \sim \begin{pmatrix} 6 & -10 & 2 & 8 \\ 0 & 22 & -5 & 1 \\ 7 & -11 & 2 & 9 \end{pmatrix} \sim \begin{pmatrix} 42 & -70 & 14 & 56 \\ 0 & 22 & -5 & 1 \\ 42 & -66 & 12 & 54 \end{pmatrix}$$

$$\sim\begin{pmatrix}42 & -70 & 14 & 56\\ 0 & 22 & -5 & 1\\ 0 & 4 & -2 & -2\end{pmatrix}\sim\begin{pmatrix}3 & -5 & 1 & 4\\ 0 & 22 & -5 & 1\\ 0 & 2 & -1 & -1\end{pmatrix}\sim\begin{pmatrix}3 & -5 & 1 & 4\\ 0 & 22 & -5 & 1\\ 0 & 0 & -6 & -12\end{pmatrix}\sim\begin{pmatrix}3 & -5 & 1 & 4\\ 0 & 22 & -5 & 1\\ 0 & 0 & 1 & 2\end{pmatrix}$$

$$\sim\begin{pmatrix}3 & -5 & 0 & 2\\ 0 & 22 & 0 & 11\\ 0 & 0 & 1 & 2\end{pmatrix}$$

$$c = 2, b = \frac{1}{2},\ \ 3a - 5\left(\frac{1}{2}\right) = 2$$

$$3a = 2 + \frac{5}{2} = \frac{9}{2}$$

$$a = \frac{3}{2}$$

$$\boxed{\boldsymbol{a + b = \frac{3}{2} + \frac{1}{2} = 2.}}$$

■

1. The sum of the squares of two consecutive odd integers is 130. What are the integers?

(Source: Minnesota State High School Mathematics League, *Individual Event*, 3*A*, 1980-1981, Problem #1)

<u>Solution</u>

7, 9 **1.** The sum of the squares of two consecutive odd integers is 130. What are the integers?

$x^2 + (x+2)^2 = 130$

$2x^2 + 4x + 4 = 130$

$x^2 + 2x + 2 = 65$

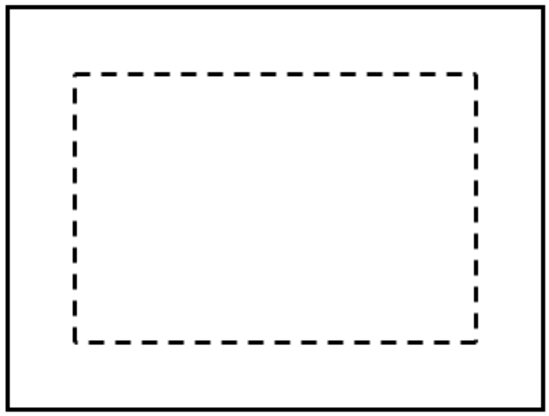
$x^2 + 2x - 63 = 0$

$(x - 7)(x + 9)$

$x = 7$

49
81
$\overline{130}$

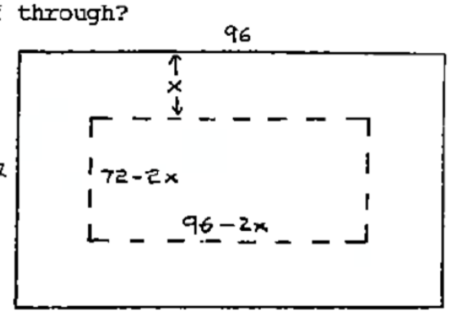
2. Homer cuts a $72' \times 96'$ lawn, starting along one outside edge and continuing to cut a border around it as indicated in the figure. How wide is the border when Homer is half through?



(Source: Minnesota State High School Mathematics League, *Individual Event*, 3*A*, 1980-1981, Problem #2)

<u>Solution</u>

<u>12 ft.</u> 2. Homer cuts a 72' x 96' lawn, starting along one outside edge and continuing to cut a border around it as indicated in the figure. How wide is the border when Homer is half through?



$$(72-2x)(96-2x) = (72\cdot 96)\,\tfrac{1}{2}$$
$$\tfrac{1}{2}72\cdot 96 - (72\cdot 2 + 96\cdot 2)x + 4x^2 = 0$$
$$x^2 - (36+48)x + 9\cdot 96 = 0$$
$$x^2 - 84x + 9\cdot 8\cdot 12 = 0$$
$$(x-12)(x-72) = 0$$
$$x = 12 \qquad \cancel{x = 72}$$

<u>Ck</u> $72 - 24 = 48$ $\qquad \dfrac{\cancel{72}\cdot 96}{48\cdot \cancel{72}} = \dfrac{1}{2}$

$96 - 24 = 72$

3. The rate of one train is 21 mph more than that of another. If the faster train takes two hours less than the other for a trip of 252 miles, at what rate does the faster train travel?

