MSHSML Meet 3, Event B
Study Guide

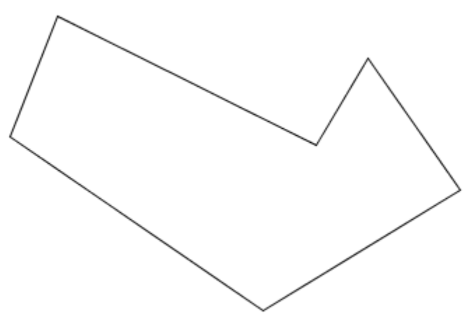
3B Polygonal Figures and Solids

Special quadrilaterals and regular polygons (including area formulas)
Intersecting diagonals
Ptolemy's Theorem
Polygonal prisms & pyramids (including volume and surface area)

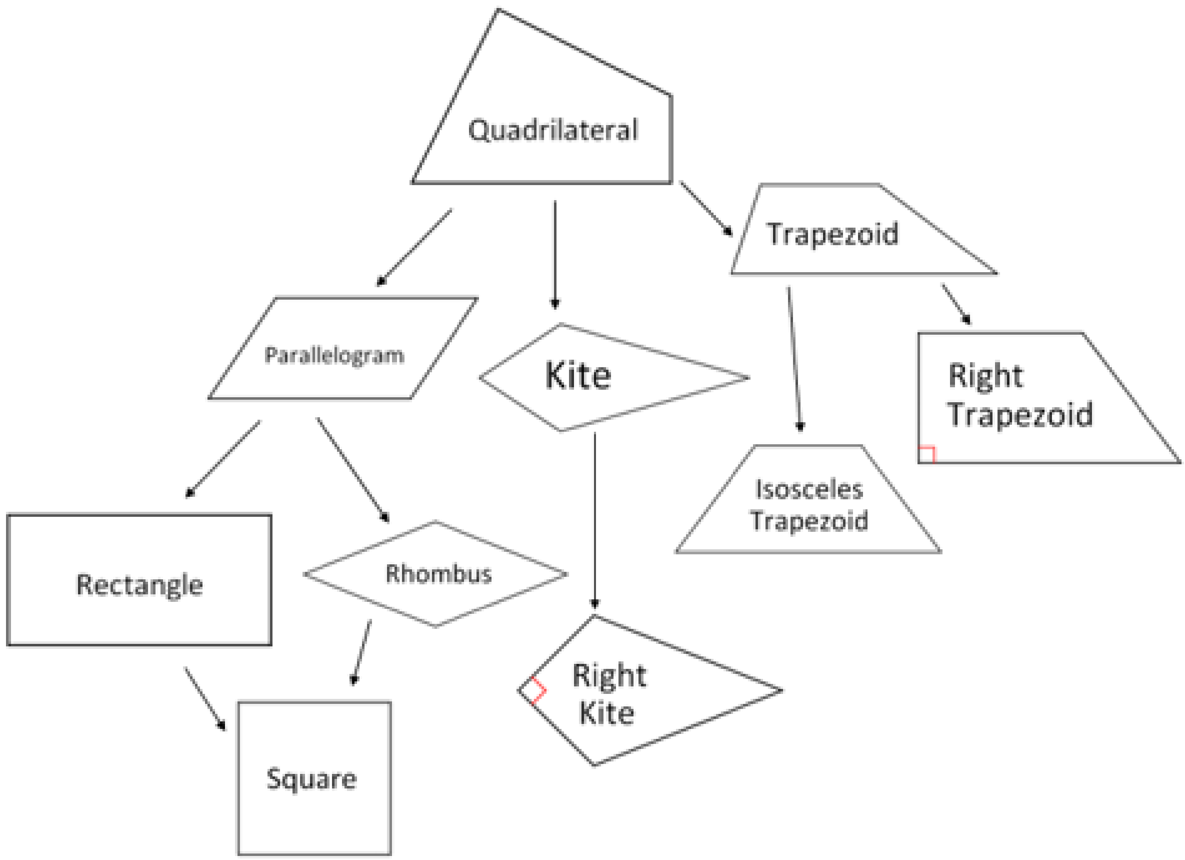
1 Contents

2 Polygons

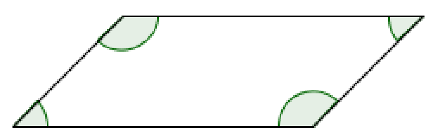
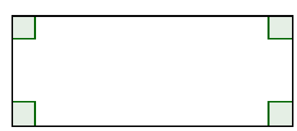
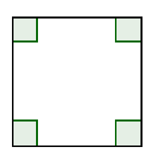
Polygon – a closed two-dimensional figure whose sides are line segments

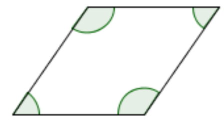
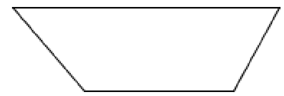
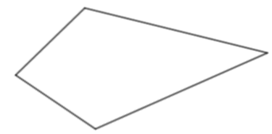


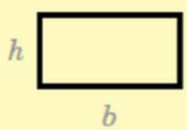
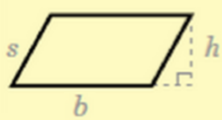
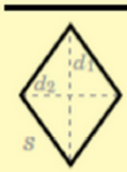
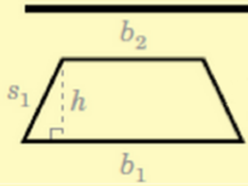
Quadrilateral – a four-sided polygon



Special Quadrilaterals

Parallelogram	Opposite sides are parallel. Opposite sides are equal in length. Opposite angles are equal in size.	
Rectangle	Opposite sides are parallel. Opposite sides are equal in length. All angles all equal 90°.	
Square	Opposite sides are parallel. All sides are equal in length. All angles all equal 90°.	

Rhombus	Opposite sides are parallel. All sides are equal in length. Opposite angles are equal in size.	
Trapezoid	One pair of opposite sides are parallel, the other pair of opposite sides are not parallel.	
Kite	A quadrilateral with two pairs of equal adjacent sides.	

	SHAPE	PERIMETER	AREA
	**Square**	$P = 4s$	$A = s^2$
	**Rectangle**	$P = 2(b + h)$	$A = bh$
	**Parallelogram**	$P = 2(b + s)$	$A = bh$
	**Rhombus**	$P = 4s$	$A = \frac{1}{2}d_1 d_2$
	**Trapezoid**	$P = b_1 + b_2 + s_1 + s_2$	$A = \frac{1}{2}(b_1 + b_2)h$

Property	Square	Rhombus	Rect-angle	Parallel-ogram	Trape-zoid
Diagonals are ≅ (congruent)	✓		✓		
Diagonals are ⊥ (perpendicular)	✓	✓			
Diagonals bisect each other	✓	✓	✓	✓	
Diagonals form two ≅ Δ's (triangles)	✓	✓	✓	✓	
Each diagonal bisects opposite ∠'s (angles)	✓	✓			
Opposite sides are ∥ (parallel)	✓	✓	✓	✓	
Opposite sides are ≅	✓	✓	✓	✓	
Opposite ∠'s are ≅	✓	✓	✓	✓	
All ∠'s are right ∠'s.	✓		✓		
All sides are ≅	✓	✓			
Consecutive angles are supplementary	✓	✓	✓	✓	
Exactly one pair of opposite sides are ∥.					✓

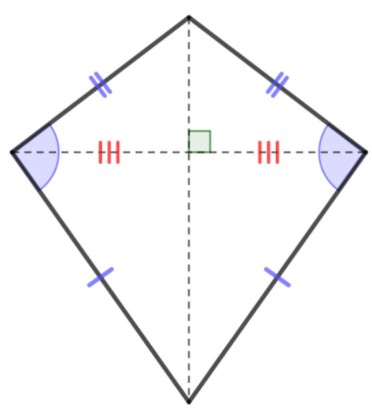
Kite

Two pairs of adjacent sides are equal in length.

One pair of opposite angles (the ones that are between the sides of unequal length) are equal in size.

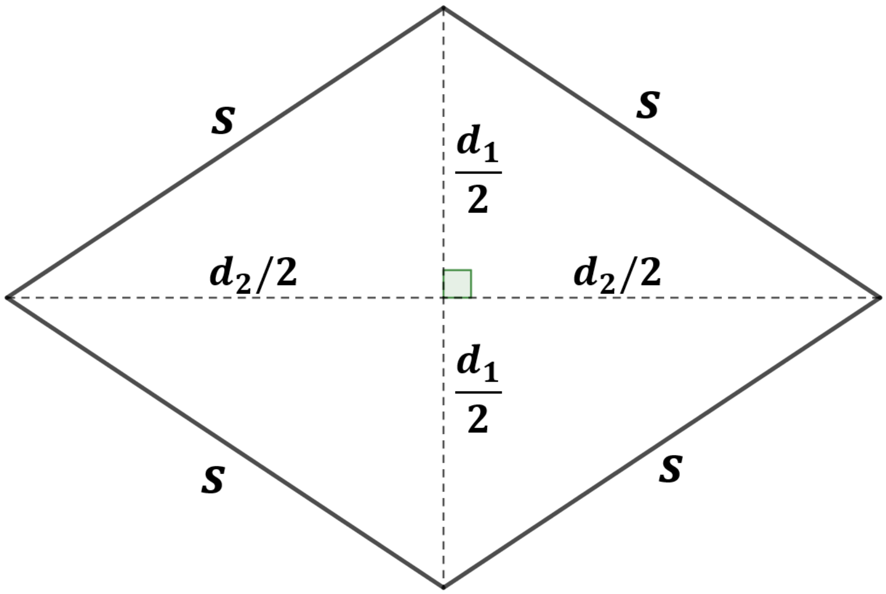
One diagonal bisects the other.

Diagonals intersect at right angles.



