MSHSML Meet 3, Event C
Study Guide

3C Trigonometry

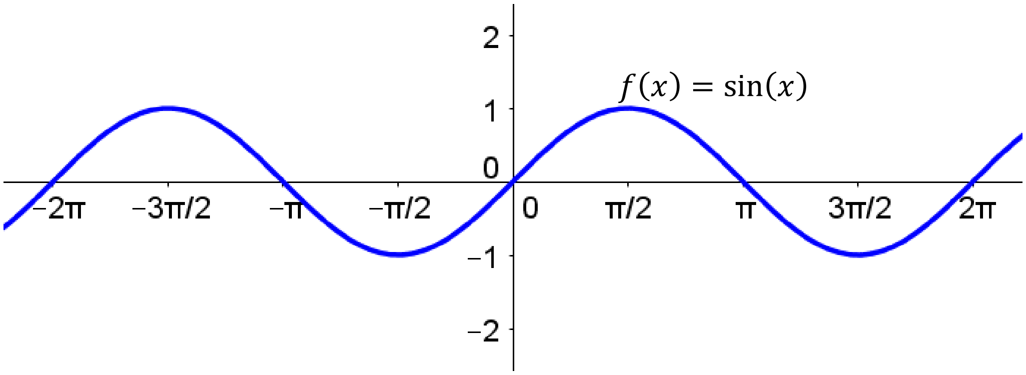
Law of sines, law of cosines
Inverse functions and their graphs
Solving trigonometric equations
De Moivre's Theorem and the roots of unity

Contents

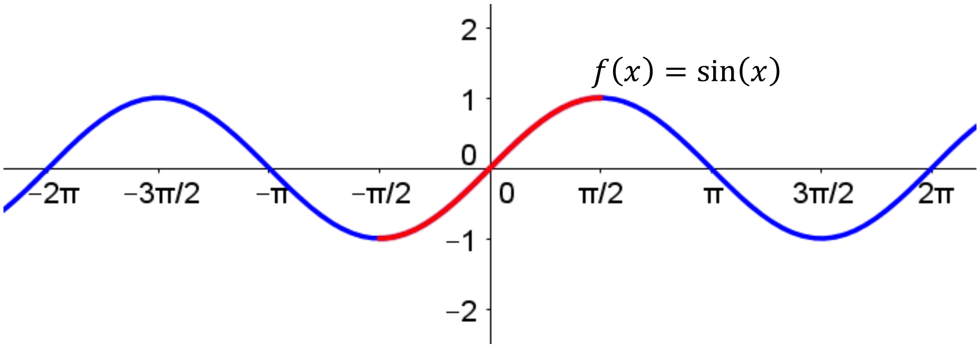
1 Inverse Trig Functions and Their Graphs

1.1 Inverse Sine Function

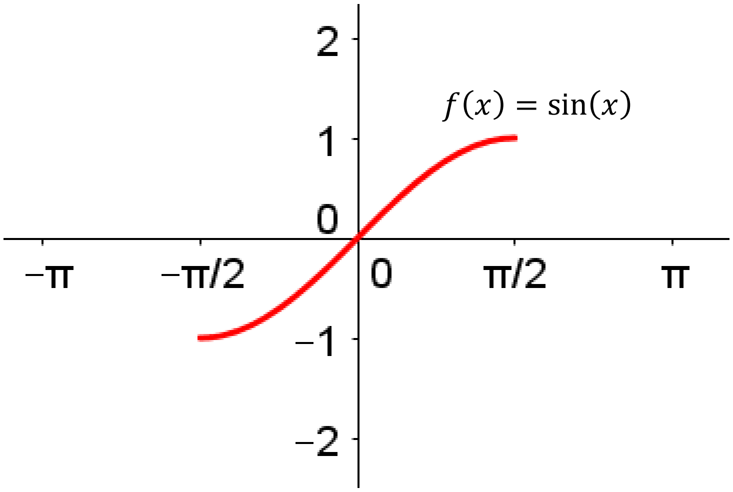
Clearly the function $f(x) = \sin(x)$ does not pass the horizontal line test over $(-\infty, \infty)$.



But if we only look at the part of $\sin(x)$ for $x \in [-\pi/2, \pi/2]$ (shown in red)



that is, the part



then $\sin(x)$ does pass the horizontal line test. So we can say that $\sin(x)$ has an inverse for $x \in [-\pi/2, \pi/2]$.

We use the notation $\sin^{-1}(x)$ to denote this inverse. Be careful not to confuse this with the reciprocal of $\sin(x)$ which is denoted by $(\sin(x))^{-1}$.

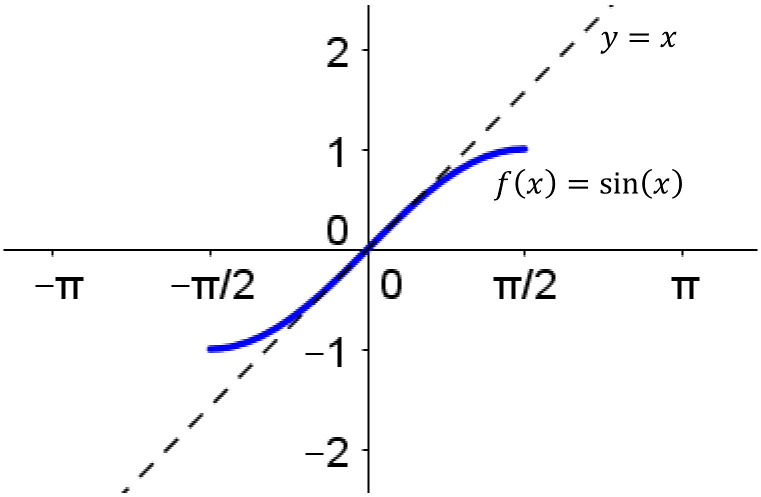
In particular,

$$(\sin(x))^{-1} = \frac{1}{\sin(x)} = \csc(x) \neq \sin^{-1}(x).$$

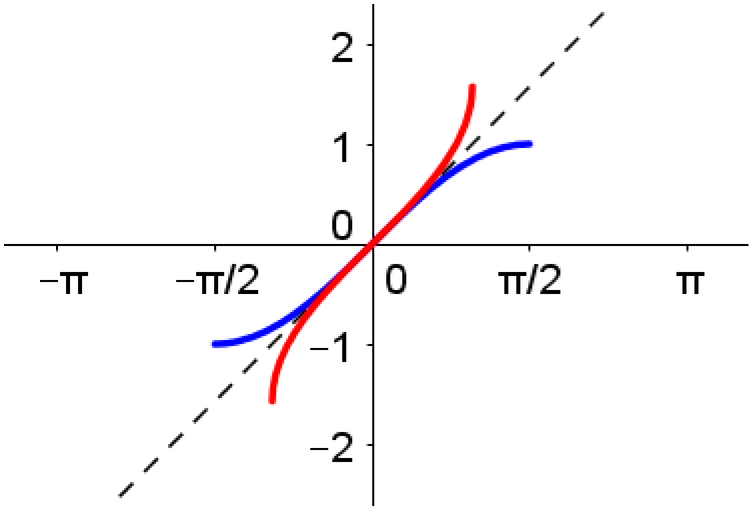
Note: Another name for the inverse sine function is **arc sine**. We can use the two terms inverse sine and arc sine interchangeably. Two terms for the same thing.

1.1.1 Graph of the Inverse Sine Function

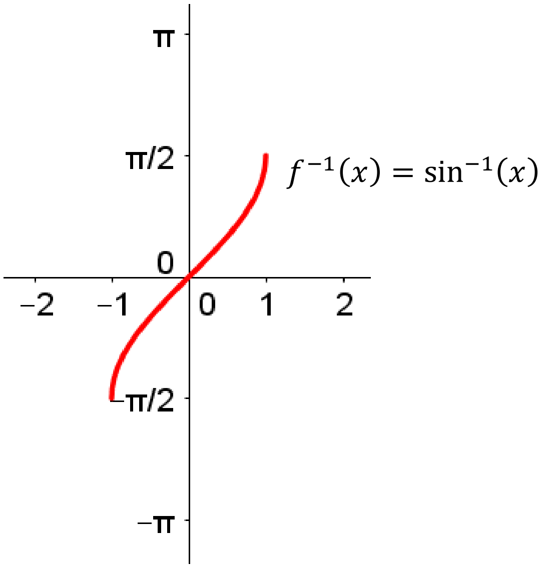
What does the graph of $\sin^{-1}(x)$ look like? Recall that we can find the graph of $f^{-1}(x)$ by reflecting the graph of $f(x)$ over the line $y = x$.



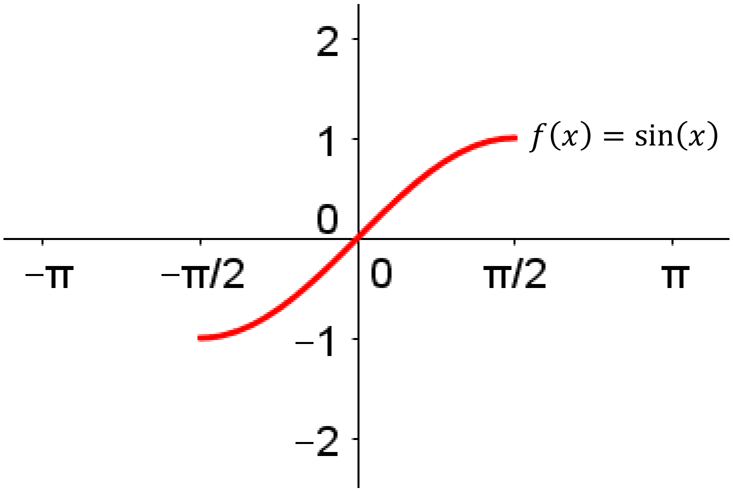
That is, to find the graph of $\sin^{-1}(x)$ we need to reflect $\sin(x)$, shown in blue above, over the dotted line $y = x$. We see that what we will get is



So the graph of $f^{-1}(x) = \sin^{-1}(x)$ looks like



Let's contrast this graph with the graph of $f(x) = \sin(x)$.



We can see that the domain of $f(x) = \sin(x)$, namely $[-\pi/2, \pi/2]$, is the range of $f^{-1}(x) = \sin^{-1}(x)$ and the range of $f(x) = \sin(x)$, namely $[-1,1]$, is the domain of $f^{-1}(x) = \sin^{-1}(x)$.

This is a general property of functions and their inverses. The domain of $f(x)$ is the range of $f^{-1}(x)$ and the range of $f(x)$ is the domain of $f^{-1}(x)$.

1.1.2 Calculating the Inverse Sine Function

There is no convenient formula for calculating $\sin^{-1}(x)$. To calculate $\sin^{-1}(x)$ we have to work backwards from the definition of an inverse function. We know (by definition of an inverse function) that

$$f(a) = b \iff f^{-1}(b) = a.$$

In our case this means that

$$\sin(a) = b \iff \sin^{-1}(b) = a.$$

So, for example, we know that

$$\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$$

because

$$\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}.$$

In general, to find the number c such that $\sin^{-1}(x) = c$ we have to think backwards and ask ourselves, “What is the number c such that $\sin(c) = x$?”.

For any $x \in [-1,1]$,

$$\sin^{-1}(x) = c \text{ provided } \sin(c) = x.$$

For any $x \notin [-1,1]$,

$$\sin^{-1}(x) \text{ is not defined.}$$

1.1.3 Cancellation Properties of the Inverse Sine Function

From the general properties of inverse functions, we have the following **cancellation properties**.

$$\sin^{-1}(\sin(x)) = \begin{cases} x & x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \\ x^* & x \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \end{cases}$$

where x^* is that unique number such that $\sin(x) = \sin(x^*)$ and $x^* \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

and

$$\sin(\sin^{-1}(x)) = \begin{cases} x & x \in [-1,1] \\ \text{undefined} & x \notin [-1,1] \end{cases}$$

Exercise 1. Simplify.

(a) $\sin^{-1}(1)$ (b) $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ (c) $\sin^{-1}(2)$

Solution

(a) $\sin^{-1} 1 = \frac{\pi}{2}$ because $\sin \frac{\pi}{2} = 1$ and $\frac{\pi}{2}$ lies in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

(b) $\sin^{-1} \frac{\sqrt{3}}{2} = \frac{\pi}{3}$ because $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$ and $\frac{\pi}{3}$ lies in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

(c) $\sin^{-1} 2$ is undefined because there is no real number x such that $\sin x = 2$.

■

Exercise 2. Simplify.

(a) $\sin^{-1}(-1)$ (b) $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$ (c) $\sin^{-1}(-2)$

Solution

(a) $\sin^{-1}(-1) = -\frac{\pi}{2}$ because $\sin\left(-\frac{\pi}{2}\right) = -1$ and $-\frac{\pi}{2}$ lies in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(b) $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$ because $\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$ and $\frac{\pi}{4}$ lies in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(c) $\sin^{-1}(-2)$ is undefined because there is no real number x such that $\sin(x) = -2$.

■

Exercise 3. Simplify.

$$\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$$

Solution

$\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$ because $\sin\left(-\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$ and $-\frac{\pi}{4}$ lies in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

■

Exercise 4. Simplify.

(a) $\sin^{-1}(0)$ (b) $\sin^{-1}\left(-\frac{1}{2}\right)$

Solution

(a) $\sin^{-1}(0) = 0$ because $\sin(0) = 0$ and 0 lies in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(b) $\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$ because $\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$ and $-\frac{\pi}{6}$ lies in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

■

Exercise 5. Simplify.

(a) $\sin\left(\sin^{-1}\left(\frac{1}{4}\right)\right)$

(b) $\sin\left(\sin^{-1}\left(\frac{3}{2}\right)\right)$

(c) $\sin\left(\sin^{-1}\left(-\frac{3}{4}\right)\right)$

Solution

Recall that $\sin(\sin^{-1}(x)) = \begin{cases} x & x \in [-1,1] \\ \text{undefined} & x \notin [-1,1] \end{cases}$

So,

(a) $\sin(\sin^{-1}(1/4)) = \frac{1}{4}$ because $\frac{1}{4}$ lies in $[-1,1]$

(b) $\sin(\sin^{-1}(3/2))$ is undefined because $\frac{3}{2}$ does not lie in $[-1,1]$

(c) $\sin(\sin^{-1}(-3/4)) = \frac{-3}{4}$ because $\frac{-3}{4}$ lies in $[-1,1]$

■

Exercise 6. Simplify.

(a) $\sin^{-1}\left(\sin\left(\frac{\pi}{4}\right)\right)$

(b) $\sin^{-1}\left(\sin\left(\frac{3\pi}{4}\right)\right)$

(c) $\sin^{-1}\left(\sin\left(\frac{7\pi}{6}\right)\right)$

Solution

(a) Recall that

$$\sin^{-1}(\sin(x)) = \begin{cases} x & x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \\ x^* & x \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \end{cases}$$

where x^* is that unique number such that $\sin(x) = \sin(x^*)$ and $x^* \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. So,

$$\sin^{-1}\left(\sin\left(\frac{\pi}{4}\right)\right) = \frac{\pi}{4} \text{ because } \frac{\pi}{4} \text{ lies in } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

(b)

$$\frac{3\pi}{4} \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \quad \text{so} \quad \sin^{-1}\left(\sin\left(\frac{3\pi}{4}\right)\right) \neq \frac{3\pi}{4}$$

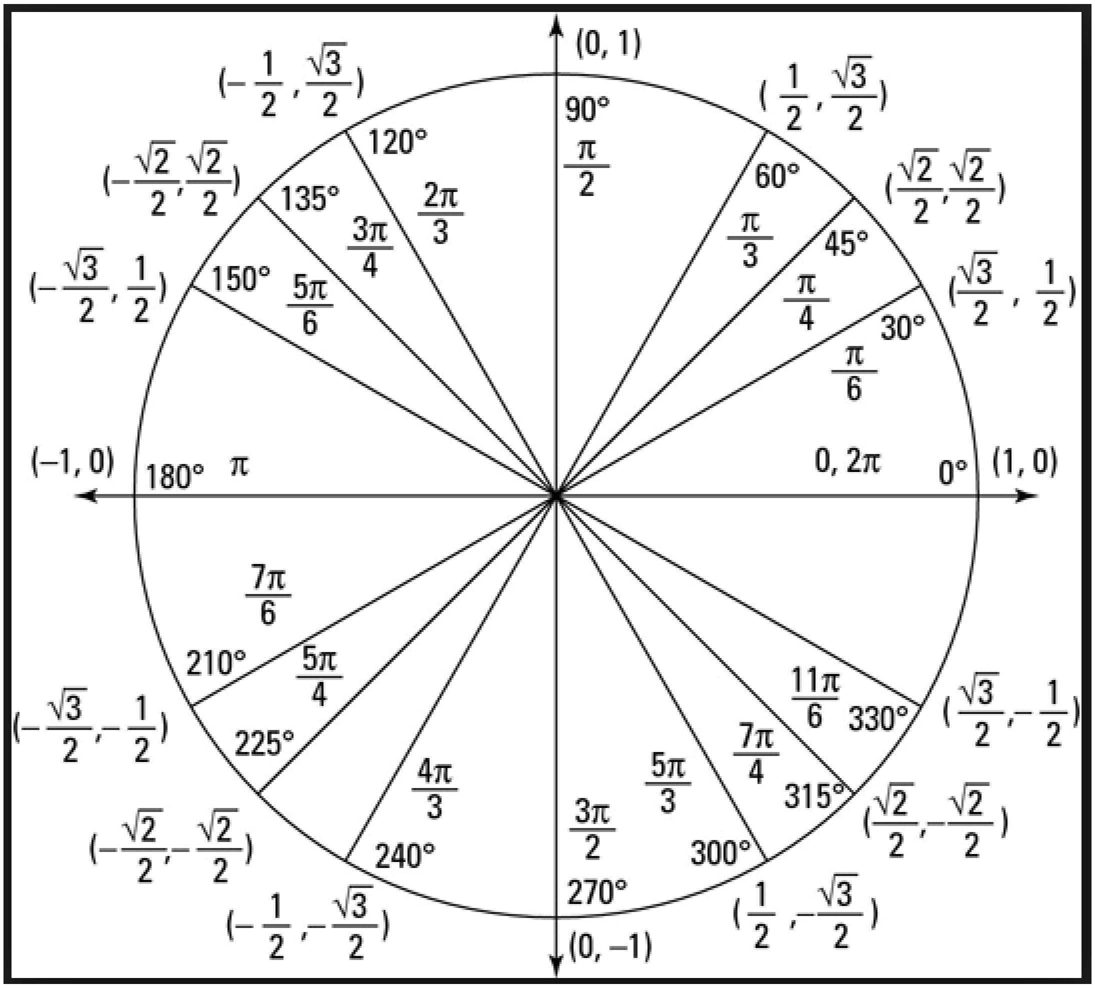
but rather

$$\sin^{-1}\left(\sin\left(\frac{3\pi}{4}\right)\right) = x^*$$

where x^* in the unique value in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that

$$\sin(x^*) = \sin\left(\frac{3\pi}{4}\right).$$

From our unit circle chart, we can see that the terminal point associated with $t = 3\pi/4$ is $\left(-\sqrt{2}/2, \sqrt{2}/2\right)$. Also remember that $\sin(t)$ equals the y-coordinate of the terminal point for t.