(Source: Minnesota State High School Mathematics League, *Individual Event*, 3*A*, 1980-1981, Problem #3)

Solution

63 3. The rate of one train is 21 mph more than that of another. If the faster train takes two hours less than the other for a trip of 252 miles, at what rate does the faster train travel?

	mph	time
slow	x	$t+2$
fast	$x+21$	t

$x(t+2) = 252$

$x(t+2) = (x+21)t$

$\cancel{xt} + 2x = \cancel{xt} + 21t$

$2x(t+2) = 2 \cdot 252$

$21t^2 + 42t - 2\cdot 252 = 0$

$t^2 + 2t - 24 = 0$

$(t-4)(t+6) = 0$

$2x = 21(4)$

$\boxed{x = 42}$

$x + 21 = 63$

Ck

$$\begin{array}{r} 63 \\ 4 \\ \hline 252 \end{array} \qquad \begin{array}{r} 42 \\ 6 \\ \hline 252 \end{array}$$

■

______ 2. Solve the system

$$\begin{aligned} 6x + 20y &= 9 \\ -3x + 12y &= 1 \end{aligned}$$

(Source: Minnesota State High School Mathematics League, *Individual Event*, 3*A*, 1986-1987, Problem #2)

Solution

a) The second system will certainly

b) The second system will certainly

c) No conclusion can be drawn with

$x = \frac{2}{3}$

$y = \frac{1}{4}$ 2. Solve the system

$$\begin{aligned} 6x + 20y &= 9 \\ -3x + 12y &= 1 \end{aligned}$$

2. For instructive purposes, I'll solve using determinants.

$$x = \frac{\begin{vmatrix} 9 & 20 \\ 1 & 12 \end{vmatrix}}{\begin{vmatrix} 6 & 20 \\ -3 & 12 \end{vmatrix}} = \frac{88}{132} = \frac{2}{3} \qquad y = \frac{\begin{vmatrix} 6 & 9 \\ -3 & 1 \end{vmatrix}}{132} = \frac{33}{132} = \frac{1}{4}$$

■

3. Find x_1, x_2, and x_3 in terms of a,b, and c if

$$\begin{aligned} x_1 + x_2 \phantom{{}+x_3} &= a \\ x_2 + x_3 &= b \\ x_1 \phantom{{}+x_2} + x_3 &= c \end{aligned}$$

$x_1 =$ ______

$x_2 =$ ______

$x_3 =$ ______

(Source: Minnesota State High School Mathematics League, *Individual Event*, 3*A*, 1986-1987, Problem #3)

<u>Solution</u>

3. Find x_1, x_2, and x_3 in terms of a,b, and c if

$$\begin{aligned} x_1 + x_2 \phantom{{}+x_3} &= a \\ x_2 + x_3 &= b \\ + x_3 &= c \end{aligned}$$

$x_1 = \frac{a-b+c}{2}$

$x_2 = \frac{a+b-c}{2}$

$x_3 = \frac{-a+b+c}{2}$

3. $\begin{cases} x_1 + x_2 = a & \text{copied} \\ x_2 + x_3 = b & \text{copied} \\ -x_2 + x_3 = c - a & -(\text{first eq.}) + (\text{third eq.}) \end{cases}$

$\begin{cases} x_1 + x_2 = a & \text{copied} \\ 2x_2 = a+b-c & -(\text{third eq}) + (\text{second eq}) \\ -x_2 + x_3 = c-a & \text{copied} \end{cases}$

$\begin{cases} x_1 = \dfrac{a-b+c}{2} & -\frac{1}{2}(\text{second eq}) + (\text{first eq}) \\ x_2 = \dfrac{a+b-c}{2} & \frac{1}{2}(\text{second eq}) \\ x_3 = \dfrac{-a+b+c}{2} & \frac{1}{2}(\text{second eq}) + (\text{third eq}) \end{cases}$

HINT Keep a system of three equations together as you move from step to step.