Rhombus

Recalling the properties of a rhombus, we have the following diagram.



$$\textbf{Perimeter} = 4s = 4\sqrt{(d_1/2)^2 + (d_2/2)^2} = \mathbf{2\sqrt{(d_1)^2 + (d_2)^2}}$$

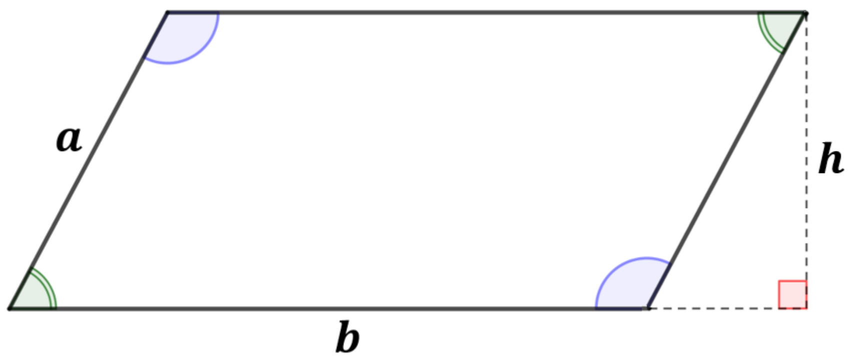
$$\textbf{Area} = 4(\text{Area small } \Delta) = 4\left(\frac{1}{2}\cdot\left(\frac{d_1}{2}\right)\left(\frac{d_2}{2}\right)\right) = \mathbf{\frac{d_1 d_2}{2}}$$

$$\sin(\alpha/2) = \frac{d_1/2}{s} \Longrightarrow \mathbf{d_1 = 2s \cdot \sin(\alpha/2)}$$
$$\cos(\alpha/2) = \frac{d_2/2}{s} \Longrightarrow \mathbf{d_2 = 2s \cdot \cos(\alpha/2)}$$

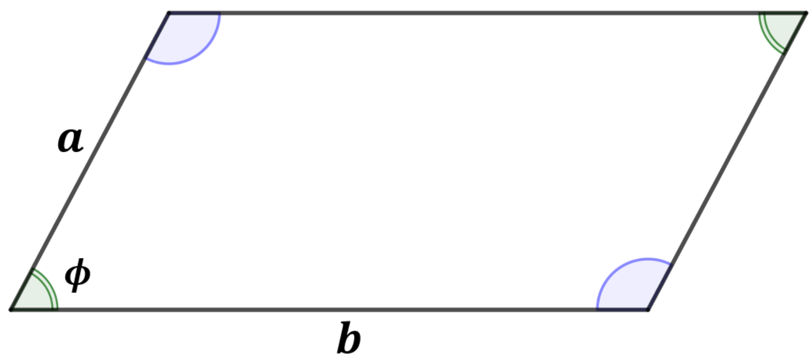
$$\textbf{Area} = \frac{d_1 d_2}{2} = 4s^2 \sin(\alpha/2)\cos(\alpha/2) = 2s^2\big(2\sin(\alpha/2)\cos(\alpha/2)\big) = \mathbf{2s^2 \sin(\alpha)}$$

Parallelogram

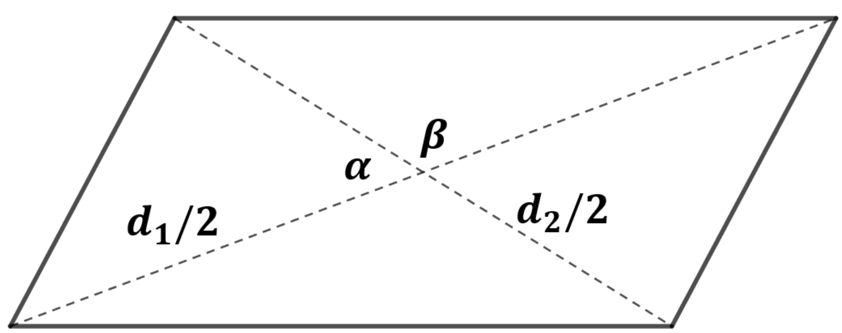
(a) The area of a parallelogram with adjacent side lengths of a and b and a distance of h between the two sides of length b is $\mathbf{b \cdot h}$.



(b) The area of a parallelogram with adjacent side lengths of a and b if it is known that one of the angles has measure ϕ degrees is $\boldsymbol{ab\sin(\phi)}$.



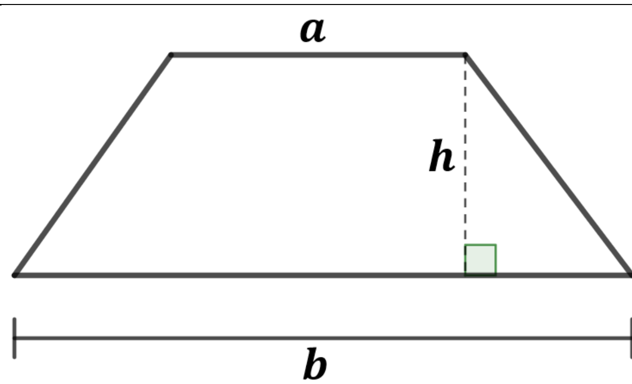
(c) The area of a parallelogram whose diagonals have lengths d_1 and d_2 if the lesser of the two angles between the diagonals has measure $\psi = \min(\alpha, \beta)$ degrees is $\boldsymbol{d_1 d_2 \sin(\psi)/2}$.



Five Ways to Prove that a Quadrilateral Is a Parallelogram

1. Show that *both* pairs of opposite sides are parallel.
2. Show that *both* pairs of opposite sides are congruent.
3. Show that *one* pair of opposite sides are both congruent and parallel.
4. Show that both pairs of opposite angles are congruent.
5. Show that the diagonals bisect each other.

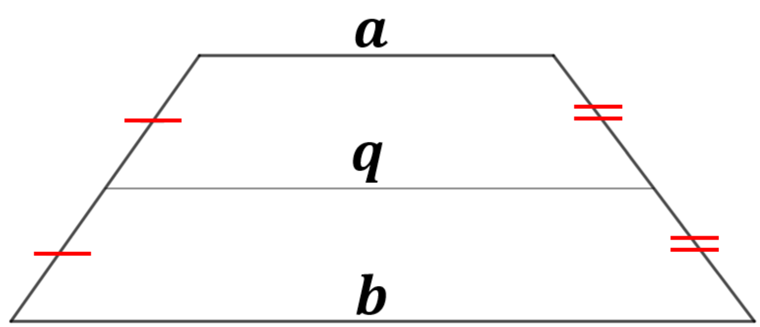
Trapezoid

<table>
<tr><td colspan="2">(1) Area of a Trapezoid</td></tr>
<tr><td>Area $= \left(\frac{a+b}{2}\right)h$</td><td>
</td></tr>
<tr><td colspan="2">Of course this formula is only useful if we know h, the height. In (5), see below, we show how to find h knowing the length of all four sides.</td></tr>
</table>

(2) Midline of a Trapezoid

The midline is the line segment connecting the midpoints of the two nonparallel sides in a trapezoid. (Note: some textbooks refer to the midline of a trapezoid as the median of the trapezoid.)

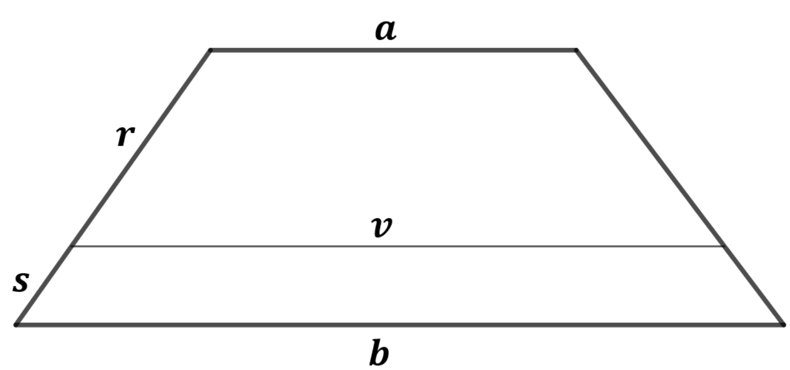
$(\boldsymbol{i})$	$\boldsymbol{q = \frac{a+b}{2}}$
$(\boldsymbol{ii})$	$\boldsymbol{q \parallel a}$
$(\boldsymbol{iii})$	$\boldsymbol{q \parallel b}$
$(\boldsymbol{iv})$	$\therefore$ **Area** $\boldsymbol{= \left(\frac{a+b}{2}\right)h = qh.}$



(3) Weighted Average Line of a Trapezoid

Suppose the line segment of labeled v in the diagram below is parallel to the two parallel sides of lengths a and b shown in the trapezoid below. Then the length of v is a weighted average of b and a with weights $r/(r+s)$ and $s/(r+s)$.

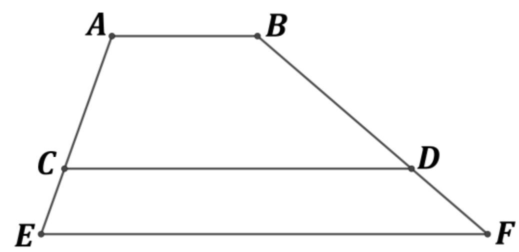
$$\boldsymbol{v = \frac{rb + sa}{r+s}}$$



(4) Side Splitter Type Results

(a) $\frac{\overline{AC}}{\overline{CE}} = \frac{\overline{BD}}{\overline{DF}} \Longrightarrow AB \parallel CD$ and $CD \parallel EF$

(b) $AB \parallel CD$ and $CD \parallel EF \Longrightarrow \frac{\overline{AC}}{\overline{CE}} = \frac{\overline{BD}}{\overline{DF}}$

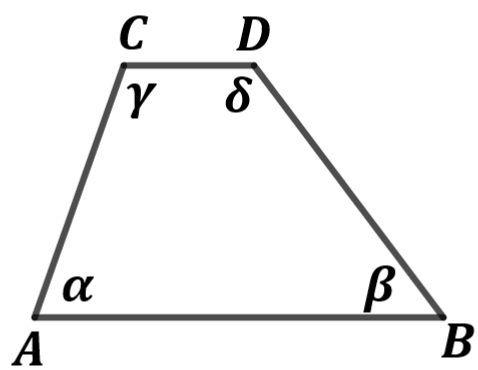


(5) Supplementary Angles within Trapezoids

Each lower base angle is supplemental to the upper base angle on the same side.

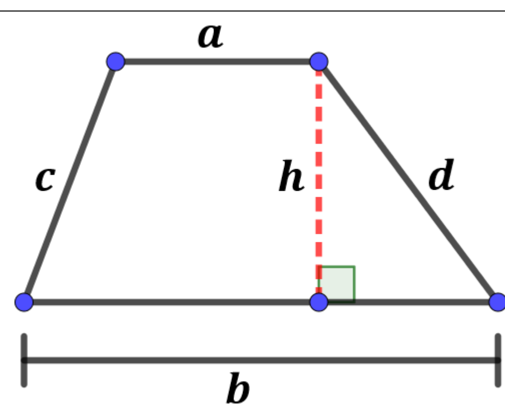
$\alpha + \gamma = 180°$

$\beta + \delta = 180°$



(6) Height of a Trapezoid Knowing all Four Sides

$$\boldsymbol{h} = \sqrt{\boldsymbol{c}^2 - \left(\frac{\boldsymbol{c}^2 - \boldsymbol{d}^2 + (\boldsymbol{b} - \boldsymbol{a})^2}{2(\boldsymbol{b} - \boldsymbol{a})}\right)^2}$$

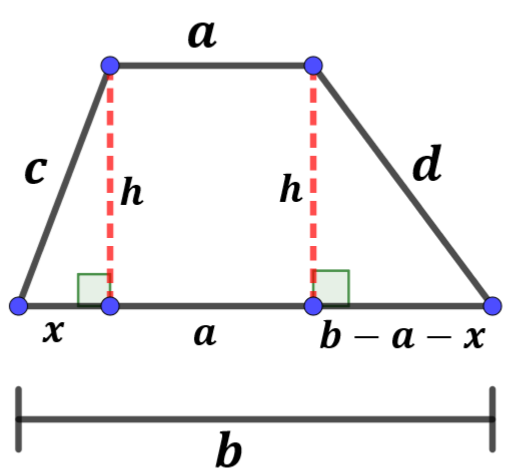


This formula looks like it would give different answers if you flipped the role of c and d. However, flipping the role of c and d will not change the value of h in the case where c and d are the lengths of the nonparallel sides of a trapezoid with bases a and b,

Bottom Line – It does not matter which nonparallel side you label c and which side you label d when using this formula.