So, $\sin(3\pi/4) = \sqrt{2}/2$. What value of t in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ gives us $\sin(t) = \sqrt{2}/2$? From our unit circle chart we can see that $\sin(\pi/4) = \sqrt{2}/2 = \sin(3\pi/4)$ and it is also true that $\pi/4$ is in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

So,

$$\sin^{-1}\left(\sin\left(\frac{3\pi}{4}\right)\right) = \sin^{-1}\left(\sin\left(\frac{\pi}{4}\right)\right) = \frac{\pi}{4}.$$

(c)

$$\frac{7\pi}{6} \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \quad \text{so} \quad \sin^{-1}\left(\sin\left(\frac{7\pi}{6}\right)\right) \neq \frac{7\pi}{6}$$

but rather

$$\sin^{-1}\left(\sin\left(\frac{7\pi}{6}\right)\right) = x^*$$

where x^* in the unique value in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that

$$\sin(x^*) = \sin\left(\frac{7\pi}{6}\right).$$

From our unit circle chart the terminal point of $t = 7\pi/6$ has coordinates $(x, y) = \left(-\sqrt{3}/2, -1/2\right)$ and $\sin(t)$ is defined as the y – coordinate of the terminal point of t. So

$$\sin\left(\frac{7\pi}{6}\right) = -\frac{1}{2}.$$

We need to find that unique value x^* such that $x^* \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and

$$\sin(x^*) = \sin\left(\frac{7\pi}{6}\right) = -\frac{1}{2}.$$

From our unit circle chart we can see that

$$\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$$

and it is also true that

$$-\frac{\pi}{6} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

So,

$$\sin^{-1}\left(\sin\left(\frac{7\pi}{6}\right)\right) = \sin^{-1}\left(\sin\left(-\frac{\pi}{6}\right)\right) = -\frac{\pi}{6}.$$

Exercise 7. Simplify.

$$\tan\left(\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)\right)$$

Solution

Recall that we showed in Exercise 2b that

$$\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4} \text{ because } \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \text{ and } \frac{\pi}{4} \text{ lies in } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

So

$$\tan\left(\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)\right) = \tan\left(\frac{\pi}{4}\right).$$

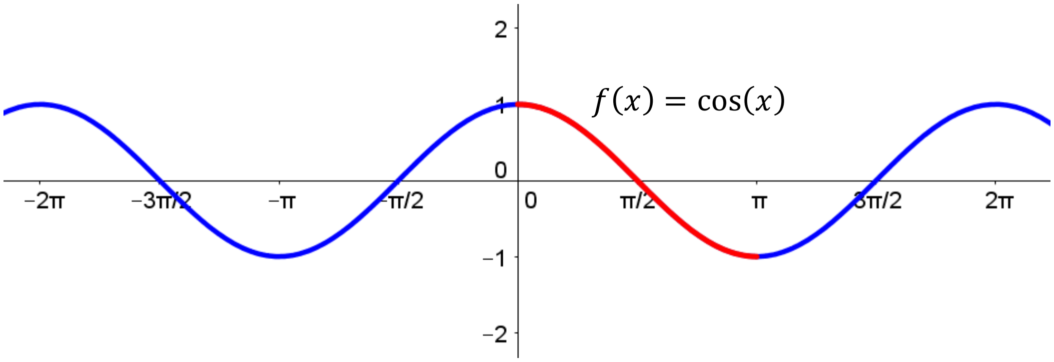
Recall that if the terminal point of t has coordinates (x, y), then $\tan(t) = y/x$. From our unit circle chart we see that the terminal point of $t = \pi/4$ has coordinates $(x, y) = (\sqrt{2}/2, \sqrt{2}/2)$. So

$$\tan\left(\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)\right) = \tan\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}/2}{\sqrt{2}/2} = 1.$$

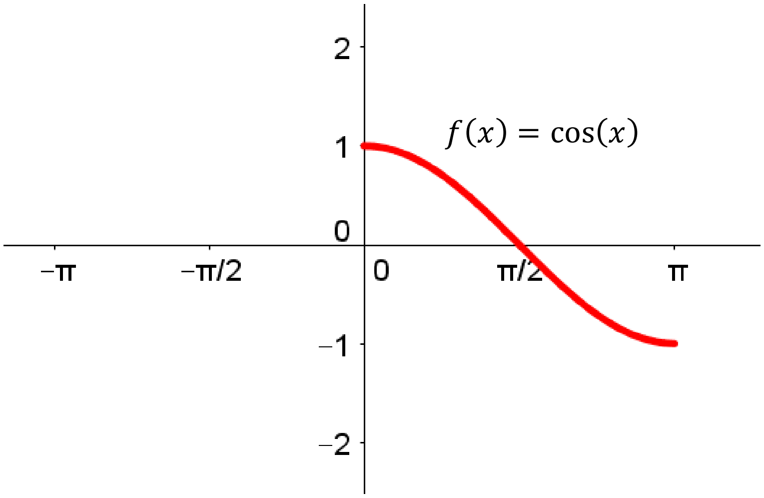
■

1.2 Inverse Cosine Function

If only look at the part of $\cos(x)$ for $x \in [0, \pi]$ (shown in red)



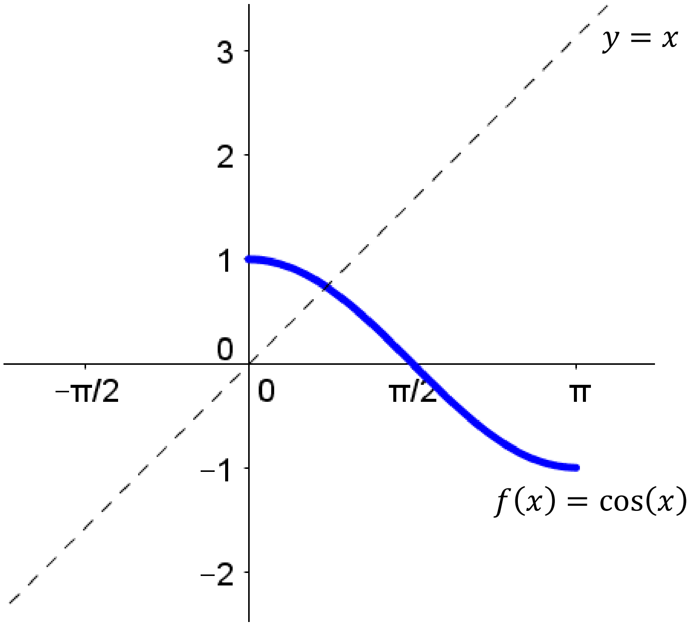
that is, the part



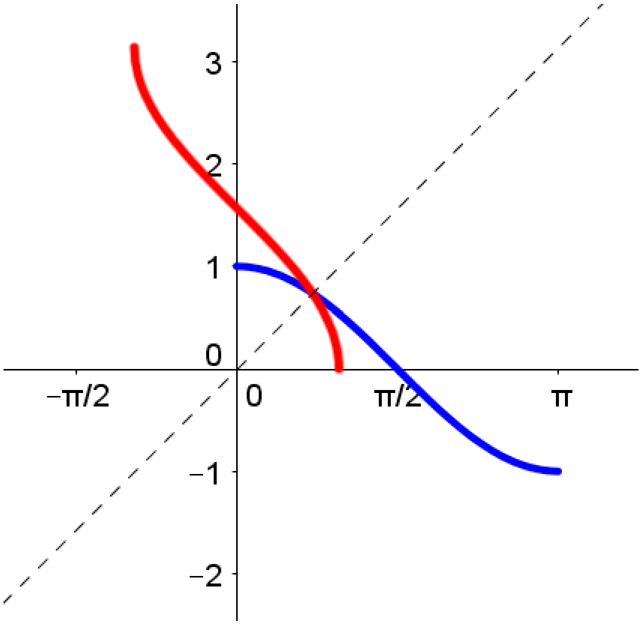
then $\cos(x)$ does pass the horizontal line test. So we can say that $\cos(x)$ has an inverse for $x \in [0, \pi]$.

1.2.1 Graph of the Inverse Cosine Function

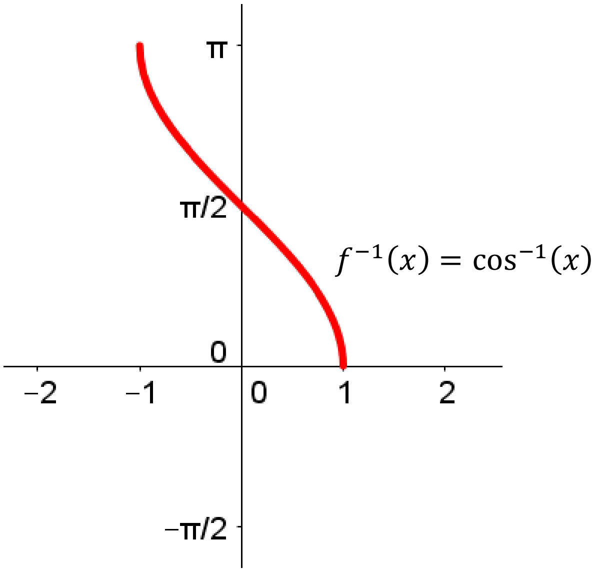
What does the graph of $\cos^{-1}(x)$ look like? Reflect the graph of $\cos(x)$ over the line $y = x$.



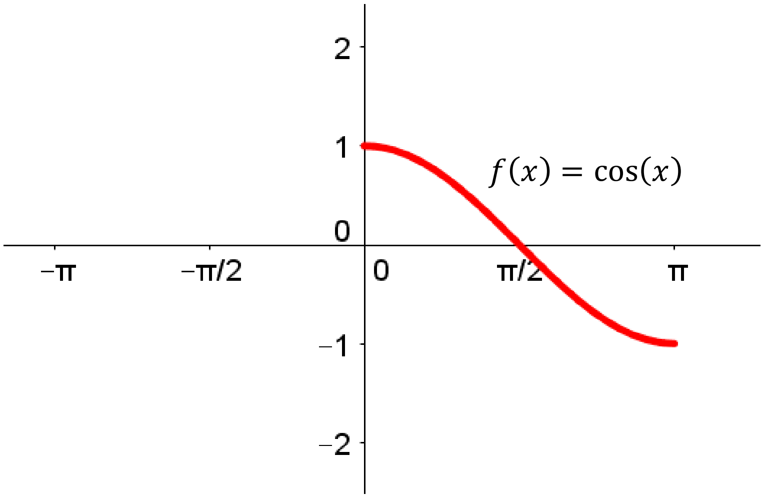
We see that what we will get is



So, the graph of $f^{-1}(x) = \cos^{-1}(x)$ looks like



Let's contrast this graph with the graph of $f(x) = \cos(x)$.



We can see that the domain of $f(x) = \cos(x)$, namely $[0, \pi]$, is the range of $f^{-1}(x) = \cos^{-1}(x)$ and the range of $f(x) = \cos(x)$, namely $[-1,1]$, is the domain of $f^{-1}(x) = \cos^{-1}(x)$.

1.2.2 Calculating the Inverse Cosine Function

By the meaning of an inverse function,

$$\cos(a) = b \iff \cos^{-1}(b) = a.$$

In general, to find the number c such that $\cos^{-1}(x) = c$ we have to think backwards and ask ourselves, “What is the number c such that $\cos(c) = x$?”.

For any $x \in [-1,1]$,

$$\cos^{-1}(x) = c \text{ provided } \cos(c) = x.$$

For any $x \notin [-1,1]$,

$$\cos^{-1}(x) \text{ is not defined.}$$

1.2.3 Cancellation Properties of the Inverse Cosine Function

From the general properties of inverse functions, we have the following **cancellation properties**.

$$\cos^{-1}(\cos(x)) = \begin{cases} x & x \in [0,\pi] \\ x^* & x \notin [0,\pi] \end{cases}$$

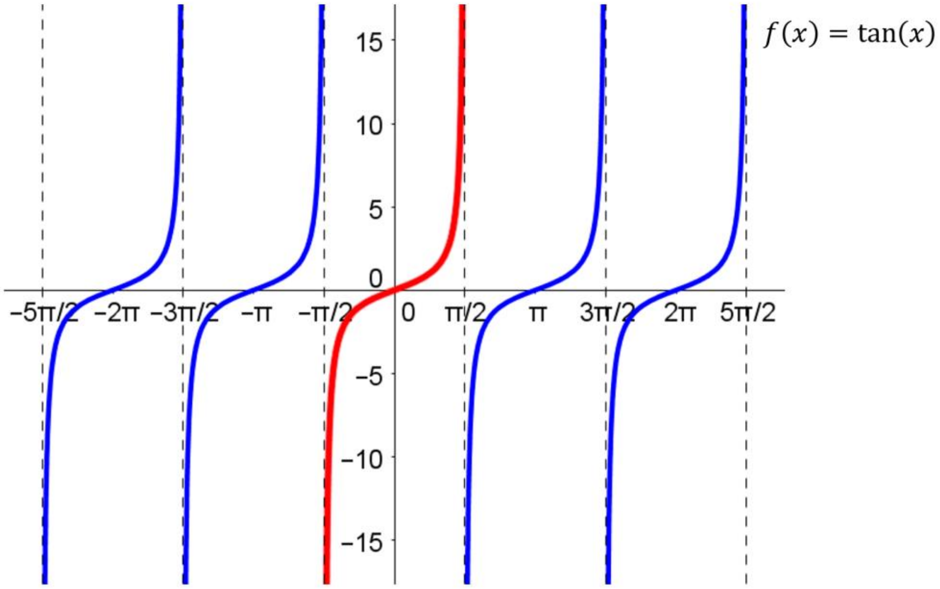
where x^* is that unique number such that $\cos(x) = \cos(x^*)$ and $x^* \in [0,\pi]$

and

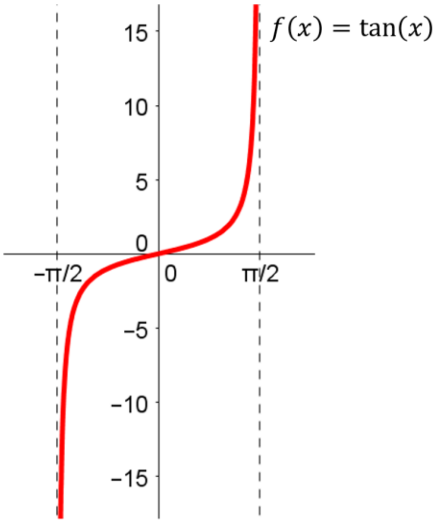
$$\cos(\cos^{-1}(x)) = \begin{cases} x & x \in [-1,1] \\ \text{undefined} & x \notin [-1,1]. \end{cases}$$

1.3 Inverse Tangent Function

If only look at the part of $\tan(x)$ for $x \in [-\pi/2, \pi/2]$ (shown in red)



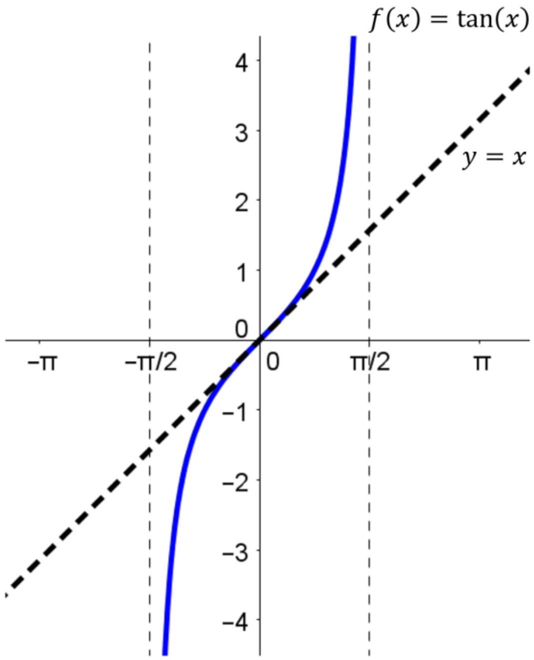
that is, the part



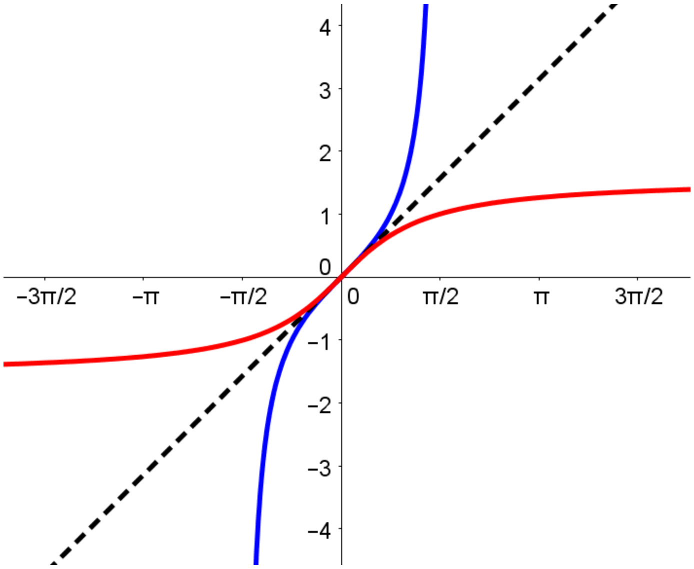
then $\tan(x)$ does pass the horizontal line test. So we can say that $\tan(x)$ has an inverse for $x \in [-\pi/2, \pi/2]$.

1.3.1 Graph of the Inverse Tangent Function

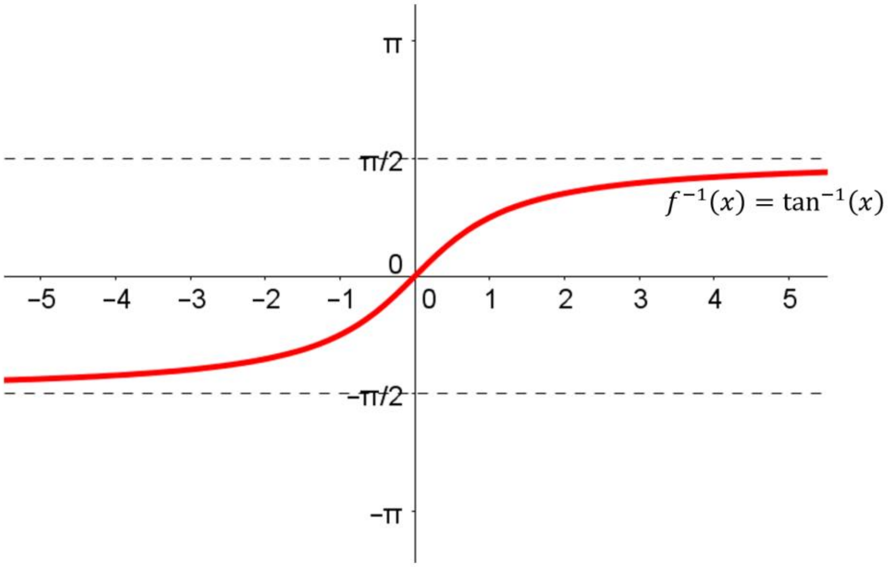
What does the graph of $\tan^{-1}(x)$ look like? Reflect the graph of $\tan(x)$ over the line $y = x$.



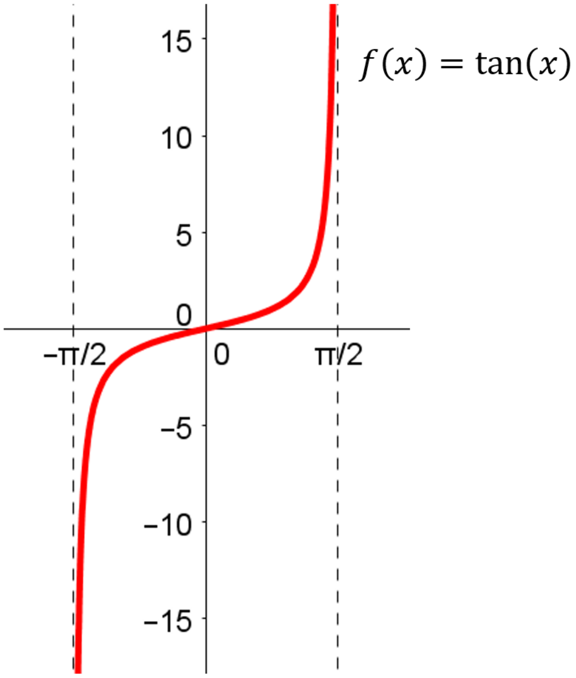
We see that what we will get is



So, the graph of $f^{-1}(x) = \tan^{-1}(x)$ looks like



Let's contrast this graph with the graph of $f(x) = \tan(x)$.