■

———— 4. With the aid of a steady current, the motor on Alice's boat took her downstream from River City to Hibanks in 6 hours and 40 minutes. Brenda, using her motor on the same type of boat, took 7 hours for the same trip. With the same current working against them the next day, Alice took 9 hours and 20 minutes while Brenda took 10 hours for the return trip. In still water, Alice's boat moves 1/2 mph faster than does Brenda's. How far is it from River City to Hibanks?

(Source: Minnesota State High School Mathematics League, *Individual Event*, 3*A*, 1986-1987, Problem #4)

Solution

4. Let b = Brenda's speed with no current (so $b+\frac{1}{2}$ = Alice's speed); let w = change in speed due to current.

$$\begin{pmatrix}\text{Alice's}\\ \text{dist.}\end{pmatrix} = \begin{pmatrix}\text{Brenda's}\\ \text{dist.}\end{pmatrix}$$

Going $(b+\frac{1}{2}+w)\frac{20}{3} = (b+w)7$

Return $(b+\frac{1}{2}-w)\frac{28}{3} = (b-w)10$

Solve to get $b=\frac{17}{2}$, $w=\frac{3}{2}$

Then dist $= (10)(7) = 70$ miles

■

Test 3A, Determinants

41. 3T171	The vertices of $\triangle AOB$ are $A=(a,b), O=(0,0)$, and $B=(c,d)$. The values of a, b, c, and d are all positive integers and they are all different. If the area of $\triangle AOB$ is 13, what is the smallest possible of the sum $a+b+c+d$?
Solution	
The area of $\triangle AOB$ *is the determinant* $\frac{1}{2}\begin{vmatrix}0&0&1\\a&b&1\\c&d&1\end{vmatrix}$ *, making* $ad-bc=26$. *To minimize the sum chose bc as small as possible, making ad as small as possible. Let b = 1 and c = 2. then ad = 28. Because 1 and 2 have already been assigned, the smallest a can be is 4, making d = 7. Therefore, the minimum value of a + b + c+ d = 1 + 2 + 4 + 7 = 14.*	
Key	

	Area of a Triangle in the *xy*-Plane The area of a triangle with vertices $(x_1, y_1), (x_2, y_2)$, and (x_3, y_3) is $\text{Area} = \pm\frac{1}{2}\det\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}$ where the sign $(\pm)$ is chosen to give a positive area.

42. 3A162	Given $\begin{vmatrix} 2 & 9 \\ 3 & b \end{vmatrix} = 2$, determine exactly $\begin{vmatrix} 9 & 2 \\ b & 3 \end{vmatrix}$.
Solution *Since* $\begin{vmatrix} 2 & 9 \\ 3 & b \end{vmatrix} = 2$, *2b – 27 = 2* $\Rightarrow b = \frac{29}{2}$. *Substituting this into our other matrix and solving for the determinant gives* $27 - 2\left(\frac{29}{2}\right) = -2$. ***Note: swapping columns of a 2x2 matrix flips the sign of the determinant***	

43. 3A164	Find the <u>sum</u> of all positive integers a and b where $a > b$ and the determinant $\begin{vmatrix} 1 & 0 & a \\ 0 & 1 & a \\ b & b & 1 \end{vmatrix} = -13$.
Solution *The determinant of* $\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & a \\ b & b & 1 \end{bmatrix} = -13$ *when* $1(1\cdot 1 - a\cdot b) - 0(0\cdot 1 - a\cdot b) + a(0\cdot b - b\cdot 1) = -13 \Rightarrow 1 - 2ab = -13 \Rightarrow ab = 7$. *Since 7 is prime and a > b, a = 7 and b = 1, so a + b is 8.*	

44. 3A153	Determine exactly <u>all</u> values of the real number c for which the determinant of $\begin{bmatrix} 1 & 2 & 3 \\ 4 & c & 6 \\ 7 & 8 & 9 \end{bmatrix}$ is positive.

Solution

The determinant of $\begin{bmatrix} 1 & 2 & 3 \\ 4 & c & 6 \\ 7 & 8 & 9 \end{bmatrix}$ *is positive when*

$$1(9c-48)-2(4\cdot 9-7\cdot 6)+3(32-7c)>0 \Rightarrow 60-12c>0 \Rightarrow c<5.$$

45. 3T151	Determine exactly the largest value for tan α, given the system $\begin{cases} 1=\frac{x}{2}+\frac{y}{2}+z\cdot\sin\alpha \\ 1=\frac{y}{3}+\frac{z}{3}+x\cdot\cos\alpha \\ 1=z+x+y \end{cases}$ has no solutions (x, y, z).