This formula for the height h is not intuitive and not easy to remember. I suggest you focus on remembering how to derive h instead of memorizing the above formula.

Derivation for the height h of a trapezoid



By the Pythagorean Theorem we have

$$x^2 + h^2 = c^2$$

and also

$$(b - a - x)^2 + h^2 = d^2.$$

We can eliminate h^2 from this pair of equations by subtracting.

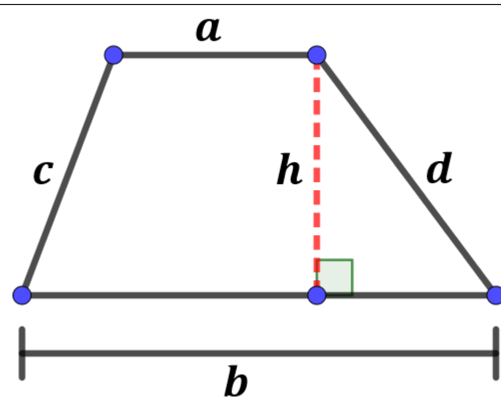
$$(b - a - x)^2 - x^2 = d^2 - c^2.$$

This allows us to simplify and solve for x. Note the x^2 terms cancel out in above equation which makes solving for x much easier.

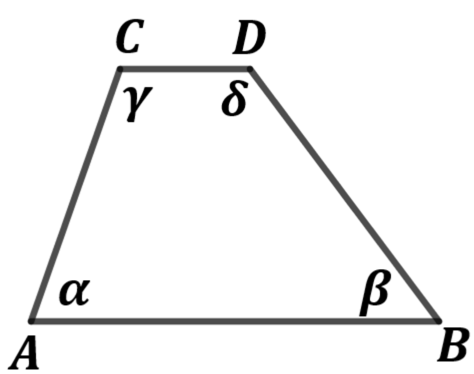
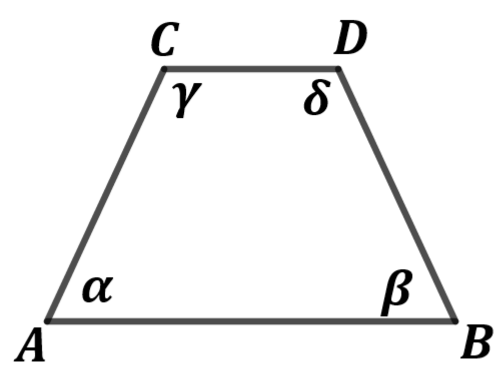
$$x = \frac{c^2 - d^2 + (b - a)^2}{2(b - a)}.$$

Knowing x we can find the height h because we know $x^2 + h^2 = c^2$.

$$h = \sqrt{c^2 - \left(\frac{c^2 - d^2 + (b-a)^2}{2(b-a)}\right)^2}.$$

(7) Area of a Trapezoid Knowing all Four Sides	
$$\textbf{Area} = \left(\frac{\boldsymbol{a}+\boldsymbol{b}}{2}\right)\boldsymbol{h}$$ where $$\boldsymbol{h} = \sqrt{\boldsymbol{c}^2 - \left(\frac{\boldsymbol{c}^2 - \boldsymbol{d}^2 + (\boldsymbol{b}-\boldsymbol{a})^2}{2(\boldsymbol{b}-\boldsymbol{a})}\right)^2}$$	
This is just the formula for area given in (1) with the value of h given in (6).	

Isosceles Trapezoid

	
(Regular) Trapezoid	**Isosceles Trapezoid**
$\boldsymbol{CD} \parallel \boldsymbol{AB}$	$\boldsymbol{CD} \parallel \boldsymbol{AB}$ **and** $\boldsymbol{\alpha} = \boldsymbol{\beta}$
This is a trapezoid but not an isosceles trapezoid because $\boldsymbol{\alpha} \neq \boldsymbol{\beta}$**.**	

Properties of Isosceles Trapezoids

(1-7)

All properties of a (regular) trapezoid carry over because an isosceles trapezoid is still a trapezoid. So results (1) – (7) given above for a (regular) trapezoid apply to an isosceles trapezoid as well.

(8)

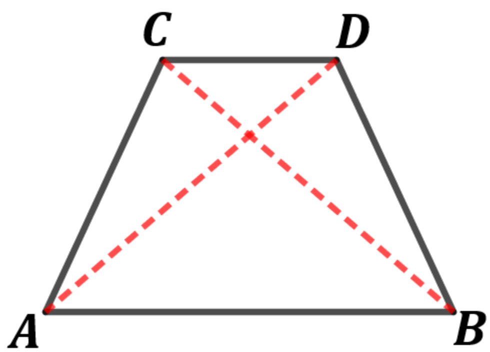
The pair of non-parallel sides are equal in length ($\overline{AC} = \overline{BD}$).

(9)

The pair of top angles are equal in measure ($\gamma = \delta$).

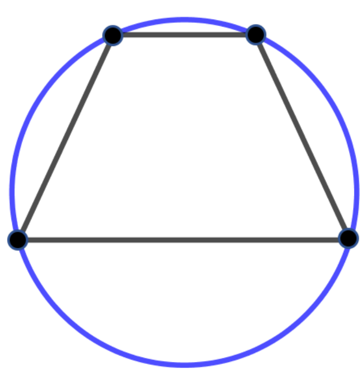
(10)

The diagonals are equal in length ($\overline{AD} = \overline{BD}$).



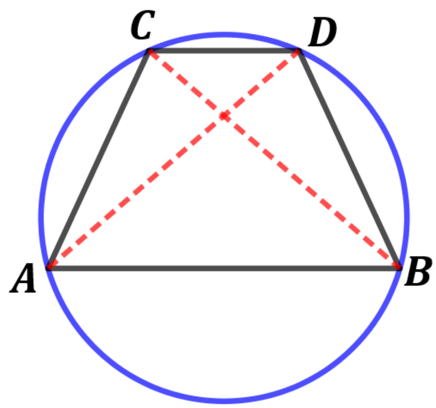
(11)

Every isosceles trapezoid is cyclic (*i.e.* there exists a circle that contains all four vertices).



(12)

Ptolemy's Theorem (which applies to any cyclic quadrilateral) states that the product of the lengths of the two diagonals equals the sum of the products of opposite sides.



$$(\overline{AD} \cdot \overline{BC}) = (\overline{AB} \cdot \overline{CD}) + (\overline{AC} \cdot \overline{BD})$$

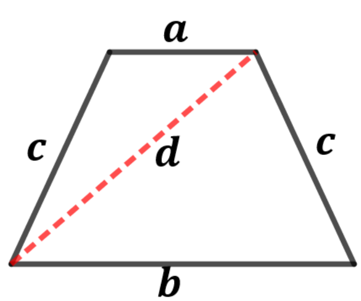
(13) Ptolemy's Theorem for Isosceles Trapezoids

In an isosceles trapezoid the diagonals are equal in length and the pair of nonparallel sides are equal in length. Hence for an isosceles trapezoid (as labeled in the figure below) what Ptolemy's Theorem tells us is

$$(\boldsymbol{d} \cdot \boldsymbol{d}) = (\boldsymbol{a} \cdot \boldsymbol{b}) + (\boldsymbol{c} \cdot \boldsymbol{c})$$

or

$$\boldsymbol{d}^2 = \boldsymbol{ab} + \boldsymbol{c}^2.$$



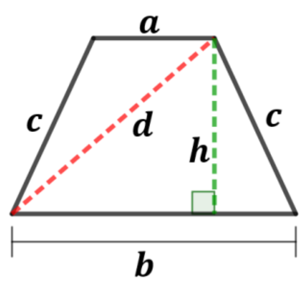
So for an isosceles trapezoid the formula for the length of a diagonal is

$$\boldsymbol{d} = \sqrt{\boldsymbol{ab} + \boldsymbol{c}^2}.$$

(14)

In an isosceles trapezoid there are two formulas for the height h.

$$h = \sqrt{d^2 - \left(\frac{a+b}{2}\right)^2} \quad \text{and} \quad h = \sqrt{c^2 - \left(\frac{a-b}{2}\right)^2}$$



Both of these formulas are derived as direct consequences of the Pythagorean Theorem.

(15)

We can combine the either formula for h from (13) with the formula in (1) for the area of *any* (not necessarily isosceles) trapezoid to find that for an isosceles trapezoid

$$\text{Area} = \left(\frac{a+b}{2}\right) \cdot h.$$

(16)

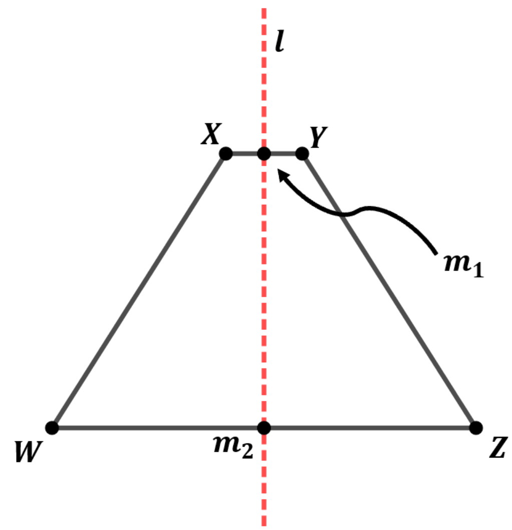
Brahmagupta's Formula (which applies to any cyclic quadrilateral) gives another way to find the area of an isosceles trapezoid:

$$\text{Area} = \sqrt{(s-a)(s-b)(s-c)(s-c)}$$

where $s = (a+b+c+c)/2$.

(17)

Suppose $WXYZ$ is a trapezoid with $XY \parallel WZ$. Let m_1 and m_2 are the midpoints of XY and WZ respectively.