We can see that the domain of $f(x) = \tan(x)$, namely $[-\pi/2, \pi/2]$, is the range of $f^{-1}(x) = \tan^{-1}(x)$ and the range of $f(x) = \tan(x)$, namely $[-\infty, \infty]$, is the domain of $f^{-1}(x) = \tan^{-1}(x)$.

1.3.2 Calculating the Inverse Tangent Function

By the meaning of an inverse function,

$$\tan(a) = b \iff \tan^{-1}(b) = a.$$

In general, to find the number c such that $\tan^{-1}(x) = c$ we have to think backwards and ask ourselves, “What is the number c such that $\tan(c) = x$?”.

For any $x \in [-\infty, \infty]$,

$$\tan^{-1}(x) = c \text{ provided } \tan(c) = x.$$

1.3.3 Cancellation Properties of the Inverse Tangent Function

From the general properties of inverse functions, we have the following **cancellation properties**.

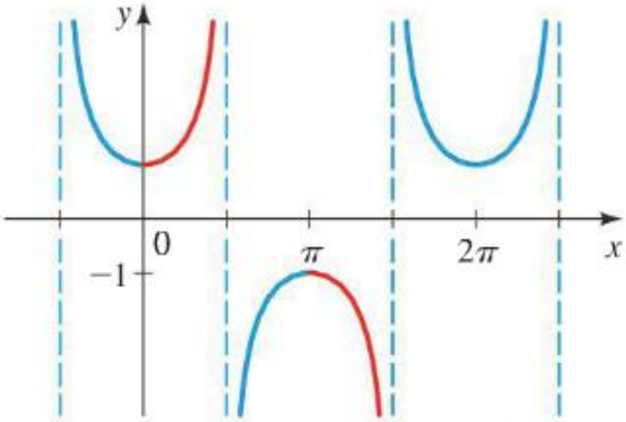
$$\tan^{-1}(\tan(x)) = \begin{cases} x & x \in (-\pi/2, \pi/2) \\ x^* & x \notin (-\pi/2, \pi/2) \text{ and } x \neq \dfrac{(2n+1)\pi}{2} \text{ for any integer } n \\ \text{undefined} & x = \dfrac{(2n+1)\pi}{2} \text{ for some integer } n \end{cases}$$

where x^* is that unique number such that $\tan(x) = \tan(x^*)$ and $x^* \in (-\pi/2, \pi/2)$

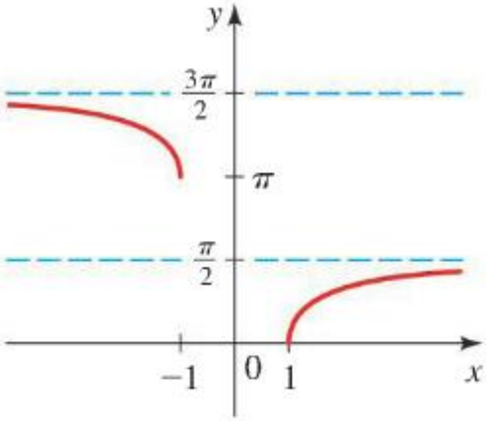
and

$$\tan(\tan^{-1}(x)) = x \text{ for all } x \in (-\infty, \infty)$$

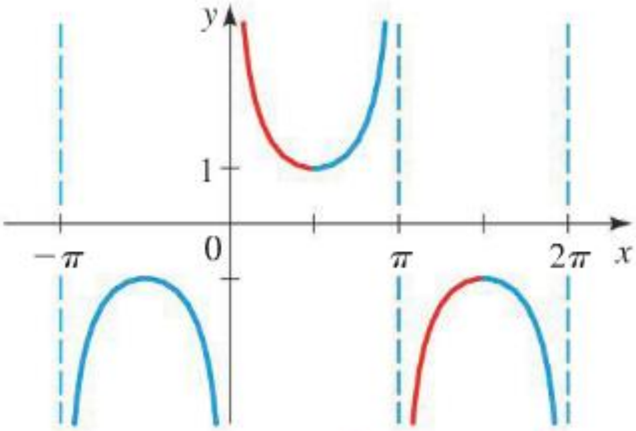
1.4 Inverse Secant, Cosecant and Cotangent Functions



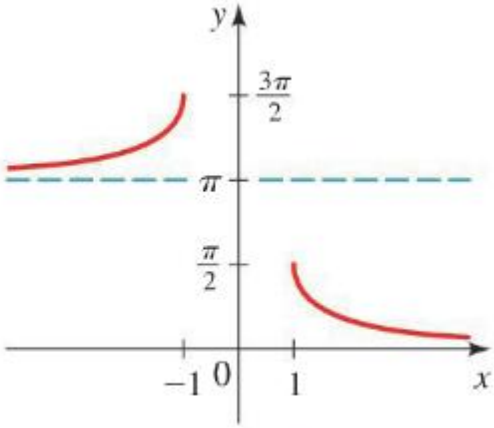
$y = \sec x,\ 0 \le x < \frac{\pi}{2},\ \pi \le x < \frac{3\pi}{2}$



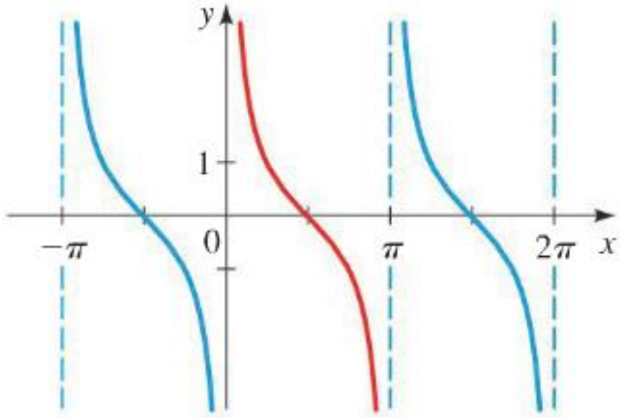
$y = \sec^{-1} x$



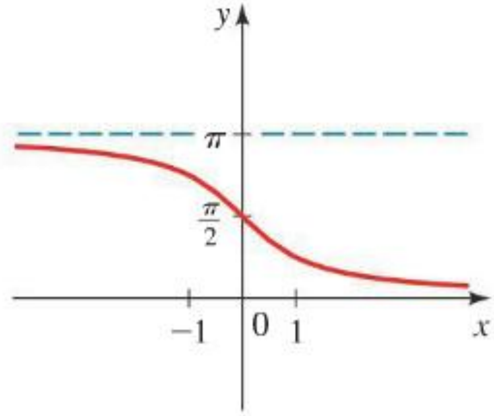
$y = \csc x,\ 0 < x \le \frac{\pi}{2},\ \pi < x \le \frac{3\pi}{2}$



$y = \csc^{-1} x$



$y = \cot x,\ 0 < x < \pi$



$y = \cot^{-1} x$

Exercise 7. Simplify.

(a) $\tan^{-1}(-1)$ (b) $\tan^{-1}(\sqrt{3})$ (c) $\tan^{-1}\left(\frac{\sqrt{3}}{3}\right)$

Solution

(a) $\tan^{-1}(-1) = -\frac{\pi}{4}$ because $\tan\left(-\frac{\pi}{4}\right) = -1$ and $-\frac{\pi}{4}$ lies in $(-\infty, \infty)$

(b) $\tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$ because $\tan\left(\frac{\pi}{3}\right) = \sqrt{3}$ and $\frac{\pi}{3}$ lies in $(-\infty, \infty)$

(c) $\tan^{-1}\left(\frac{\sqrt{3}}{3}\right) = \frac{\pi}{6}$ because $\tan\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{3}$ and $\frac{\pi}{6}$ lies in $(-\infty, \infty)$

1.5 Properties of the Inverse Trig Functions

Functions	Domain	Range
$\sin^{-1}(x)$	$[-1,1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1}(x)$	$[-1,1]$	$[0, \pi]$
$\tan^{-1}(x)$	$\mathbb{R}$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$\cot^{-1}(x)$	$\mathbb{R}$	$(0, \pi)$

$\sec^{-1}(x)$	$\mathbb{R} \setminus (-1,1)$	$(0,\pi] \setminus \left\{\frac{\pi}{2}\right\}$
$\csc^{-1}(x)$	$\mathbb{R} \setminus (-1,1)$	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right] \setminus \{0\}$

Property I		
(i)	$\sin^{-1}\left(\sin(\theta)\right) = \theta$	if $\theta \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$
(ii)	$\cos^{-1}\left(\cos(\theta)\right) = \theta$	if $\theta \in [0,\pi]$
(iii)	$\tan^{-1}\left(\tan(\theta)\right) = \theta$	if $\theta \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right)$
(iv)	$\csc^{-1}\left(\csc(\theta)\right) = \theta$	if $\theta \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right], \theta \neq 0$
(v)	$\sec^{-1}\left(\sec(\theta)\right) = \theta$	if $\theta \in [0,\pi], \theta \neq \frac{\pi}{2}$
(vi)	$\cot^{-1}\left(\cot(\theta)\right) = \theta$	if $\theta \in (0,\pi)$

Property II		
(i)	$\sin\left(\sin^{-1}(x)\right) = x$	if $x \in [-1,1]$
(ii)	$\cos\left(\cos^{-1}(x)\right) = x$	if $x \in [-1,1]$
(iii)	$\tan(\tan^{-1}(x)) = x$	if $x \in \mathbb{R}$
(iv)	$\csc(\csc^{-1}(x)) = x$	if $x \in (-\infty,-1] \cup [1,\infty)$
(v)	$\sec\left(\sec^{-1}(x)\right) = x$	if $x \in (-\infty,-1] \cup [1,\infty)$
(vi)	$\cot\left(\cot^{-1}(x)\right) = x$	if $x \in \mathbb{R}$

Property III		
(i)	$\sin^{-1}(-x) = -\sin^{-1}(x)$	if $x \in [-1,1]$
(ii)	$\cos^{-1}(-x) = \pi - \cos^{-1}(x)$	if $x \in [-1,1]$
(iii)	$\tan^{-1}(-x) = -\tan^{-1}(x)$	if $x \in \mathbb{R}$
(iv)	$\csc^{-1}(-x) = \pi - \csc^{-1}(x)$	if $x \in (-\infty,-1] \cup [1,\infty)$
(v)	$\sec^{-1}(-x) = \pi - \sec^{-1}(x)$	if $x \in (-\infty, 1] \cup [1,\infty)$
(vi)	$\cot^{-1}(-x) = \pi - \cot^{-1}(x)$	if $x \in \mathbb{R}$

Property IV		
(i)	$\sin^{-1}\left(\frac{1}{x}\right) = \csc^{-1}(x)$	if $x \in (-\infty, 1] \cup [1,\infty)$
(ii)	$\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1}(x)$	if $x \in (-\infty, 1] \cup [1,\infty)$
(iii)	$\tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} \cot^{-1}(x) & \text{if } x > 0 \\ -\pi + \cot^{-1}(x) & \text{if } x < 0 \end{cases}$	

Property V		
(i)	$\sin^{-1}(x) + \cos^{-1}(x) = \frac{\pi}{2}$	if $x \in [-1,1]$
(ii)	$\tan^{-1}(x) + \cot^{-1}(x) = \frac{\pi}{2}$	if $x \in \mathbb{R}$
(ii)	$\sec^{-1}(x) + \csc^{-1}(x) = \frac{\pi}{2}$	if $x \in (-\infty,-1] \cup [1,\infty)$

Property VI	
(i)	$\sin^{-1}(x)+\sin^{-1}(y)=\begin{cases}\sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right)\\ \pi-\sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right)\\ -\pi-\sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right)\end{cases}$ (if $-1\leq x,y\leq 1$ and $x^2+y^2\leq 1$) or (if $xy<0$ and $x^2+y^2>1$) if $0\leq x,y\leq 1$ and $x^2+y^2>1$ if $-1\leq x,y\leq 0$ and $x^2+y^2>1$
(ii)	$\sin^{-1}(x)-\sin^{-1}(y)=\begin{cases}\sin^{-1}\left(x\sqrt{1-y^2}-y\sqrt{1-x^2}\right)\\ \pi-\sin^{-1}\left(x\sqrt{1-y^2}-y\sqrt{1-x^2}\right)\\ -\pi-\sin^{-1}\left(x\sqrt{1-y^2}-y\sqrt{1-x^2}\right)\end{cases}$ (if $-1\leq x,y\leq 1$ and $x^2+y^2\leq 1$) or (if $xy>0$ and $x^2+y^2>1$) if $0\leq x\leq 1,-1\leq y\leq 0$ and $x^2+y^2>1$ if $-1\leq x<0,0<y\leq 1$ and $x^2+y^2>1$

Property VII	
(i)	$\cos^{-1}(x)+\cos^{-1}(y)$ $=\begin{cases}\cos^{-1}\left(xy-\sqrt{1-x^2}\,\sqrt{1-y^2}\right) & \text{if}-1\leq x,y\leq 1\text{ and }x+y\geq 0\\ 2\pi-\cos^{-1}\left(xy-\sqrt{1-x^2}\,\sqrt{1-y^2}\right) & \text{if}-1\leq x,y\leq 1\text{ and }x+y\leq 0\end{cases}$

(ii)	$\cos^{-1}(x) - \cos^{-1}(y)$ $= \begin{cases} \cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right) & \text{if} -1 \le x, y \le 1 \text{ and } x \le y \\ -\cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right) & \text{if} -1 \le y \le 0, 0 < x \le 1 \text{ and } x \ge y \end{cases}$

Property VIII	
(i)	$\tan^{-1}(x) + \tan^{-1}(y)$ $= \begin{cases} \tan^{-1}\left(\frac{x+y}{1-xy}\right) & \text{if } xy < 1 \\ \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right) & \text{if } x > 0, y > 0 \text{ and } xy > 1 \\ -\pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right) & \text{if } x < 0,\ y < 0 \text{ and } xy > 1 \end{cases}$
(ii)	$\tan^{-1}(x) - \tan^{-1}(y)$ $= \begin{cases} \tan^{-1}\left(\frac{x-y}{1+xy}\right) & \text{if } xy > -1 \\ \pi + \tan^{-1}\left(\frac{x-y}{1+xy}\right) & \text{if } x > 0, y < 0 \text{ and } xy < -1 \\ -\pi + \tan^{-1}\left(\frac{x-y}{1+xy}\right) & \text{if } x < 0,\ y > 0 \text{ and } xy < -1 \end{cases}$

Property IX	
(i)	$\sin^{-1}(x) = \cos^{-1}\left(\sqrt{1-x^2}\right) = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = \cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$ $= \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) = \csc^{-1}\left(\frac{1}{x}\right)$

(ii)	$\cos^{-1}(x) = \sin^{-1}\left(\sqrt{1-x^2}\right) = \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) = \cot^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$ $= \sec^{-1}\left(\frac{1}{x}\right) = \csc^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$
(iii)	$\tan^{-1}(x) = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right) = \cot^{-1}\left(\frac{1}{x}\right)$ $= \csc^{-1}\left(\frac{\sqrt{1+x^2}}{x}\right) = \sec^{-1}\left(\sqrt{1+x^2}\right)$

Property X	
(i)	$2\sin^{-1}(x) = \begin{cases} \sin^{-1}\left(2\sqrt{1-x^2}\right) & \text{if } -\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}} \\ \pi - \sin^{-1}\left(2\sqrt{1-x^2}\right) & \text{if } \frac{1}{\sqrt{2}} \le x \le 1 \\ -\pi - \sin^{-1}\left(2\sqrt{1-x^2}\right) & \text{if } -1 \le x \le -\frac{1}{\sqrt{2}} \end{cases}$
(ii)	$2\cos^{-1}(x) = \begin{cases} \cos^{-1}(2x^2-1) & \text{if } 0 \le x \le 1 \\ 2\pi - \cos^{-1}(2x^2-1) & \text{if} -1 \le x \le 0 \end{cases}$
(iii)	$2\tan^{-1}(x) = \begin{cases} \tan^{-1}\left(\frac{2x}{1-x^2}\right) & \text{if } -1 < x \le 1 \\ \pi + \tan^{-1}\left(\frac{2x}{1-x^2}\right) & \text{if } x > 1 \\ -\pi + \tan^{-1}\left(\frac{2x}{1-x^2}\right) & \text{if } x < -1 \end{cases}$

Property XI	
(i)	$3\sin^{-1}(x) = \begin{cases} \sin^{-1}(3x - 4x^3) & \text{if } -\frac{1}{2} \leq x \leq \frac{1}{2} \\ \pi - \sin^{-1}(3x - 4x^3) & \text{if } \frac{1}{2} < x \leq 1 \\ -\pi - \sin^{-1}(3x - 4x^3) & \text{if } -1 \leq x < -\frac{1}{2} \end{cases}$
(ii)	$3\cos^{-1}(x) = \begin{cases} \cos^{-1}(4x^3 - 3x) & \text{if } \frac{1}{2} \leq x \leq 1 \\ 2\pi - \cos^{-1}(4x^3 - 3x) & \text{if } -\frac{1}{2} \leq x \leq \frac{1}{2} \\ 2\pi + \cos^{-1}(4x^3 - 3x) & \text{if } -1 \leq x \leq -\frac{1}{2} \end{cases}$
(iii)	$3\tan^{-1}(x) = \begin{cases} \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right) & \text{if } -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}} \\ \pi + \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right) & \text{if } x > \frac{1}{\sqrt{3}} \\ -\pi + \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right) & \text{if } x < -\frac{1}{\sqrt{3}} \end{cases}$

Property XII	
(i)	$2\tan^{-1}(x) = \begin{cases} \sin^{-1}\left(\frac{2x}{1 + x^2}\right) & \text{if } -1 \leq x \leq 1 \\ \pi - \sin^{-1}\left(\frac{2x}{1 + x^2}\right) & \text{if } x > 1 \\ -\pi - \sin^{-1}\left(\frac{2x}{1 + x^2}\right) & \text{if } x < -1 \end{cases}$

(ii)	$2\tan^{-1}(x) = \begin{cases} \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right) & \text{if } 0 \le x < \infty \\ -\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right) & \text{if} -\infty \le x \le 0 \end{cases}$

Property XIII	
(i)	$\tan^{-1}(x) + \tan^{-1}(y) + \tan^{-1}(z) = \tan^{-1}\left(\dfrac{x+y+z-xyz}{1-xy-yz-zx}\right)$
(ii)	$\tan^{-1}(w) + \tan^{-1}(x) + \tan^{-1}(y) + \tan^{-1}(z)$ $= \tan^{-1}\left(\dfrac{w+x+y+z-wxy-wxz-wyz-xyz}{1-wx-wy-wz-xy-yz-zx+wxyz}\right)$
(iii)	$\tan^{-1}(x_1) + \tan^{-1}(x_2) + \cdots + \tan^{-1}(x_n) = \tan^{-1}\left(\dfrac{S_1 - S_3 + S_5 - \cdots}{1 - S_2 + S_4 - S_6 + \cdots}\right)$ where S_k denotes the sum of the product of $x_1, x_2, \dots, x_n$ taken k at a time.
(iv)	If $\tan^{-1}(x) + \tan^{-1}(y) = \dfrac{\pi}{2}$, then $xy = 1$
(v)	If $\tan^{-1}(x) + \tan^{-1}(y) + \tan^{-1}(z) = \dfrac{\pi}{2}$, then $xy + yz + zx = 1$
(vi)	If $\tan^{-1}(x) + \tan^{-1}(y) + \tan^{-1}(z) = \pi$, then $x + y + z = xyz$