Solution

The system of equations will have no solutions when the determinant of the coefficient matrix is 0. The determinant is $\frac{1}{2}\left(\frac{1}{3}\cdot 1-1\cdot\frac{1}{3}\right)-\frac{1}{2}\left(\cos\alpha-\frac{1}{3}\right)+\sin\alpha\left(\cos\alpha-\frac{1}{3}\right)=\sin\alpha\cos\alpha-\frac{\sin\alpha}{3}-\frac{\cos\alpha}{2}+\frac{1}{6}=\left(\cos\alpha-\frac{1}{3}\right)\left(\sin\alpha-\frac{1}{2}\right)$. *So we need* $\cos\alpha=\frac{1}{3}$ *or* $\sin\alpha=\frac{1}{2}$. *We have two choices* $\sin\alpha=\frac{1}{2}\Rightarrow\cos\alpha=\pm\frac{\sqrt{3}}{2}\Rightarrow\tan\alpha=\pm\frac{\sqrt{3}}{3}$ *or* $\cos\alpha=\frac{1}{3}\Rightarrow\sin\alpha=\pm\frac{2\sqrt{2}}{3}\Rightarrow\tan\alpha=\pm 2\sqrt{2}$. *The largest value of tan α =* $\boxed{2\sqrt{2}}$.

46. 3A143	$A=\begin{bmatrix} x & y \\ 5 & 8 \end{bmatrix}$ and $B=\begin{bmatrix} x & y \\ 2 & 3 \end{bmatrix}$. If det A = 10 and det B = 36, determine x and y exactly.

Solution

det A = 8x – 5y = 10. Likewise, det B = 3x – 2y = 36. We now have a system of two linear equations: $\begin{cases} 8x-5y=10 \\ 3x-2y=36 \end{cases}$. *Multiply the first equation by –2 and the second by 5 to get* $\begin{cases} -16x+10y=-20 \\ 15x-10y=180 \end{cases}$. *Add the two equations to obtain x = –160; substitution yields y = –258.*

47. 3A131	Determinants of matrices can be used to represent linear expressions. For example, $\begin{vmatrix} 2 & 3 \\ x & y \end{vmatrix} = 2$ describes a line in the xy-plane. What is this line's y-intercept?

Solution

$$\begin{vmatrix} 2 & 3 \\ x & y \end{vmatrix} = 2 \Rightarrow 2y - 3x = 2. \text{ Setting } x = 0\text{, we have } 2y = 2 \Rightarrow y\text{-int} = 1.$$

48. 3T133	The system $\begin{cases} 10x - 4y \le 7 \\ 3x + ky \,\square\, 11 \end{cases}$ is to have no solution. Given this condition, determine exactly the value of k, <u>and</u> two possible relation symbols that could be written in the box.

Solution

We need the two lines that form the structure of this system to be parallel. In other words, thinking of the system as the matrix $\left[\begin{array}{cc|c} 10 & -4 & 7 \\ 3 & k & 11 \end{array}\right]$, *we need* $\begin{vmatrix} 10 & -4 \\ 3 & k \end{vmatrix} = 0$, so $10k + 12 = 0 \Rightarrow k = \boxed{-\frac{6}{5}}$. *Now graph the first inequality, and the line suggested by the second statement (Figure 3). In order for there to be no shared points, the second statement's symbol must be* $\boxed{>}$, $\boxed{\ge}$, *or* $\boxed{=}$.

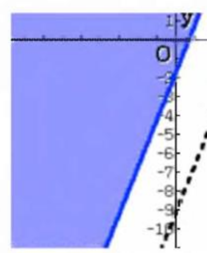


Figure 3

49. 3A124	It is known that the system of equations $\begin{cases} Ax + y + z = 1 \\ x + By + z = 1 \\ x + y + Cz = 1 \end{cases}$ has no solution. Determine exactly the value of ABC - $(A + B + C)$.