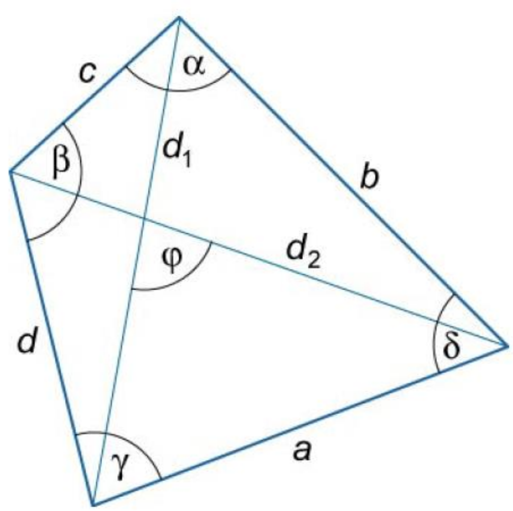


(a) If the line l that passes through the midpoints m_1 and m_2 is perpendicular to both XY and WZ then trapezoid $WXYZ$ is isosceles.

(b) If $WXYZ$ is isosceles then the line l that passes through the midpoints m_1 and m_2, then l is perpendicular to both XY and WZ.

Area of General Convex Quadrilateral

The area of the general quadrilateral



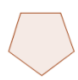
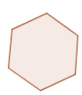
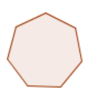
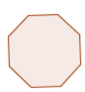
as shown in the figure above can be expressed by

$$\frac{d_1 d_2 \sin(\varphi)}{2}.$$

<u>Recommendation</u> - this result is also useful enough to commit it to memory.

Regular Polygons

Regular Polygon – a polygon which is equiangular (all angles are the same) and equilateral (all sides have the same length).

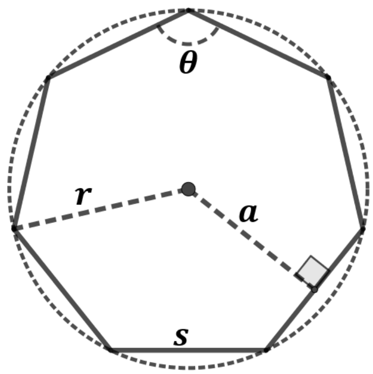
	Equilateral Triangle
	Square
	Regular Pentagon
	Regular Hexagon
	Regular Heptagon
	Regular Octagon

We will denote by $\boldsymbol{s}$ the common length of each side of a regular polygon and we will denote by $\boldsymbol{\theta}$ the common measure of all interior angles of a regular polygon.

Every regular polygon can be circumscribed by a circle. The center of the circle that circumscribes a regular polygon is defined as the **center point** $\boldsymbol{C}$ of that regular polygon.

The **radius** $\boldsymbol{r}$ is the distance from the center point to a vertex of a regular polygon.

The **apothem** $\boldsymbol{a}$ is the perpendicular distance from the center point to a side regular polygon.



Angles in a Regular n-gon

The measure of each angle in a regular n-sided polygon equals

$$\boldsymbol{\theta = \left(\frac{n-2}{n}\right) 180°}.$$

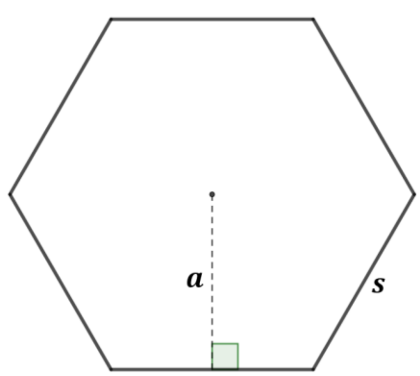
This result follows directly from the study notes for Test 1B where we noted that the degree sum of the n angles in any n-sided polygon equals $(n-2)180°$. So, in a regular n-sided polygon each angle equals $\left(\frac{n-2}{n}\right) 180°$.

Area of a Regular n-gon

The area of a regular polygon with n sides, each of length s, with an apothem of length a can be expressed by

$$\textbf{Area} = \frac{1}{2}(sn)a = \frac{1}{2}Pa$$

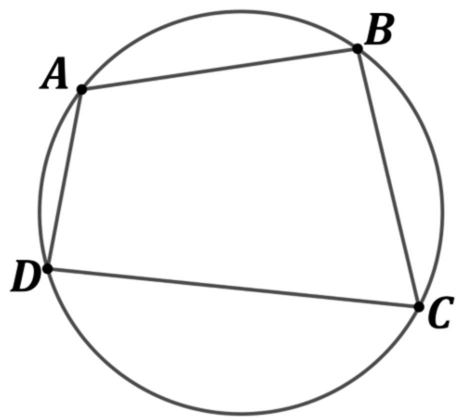
where $P = sn$ is the perimeter of the n-gon



3 Cyclic Quadrilaterals and Ptolemy's Theorem

Cyclic Quadrilateral

A convex quadrilateral is cyclic when its four vertices lie on a circle.

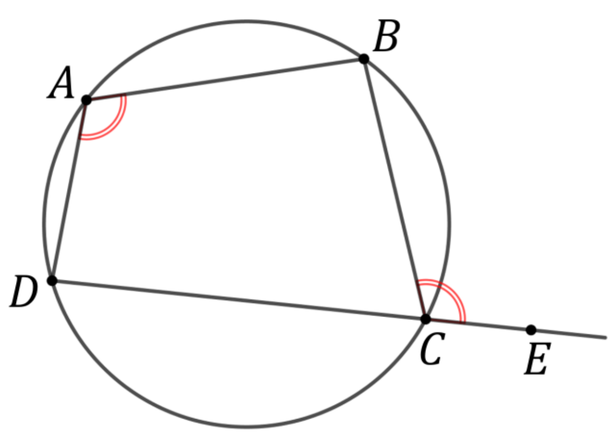


Convex quadrilateral $ABCD$ is cyclic.

Properties

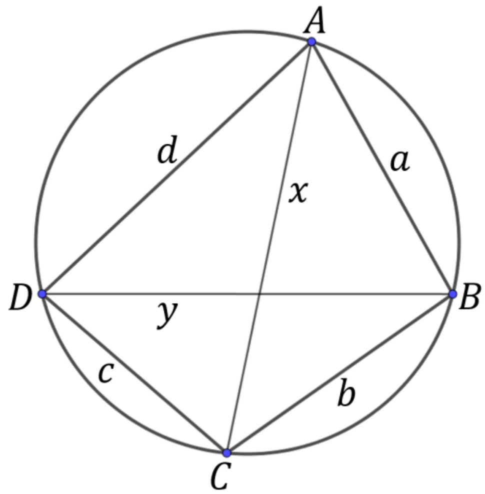
A necessary and sufficient condition for a convex quadrilateral to be cyclic, is that the sum of a pair of opposite angles be equal to $180°$ (*i.e.* either $m\angle A + m\angle C = 180°$ or $m\angle B + m\angle D = 180°$.)

An immediately corollary is that a convex quadrilateral is cyclic if and only if an exterior angle (for example $\angle BCE$) equal to the interior opposite angle (for example $\angle DAB$).



Ptolemy's Theorem

Let x and y be the lengths of the diagonals in a cyclic quadrilateral with sides a, b, c and d.



Then Ptolemy's Theorem states that

$$xy = ac + bd.$$

It is also true that

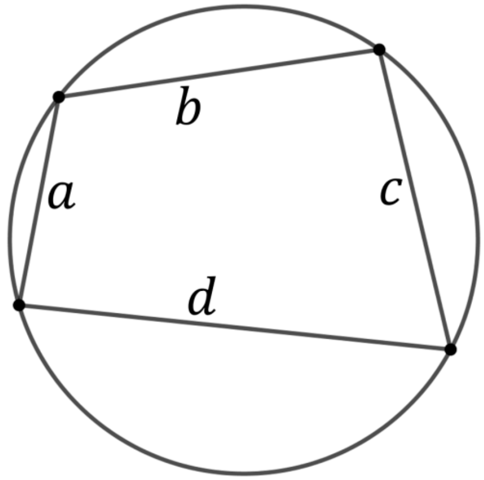
$$x^2 = \frac{(ac+bd)(ad+bc)}{ab+cd} \text{ and } y^2 = \frac{(ac+bd)(ab+cd)}{ad+bc}$$

$$\frac{x}{y} = \frac{ad+bc}{ab+cd}$$

and

$$\cos(\angle ABC) = \frac{a^2+b^2-c^2-d^2}{2(ab+cd)}.$$

Brahamagupta's Formula (Area of a Cyclic Quadrilateral)



The area K of a cyclic convex quadrilateral with sides a, b, c and d equals

$$K = \sqrt{(s-a)(s-b)(s-c)(s-d)}$$

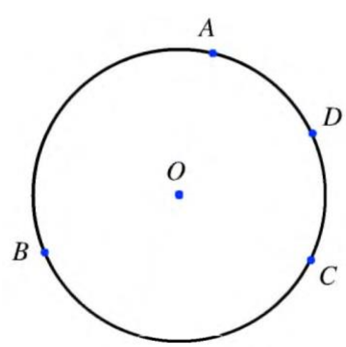
where s is the semiperimeter (half the perimeter) of the quadrilateral. That is,

$$s = \frac{a+b+c+d}{2}.$$

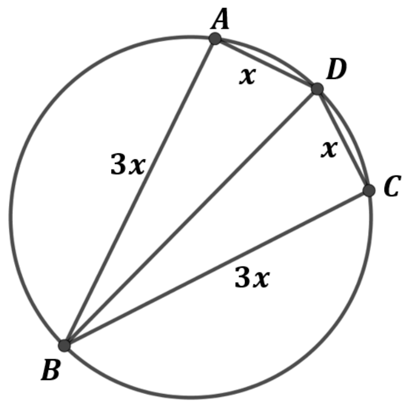
Problems

2016-17, Meet 3B, Problem 3.

Points A, B, C and D are located on circle O as shown in the figure below. If chords $AB = BC = 3CD = 3AD$, determine exactly the ratio $AC\colon BD$.



Solution



First notice that $ADCB$ is a cyclic quadrilateral (all four vertices are on a circle), which implies that $m\angle BAC + m\angle BCD = 180°$.