Property XIV	
(i)	If $\sin^{-1}(x) + \sin^{-1}(y) + \sin^{-1}(z) = \dfrac{\pi}{2}$, then $x^2 + y^2 + z^2 + 2xyz = 1$

(ii)	If $\sin^{-1}(x) + \sin^{-1}(y) + \sin^{-1}(z) = \pi$, then $x\sqrt{1-x^2} + y\sqrt{1-y^2} + z\sqrt{1-z^2} = 2xyz$
(iii)	If $\sin^{-1}(x) + \sin^{-1}(y) + \sin^{-1}(z) = \frac{3\pi}{2}$, then $xy + yz + zx = 3$

Property XV

(i)	If $\cos^{-1}(x) + \cos^{-1}(y) + \cos^{-1}(z) = \pi$, then $x^2 + y^2 + z^2 + 2xyz = 1$
(ii)	If $\cos^{-1}(x) + \cos^{-1}(y) + \cos^{-1}(z) = 3\pi$, then $xy + yz + zx = 3$

Property XVI

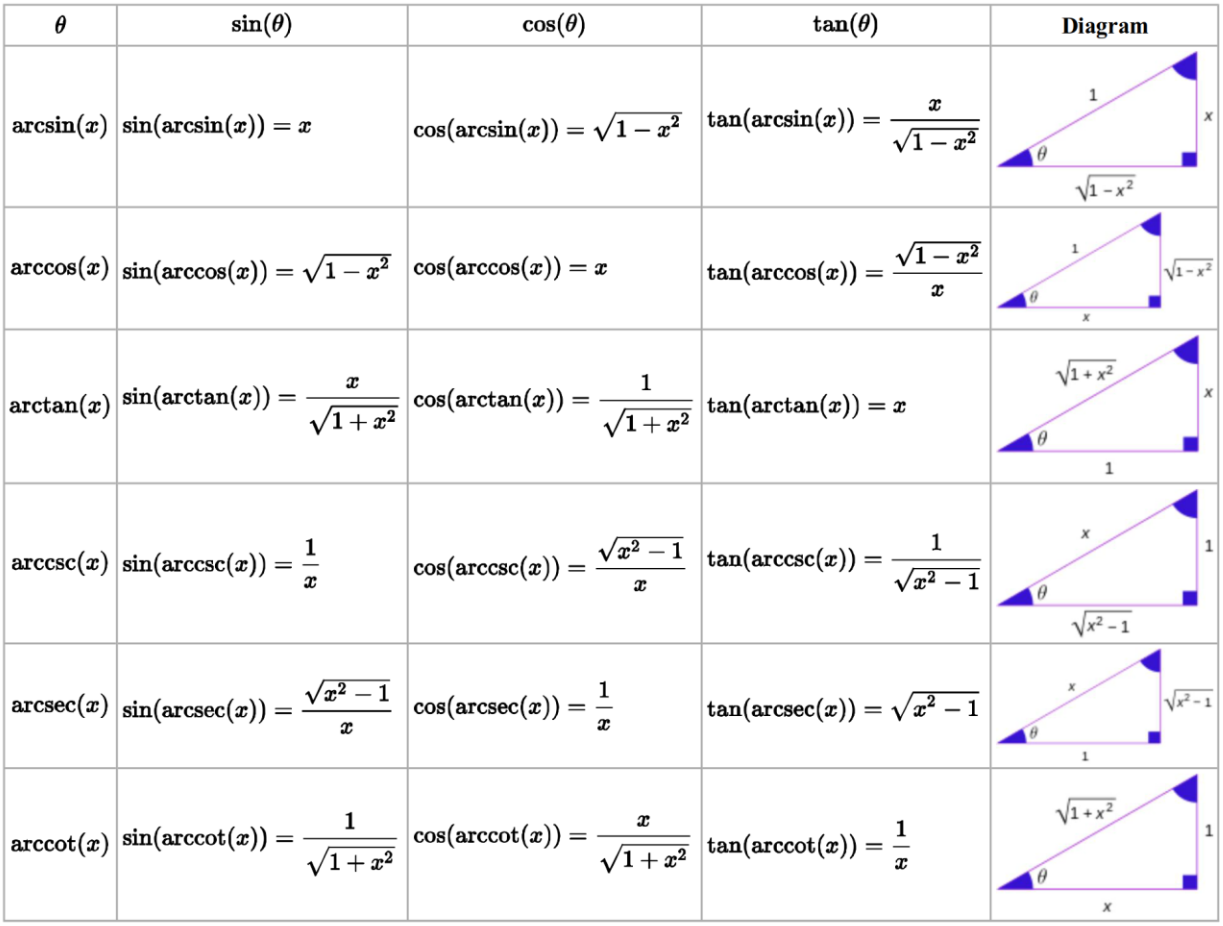
(i)	If $\sin^{-1}(x) + \sin^{-1}(y) = \theta$, then $\cos^{-1}(x) + \cos^{-1}(y) = \pi - \theta$
(ii)	If $\cos^{-1}(x) + \cos^{-1}(y) = \theta$, then $\sin^{-1}(x) + \sin^{-1}(y) = \pi - \theta$

Property XVII

(i)	If $\cos^{-1}\left(\frac{x}{a}\right) + \cos^{-1}\left(\frac{y}{b}\right) = \theta$, then $\frac{x^2}{a^2} - \frac{2xy}{ab}\cos(\theta) + \frac{y^2}{b^2} = \sin^2(\theta)$

Property XVIII

(i)	If $\cot^{-1}(x) + \cot^{-1}(y) = \frac{\pi}{2}$, then $xy = 1$



θ	$\sin(\theta)$	$\cos(\theta)$	$\tan(\theta)$	Diagram
$\arcsin(x)$	$\sin(\arcsin(x)) = x$	$\cos(\arcsin(x)) = \sqrt{1-x^2}$	$\tan(\arcsin(x)) = \dfrac{x}{\sqrt{1-x^2}}$	
$\arccos(x)$	$\sin(\arccos(x)) = \sqrt{1-x^2}$	$\cos(\arccos(x)) = x$	$\tan(\arccos(x)) = \dfrac{\sqrt{1-x^2}}{x}$	
$\arctan(x)$	$\sin(\arctan(x)) = \dfrac{x}{\sqrt{1+x^2}}$	$\cos(\arctan(x)) = \dfrac{1}{\sqrt{1+x^2}}$	$\tan(\arctan(x)) = x$	
$\text{arccsc}(x)$	$\sin(\text{arccsc}(x)) = \dfrac{1}{x}$	$\cos(\text{arccsc}(x)) = \dfrac{\sqrt{x^2-1}}{x}$	$\tan(\text{arccsc}(x)) = \dfrac{1}{\sqrt{x^2-1}}$	
$\text{arcsec}(x)$	$\sin(\text{arcsec}(x)) = \dfrac{\sqrt{x^2-1}}{x}$	$\cos(\text{arcsec}(x)) = \dfrac{1}{x}$	$\tan(\text{arcsec}(x)) = \sqrt{x^2-1}$	
$\text{arccot}(x)$	$\sin(\text{arccot}(x)) = \dfrac{1}{\sqrt{1+x^2}}$	$\cos(\text{arccot}(x)) = \dfrac{x}{\sqrt{1+x^2}}$	$\tan(\text{arccot}(x)) = \dfrac{1}{x}$	

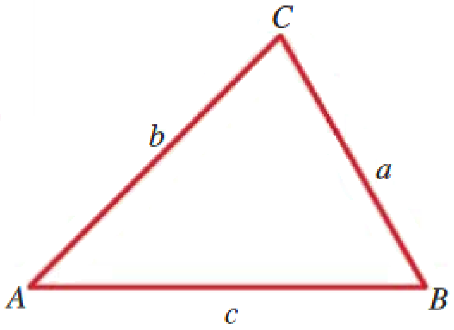
2 Law of Sines, Law of Cosines, and the Law of Tangents

Trigonometric functions can be used directly to solve (finding missing sides and/or missing angles) right triangles.

But the trigonometric functions can also be used to solve *oblique triangles*, that is, triangles with no right angles.

To do this, we need to understand how to use the Law of Sines and the Law of Cosines.

To state these laws (or formulas) more easily, we follow the convention of labeling the angles of a triangle as A, B, C and the lengths of the corresponding opposite sides as a, b, c, as in Figure 1.



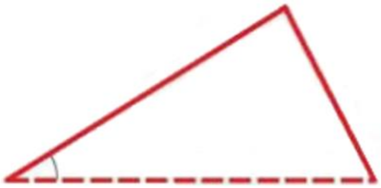
In general, a triangle is determined by three of its six parts (angles and sides) as long as at least one of these three parts is a side.

So the possibilities are as follows.

Case 1 One side and two angles (ASA or SAA)

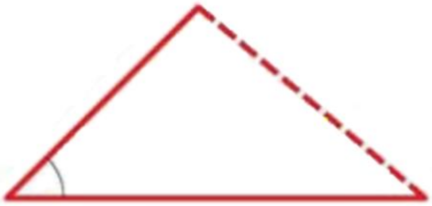
ASA or SAA

Case 2 Two sides and the angle opposite one of those sides (SSA)



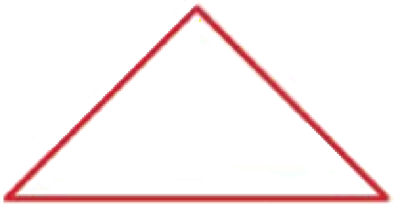
SSA

Case 3 Two sides and the included angle (SAS)



SAS

Case 4 Three sides (SSS)



SSS

Notation

When we use the notation SAS (for example) with S for side and A for angle, it denotes that we know two sides and one angle. But it tells us something else. This notation means that the angle A we know is not just any of the three angles of a triangle but is specifically the angle between the two sides we know.

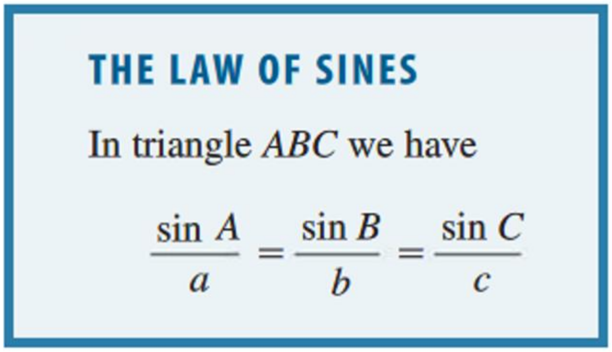
And when we use ASA it means we know two angles and one side and the side we know is between the two we know.

And when we use SAA it also means we know two angles and one side but in this case the side we know is NOT between the two angles we know but rather is immediately followed by the two angles we know.

The **ASA**, **SAA**, **SSA** cases are solved by using the **Law of Sines**.

The **SAS** and **SSS** cases are solved by using the **Law of Cosines**.

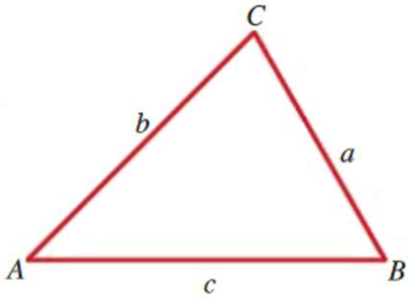
2.1 The Law of Sines



THE LAW OF SINES

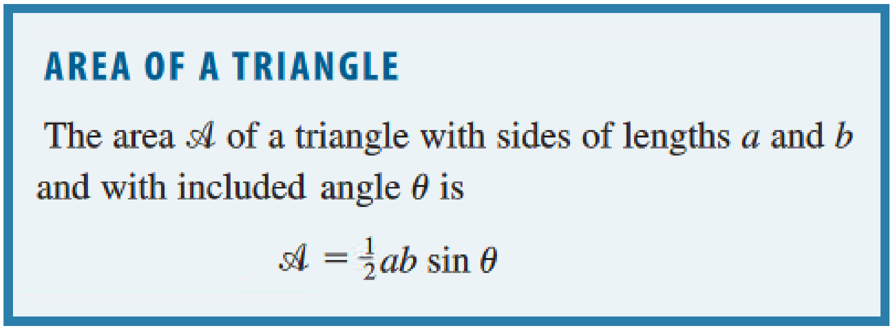
In triangle ABC we have

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$



PROOF

We have already stated the following result for the area of a triangle.

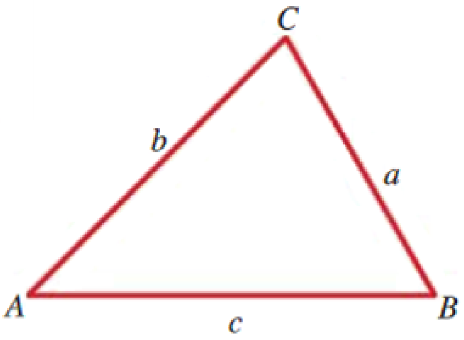


AREA OF A TRIANGLE

The area $\mathcal{A}$ of a triangle with sides of lengths a and b and with included angle θ is

$$\mathcal{A} = \tfrac{1}{2}ab\sin\theta$$

Applying this formula to the triangle below, we see that



$$\mathcal{A} = \frac{1}{2}bc\sin(A) \qquad \mathcal{A} = \frac{1}{2}ac\sin(B) \qquad \mathcal{A} = \frac{1}{2}ab\sin(C)$$

So,

$$\frac{1}{2}\,bc\sin(A) = \frac{1}{2}\,ac\sin(B) = \frac{1}{2}\,ab\sin(C).$$

Multiplying all parts by the same quantity $2/abc$, we have

$$\left(\frac{2}{abc}\right)\left(\frac{1}{2}\,bc\sin(A)\right) = \left(\frac{2}{abc}\right)\left(\frac{1}{2}\,ac\sin(B)\right) = \left(\frac{2}{abc}\right)\left(\frac{1}{2}\,ab\sin(C)\right).$$

Simplifying we immediately have the Law of Sines,

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c}.$$

■

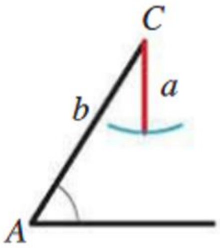
2.1.1 SSA, The Ambiguous Case

The Law of Sines can be used in the ASA or SAA to solve the triangle and find the ***unique*** (one and only one) triangle with the given properties.

However, in the **SSA** case there may be **two** triangles, **one** triangle, or **no** triangles with the given properties.

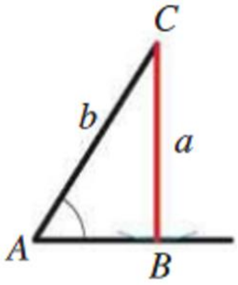
For this reason the SSA case is referred to as the "ambiguous case".

We illustrate the possibilities when angle A and sides a and b are given. In (a), no solution is possible, because side a is too short to complete the triangle.



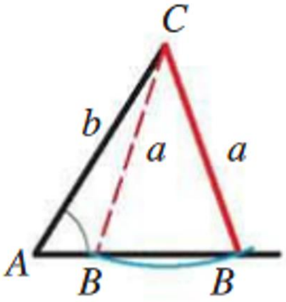
(a)

In (b) there is one solution and it is a right triangle.



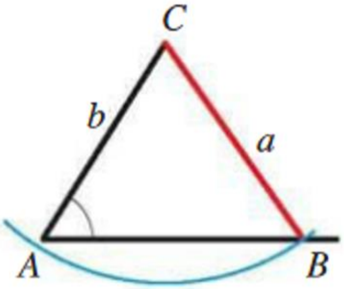
(b)

In part (c) two solutions are possible.



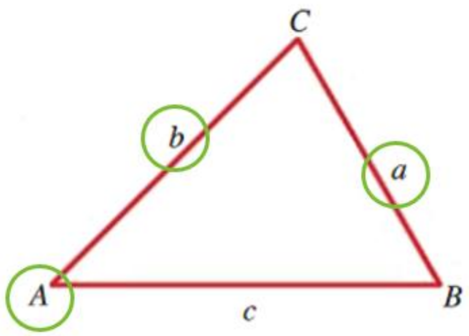
(c)

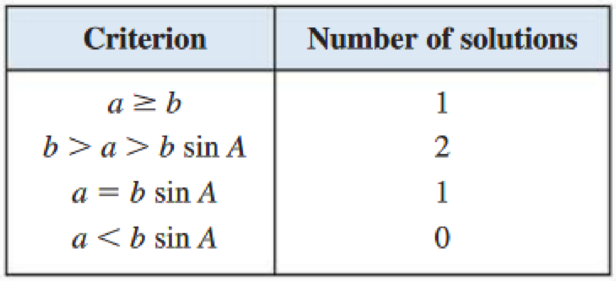
In part (d) only one solution is possible.



(d)

Suppose we know sides a, b and angle A as shown in the figure below. The chart shown below tells us whether there will be 2 or 1 or no triangles that can be formed with these two sides and one angle.





Criterion	Number of solutions
$a \geq b$	1
$b > a > b \sin A$	2
$a = b \sin A$	1
$a < b \sin A$	0

Caution!
When using this chart in a SSA situation be sure that **what you use for a is the known side that is opposite the known angle in your problem** – regardless of how the side and angle labels are placed in your problem.

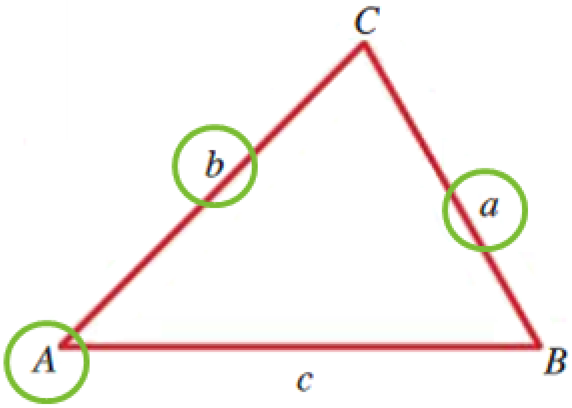
(*e.g.* In your problem you may be given sides b and c and the angle B not between these two sides. In this case, your b would be the a in the chart above and your c would be the b in the chart above. Be careful!)

EXAMPLE | SSA, the Two-Solution Case

Solve triangle ABC if $\angle A = 43.1°$, $a = 186.2$, and $b = 248.6$.

SOLUTION

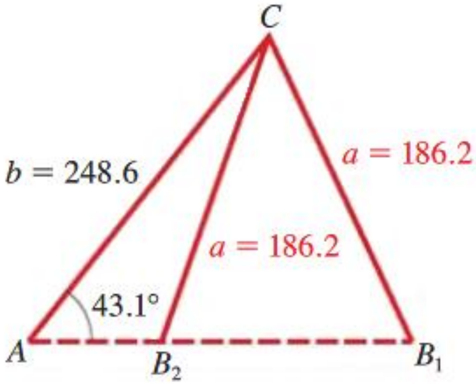
First note that we are the SSA situation.



Secondly, note that

$$b = 248.6 > a = 186.2 > b\sin(A) = 248.6 \cdot \sin(43.1°) = 169.862$$

So, we are in the situation of SSA where there are two solutions.