Solution

For the system to be inconsistent, the determinant of $\begin{bmatrix} A & 1 & 1 \\ 1 & B & 1 \\ 1 & 1 & C \end{bmatrix}$ *must equal zero. Using the cofactors of row 1:* $A \cdot \begin{vmatrix} B & 1 \\ 1 & C \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 1 \\ 1 & C \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & B \\ 1 & 1 \end{vmatrix} = A(BC - 1) - (C - 1) + (1 - B)$

$= ABC - A - C + 1 + 1 - B = 0$, *so* $ABC - A - B - C = ABC - (A + B + C) = -2$.

50. 3A113	Determine exactly **all** values of x for which the determinant of $\begin{bmatrix} 1 & x & 2 \\ 3 & 4 & 5 \\ 6 & x & 7 \end{bmatrix}$ equals zero.

Solution

Expand the determinant across column 2:

$$\text{determinant} = -x \cdot \begin{vmatrix} 3 & 5 \\ 6 & 7 \end{vmatrix} + 4 \cdot \begin{vmatrix} 1 & 2 \\ 6 & 7 \end{vmatrix} - x \cdot \begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix} = -x(21-30) + 4(7-12) - x(5-6)$$

$= 9x - 20 + x = 10x - 20$. *Setting this equal to zero and solving yields only* $x = 2$.

51. 3A083	If $\begin{vmatrix} n & 2n & 3n \\ 1 & 0 & 2 \\ 0 & 3 & 0 \end{vmatrix} = 6$, find n.

Solution

Expanding the determinant along row 3, we have

$$(-1)^{3+2} \cdot 3 \cdot \begin{vmatrix} n & 3n \\ 1 & 2 \end{vmatrix} = 6 \Rightarrow -3 \cdot (2n - 3n) = 6 \Rightarrow 3n = 6$$

52. 3A063	The equation $$\begin{vmatrix} x & y & 1 \\ 4 & 1 & 1 \\ -1 & 7 & 1 \end{vmatrix} = 0$$ has a graph that is a straight line. What is the slope of the line?

Solution

Both $(4,1)$ and $(-1,7)$ lie on the line, so its slope $= \dfrac{7-1}{-1-4} = -\dfrac{6}{5}$

53. 3A064	A company makes school sweatshirts and sweatpants. Five sweatshirts and six sweatpants cost a total of $147. For orders totaling more than 30 items, the company reduces the total price of the sweatshirts by 40% and cuts the total price of the sweatpants in half. Forty sweatshirts and forty sweatpants, therefore, cost a total of $578. Find the original cost of a single pair of sweatpants. [Massachusetts Math League, March 2006].

Solution

From the given information, with
s = price of a sweatshirt, p = price of pants,

$$5s + 6p = 147$$
$$(40)(.6s) + 40(.5p) = 578$$

$$5s + 6p = 147$$
$$12s + 10p = 289$$

$$p = \frac{\begin{vmatrix} 5 & 147 \\ 12 & 289 \end{vmatrix}}{\begin{vmatrix} 5 & 6 \\ 12 & 10 \end{vmatrix}} = \$14.50$$

54. 3A052	Find (x,y), given $\frac{3}{8}x - \frac{4}{7}y = 1$, $-\frac{2}{3}x + \frac{3}{2}y = 5$

Solution

$$21x - 32y = 56$$
$$-4x + 9y = 30$$

$$x = \frac{\begin{vmatrix} 56 & -32 \\ 30 & 9 \end{vmatrix}}{\begin{vmatrix} 21 & -32 \\ -4 & 9 \end{vmatrix}} = \frac{504+960}{189-128}$$
$$= \frac{1464}{61} = 24$$

$$y = \frac{\begin{vmatrix} 21 & 56 \\ -4 & 30 \end{vmatrix}}{61} = \frac{630+224}{61}$$
$$= \frac{854}{61} = 14$$

2. Balance the following chemical equations:

a) $\mathrm{Pb(NO_3)_2} \rightarrow \mathrm{PbO} + \mathrm{NO_2} + \mathrm{O_2}$

Solution 2(a)

$$ax + 2ay + 6az = bx + bz + cy + 2cz + 2dz$$

$$a = b$$

$$2a = c$$

$$6a = b + 2c + 2d$$

so

$$6a = a + 4a + 2d$$

or

$$a = 2d.$$

So

$$\begin{aligned} a &= a \\ b &= a \\ c &= 2a \\ d &= a/2. \end{aligned}$$

To make them all integers (and as small as possible), let $a = b = 2, c = 4, d = 1$.

■