Next notice that ΔBAD and ΔBCD are congruent triangles by SSS. Therefore $m\angle BAC = m\angle BCD$ and hence both are right angles and ΔBAD and ΔBCD are right triangles. Hence we can solve for BD.

$$BD = \sqrt{x^2 + (3x)^2} = x\sqrt{10}.$$

Let y be the length of the diagonal AC. We can apply Ptolemy's Theorem because this is a cyclic quadrilateral. So we have that

$$AC \cdot BD = AD \cdot BC + DC \cdot AB.$$

That is,

$$y \cdot x\sqrt{10} = x \cdot 3x + x \cdot 3x = 6x^2.$$

or

$$y = \frac{6x}{\sqrt{10}}.$$

The problem asks for

$$\frac{AC}{BD} = \frac{y}{x\sqrt{10}} = \frac{\left(\frac{6x}{\sqrt{10}}\right)}{x\sqrt{10}} = \frac{6x}{10x} = \frac{3}{5}.$$

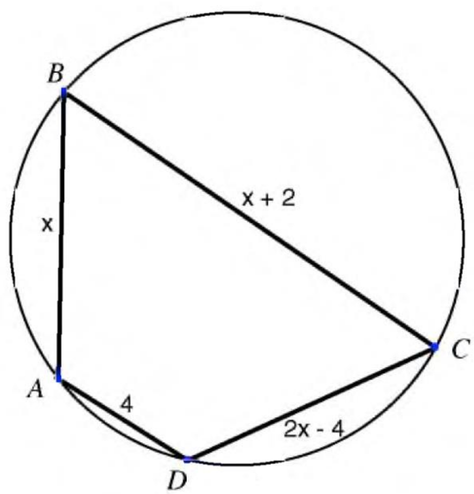
So,

$$AC{:}\,BD = 3{:}\,5.$$

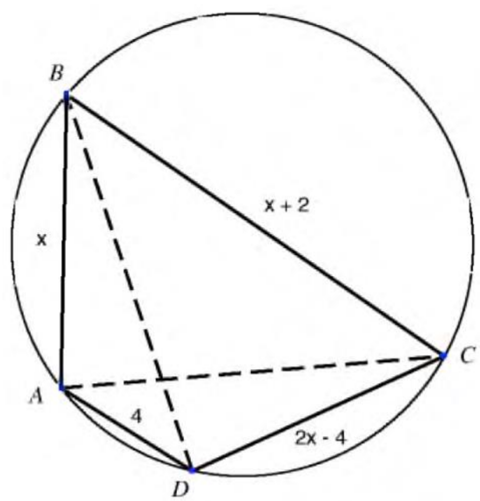
■

2015-16 Meet 3B, Problem 3

Cyclic quadrilateral $ABCD$ is shown in the figure below. If $AC = 6.25$ and $BD = 6.4$, determine exactly the perimeter of quadrilateral $ABCD$.



<u>Solution</u>



By Ptolemy's Theorem,

$$6.25(6.4) = AC \cdot BD = x(2x - 4) + 4(x + 2)$$

or

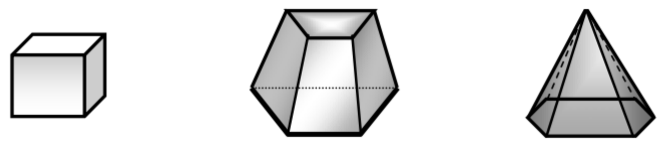
$$40 = 2x^2 + 8.$$

Therefore, $x^2 = 16$ and hence $x = -4$ or 4. But $x = -4$ is not possible so $x = 4$. Therefore, the perimeter must be $4 + 4 + 6 + 4 = 18$.

■

4 Prisms and Pyramids

A polyhedron is a closed solid whose faces are polygons.



In these notes we will consider two classes of polyhedron – prisms and pyramids.

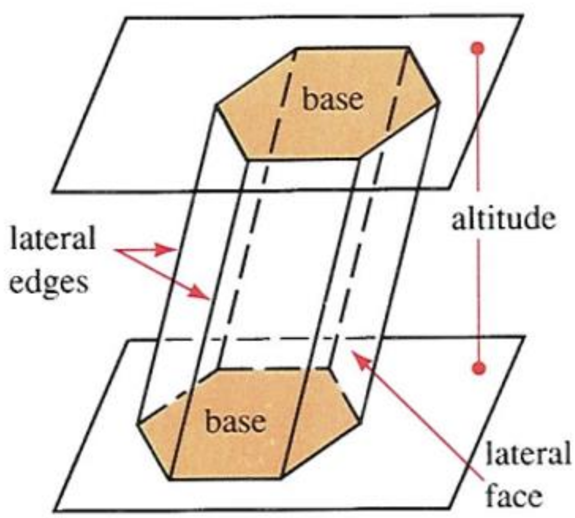
Prisms

A prism is a polyhedron with two congruent faces in parallel planes which are connected with parallelograms.

The top and bottom shaded faces of a prism are called its **bases**. Bases of a prism lie in parallel planes and are congruent polygons.

The faces of a prism that are not bases are called **lateral faces**.

The line segments where the lateral faces intersect each other are called **lateral edges**. Notice that lateral edges are necessarily parallel to each other.



A line segment joining the two base planes of a prism that is perpendicular to both bases is called an **altitude** of that prism.

In a right prism, each lateral edge is an altitude.

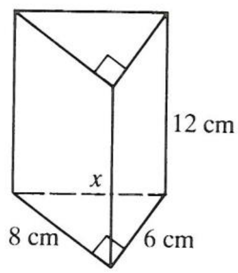
The length of an altitude is the **height**, h, of a prism.

The **lateral area** (L.A.) of a prism is the sum of the areas of its lateral faces.

The **total area** (T.A.) of a prism is the sum of its lateral area and the area of its two bases.

Example 1

Find the lateral area of the right triangular prism.



Solution

First find the value of x:

$x^2 = 6^2 + 8^2 \qquad x = 10$

perimeter of base: $p = 6 + 8 + 10 = 24$

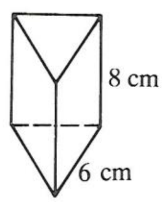
height of prism: $h = 12$

L.A. $= ph = 24 \cdot 12 = 288$

The lateral area is 288 cm^2.

Example 2

Find the total area of the right triangular prism with bases that are equilateral triangles.



Solution

T.A. $=$ L.A. $+ 2B$

$p = 3 \cdot 6 = 18; h = 8$

L.A. $= ph = 18 \cdot 8 = 144$

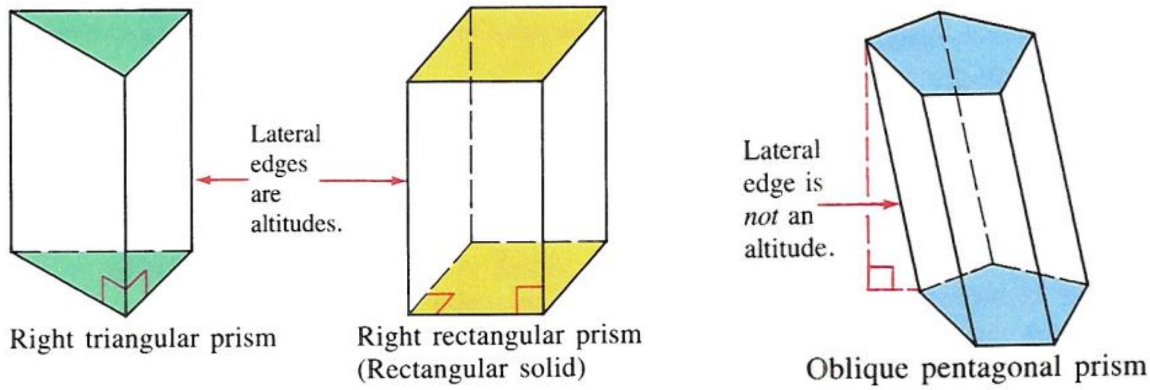
$B = \frac{1}{2}bh = \frac{1}{2} \cdot 6 \cdot 3\sqrt{3} = 9\sqrt{3}$

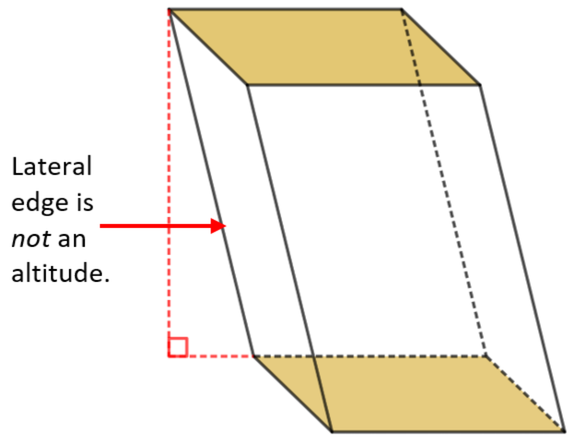
T.A. $=$ L.A. $+ 2B = 144 + 18\sqrt{3}$

The total area is $(144 + 18\sqrt{3})$ cm^2.

The lateral faces of a prism are parallelograms. If they are rectangles (a parallelogram where all four angles are right angles), the prism is a right prism. Otherwise the prism is an oblique prism. In a right prism, the lateral edges are also altitudes.

The diagrams below show that a prism is also classified by the shape of its bases.



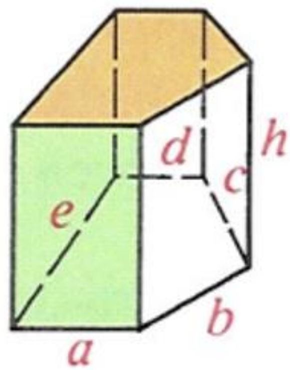


Oblique parallelogram solid
(a parallelepiped)

Lateral Surface Area of a Right Prism

The lateral area of a right prism (all lateral edges are perpendicular to top and bottom base of the prism) equals the perimeter of a base p times the height of the prism (length of a lateral edge) h.

$$\text{Lateral Area} = ph.$$



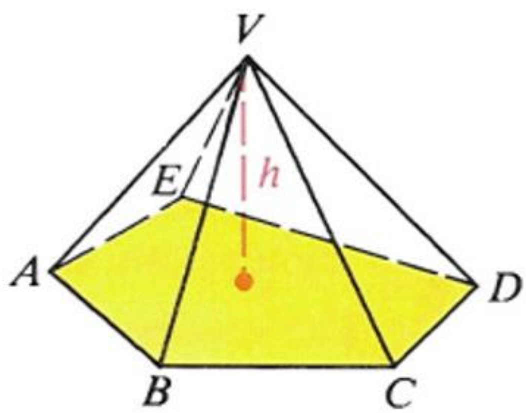
$$\begin{aligned} \text{L.A.} &= ah + bh + ch + dh + eh \\ &= (a + b + c + d + e)h \\ &= \text{perimeter} \cdot h \\ &= ph \end{aligned}$$

Take caution in using this result. It is only true when the prism is a right prism. We take up the case of the lateral surface area of an oblique prism as a "Challenge" at the end of these notes.

Pyramids

A pyramid is a polyhedron with a polygonal base and triangular faces that meet at a common point (called the vertex) not in the plane containing the base.

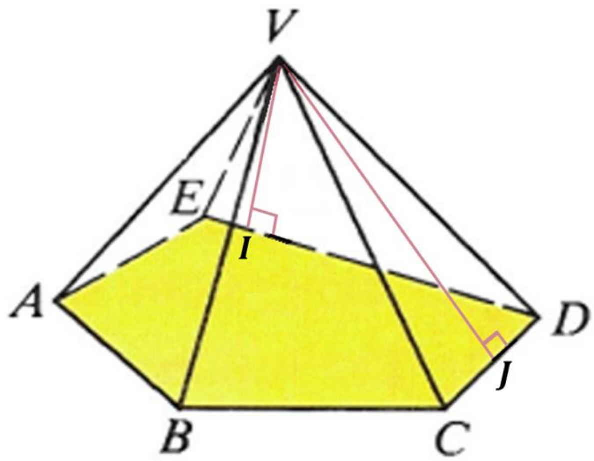
The diagram below shows **pyramid** $V - ABCDE$. Point V is the **vertex** of the pyramid and **base** $ABCDE$. The segment from the vertex perpendicular to the plane containing the base is the **altitude** and its length is the height, h, of the pyramid.