From the Law of Sines

$$\sin B = \frac{b\sin A}{a} = \frac{248.6 \sin 43.1°}{186.2} \approx 0.91225$$

There are two possible angles B between 0° and 180° such that $\sin B = 0.91225$.

Using a calculator, we find that one of the angles is

$$\sin^{-1}(0.91225) \approx 65.8°.$$

The other angle is approximately $180° - 65.8° = 114.2°$.

We denote these two angles by B_1 and B_2 so that

$$\angle B_1 \approx 65.8° \qquad \text{and} \qquad \angle B_2 \approx 114.2°$$

Note: It is always the case the $\boldsymbol{\angle B_1 + \angle B_2 = 180°}$. You can use this fact to find $\angle B_2$ once you have $\angle B_1$.

Thus two triangles satisfy the given conditions: triangle $A_1B_1C_1$ and triangle $A_2B_2C_2$.

Solve triangle $A_1B_1C_1$:

$$\angle C_1 \approx 180° - (43.1° + 65.8°) = 71.1° \qquad \text{Find } \angle C_1$$

Thus

$$c_1 = \frac{a_1 \sin C_1}{\sin A_1} \approx \frac{186.2 \sin 71.1°}{\sin 43.1°} \approx 257.8 \qquad \text{Law of Sines}$$

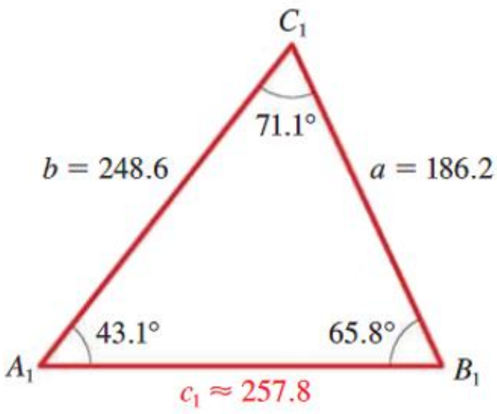
Solve triangle $A_2B_2C_2$:

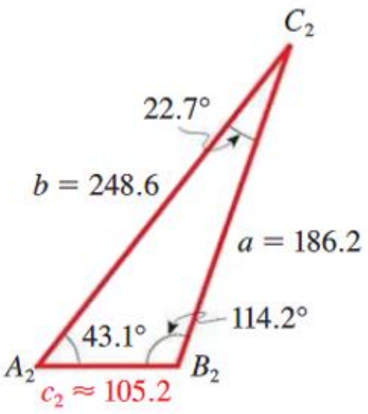
$$\angle C_2 \approx 180^\circ - (43.1^\circ + 114.2^\circ) = 22.7^\circ \qquad \text{Find } \angle C_2$$

Thus

$$c_2 = \frac{a_2 \sin C_2}{\sin A_2} \approx \frac{186.2 \sin 22.7^\circ}{\sin 43.1^\circ} \approx 105.2 \qquad \text{Law of Sines}$$

The two solved triangles $A_1B_1C_1$ and $A_2B_2C_2$ are shown below.





2.2 The Law of Cosines

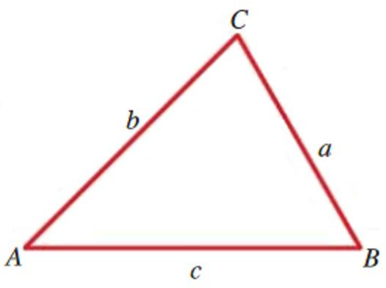
THE LAW OF COSINES

In any triangle ABC, we have

$$a^2 = b^2 + c^2 - 2bc \cos A$$

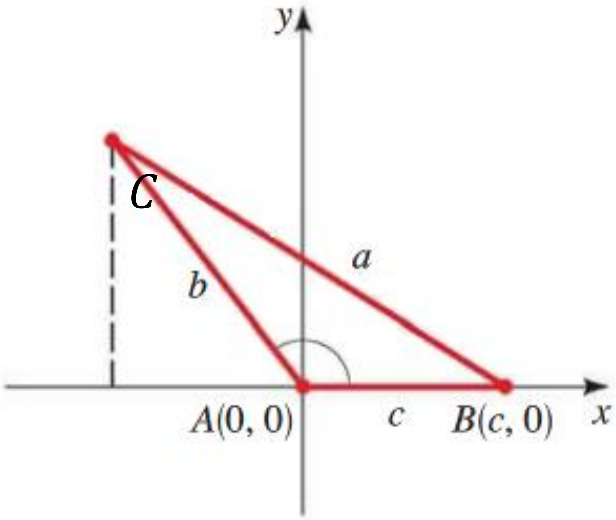
$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$



PROOF

To prove the Law of Cosines, place triangle ABC so that $\angle A$ is at the origin.



The coordinates of B will then be $(c, 0)$.

Suppose we let (x_C, y_C) represent the coordinates of C.

If we drop a perpendicular from C to the x-axis then using the triangle formed on the left we see that

$$\sin(180° - A) = \frac{y_C}{b}$$

and

$$\cos(180° - A) = \frac{-x_C}{b}.$$

But

$$\sin(180° - A) = \sin(A) \text{ for all } A \in [0°, 180°]$$

and

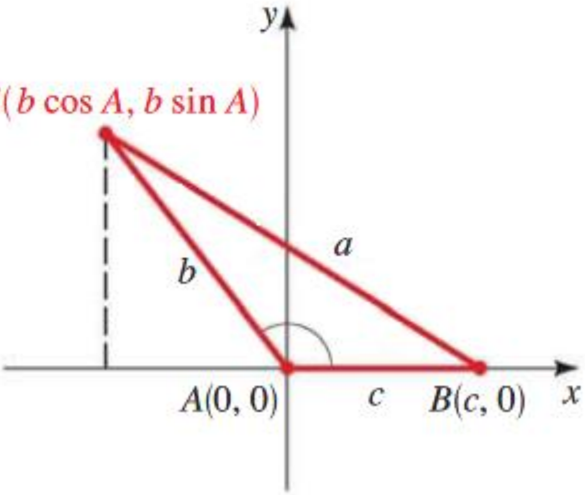
$$\cos(180° - A) = -\cos(A) \text{ for all } A \in [0°, 180°].$$

So,

$$\sin(A) = \frac{y_C}{b} \Rightarrow y_C = b\sin(A)$$

and

$$-\cos(A) = -\frac{x_C}{b} \Rightarrow x_C = b\cos(A).$$

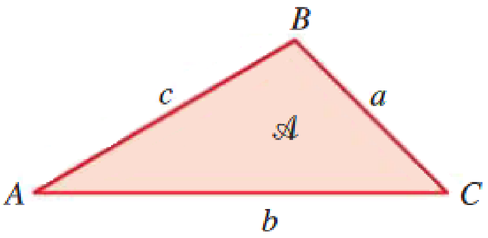


Using the Distance Formula, we get

$$\begin{aligned} a^2 &= (b\cos A - c)^2 + (b\sin A - 0)^2 \\ &= b^2\cos^2 A - 2bc\cos A + c^2 + b^2\sin^2 A \\ &= b^2(\cos^2 A + \sin^2 A) - 2bc\cos A + c^2 \\ &= b^2 + c^2 - 2bc\cos A \qquad \text{Because } \sin^2 A + \cos^2 A = 1 \end{aligned}$$

This proves the first formula. The other two formulas are obtained in the same way by placing each of the other vertices of the triangle at the origin and repeating the preceding argument. ■

2.2.1 The Area of a Triangle in the SSS Case – Heron's Formula



HERON'S FORMULA

The area $\mathcal{A}$ of triangle ABC is given by

$$\mathcal{A} = \sqrt{s(s-a)(s-b)(s-c)}$$

where $s = \frac{1}{2}(a+b+c)$ is the **semiperimeter** of the triangle; that is, s is half the perimeter.

PROOF We start with the area of a triangle formula $\mathcal{A} = ab\sin(C)/2$. Squaring both sides we have

$$\begin{aligned} \mathcal{A}^2 &= \tfrac{1}{4}a^2b^2\sin^2 C \\ &= \tfrac{1}{4}a^2b^2(1-\cos^2 C) && \text{Pythagorean identity} \\ &= \tfrac{1}{4}a^2b^2(1-\cos C)(1+\cos C) && \text{Factor} \end{aligned}$$

Next, we write the expressions $1-\cos(C)$ and $1+\cos(C)$ in terms of a, b, and c. By the Law of Cosines we have

$$\begin{aligned}
\cos C &= \frac{a^2 + b^2 - c^2}{2ab} && \text{Law of Cosines}\\
1 + \cos C &= 1 + \frac{a^2 + b^2 - c^2}{2ab} && \text{Add 1}\\
&= \frac{2ab + a^2 + b^2 - c^2}{2ab} && \text{Common denominator}\\
&= \frac{(a + b)^2 - c^2}{2ab} && \text{Factor}\\
&= \frac{(a + b + c)(a + b - c)}{2ab} && \text{Difference of squares}
\end{aligned}$$

Similarly

$$1 - \cos C = \frac{(c + a - b)(c - a + b)}{2ab}$$

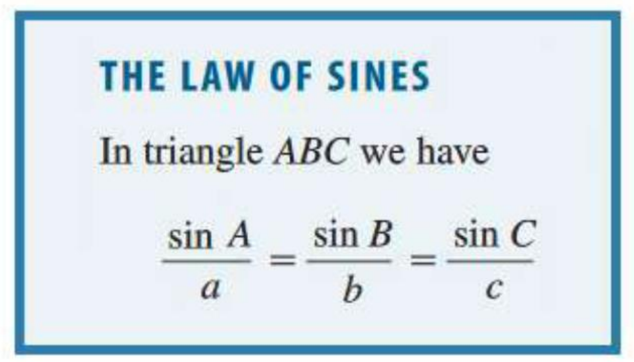
Substituting these expressions in the formula we obtained for $\mathscr{A}^2$ gives

$$\begin{aligned}
\mathscr{A}^2 &= \tfrac{1}{4}a^2b^2\frac{(a + b + c)(a + b - c)}{2ab}\,\frac{(c + a - b)(c - a + b)}{2ab}\\
&= \frac{(a + b + c)}{2}\,\frac{(a + b - c)}{2}\,\frac{(c + a - b)}{2}\,\frac{(c - a + b)}{2}\\
&= s(s - c)(s - b)(s - a)
\end{aligned}$$

Heron's Formula now follows by taking the square root of each side. ■

2.3 Examples to Illustrate the Use of the Law of Sines and the Law of Cosines

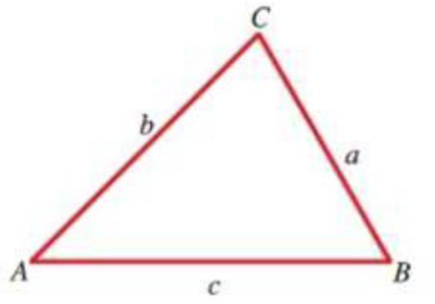
2.3.1 Using the Law of Sines to "solve" a triangle in the ASA case.

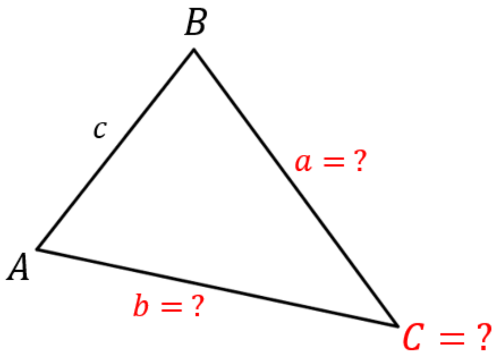


THE LAW OF SINES

In triangle ABC we have

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$





Suppose we know $m\angle A$ (the measure of angle A), c (the length of the side opposite angle C) and $m\angle B$ (the measure of angle B). Then we are in the ASA case.

The goal is to find the length of sides a and b and $m\angle C$, the measure of angle C. The first step is to solve for $m\angle C$ by subtraction from $180°$.

$$\boxed{\boldsymbol{m\angle C} = 180 - m\angle A - m\angle B.}$$

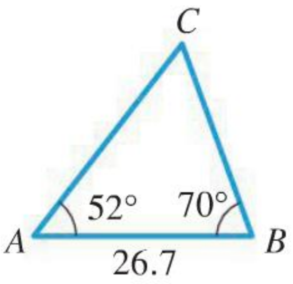
Now that we know C, we can find $\boldsymbol{a}$ and $\boldsymbol{b}$ using the Law of Sines.

$$\frac{\sin(A)}{\boldsymbol{a}} = \frac{\sin(C)}{c} \Rightarrow \boxed{\boldsymbol{a} = c \cdot \frac{\sin(A)}{\sin(C)}}$$

and

$$\frac{\sin(B)}{\boldsymbol{b}} = \frac{\sin(C)}{c} \Rightarrow \boxed{\boldsymbol{b} = c \cdot \frac{\sin(B)}{\sin(C)}.}$$

Example 1. Solve the following triangle.



<u>Solution</u>

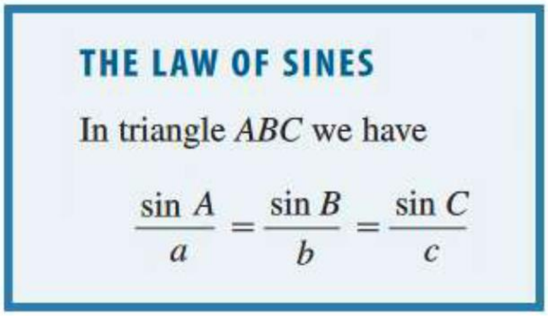
This is an ASA problem. So,

$$m\angle C = 180° - m\angle A - m\angle B = 180° - 52° - 70° = 58°$$

$$a = c \cdot \frac{\sin(A)}{\sin(C)} = 26.7 \cdot \frac{\sin(52°)}{\sin(58°)} \approx 24.8$$

$$b = c \cdot \frac{\sin(B)}{\sin(C)} = 26.7 \cdot \frac{\sin(70°)}{\sin(58°)} \approx 29.6.$$

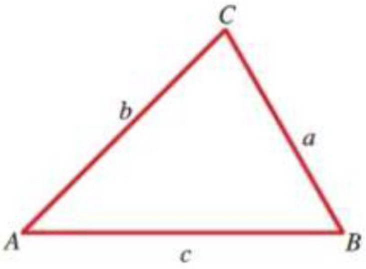
2.3.2 Using the Law of Sines to "solve" a triangle in the AAS case.

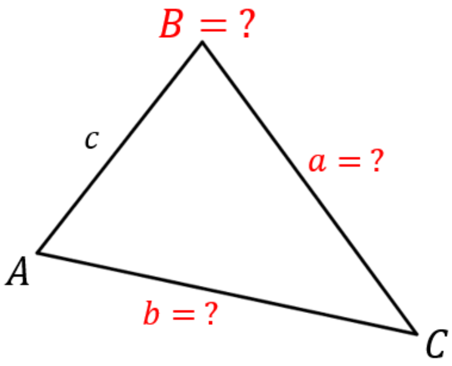


THE LAW OF SINES

In triangle ABC we have

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$





Suppose we know $m\angle C$ (the measure of angle C), $m\angle A$ (the measure of angle A) and c (the length of the side opposite angle C) and. Then we are in the AAS case. The goal is to find the length of sides a and b and $m\angle B$, the measure of angle B. The first step is to solve for $m\angle B$ by subtraction from 180°.

That is,

$$\boldsymbol{m\angle B} = 180 - m\angle A - m\angle C.$$

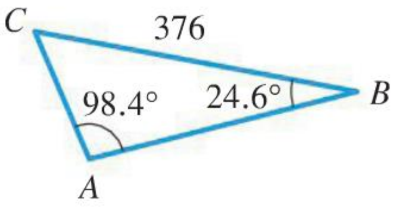
Then, just as in the ASA case, we can find $\boldsymbol{a}$ and $\boldsymbol{b}$ using the Law of Sines.

$$\frac{\sin(A)}{\boldsymbol{a}} = \frac{\sin(C)}{c} \Longrightarrow \boxed{\boldsymbol{a} = c \cdot \frac{\sin(A)}{\sin(C)}}$$

and

$$\frac{\sin(B)}{\boldsymbol{b}} = \frac{\sin(C)}{c} \Longrightarrow \boxed{\boldsymbol{b} = c \cdot \frac{\sin(B)}{\sin(C)}.}$$

Example 2. Solve the following triangle.



Solution

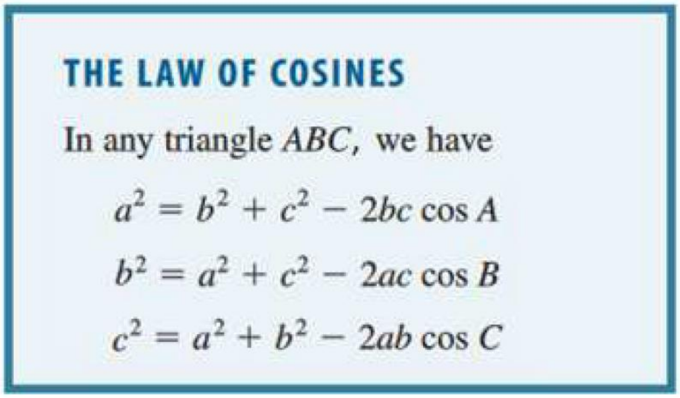
This is an AAS problem. So,

$$m\angle C = 180° - m\angle A - m\angle B = 180° - 98.4° - 24.6° = 57°$$

$$b = a \cdot \frac{\sin(B)}{\sin(A)} = 376 \cdot \frac{\sin(24.6°)}{\sin(98.4°)} \approx 158.22$$

$$c = a \cdot \frac{\sin(C)}{\sin(A)} = 376 \cdot \frac{\sin(57°)}{\sin(98.4°)} \approx 318.75$$

2.3.3 Using the Law of Cosines to “solve” a triangle in the SAS case.



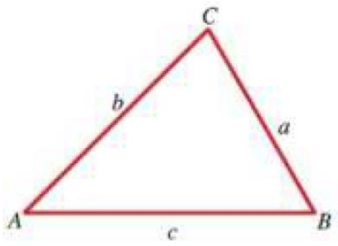
THE LAW OF COSINES

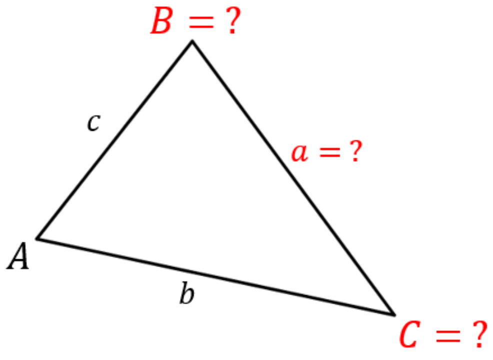
In any triangle ABC, we have

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$





Suppose we know b (the length of the side opposite angle B), $m\angle A$ (the measure of angle A) and c (the length of the side opposite angle C). Then we are in the SAS case. The goal is to find the length of side a, $m\angle B$ (the measure of angle B) and $m\angle C$ (the measure of angle C).