The five triangular faces with V in common, such as ΔVAB, are **lateral faces**. These faces intersect in segments called **lateral edges**. In the above figure, $\overline{VA}$, $\overline{VB}$, … , $\overline{VE}$ are lateral edges.

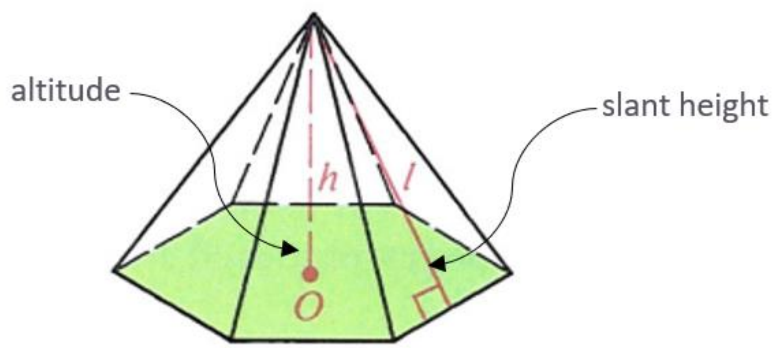
The sides of the base of a pyramid are called **base edges**. In the above figure, $\overline{AB}$, $\overline{BC}$, … , $\overline{EA}$ are base edges.

The **slant height** of a given lateral face is the (perpendicular) distance from the vertex to the base edge of that lateral face.



So, for example, in the above figure, the slant height of lateral face ΔVED is VI, the distance from V to the base edge $\overline{ED}$ and the slant height of lateral face ΔVCD is VJ, the distance from V to the base edge $\overline{CD}$.

Regular Pyramid



Regular hexagonal pyramid

A regular pyramid is a pyramid with the following properties:

(1) The base is a regular polygon.

(2) All lateral edges are congruent.

(3) All lateral faces are congruent isosceles triangles and the **slant height** l of each lateral face is the same. (So it makes sense to talk about ***the*** slant height of a regular pyramid.)

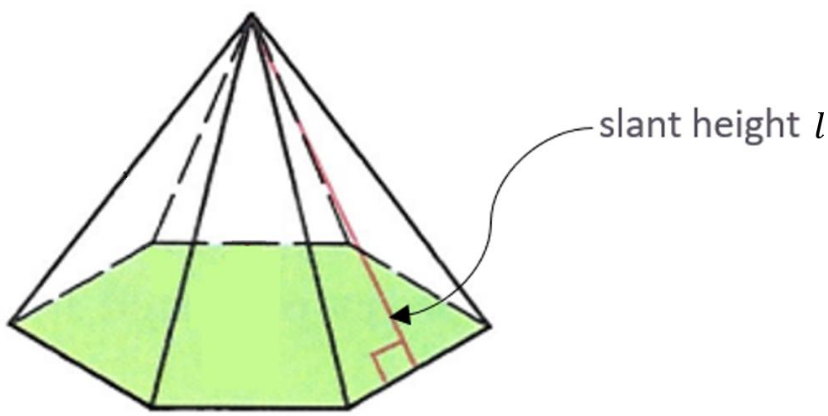
(4) The **altitude** h extends from the vertex to the base and is perpendicular to the base.

The altitude of a regular pyramid meets the base at its center, O.

Lateral Surface Area of a Regular Pyramid

Suppose we have a regular n-sided pyramid with slant height l and base perimeter p. Show that the lateral (just the sides, not the base) surface area equals

$$\frac{pl}{2}.$$



Solution

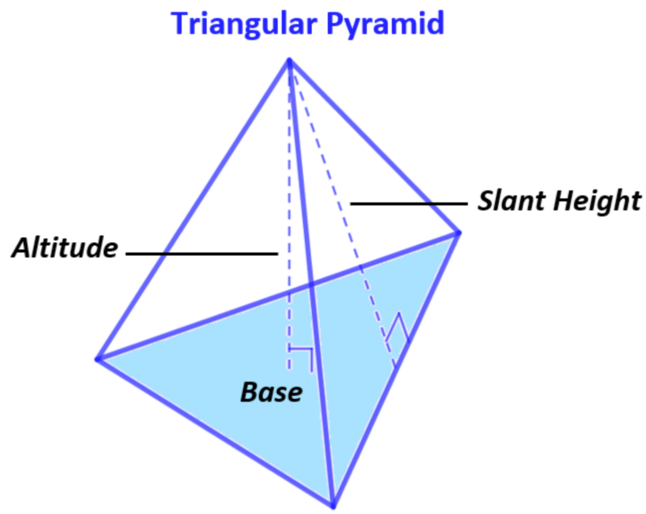
Each of the n lateral faces is an isosceles triangle with height l and because this is a regular pyramid all lateral faces are congruent. If we denote the common base edge length of each of these n triangles by b then the area of each of these triangles is just $(1/2)bl$. Therefore

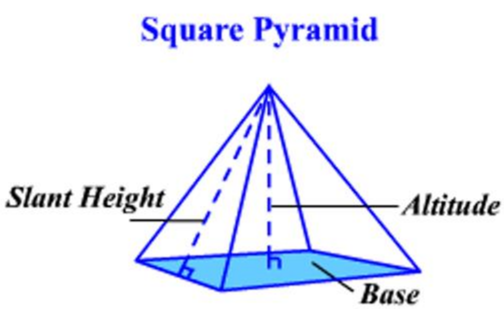
$$\text{pyramid lateral surface area} = n \cdot \left(\frac{1}{2}bl\right) = \frac{(nb)l}{2} = \frac{pl}{2}$$

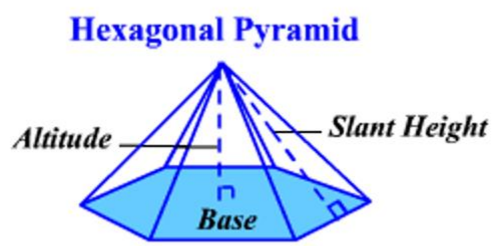
because the perimeter p equals nb.

Classifying Pyramids

Like prisms, pyramids are classified according to the shape of their base.

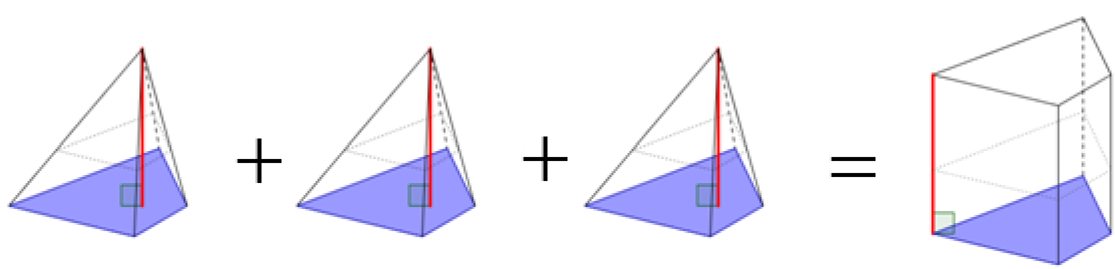




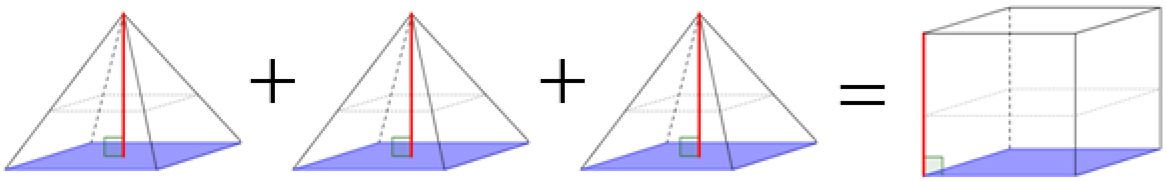


Volume of Prisms and Pyramids

There is an interesting connection between the volume of a prism and a pyramid with the same height and congruent bases.



and



In particular, we have the two formulas

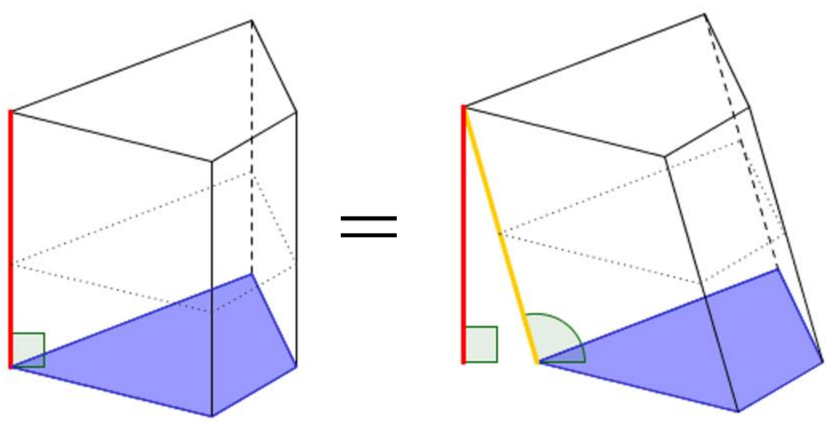
$$\text{Volume}_{\text{Pyramid}} = \frac{1}{3}(\text{Base Area})(\text{Height})$$

$$\text{Volume}_{\text{Prism}} = (\text{Base Area})(\text{Height}).$$

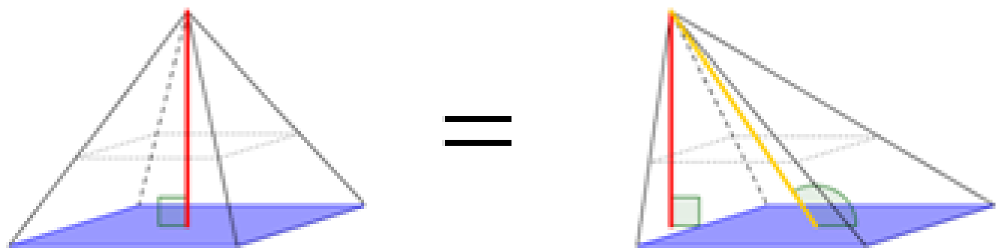
Cavalieri's Principle

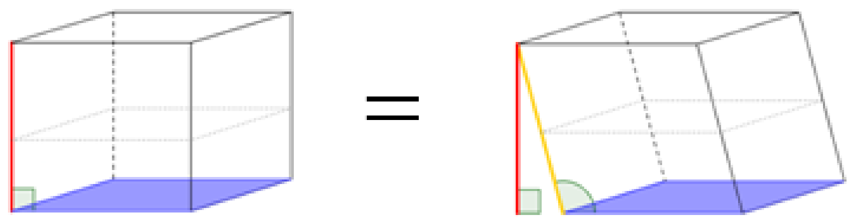
Two solids with the same height and equal cross-sectional area at *all* values of that height will have equal volumes.

It follows from Cavalieri's Principle the following right prism and oblique prism will have equal volumes because they have the same cross sections at each value of their common height.



Other examples of the same principle are



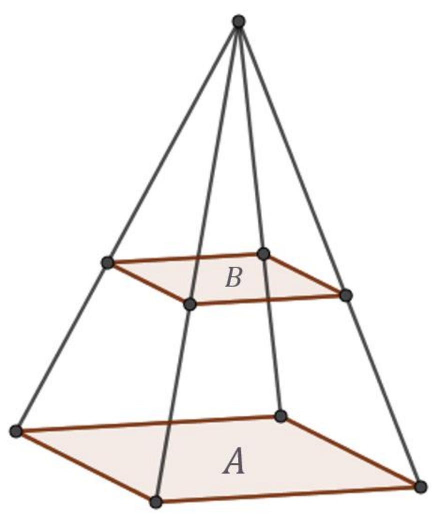


The objects on the left and right have the same volume in each of these cases because they have the same height and the same cross-sectional area at each value of that height.

Cross Sections

Cross sections of prisms are identical (congruent).

However, cross sections of pyramids get smaller and smaller but remain similar (corresponding angles are equal and the ratio of corresponding sides are equal.)



Cross sections A and B of a pyramid are similar objects (all corresponding angles are equal and the ratios of corresponding sides are equal).

Frustums

If you remove the top of a pyramid (which is itself a smaller pyramid) the part that remains is called a **frustum** of that pyramid.

We define the **height of a frustum** as the difference between the height of the full pyramid and the height of the smaller top pyramid that we have removed.

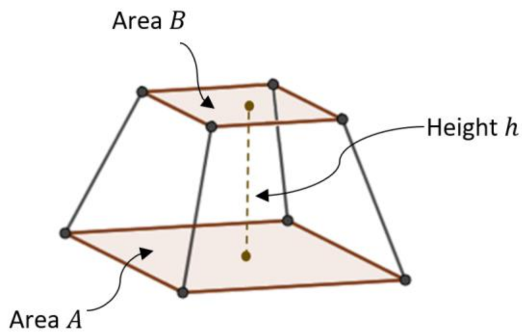
Similarly, we define the **slant height of a lateral face of a frustum** as the difference between the slant height of that lateral face of the full pyramid and the slant height of that same lateral face of just the top pyramid.

Exercise

Show that the formula for the volume of a frustum of a pyramid if the bottom base of that frustum has area A, the top base has area B and if the frustum has height h can be expressed in the form

$$\textbf{Volume of Frustum} = \frac{1}{3}h\left(A + B + \sqrt{AB}\right).$$

Frustum



Solution

If we let y be the height of the original pyramid (before the top is removed), then this original pyramid has volume $(1/3)Ay$. In this case the pyramid that is removed has volume $(1/3)B(y-h)$. Therefore,

$$\text{Volume of Frustum} = \frac{1}{3}Ay - \frac{1}{3}B(y-h).$$

The original pyramid and the top pyramid are similar solids. (Note: Similarity is covered in depth in Test 5B.) We know from the properties of similar objects that the ratio of two-dimensional measures (in particular base area) is the square of the ratio of one-dimensional measures (in particular height). So, we can say, just based on similarity of the two pyramids, that

$$\frac{\text{base area of top pyramid}}{\text{base area of full pyramid}} = \left(\frac{\text{height of top pyramid}}{\text{height of full pyramid}}\right)^2.$$

That is,

$$\frac{B}{A} = \left(\frac{y-h}{y}\right)^2.$$

Solving for y we find, after simplification, that

$$y = h\left(\frac{A+\sqrt{AB}}{A-B}\right).$$

Substituting this value of y into our initial result for the area of a frustum we have

$$\text{Area Frustum} = \frac{1}{3}A\left(h\left(\frac{A+\sqrt{AB}}{A-B}\right)\right) - \frac{1}{3}B\left(\left(h\left(\frac{A+\sqrt{AB}}{A-B}\right)\right) - h\right).$$

After several steps of simplification this reduces to

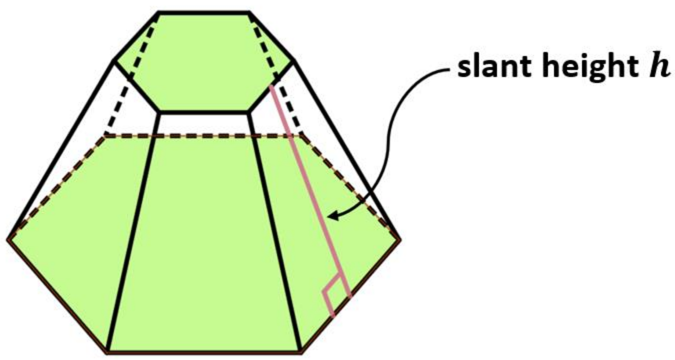
$$\text{Area Frustum} = \frac{1}{3}h\left(A + B + \sqrt{AB}\right).$$

■

Lateral Surface Area of a Frustum of a Regular Pyramid

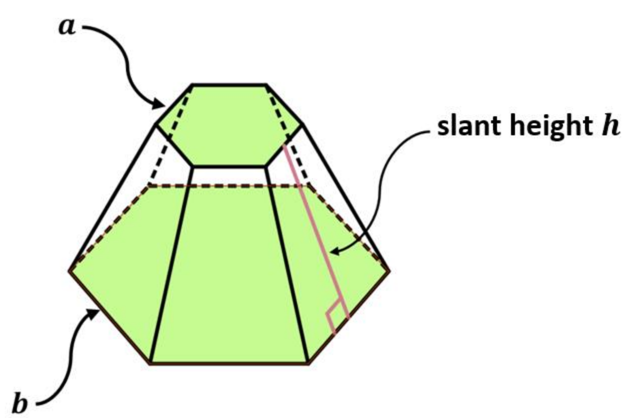
Show that the formula for the lateral surface area (just the sides, not the base) of a frustum of a regular n-sided pyramid where each face of the frustum has slant height h, with base perimeter p_1 and top perimeter p_2 can be expressed in the form

$$\left(\frac{p_1 + p_2}{2}\right)h.$$



Solution

Each of the n congruent faces of this frustum is a trapezoid with height h. Suppose we denote the common base side width of each trapezoid by b and the common top side width by a.



We know from earlier in these same study notes that the area of such a trapezoid equals

$$\left(\frac{a+b}{2}\right)h.$$

Therefore,

$$\text{frustum lateral surface area} = n \cdot \left(\frac{a+b}{2}\right)h = \left(\frac{na+nb}{2}\right)h = \left(\frac{p_1+p_2}{2}\right)h.$$

■

Extra Solved Problems

1. The sides of a rhombus have length 6, and one of the diagonals has length 8. Give the length of the other diagonal in exact terms (not a decimal representation). (Source: MSHSML 3B011)

Solution

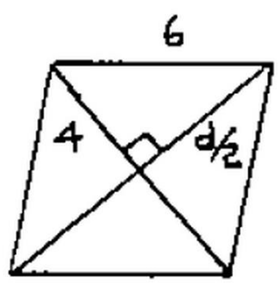


Figure 1

$$4^2 + \left(\frac{d}{2}\right)^2 = 36$$

$$\left(\frac{d}{2}\right)^2 = 20$$

$$d = 4\sqrt{5}$$

■

2. Using the short side of a parallelogram as its base, its height is 4 (Figure 2). If the long side of the parallelogram has length 5 and its long diagonal has length 8, express in exact terms the length of the short side. (Source: MSHSML 3B012)

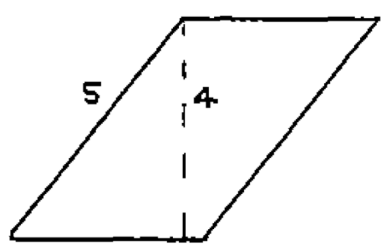


Figure 2

Solution

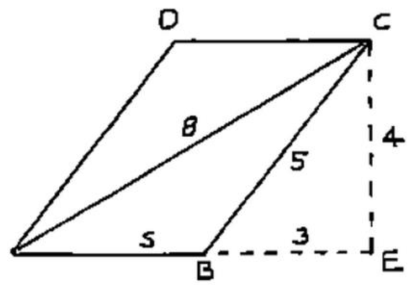


Figure 2

$CE = \frac{1}{2} AC$

$\therefore \angle CAE = 30^\circ$

$AE = CE\sqrt{3} = 4\sqrt{3}$

$AB = 4\sqrt{3} - 3$

■

3. Two 3-4-5 triangles, ΔABC and ΔABD are inscribed in a semi-circle of diameter 5 (Figure 3). What is the length of $\overline{CD}$. (Source: MSHSML 3B013)

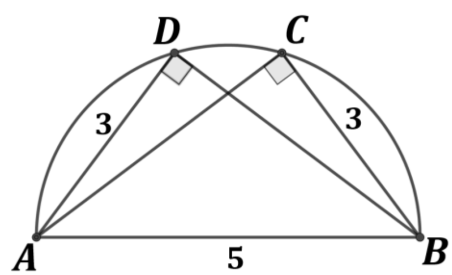
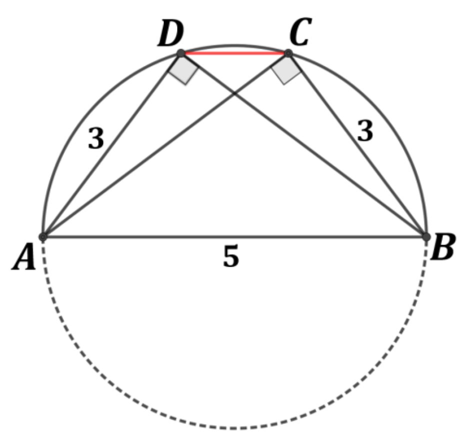


Figure 3

Solution



If we draw in $\overline{CD}$ and the lower semicircle we can see that quadrilateral $ABCD$ is **cyclic** (it's 4 vertices are points on a circle).

Therefore, we can apply Ptolemy's Theorem to this quadrilateral. Ptolemy's Theorem tells us that

$$AC \cdot BD = AB \cdot DC + AD \cdot BC.$$

In addition to the lengths already labeled in the above diagram we also know that $BD = 4$ and $AC = 4$. Filling in all the known values, we are left with

$$4 \cdot 4 = 5 \cdot DC + 3 \cdot 3$$

from which we can solve for DC.