The first step is to solve for $\boldsymbol{a}$ using the Law of Cosines.

$$\boldsymbol{a}^2 = b^2 + c^2 - 2bc\cos(A).$$

So

$$\boxed{\boldsymbol{a} = \sqrt{b^2 + c^2 - 2bc\cos(A)}\,.}$$

Now that we know the length of side a, we can continue to use the Law of Cosines to find $m\angle B$ and $m\angle C$.

$$b^2 = a^2 + c^2 - 2ac\cos(\boldsymbol{B}) \Longrightarrow \cos(\boldsymbol{B}) = \frac{a^2 + c^2 - b^2}{2ac}.$$

Now it must be that $B \in [0, \pi]$ (i.e. between $0°$ and $180°$) and in this case $\cos^{-1}(\cos(B)) = B$

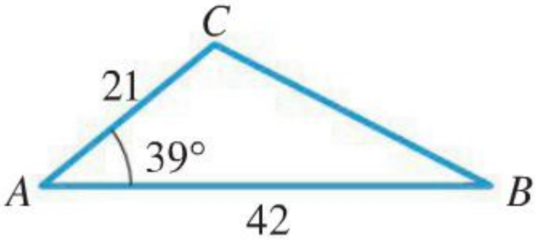
implies

$$\boxed{\boldsymbol{B} = \cos^{-1}(\cos(\boldsymbol{B})) = \cos^{-1}\left(\frac{a^2 + c^2 - b^2}{2ac}\right).}$$

By the same reasoning,

$$\boxed{\boldsymbol{C} = \cos^{-1}(\cos(\boldsymbol{C})) = \cos^{-1}\left(\frac{a^2 + b^2 - c^2}{2ab}\right).}$$

Example 3. Solve the following triangle.



Solution

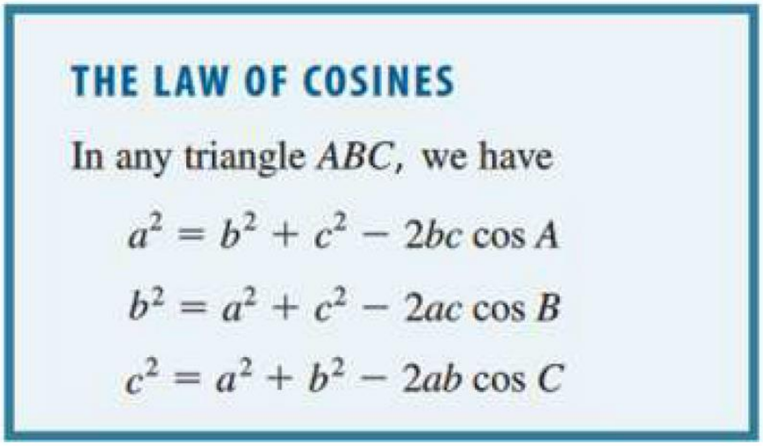
This is an SAS problem. So,

$$a = \sqrt{b^2 + c^2 - 2bc\cos(A)} = \sqrt{21^2 + 42^2 - 2(21)(42)\cos(39°)} \approx 28.881$$

$$B = \cos^{-1}\left(\frac{a^2 + c^2 - b^2}{2ac}\right) = \cos^{-1}\left(\frac{28.881^2 + 42^2 - 21^2}{2(28.881)(42)}\right) \approx 27.23°$$

$$C = \cos^{-1}\left(\frac{a^2 + b^2 - c^2}{2ab}\right) = \cos^{-1}\left(\frac{28.881^2 + 21^2 - 42^2}{2(28.881)(21)}\right) \approx 113.77°.$$

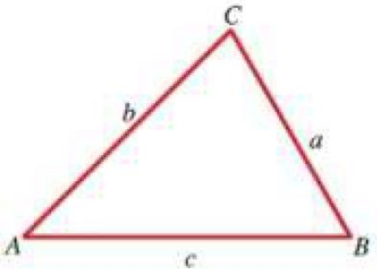
2.3.4 Using the Law of Cosines to "solve" a triangle in the SSS case.

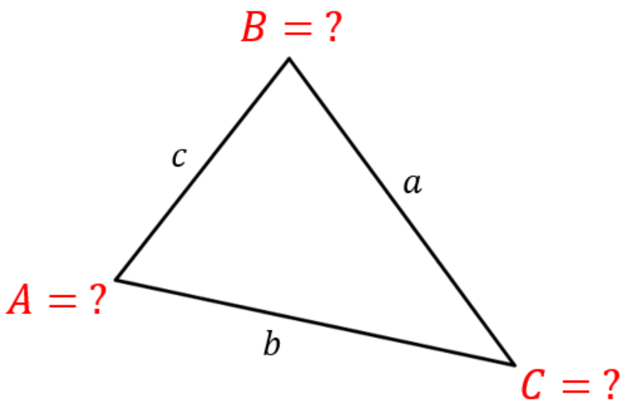


THE LAW OF COSINES

In any triangle *ABC*, we have

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$





Suppose we know a (the length of the side opposite angle A), b (the length of the side opposite angle B) and c (the length of the side opposite angle C). Then we are in the SSS case. The goal is to find $m\angle A$ (the measure of angle A), $m\angle B$ (the measure of angle B) and $m\angle C$ (the measure of angle C).

We can use the Law of Cosines to solve for each of the unknown angles. By the Law of Cosines,

$$a^2 = b^2 + c^2 - 2bc\cos(\boldsymbol{A}).$$

But

$$a^2 = b^2 + c^2 - 2bc\cos(\boldsymbol{A}) \Longrightarrow \cos(\boldsymbol{A}) = \frac{b^2 + c^2 - a^2}{2bc}.$$

Now it must be that $A \in [0, \pi]$ (i.e. between $0°$ and $180°$) and in this case $\cos^{-1}(\cos(A)) = A$ implies

$$\boxed{\boldsymbol{A} = \cos^{-1}(\cos(\boldsymbol{A})) = \cos^{-1}\left(\frac{b^2 + c^2 - a^2}{2bc}\right).}$$

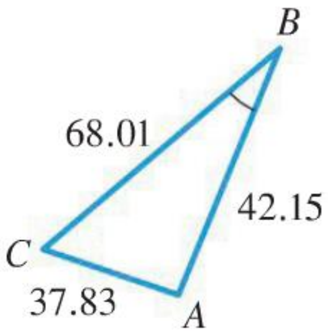
By the same reasoning,

$$\boxed{\boldsymbol{B} = \cos^{-1}(\cos(\boldsymbol{B})) = \cos^{-1}\left(\frac{a^2 + c^2 - b^2}{2ac}\right)}$$

and

$$C = \cos^{-1}(\cos(C)) = \cos^{-1}\left(\frac{a^2+b^2-c^2}{2ab}\right).$$

Example 4. Solve the following triangle.



Solution

This is an SSS problem. So,

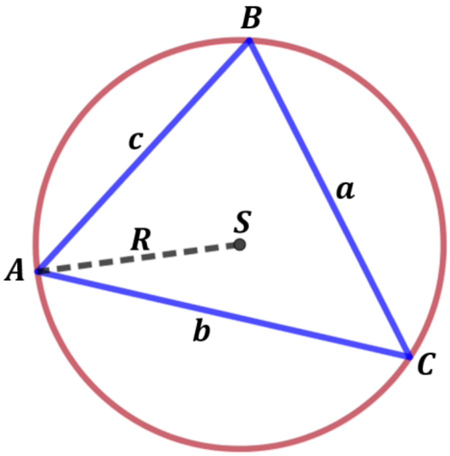
$$A = \cos^{-1}\left(\frac{b^2+c^2-a^2}{2bc}\right) = \cos^{-1}\left(\frac{37.83^2+42.15^2-68.01^2}{2(37.83)(42.15)}\right) \approx 116.39°$$

$$B = \cos^{-1}\left(\frac{a^2+c^2-b^2}{2ac}\right) = \cos^{-1}\left(\frac{68.01^2+42.15^2-37.83^2}{2(68.01)(42.15)}\right) \approx 29.89°$$

$$C = \cos^{-1}\left(\frac{a^2+b^2-c^2}{2ab}\right) = \cos^{-1}\left(\frac{68.01^2+37.83^2-42.15^2}{2(68.01)(37.83)}\right) \approx 33.72°$$

2.4 Extended Law of Sines

For any triangle ΔABC we can circumscribe a unique circle with center S because any three non-colinear points uniquely determine a circle.



We stated in the Study Guide for Meet 2, Event B that the radius R of the circumscribing circle can be expressed by the formula

$$R = \frac{abc}{4\,\text{Area}(\Delta ABC)}.$$

We also stated in Meet 2B study guide that the area of ΔABC can be expressed by the Side Angle Side Formula

$$\text{Area}(\Delta ABC) = \frac{1}{2} a\, b \sin(C).$$

It follows by substitution that

$$2R = \frac{abc}{2\,\text{Area}(\Delta ABC)} = \frac{abc}{2\left(\frac{1}{2} ab \sin(C)\right)} = \frac{abc}{ab \sin(C)} = \frac{c}{\sin(C)}$$

or

$$\frac{\sin(C)}{c} = \frac{1}{2R}.$$

Combining this result with the Law of Sines we have the **Extended Law of Sines**

$$\boxed{\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c} = \frac{1}{2R}.}$$

3 Solving Trigonometric Equations

3.1 $\sin(x) = \sin(y)$, $\cos(x) = \cos(y)$, etc.

Equality of Same Trig Functions
$\sin(x) = \sin(y) \Leftrightarrow x = 2n\pi + (\pi - y)$ or $x = 2n\pi + y$ for some integer n
$\cos(x) = \cos(y) \Leftrightarrow x = 2n\pi - y$ or $x = 2n\pi + y$ for some integer n
$\tan(x) = \tan(y) \Leftrightarrow x = n\pi + y$ for some integer n
$\cot(x) = \cot(y) \Leftrightarrow \tan(x) = \tan(y)$
$\sec(x) = \sec(y) \Leftrightarrow \cos(x) = \cos(y)$
$\csc(x) = \csc(y) \Leftrightarrow \sin(x) = \sin(y)$

1.	Find the general values of x which satisfy the equation $\sin 2x = -\frac{1}{2}$. (Source: Math-Only-Math.com)

Solution

$$\sin(2x) = -\frac{1}{2}$$

$$\Leftrightarrow \sin(2x) = -\sin\left(\frac{\pi}{6}\right)$$

$$\Leftrightarrow \sin(2x) = \sin\left(\pi + \frac{\pi}{6}\right) = \sin\left(\frac{7\pi}{6}\right)$$

In general, $\sin(x) = \sin(y)$ implies $x = 2n\pi + (\pi - y)$ or $x = 2n\pi + y$ for some integer n.

$$\Leftrightarrow 2x = 2n\pi + \left(\pi - \frac{7\pi}{6}\right) \quad \text{or} \quad 2x = 2n\pi + \frac{7\pi}{6} \quad \text{for some integer } n$$

$$\Leftrightarrow x = n\pi - \frac{\pi}{12} \quad \text{or} \quad x = n\pi + \frac{7\pi}{12} \quad \text{for some integer } n.$$

■

3.2 Avoid Division by Zero and Creating Missing Solutions

2.	Solve for x if $$\frac{1+\sin(x)}{\cos(x)}+\frac{\cos(x)}{1+\sin(x)}=4.$$

Solution

Because denominators of fractions cannot equal zero, real numbers that cause the denominators to equal zero must be eliminated from the set of possible solutions.

$$\cos(x) \neq 0 \rightarrow x \neq \pm\frac{\pi}{2} \pm n\pi$$

and

$$\sin(x) \neq -1 \rightarrow x \neq \frac{3\pi}{2} \pm 2n\pi.$$

Therefore, before we even start to solve the problems, the set of real numbers in the set

$$\left\{\pm\frac{\pi}{2} \pm n\pi\right\}$$

must be excluded from the possible set of solutions.

To simplify the equation, let's multiple the second fraction by 1 in the form

$$\frac{1-\sin(x)}{1-\sin(x)},$$

then simplify and solve. The result will be an equation that may be equivalent to the original equation, but an equation where we can solve for x. With this type of manipulations, there may be extraneous solutions. In other words, you may come up with solutions for the new equation that are not solutions to the original equation. Be sure to check your answers with the original equation.

$$\frac{1+\sin(x)}{\cos(x)}+\frac{\cos(x)}{1+\sin(x)}=4$$

$$\frac{1+\sin(x)}{\cos(x)}+\frac{\cos(x)}{1+\sin(x)}\cdot\left(\frac{1-\sin(x)}{1-\sin(x)}\right)=4$$

$$\frac{1+\sin(x)}{\cos(x)}+\frac{\cos(x)\left(1-\sin(x)\right)}{\cos^2(x)}=4$$

$$\frac{1+\sin(x)}{\cos(x)}+\frac{\cos(x)\left(1-\sin(x)\right)}{\cos^2(x)}=4$$

$$\frac{1+\sin(x)}{\cos(x)}+\frac{1-\sin(x)}{\cos(x)}=4$$

$$\frac{2}{\cos(x)}=4$$

$$\cos(x)=\frac{1}{2}.$$

First let's find all answers for $0 \leq x \leq 2\pi$. The cosine function is only positive in the 1st and 4th quadrants. In the first quadrant $\cos(\pi/3)=\cos(60°)=1/2$. By symmetry we know that $\cos(2\pi-\pi/3)=\cos(300°)=1/2$ in the fourth quadrant.

Therefore, the set of all possible answers would be

$$x=\frac{\pi}{3}+2n\pi \text{ or } x=\frac{5\pi}{3}+2n\pi$$

for $n=0,\pm1,\pm2,\ldots$. These answers are never one of the excluded values $\left\{\pm\frac{\pi}{2}\pm n\pi\right\}$ so we don't have to worry about that possibility.

■

3.	Find the general solution for $\sin\theta\cos^2\theta=\sin^3\theta$.

Solution

First and foremost – Don't try to solve this by dividing both sides of the equation by $\sin\theta$. Why not?

It is possible that you will lose solutions when you divide both sides of an equation by the same quantity. It is the problem of ***missing solutions*** – the flip side of ***extraneous solutions***. In this problem you will lose solutions by dividing both sides by $\sin(\theta)$.

Furthermore, to make the division valid we could not consider those values of θ where $\sin\theta = 0$.

The correct approach is to subtract $\sin^3\theta$ from both sides and then proceed to simplify.

$$\begin{aligned}
&\sin\theta\cos^2\theta = \sin^3\theta \\
&\quad \Leftrightarrow \sin\theta\cos^2\theta - \sin^3\theta = 0 \\
&\quad \Leftrightarrow \sin\theta\,(\cos^2\theta - \sin^2\theta) = 0 \\
&\quad \Leftrightarrow \sin\theta\,(\cos\theta - \sin\theta)(\cos\theta + \sin\theta) = 0 \\
&\quad \Leftrightarrow \sin\theta = 0 \quad \text{or} \quad (\cos\theta - \sin\theta) = 0 \quad \text{or} \quad (\cos\theta + \sin\theta) = 0.
\end{aligned}$$

Case 1. $\sin\theta = 0$

$$\sin\theta = 0 \Leftrightarrow \theta = n\pi \text{ for some integer } n.$$

Case 2. $\cos\theta - \sin\theta = 0$

$$\begin{aligned}
\cos\theta - \sin\theta = 0 &\Leftrightarrow \cos\theta = \sin\theta \\
&\Leftrightarrow \cos\theta = \cos\left(\frac{\pi}{2} - \theta\right).
\end{aligned}$$

In general, $\cos(x) = \cos(y)$ implies $x = 2n\pi - y$ or $x = 2n\pi + y$ for some integer n.

In general, $\cos(x) = \cos(y)$ implies $x = 2n\pi \pm y$ for some integer n. So, in this problem we have

$$\theta = 2n\pi \pm \left(\frac{\pi}{2} - \theta\right).$$

But

$$\theta = 2n\pi - \left(\frac{\pi}{2} - \theta\right) \Leftrightarrow \pi\left(2n - \frac{1}{2}\right) = 0$$

which is impossible. Now

$$\theta = 2n\pi + \left(\frac{\pi}{2} - \theta\right) \Leftrightarrow 2\theta = 2n\pi + \frac{\pi}{2} \Leftrightarrow \theta = n\pi + \frac{\pi}{4}.$$

Thus,

$$\cos\theta - \sin\theta = 0 \Leftrightarrow \theta = \frac{\pi}{4} + n\pi$$

for some integer n.

Case 3. $\cos\theta + \sin\theta = 0$

$$\cos\theta + \sin\theta = 0 \Leftrightarrow \cos\theta = -\sin\theta.$$

We know that

$$\cos\theta = \sin\left(\frac{\pi}{2} + \theta\right) \text{ and } -\sin\theta = \sin(-\theta).$$

Therefore

$$\cos\theta = -\sin\theta \Leftrightarrow \sin\left(\frac{\pi}{2} + \theta\right) = \sin(-\theta).$$

In general, $\sin(x) = \sin(y)$ implies $x = 2n\pi + (\pi - y)$ or $x = 2n\pi + y$ for some integer n.

So, in this problem we have

$$\frac{\pi}{2} + \theta = 2n\pi + \big(\pi - (-\theta)\big) \text{ or } \frac{\pi}{2} + \theta = 2n\pi + (-\theta).$$

But

$$\frac{\pi}{2} + \theta = 2n\pi + (\pi + \theta) \Leftrightarrow \left(2n + \frac{1}{2}\right)\pi = 0$$

which is impossible. Now

$$\begin{aligned} \frac{\pi}{2} + \theta = 2n\pi + (-\theta) &\Leftrightarrow 2\theta = 2n\pi - \frac{\pi}{2} \\ &\Leftrightarrow \theta = n\pi - \frac{\pi}{4}. \end{aligned}$$

Thus,

$$\cos\theta + \sin\theta = 0 \Leftrightarrow \theta = n\pi - \frac{\pi}{4}$$

for some integer n.

Pulling the results from all three cases together, we have $\sin\theta\cos^2\theta = \sin^3\theta$ if

$$\theta = n\pi \quad \text{or} \quad \theta = n\pi \pm \frac{\pi}{4}$$

for some integer n.