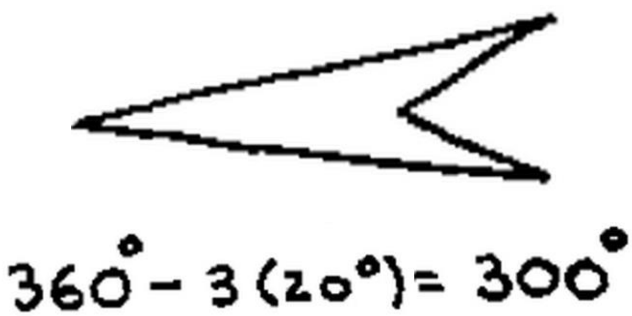
$$5 \cdot DC = 16 - 9 = 7$$

$$\boxed{DC = 7/5.}$$

■

4. What is the largest angle possible in a quadrilateral if its smallest angle is $20°$?
(Source: MSHSML 3B001)

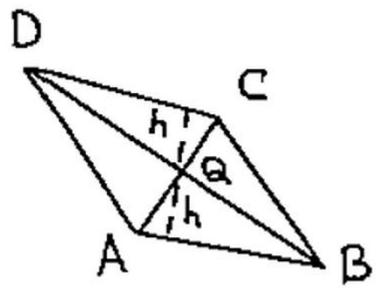
Solution



■

5. The diagonals of a quadrilateral $ABCD$ bisect each other at Q, and have lengths of $AC = 4$ and $BD = 10$. If ΔABQ has an area of 3.5, what is the area of quadrilateral $ABCD$?
(Source: MSHSML 3B004)

Solution



The diagonals of a quadrilateral bisect each other if and only if the quadrilateral is a parallelogram. Furthermore, the diagonals of a parallelogram partition the parallelogram into four triangles of equal area.

Therefore, $\text{Area}(ABCD) = 4(3.5) = 14$.

■

6. A regular hexagon $ABCDEF$ (Figure 3) has sides of length 2. Through its center K, a line segment $\overline{JK}$ of length 1 is drawn parallel to $\overline{AB}$ so that $JK = KL$. How long is $\overline{DL}$? (Source: MSHSML 3B053)

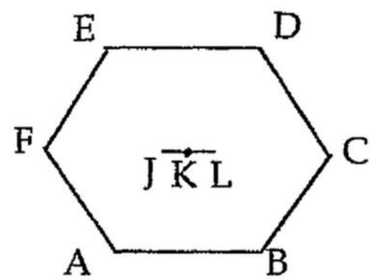
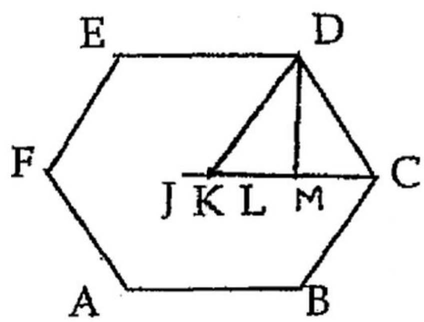


Figure 3

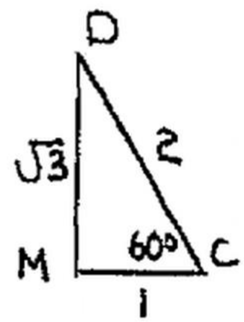
Solution

We are given that $DC = 2$. But in a regular hexagon, the six triangles, of which ΔCDK is one, are all equilateral. Hence $KC = KD = 2$.

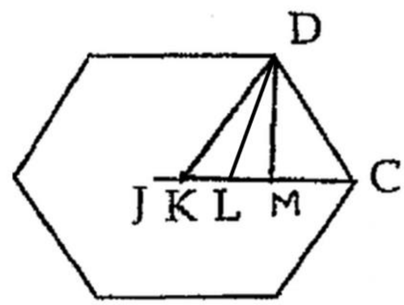
Now drop an altitude from D to $\overline{KC}$ and let M be the point of intersection.



We note that $m\angle DCM = m\angle DCK = 60°$. Therefore ΔDMC is a 30-60-90 triangle with hypotenuse length $DC = 2$. It follows that $MC = 1$ and $DM = \sqrt{3}$.



Therefore $KM = KC - MC = 2 - 1 = 1$. Furthermore we are given that $KL = 1/2$. This implies that $LM = 1/2$.



Thus ΔDLM is a right triangle where $LM = 1/2$ and $DM = \sqrt{3}$. By the Pythagorean Theorem,

$$DL^2 = LM^2 + DM^2 = \frac{1}{4} + 3 = \frac{13}{4}.$$

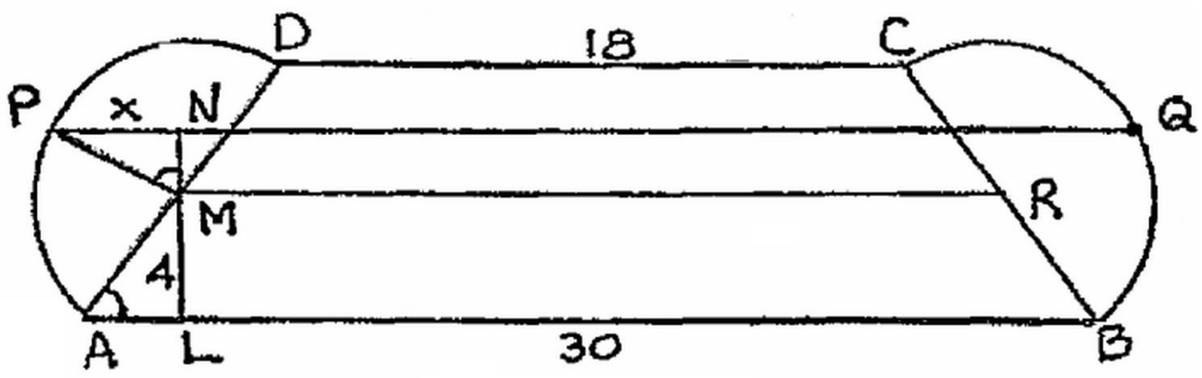
Hence

$$DL = \frac{\sqrt{13}}{2}.$$

■

7. The bases of an isosceles trapezoid are 18 and 30. Its height is 8. Using each leg as a diameter, semi-circles are drawn exterior to the trapezoid, and the midpoints of the arcs of the semicircles are designated P and Q. Find the length of $\overline{PQ}$. (Source: MSHSML 3B054)

Solution



Let M, R be centers of the semi-circles
Through M, erect LN ⊥ AB. $\angle LAM = \angle PMN$
$\therefore$ $\Delta MNP \cong \Delta ALM$, making $PN = ML = 4$
$MR = \frac{18+30}{2} = 24$, so $PQ = 24 + 2(4) = 32$

■

8. The isosceles trapezoid $ABCD$ inscribed in a semicircle (Figure 4) has sides of length 6 and $BC = 14$. How long is the diameter of the semicircle? (Source: MSHSML 3B044)

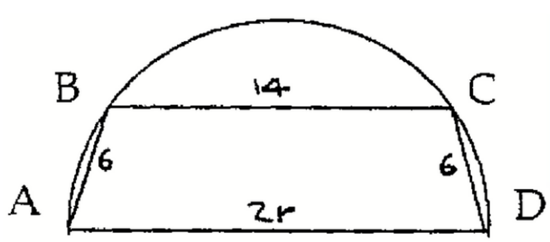


Figure 4

Solution

Let $AD = 2r$.
By Ptolemy's Theorem,
$14(2r) + 6 \cdot 6 = d^2$
ΔABD is a rt. Δ, so
$d^2 + 36 = 4r^2$
$28r + 36 = 4r^2 - 36$
$r^2 - 7r - 18 = 0$
$(r+2)(r-9) = 0$
$r \neq -2$; $r = 9$
diam $= 18$

■

9. The kite shaped quadrilateral $ABCD$ (Figure 1) has $\angle CAD = \angle ACD = 30°$, $\angle CAB = \angle ACB = 60°$, and $AC = 16$. Express the length of $\overline{BD}$ as a single fraction in rationalized form. (Source: MSHSML 3B021)

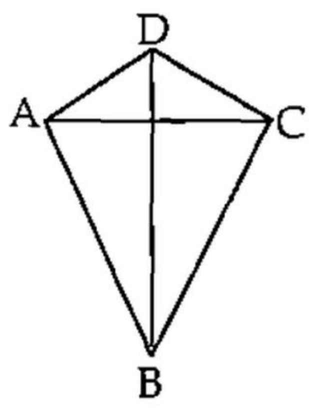


Figure 1

Solution

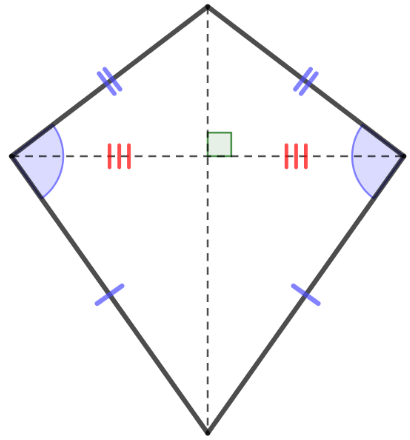
Kite

Two pairs of adjacent sides are equal in length.

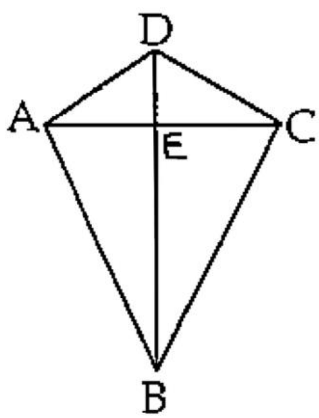
One pair of opposite angles (the ones that are between the sides of unequal length) are equal in size.

The longer diagonal bisects the shorter diagonal.

Diagonals intersect at right angles.



Let E be the point of intersection of the two diagonals.



Then by the properties of a kite stated above we know the following facts:

$$AD = CD,\ \ AB = CB,\ \ AE = CE,\ \ AC \perp BD,\ \ \angle DAB = \angle DCB.$$

From these properties we have that $\Delta AED \cong \Delta CED$ (these two triangles are congruent) by the SSS (side-side-side) criteria. In the same criteria, $\Delta AEB \cong \Delta CEB$.

We are also given the information that $\angle CAD = \angle ACD = 30°$, $\angle CAB = \angle ACB = 60°$, and $AC = 16$.

Hence, $AE = 16/2 = 8$, ΔADE and ΔBAE are 30-60-90 triangles. By the properties of a 30-60-90 triangle, we can determine that $DE = AE/\sqrt{3} = 8/\sqrt{3}$ and $BE = AE\sqrt{3} = 8\sqrt{3}$.

Therefore,

$$\begin{aligned} BD &= DE + BE \\ &= \frac{8}{\sqrt{3}} + 8\sqrt{3} \\ &= \frac{8\sqrt{3} + 24\sqrt{3}}{3} \\ &= \frac{32\sqrt{3}}{3}. \end{aligned}$$

■