■

3.3 Equations of Quadratic Type

4.	Solve $\cos(4x) = \sin(2x)$. (Source: https://brownmath.com)

Solution

Here we can make good use of the identity $\cos(2\theta) = 1 - 2\sin^2(\theta)$. Let $\theta = 2x$. Then

$$\cos(4x) = 1 - 2\sin^2(2x).$$

Making this substitution we are left with the equation

$$1 - 2\sin^2(2x) = \sin(2x)$$

or

$$2\sin^2(2x) + \sin(2x) - 1 = 0.$$

This is a quadratic equation in the variable $\sin(2x)$ that factors nicely into

$$(2\sin(2x) - 1)(\sin(2x) + 1) = 0.$$

Therefore,

$$\begin{aligned}\cos(4x) = \sin(2x) &\Leftrightarrow \sin(2x) = \frac{1}{2} \text{ or } \sin(2x) = -1\\ &\Leftrightarrow \sin(2x) = \sin\left(\frac{\pi}{6}\right) \text{ or } \sin(2x) = \sin\left(\frac{3\pi}{2}\right).\end{aligned}$$

Now recall that the general rule of equality for sine functions:

$$\boxed{\sin(x) = \sin(y) \text{ implies } x = 2n\pi + (\pi - y) \text{ or } x = 2n\pi + y \text{ for some integer } n}$$

So,

$$\begin{aligned}\sin(2x) = \sin\left(\frac{\pi}{6}\right) &\Leftrightarrow 2x = 2n\pi + \left(\pi - \frac{\pi}{6}\right) \text{ or } 2x = 2n\pi + \frac{\pi}{6} \text{ for some integer } n\\ &\Leftrightarrow x = n\pi + \frac{5\pi}{12} \text{ or } x = n\pi + \frac{\pi}{12}\end{aligned}$$

and

$$\begin{aligned}\sin(2x) = \sin\left(\frac{3\pi}{2}\right) &\Leftrightarrow 2x = 2n\pi + \left(\pi - \frac{3\pi}{2}\right) \text{ or } 2x = 2n\pi + \frac{3\pi}{2}\\ &\Leftrightarrow x = n\pi - \frac{\pi}{4} \text{ or } x = n\pi + \frac{3\pi}{4}.\end{aligned}$$

But $n\pi - (\pi/4)$ is superfluous because

$$n\pi - \frac{\pi}{4} = m\pi + \frac{3\pi}{4} \ \text{ for } \ m = n + 1.$$

So, the set of all possible solutions is

$$x = \frac{5\pi}{2} + n\pi \ \text{ or } \ x = \frac{\pi}{12} + n\pi \ \text{ or } \ x = \frac{3\pi}{4} + n\pi \ \text{ for some integer } n.$$

■

3.4 Recasting in Terms of Sine and Cosine Only

5.	Solve $2\csc(x) - \cot(x) = \tan(x)$. (Source: mathonweb.com)

<u>Solution</u>

Rewriting the equation in terms of just $\sin(x)$ and $\cos(x)$ is a standard step to consider when several different trig functions appear in the problem.

$$\begin{aligned}
2\csc(x) - \cot(x) = \tan(x) &\Leftrightarrow \frac{2}{\sin(x)} - \frac{\cos(x)}{\sin(x)} = \frac{\sin(x)}{\cos(x)} \\
&\Leftrightarrow \frac{2 - \cos(x)}{\sin(x)} = \frac{\sin(x)}{\cos(x)} \\
&\Leftrightarrow 2\cos(x) - \cos^2(x) = \sin^2(x) \\
&\Leftrightarrow 2\cos(x) = \sin^2(x) + \cos^2(x) = 1 \\
&\Leftrightarrow \cos(x) = \frac{1}{2} \\
&\Leftrightarrow \cos(x) = \cos\left(\frac{\pi}{3}\right) \\
&\Leftrightarrow x = 2n\pi - \frac{\pi}{3} \ \text{ or } \ 2n\pi + \frac{\pi}{3}.
\end{aligned}$$

■

3.5 Matching with a Pythagorean Relationship

6.	Solve $\sec(x) = \tan(x) + 1$ for $0 \le x < 2\pi$.

Solution

Knowing the Pythagorean relationship $\sec^2(\theta) = \tan^2(\theta) + 1$ is a hint that we should square both sides. But squaring can introduce extraneous solutions so we will have to check for this at the end of the problem.

$$\begin{gathered}\sec(x) = \tan(x) + 1 \\ \sec^2(x) = (\tan(x) + 1)^2 = \tan^2(x) + 2\tan(x) + 1 \\ \tan^2(x) + 1 = \tan^2(x) + 2\tan(x) + 1 \\ 2\tan(x) = 0 \\ \tan(x) = 0 \\ \tan(x) = \tan(0) \\ x = n\pi \text{ for some integer } n.\end{gathered}$$

So, given the restriction $0 \le x < 2\pi$, our only two candidate solutions are $x = 0$ and $x = \pi$. But as we mentioned above we must check for extraneous solutions.

Is $x = 0$ an actual solution? Does $\sec(0) \overset{?}{=} \tan(0) + 1$? Yes, because $\sec(0) = 1$ and $\tan(0) = 0$.

Is $x = \pi$ an actual solution? Does $\sec(\pi) \overset{?}{=} \tan(\pi) + 1$? No, because $\sec(\pi)$ is undefined and $\tan(\pi) = 0$.

So, $x = 0$ is the only solution on $[0, 2\pi)$.

■

3.6 Using the "Sum to Product" Identity

Sum to Product	
$\cos(\alpha) + \cos(\beta) = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$	(4)
$\cos(\alpha) - \cos(\beta) = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$	(5)
$\sin(\alpha) + \sin(\beta) = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$	(6)
$\sin(\alpha) - \sin(\beta) = 2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)$	(7)

7.	Solve $\sin x + \sin 5x = \sin 3x$ for x in $[0, \pi/2]$. (Source: Math-Only-Math.com)

Solution

$$\sin(5x) + \sin(x) = \sin(3x)$$

$$\Leftrightarrow 2\sin\left(\frac{5x+x}{2}\right)\cos\left(\frac{5x-x}{2}\right) = \sin(3x)$$

(using Identity (6) above)

$$\Leftrightarrow 2\sin(3x)\cos(2x) = \sin(3x)$$

(remember not to divide out a variable on both sides)

$$\Leftrightarrow 2\sin(3x)\cos(2x) - \sin(3x) = 0$$

$$\Leftrightarrow \sin(3x)\,(2\cos(2x) - 1) = 0$$

Case 1. $\sin(3x) = 0$

$$\sin(3x) = 0 \Leftrightarrow 3x = n\pi$$

$$\Leftrightarrow x = \frac{n\pi}{3}.$$

Case 2. $2\cos(2x) - 1 = 0$

$$2\cos(2x) - 1 = 0$$
$$2\cos(2x) = 1$$
$$\cos(2x) = \frac{1}{2}$$
$$\cos(2x) = \cos\left(\frac{\pi}{3}\right)$$

In general, $\cos(x) = \cos(y)$ implies $x = 2n\pi - y$ or $x = 2n\pi + y$ for some integer n.

$$2x = 2n\pi - \frac{\pi}{3} \text{ or } 2x = 2n\pi + \frac{\pi}{3}$$
$$x = n\pi - \frac{\pi}{6} \text{ or } x = n\pi + \frac{\pi}{6}$$

So, the set of all possible solutions would be all x such that

$$x = \frac{n\pi}{3},\ x = n\pi - \frac{\pi}{6},\ n\pi + \frac{\pi}{6}$$

for some integer n. But the problem restricts x to the interval $[0, \pi/2]$.

Taking $n = 0$ in the general form $x = n\pi/3$ yields $x = 0$ which is in $[0, \pi/2]$.

Taking $n = 1$ in the general form $x = n\pi/3$ yields $x = \pi/3$ which is in $[0, \pi/2]$.

Taking $n = 0$ in the general form $x = n\pi + \pi/6$ yields $x = \pi/6$ which is in $[0, \pi/2]$.

All other cases yield solutions outside $[0, \pi/2]$.

So, the set of all possible solutions would be $x = 0, x = \pi/6$ and $x = \pi/3$.

■

8.	Solve $\cos(3x) + \sin(2x) - \sin(4x) = 0$. (Source: *Methods of Solving Nonstandard Problems*, Ellina Grigorieva)

Solution

$$\cos(3x) + (\sin(2x) - \sin(4x)) = 0$$

$$\Leftrightarrow \cos(3x) - 2\sin(x)\cos(3x) = 0 \qquad \text{(using Identity (7) above)}$$

$$\Leftrightarrow \cos(3x)\,(1 - 2\sin(x)) = 0$$

$$\Leftrightarrow \cos(3x) = 0 \;\text{ or }\; \sin(x) = \frac{1}{2}$$

$$\Leftrightarrow x = \frac{\pi}{6} + \frac{\pi}{3}n \;\text{ or }\; x = (-1)^k\frac{\pi}{6} + \pi k \;\text{ for some integers } n \text{ and } k.$$

However, the values $(-1)^k(\pi/6) + \pi k$ for some integer k are all special cases of $(\pi/6) + (\pi/3)n$ for some integer n. That is, we can simplify our final answer to

$$x = \frac{\pi}{6} + \frac{\pi}{3}n \;\text{ for some integer } n.$$

3.7 Using the "Product to Sum" Identity

Product to Sum	
$\cos(\alpha)\cos(\beta) = \frac{1}{2}(\cos(\alpha - \beta) + \cos(\alpha + \beta))$	(1)
$\sin(\alpha)\sin(\beta) = \frac{1}{2}(\cos(\alpha - \beta) - \cos(\alpha + \beta))$	(2)
$\sin(\alpha)\cos(\beta) = \frac{1}{2}(\sin(\alpha + \beta) + \sin(\alpha - \beta))$	(3)

9.	Solve $\sin(5x)\cos(3x) = \sin(6x)\cos(2x)$. (Source: *Methods of Solving Nonstandard Problems*, Ellina Grigorieva)

Solution

By the Identity (3) above we have

$$\sin(5x)\cos(3x) = \frac{1}{2}(\sin(5x + 3x) + \sin(5x - 3x)) = \frac{1}{2}(\sin(8x) + \sin(2x))$$

$$\sin(6x)\cos(2x) = \frac{1}{2}(\sin(6x + 2x) + \sin(6x - 2x)) = \frac{1}{2}(\sin(8x) + \sin(4x)).$$

Making these substitutions, the original equation is transformed to

$$\frac{1}{2}(\sin(8x) + \sin(2x)) = \frac{1}{2}(\sin(8x) + \sin(4x))$$

$$\Leftrightarrow \sin(2x) = \sin(4x)$$

$$\Leftrightarrow \sin(2x) - \sin(4x) = 0$$

$$\Leftrightarrow 2\sin\left(\frac{2x - 4x}{2}\right)\cos\left(\frac{2x + 4x}{2}\right) = 0$$ (using Identity (7) above)

$$\Leftrightarrow 2\sin(-x)\cos(3x) = 0$$

$$\Leftrightarrow -2\sin(x)\cos(3x) = 0$$

$$\Leftrightarrow \sin(x) = 0 \text{ or } \cos(3x) = 0$$

$$\Leftrightarrow x = \pi n \text{ or } 3x = \frac{\pi}{2} + \pi k \text{ for some integers } n \text{ and } k$$

$$\Leftrightarrow x = \pi n \text{ or } x = \frac{\pi}{6} + \frac{\pi k}{3} \text{ for some integers } n \text{ and } k$$

■

3.8 $a\ \sin(x) + b\ \cos(y) = c$

10.	Solve $\sqrt{3}\cos(x) + \sin(x) = \sqrt{2}$, for $0 \leq x \leq 2\pi$.

Solution

Useful Identity:

$$a\cos(x) + b\sin(x) = \sqrt{a^2 + b^2} \cdot \cos(x - \theta)$$

where θ is that angle in $0 \leq \theta \leq 2\pi$ such that

$$\cos(\theta) = \frac{a}{\sqrt{a^2 + b^2}} \text{ and } \sin(\theta) = \frac{b}{\sqrt{a^2 + b^2}}.$$

Note: We can determine which of the four quadrants θ belongs in according to which combination of positive and negative $\cos(\theta)$ and $\sin(\theta)$ are.

To understand why this works, notice that with these substitutions we have

$$a\cos(x) + b\sin(x) = \cos(x)\cos(\theta) + \sin(x)\sin(\theta) = \cos(x - \theta).$$

Furthermore, these are valid substitutions because with these substitutions we have $-1 \leq \cos(\theta) \leq 1$, $-1 \leq \sin(\theta) \leq 1$ and $\sin^2(\theta) + \cos^2(\theta) = 1$.

Using the above identity, we have

$$\cos(\theta) = \frac{\sqrt{3}}{\sqrt{\left(\sqrt{3}\right)^2 + 1^2}} = \frac{\sqrt{3}}{2}$$

and

$$\sin(\theta) = \frac{1}{\sqrt{\left(\sqrt{3}\right)^2 + 1^2}} = \frac{1}{2}.$$

We see that $\cos(\theta)$ and $\sin(\theta)$ are both positive so θ is in the first quadrant. In particular, we see that $\theta = \pi/6$.

So,

$$\frac{\sqrt{3}}{2}\cos(x) + \frac{1}{2}\sin(x) = \cos\left(x - \frac{\pi}{6}\right).$$

Therefore,

$$\sqrt{3}\cos(x) + \sin(x) = 2\cos\left(x - \frac{\pi}{6}\right) = \sqrt{2}, \text{ for } 0 \leq x \leq 2\pi$$

or

$$\cos\left(x - \frac{\pi}{6}\right) = \frac{1}{\sqrt{2}}, \text{ for } 0 \leq x \leq 2\pi.$$

Therefore

$$x - \frac{\pi}{6} = \frac{\pi}{4} \quad \text{or} \quad x - \frac{\pi}{6} = 2\pi - \frac{\pi}{4}$$

$$x = \frac{\pi}{4} + \frac{\pi}{6} = \frac{5\pi}{12} \quad \text{or} \quad x = 2\pi - \frac{\pi}{4} + \frac{\pi}{6} = \frac{23\pi}{12}.$$

■

3.9 Maximize (minimize) $a\ \sin(x) + b\ \cos(y)$

11.	Find the maximum value of $\sin(x) + \cos(x)$.

Solution

This is another example where the technique below can be used.

Useful Identity:

$$a\cos(x) + b\sin(x) = \sqrt{a^2 + b^2} \cdot \cos(x - \theta)$$

where θ is that angle in $0 \leq \theta \leq 2\pi$ such that

$$\cos(\theta) = \frac{a}{\sqrt{a^2 + b^2}} \text{ and } \sin(\theta) = \frac{b}{\sqrt{a^2 + b^2}}.$$

Note: We can determine which of the four quadrants θ belongs in according to which combination of positive and negative $\cos(\theta)$ and $\sin(\theta)$ are.

$$\sin(x) + \cos(x) = 1 \cdot \cos(x) + 1 \cdot \sin(x) \Longrightarrow a = 1, b = 1.$$

Let θ be that angle in $[0,2\pi]$ such that

$$\cos(\theta) = \frac{1}{\sqrt{1^2 + 1^2}} = \frac{1}{\sqrt{2}} \text{ and } \sin(\theta) = \frac{1}{\sqrt{1^2 + 1^2}} = \frac{1}{\sqrt{2}}.$$

Because both $\sin(\theta)$ and $\cos(\theta)$ are positive, we are looking for an angle in the 1^{st} quadrant. By inspection we see that $\theta = \pi/4$.

Therefore,

$$\sin(x) + \cos(x) = \sqrt{1^2 + 1^2} \cdot \cos\left(x - \frac{\pi}{4}\right) = \sqrt{2}\cos\left(x - \frac{\pi}{4}\right).$$

In this form we can clearly see that the maximum value of $\sin(x) + \cos(x)$ is $\sqrt{2}$ and this occurs when $x = \pi/4$.

■

3.10 System of Trigonometric Equations

12.	Solve the system $\sin(x) + \sin(y) = 1$ and $\cos(x) + \cos(y) = 0$ for x and y. (Source: math.stackexchange.com)

Solution

First a couple notes about steps frequently needed in problem similar to this one. To keep the notation simple, let A represent some function of x and y (e.g. $\sin(x) + \sin(y)$) and let B represent some function of x and y and consider solving the system $A = a$ and $B = b$ for x and y for some constants a and b.

(i) If you replace the equation $A = a$ with $A^2 = a^2$ you will not lose any solutions but you may gain some false (extraneous) ones. So, you will need to check all solutions to see if they really solve the original equations $A = a$ and $B = b$.

(ii) sometimes it helps to replace the equation $A = a$ with some linear function of A and B (e.g. $c_1A + c_2B = c_1a + c_2b, c_1 \neq 0$). This does not change the solution set. That is, the solution sets to the two systems

$$\begin{Bmatrix} A = a \\ B = b \end{Bmatrix} \text{ and } \begin{Bmatrix} c_1A + c_2B = c_1a + c_2b, c_1 \neq 0 \\ B = b \end{Bmatrix}$$

are the same because this is an invertible transformation. A typical case is where you replace $A = a$ with $A - B = a - b$ or $A + B = a + b$.

We will begin by squaring both equations. This will allow us to take advantage of the Pythagorean relationship $\sin^2(\theta) + \cos^2(\theta) = 1$.

So, we have

$$\sin^2(x) + \sin^2(y) + 2\sin(x)\sin(y) = 1$$

$$\cos^2(x) + \cos^2(y) + 2\cos(x)\cos(y) = 0.$$

Now we will replace the first equation with the sum of these two equations. The sum of these two equations, after simplification, is $\sin(x)\sin(y) + \cos(x)\cos(y) = -1/2$. But we can further simplify by applying the angle addition formula

$$\sin(\theta)\sin(\beta) + \cos(\theta)\cos(\beta) = \cos(\theta - \beta).$$

We can then rewrite the equation $\cos(x-y)=-1/2$ as $\cos(x-y)=\cos(2\pi/3)$ in as much as $\cos(2\pi/3)=-1/2$. This rewrite will make it plain how to use our result on the "Equality of Same Trig Functions" given in the previous table.

Finally, it will be useful to note that

$$\cos(x)+\cos(y)=0 \Leftrightarrow \cos(x)=-\cos(y) \Leftrightarrow \cos(x)=\cos(\pi-y).$$

So, the original system of equations has been transformed to the equivalent (up to potential extraneous solutions) system:

$$\cos(x-y)=\cos\left(\frac{2\pi}{3}\right)$$
$$\cos(x)=\cos(\pi-y).$$

The solution to the first equation is

$$x-y=2n_1\pi-\frac{2\pi}{3} \;\text{ or }\; x-y=2n_2\pi+\frac{2\pi}{3}$$

and the solution to the second equation is

$$x=2n_3\pi-(\pi-y) \;\text{ or }\; x=2n_4\pi+(\pi-y)$$

where n_1, n_2, n_3 and n_4 can be any (not necessarily equal) integers.

These leads to four cases:

Case 1	$x=2n_1\pi-\frac{2\pi}{3}+y$ $x=2n_3\pi-\pi+y$	Case 2	$x=2n_1\pi-\frac{2\pi}{3}+y$ $x=2n_4\pi+\pi-y$
Case 3	$x=2n_2\pi+\frac{2\pi}{3}+y$ $x=2n_3\pi-\pi+y$	Case 4	$x=2n_2\pi+\frac{2\pi}{3}+y$ $x=2n_4\pi+\pi-y$

Cases 1 and 3 lead to contradictions because the variable y cancels out and we are left with an equality that is never true.

Case 2	$x=2n_1\pi-\frac{2\pi}{3}+y$ $x=2n_4\pi+\pi-y$

$$2n_1\pi - \frac{2\pi}{3} + y = 2n_4\pi + \pi - y$$

$$2y = 2\pi(n_4 - n_1) + \frac{5\pi}{3}$$

$$y = (n_4 - n_1)\pi + \frac{5\pi}{6}$$

$$x = 2n_1\pi - \frac{2\pi}{3} + (n_4 - n_1)\pi + \frac{5\pi}{6} = (n_4 + n_1)\pi + \frac{\pi}{6}$$

$$(x, y) = \left((n_4 + n_1)\pi + \frac{\pi}{6}, (n_4 - n_1)\pi + \frac{5\pi}{6}\right) \text{ for some integers } n_1 \text{ and } n_4.$$

Case 4

$$x = 2n_2\pi + \frac{2\pi}{3} + y$$
$$x = 2n_4\pi + \pi - y$$

$$2n_2\pi + \frac{2\pi}{3} + y = 2n_4\pi + \pi - y$$

$$2y = 2\pi(n_4 - n_2) + \pi - \frac{2\pi}{3}$$

$$y = (n_4 - n_2)\pi + \frac{\pi}{6}$$

$$x = 2n_2\pi + \frac{2\pi}{3} + \left((n_4 - n_2)\pi + \frac{\pi}{6}\right) = (n_4 + n_2)\pi + \frac{5\pi}{6}$$

$$(x, y) = \left((n_4 + n_2)\pi + \frac{5\pi}{6}, (n_4 - n_2)\pi + \frac{\pi}{6}\right) \text{ for some integers } n_2 \text{ and } n_4.$$

Are any of these solutions extraneous? Fortunately, we don't have to plug each solution back into the original equations. Looking at Case 2, the only relevant question is whether $(n_4 + n_1)$, $(n_4 - n_1)$, $(n_4 + n_2)$ and $(n_4 - n_2)$ are even or odd. All cases where n_1 and n_4 have the same parity (*i.e.* are both even or both odd) reduce to $(x, y) = (5\pi/6, \pi/6)$. All cases where n_1 and n_4 have different parity reduce to $(x, y) = (11\pi/6, 7\pi/6)$. So in fact there are just two cases to check. It turns out that $(x, y) = (11\pi/6, 7\pi/6)$ does not satisfy the equation $\sin(x) + \sin(y) = 1$. The same analysis holds for the solutions that arise from Case 4.

In summary all valid (non-extraneous) solutions will have the form

$$(x, y) = \left(2n\pi + \frac{5\pi}{6}, 2m\pi + \frac{\pi}{6}\right) \text{ for some integers } n \text{ and } m$$

or

$$(x, y) = \left(2n\pi + \frac{\pi}{6}, 2m\pi + \frac{5\pi}{6}\right) \text{ for some integers } n \text{ and } m.$$

■

13.	Determine all values of A, if $\cos A - \cos B = -\sin 80°$ and $A + B = 60°$, where $0° \leq A \leq 180°$. (Source: MSHSML 3C154)

Solution

Because $B = 60° - A$ we have

$$\begin{aligned}
\cos A - \cos B &= \cos A - \cos(60° - A) \\
&= \cos A - (\cos 60° \cos A + \sin 60° \sin A) \\
&= \cos A\,(1 - \cos 60°) - \sin 60° \sin A \\
&= \cos A\left(1 - \frac{1}{2}\right) - \frac{\sqrt{3}}{2}\sin A \\
&= \sin 30° \cos A - \cos 30° \sin A \\
&= \sin(30° - A) \\
&= -\sin(A - 30°).
\end{aligned}$$

Note: In the above argument we have used the following two Subtraction Identities

$$\cos(x - y) = \cos x \cos y + \sin x \sin y$$
$$\sin(x - y) = \sin x \cos y - \cos x \sin y$$

and the Even-Odd Identity

$$\sin(-x) = -\sin x$$

as stated in the Study Guide for Meet 2, Event C.

Therefore,

$$-\sin(A - 30°) = -\sin(80°).$$

Now recall the general equivalence result (stated in radians) stated earlier in this study guide that

$$\boxed{\sin(x) = \sin(y) \Leftrightarrow x = 2n\pi + (\pi - y) \text{ or } x = 2n\pi + y \text{ for some integer } n.}$$

From this result (after translating into degrees) it follows that for some integer n

$$A - 30° = 2n(180°) + (180° - 80°)$$

or

$$A - 30° = 2n(180°) + 80°.$$

Simplifying, we have

$$A = 110° + 360°n \quad \text{or} \quad A = 130° + 360°n$$

for some integer n.

But recall that the statement of the problem requires that $0° \leq A \leq 180°$. Therefore, the only possible values of A are $110°$ and $130°$.

■

14.	How many solutions to $\sin\theta = -\cos\theta$ exist on the interval $0 \leq \theta < 2\pi$? (Source: MSHSML 3C101)

Solution

$$\sin\theta = -\cos\theta \;\Rightarrow\; \frac{\sin\theta}{\cos\theta} = \frac{-\cos\theta}{\cos\theta} \;\Rightarrow\; \tan\theta = -1, \textit{which occurs twice on the unit circle.}$$

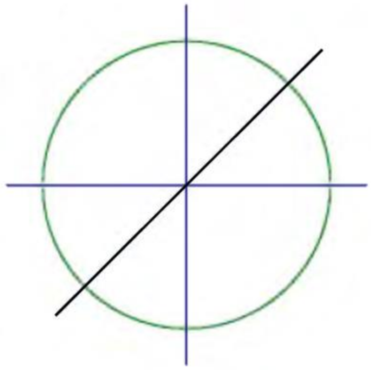
■

15.	Find the solution of the equation $\cos x - \sin x = 0$ where $\pi \leq x < 3\pi/2$. (Source: MSHSML 3C081)

Solution

Adding $\sin x$ *to both sides yields* $\cos x = \sin x$.

This certainly occurs at $x = \frac{\pi}{4}$*, but this is not in the desired domain. Reflect across the origin.*



■

16.	Find all solutions to the equation $\sec^2\theta - 3\sec\theta - 2 = 0$ on the interval $0 \le \theta < 2\pi$. (Source: MSHSML 3C082)

Solution

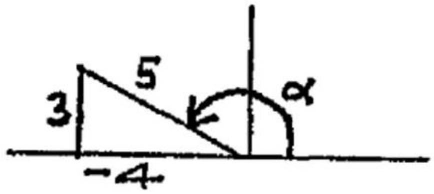
Let $y = \sec\theta$. *Then* $y^2 - 3y - 2 = 0$. *By the Quadratic Formula,* $y \in \{-0.56155, 3.56155\}$. *Since* $\sec\theta$ *must be* ≤ -1 *or* ≥ 1, *we exclude* -0.56155. $\sec\theta \approx 3.56155 \Rightarrow \cos\theta \approx 0.28078$.

$$\theta \in \{1.286, 4.997\}$$

■

17.	The obtuse angle in a triangle has a sine of $3/5$. What is the tangent of this angle? (Source: MSHSML 3C051)

Solution



$$\tan \alpha = \frac{3}{-4}$$

■

4 De Moivre's Theorem and the Roots of Unity

Complex Number	A complex number is any number that can be written in the form $a + bi$, where a and b are real numbers and i is the imaginary unit $(i^2 = -1)$. a is called the *real part*, and b is called the *imaginary part*.
Equality of Two Complex Numbers	$a + bi = c + di \Leftrightarrow a = c$ AND $b = d$ *i.e.* the real parts must be equal and the imaginary parts must be equal.
Addition and Subtraction of Complex Numbers	Combine like terms. $(a + bi) + (c + di) = (a + c) + (b + d)i$ $(a + bi) - (c + di) = (a - c) + (b - d)i$
Multiplication of Complex Numbers	Use the definition of $i^2 = -1$ and the FOIL method: $(a + bi)(c + di) = (ac - bd) + (ad + bc)i$
Multiplication of a Complex Number by a Constant	$c \cdot (a + bi) = ac + (bc)i$

Conjugate of a Complex Number	The complex conjugate of $a + bi$ is defined to be $a - bi$.
Notation for a Complex Number and its Conjugate	If z is a complex number then we denote the conjugate of z by $\overline{z}$. *e.g.* If $z = 3 + 2i$, then $\overline{z} = 3 - 2i$ and if $z = -1 - 2i$, then $\overline{z} = -1 + 2i$. We can also write this as $\overline{a + bi} = a - bi$.
Product of a Complex Number and its Conjugate, Part 1.	$(a + bi)(a - bi) = \big(a^2 - b(-b)\big) + (-ab + ab)i = a^2 + b^2$
Division of Complex Numbers	$\frac{a + bi}{c + di} = \left(\frac{a + bi}{c + di}\right)\left(\frac{c - di}{c - di}\right) = \frac{(ac + bd) + (-ad + bc)i}{c^2 + d^2}$
Absolute Value (Norm) of a Complex Number	The absolute value of a complex number is defined by $\lvert a + bi\rvert = \sqrt{a^2 + b^2}$.
Product of a Complex Number and its Conjugate, Part 2.	$z \cdot \overline{z} = \lvert z\rvert^2$ Proof: Let $z = a + bi$. Then $z \cdot \overline{z} = (a + bi)(a - bi) = a^2 + b^2 = \lvert a + bi\rvert^2 = \lvert z\rvert^2$
Multiplicative Inverse of a Complex Number	$z \cdot \left(\frac{\overline{z}}{\lvert z\rvert^2}\right) = 1$, provided $\lvert z\rvert^2 \neq 0$ That is, z and $\overline{z}/\lvert z\rvert^2$ are multiplicative inverses. Proof: We just proved that $z \cdot \overline{z} = \lvert z\rvert^2$. So, on dividing both sides by $\lvert z\rvert^2$, we have $z \cdot \left(\frac{\overline{z}}{\lvert z\rvert^2}\right) = 1$ provided $\lvert z\rvert^2 \neq 0$, *i.e.* provided $z \neq 0 + 0i$.

Absolute Value of a Product	$\lvert z_1 \cdot z_2 \rvert = \lvert z_1 \rvert \cdot \lvert z_2 \rvert$ Proof: Let $z_1 = a + bi$ and let $z_2 = c + di$. Then $z_1 \cdot z_2 = (a + bi)(c + di) = (ac - bd) + (ad + bc)i$ Therefore, $\begin{aligned} \lvert z_1 \cdot z_2 \rvert^2 &= (ac - bd)^2 + (ad + bc)^2 \\ &= a^2c^2 - 2abcd + b^2d^2 + a^2d^2 + 2abcd + b^2c^2 \\ &= a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2 \\ &= a^2(c^2 + d^2) + b^2(c^2 + d^2) \\ &= (a^2 + b^2)(c^2 + d^2) \\ &= \lvert z_1 \rvert^2 \cdot \lvert z_2 \rvert^2. \end{aligned}$ Taking square roots of both sides gives us the final result $\lvert z_1 \cdot z_2 \rvert = \lvert z_1 \rvert \cdot \lvert z_2 \rvert$.
Absolute Value of a Power	$\lvert z^n \rvert = \lvert z \rvert^n$ for any positive integer n.
Absolute Value of a Ratio	$\left\lvert \frac{z_1}{z_2} \right\rvert = \frac{\lvert z_1 \rvert}{\lvert z_2 \rvert}$
Absolute Value of a Conjugate	$\lvert z \rvert = \lvert \overline{z} \rvert$ Proof: Let $z = a + bi$ and $\overline{z} = a - bi$. Then $\lvert \overline{z} \rvert = \lvert a - bi \rvert = \sqrt{a^2 + (-b)^2} = \sqrt{a^2 + b^2} = \lvert a + bi \rvert = \lvert z \rvert$.
Conjugate of Sum, Difference, Product and Ratio	$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$ $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$ $\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}$ $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$
Conjugate of a Power	$\overline{z^n} = (\overline{z})^n$ for any positive integer n.

Powers of i	
$i^{4m+k} = i^k$ and $i^{-4m+k} = i^k$ for any integer m	
$i^0 = 1$	$i^0 = i^4 = i^8 = i^{12} = i^{16} = \cdots = i^{4k}$ for any integer k
$i^1 = i = \sqrt{-1}$	$i^1 = i^5 = i^9 = i^{13} = \cdots = i^{4k+1}$ for any integer k
$i^2 = -1$	$i^2 = i^6 = i^{10} = i^{14} = \cdots = i^{4k+2}$ for any integer k
$i^3 = -i$	$i^3 = i^7 = i^{11} = i^{15} = \cdots = i^{4k+3}$ for any integer k
$i^{-1} = -i$	$i^3 = i^{-1} = i^{-5} = i^{-9} = i^{-13} = \cdots = i^{-4k+3}$ for any integer k
$i^{-2} = -1$	$i^2 = i^{-2} = i^{-6} = i^{-10} = i^{-14} = \cdots = i^{-4k+2}$ for any integer k
$i^{-3} = i$	$i^1 = i^{-3} = i^{-7} = i^{-11} = i^{-15} = \cdots = i^{-4k+1}$ for any integer k
$i^{-4} = 1$	$i^0 = i^{-4} = i^{-8} = i^{-12} = i^{-16} = \cdots = i^{-4k}$ for any integer k
e.g. $i^{27} = i^{4(6)+3} = i^3 = -i$ $i^{-27} = i^{-4(7)+1} = i^1 = i.$	

All of the notes from **Test 1D** concerning complex roots of quadratic equations are relevant for Test 3C and are **reproduced here**.

4.1 Complex Roots of Quadratic Equations

Complex Number	A complex number is any number that can be written in the form $a + bi$, where a and b are real numbers and i is the imaginary unit $(i^2 = -1)$. a is called the real part, and b is called the imaginary part.
Equality of Two Complex Numbers	$a + bi = c + di \Leftrightarrow a = c$ <u>AND</u> $b = d$ *i.e.* the real parts must be equal and the imaginary parts must be equal.

Conjugate of a Complex Number	The complex conjugate of $a + bi$ is defined to be $a - bi$.
Notation for a Complex Number and its Conjugate	If z is a complex number then we denote the conjugate of z by $\overline{z}$. $e.g.$ If $z = 3 + 2i$, then $\overline{z} = 3 - 2i$ and if $z = -1 - 2i$, then $\overline{z} = -1 + 2i$. We can also write this as $\overline{a + bi} = a - bi$.

Complex Conjugate Roots Theorem

If $p(x)$ is any polynomial (of any degree) **with real coefficients** and if $a + bi$ is a complex root of $p(x)$, then its complex conjugate $a - bi$ is also a root of $p(x)$.

Note: This theorem does not continue to hold true when some or all coefficients of the polynomial are complex numbers.

Discriminant

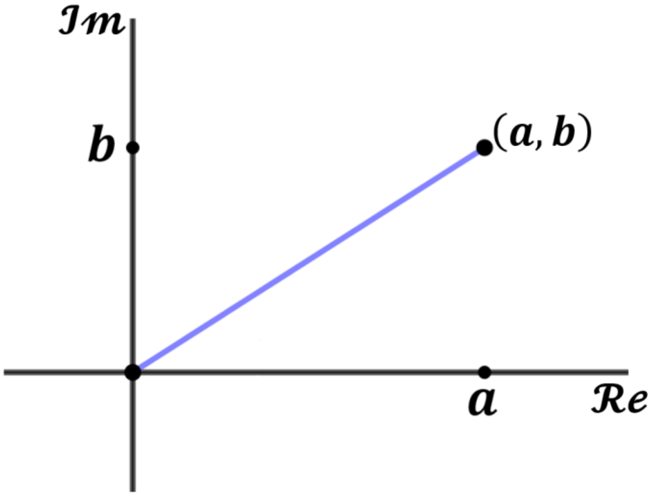
$b^2 - 4ac$ is called the discriminant of the quadratic function $f(x) = ax^2 + bx + c$.

If …

$b^2 - 4ac > 0$	$f(x)$ has two distinct real roots
$b^2 - 4ac = 0$	$f(x)$ has two equal real roots
$b^2 - 4ac < 0$	$f(x)$ has complex conjugate roots

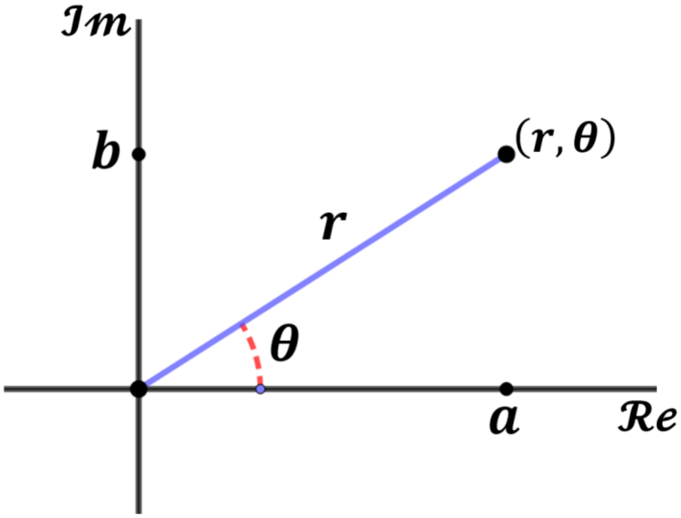
Graphing

We can represent a complex number as a point on the $\mathcal{Re}$ (Real) and $\mathcal{Im}$ (Imaginary) axes. The complex number $a + bi$ has real part a and imaginary part b and is located at the coordinates (a, b) in this coordinate system.



Polar Form of a Complex Number

But we can also represent the point (a, b) by specifying its (r, θ) values, where r is the absolute distance this point is from the origin $(0,0)$ and θ is the angle this point makes with the positive $\mathcal{Re}$ (Real) axis.



We refer to (a, b) as the rectangular form and (r, θ) as the polar form of a complex number. We can calculate one pair of coordinates from the other.

Given the rectangular coordinates (a, b) we can calculate the polar coordinates (r, θ) through

$$r = \sqrt{a^2 + b^2}$$

and

$$\theta = \begin{cases} \tan^{-1}\left(\frac{b}{a}\right) & a > 0 \\ \pi + \tan^{-1}\left(\frac{b}{a}\right) & a < 0 \\ \pi/2 & a = 0, b > 0 \\ 3\pi/2 & a = 0, b < 0. \end{cases}$$

Given the polar coordinates (r, θ) we can calculate the rectangular coordinates (a, b) through

$$a = r\cos(\theta)$$
$$b = r\sin(\theta).$$

That is, $a + bi = r\cos(\theta) + i\,r\sin(\theta) = r\big(\cos(\theta) + i\sin(\theta)\big)$. The number r is called the **modulus** and the angle θ is called the **argument** of the complex number $a + bi$. In the polar form the origin is also called the **pole**.

You might notice that the modulus r for a complex number in polar form $r\big(\cos(\theta) + i\sin(\theta)\big)$ is equivalent to the norm for a complex number expressed in rectangular form. We note that if

$$a + bi = r\big(\cos(\theta) + i\sin(\theta)\big) = \big(r\cos(\theta)\big) + i\big(r\sin(\theta)\big)$$

then

$$\sqrt{a^2 + b^2} = \sqrt{\big(r\cos(\theta)\big)^2 + \big(r\sin(\theta)\big)^2} = \sqrt{r^2\big(\cos^2(\theta) + \sin^2(\theta)\big)} = \sqrt{r^2(1)} = r.$$

Alternate Notations

The abbreviated notations

$$\cos(\theta) + i\sin(\theta) = \operatorname{cis}(\theta)$$

and

$$r\big(\cos(\theta) + i\sin(\theta)\big) = r\angle\theta$$

are commonly used so don't be thrown if you see them used.

Multiplication and Division (in Polar Form)

Let the two complex numbers z_1 and z_2 have the polar forms

$$z_1 = r_1\left(\cos(\theta_1) + i\sin(\theta_1)\right) \text{ and } z_2 = r_2\left(\cos(\theta_2) + i\sin(\theta_2)\right)$$

then

$$z_1 \cdot z_2 = r_1 r_2\left(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\right)$$

and

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right), \text{ provided } z_2 \neq 0.$$

4.2 Powers and Roots in Polar Form – DeMoivre's Theorem

$$\left(r\left(\cos(\theta) + i\sin(\theta)\right)\right)^n = r^n\left(\cos(n\theta) + i\sin(n\theta)\right) \qquad \textbf{(DeMoivre's Theorem)}$$

and

$$\left(r\left(\cos(\theta) + i\sin(\theta)\right)\right)^{-n} = \left(\frac{1}{r}\right)^n\left(\cos(n\theta) - i\sin(n\theta)\right)$$

and

$$\sqrt[n]{r\left(\cos(\theta) + i\sin(\theta)\right)} = r^{(1/n)}\left(\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right), \quad k = 0,1,\dots,n-1$$

and

$$\left(r\left(\cos(\theta) + i\sin(\theta)\right)\right)^{p/q} = r^{(p/q)}\left(\cos\left(\frac{p\theta + 2k\pi}{q}\right) + i\sin\left(\frac{p\theta + 2k\pi}{q}\right)\right), \quad k = 0,1,\dots,q-1$$

for positive integers p, q and n.

4.3 Exponential Form of Complex Number (Euler's Formula)

For θ measured in radians

$$re^{i\theta} = r\left(\cos(\theta) + i\sin(\theta)\right) \quad \textbf{(Euler's Formula)}$$

Therefore,

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

$$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos(\theta) - i\sin(\theta).$$

It follows that

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

and

$$\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

4.4 Complex Roots of Unity

A **root of unity** is a complex number z which satisfies $z^n = 1$ for some positive integer n. Equivalently, we write $z = \sqrt[n]{1}$.

Solving for the n^{th} roots of unity

Since $z^n = 1$ is a polynomial with complex coefficients and a degree of n, it must have exactly n complex roots according to the Fundamental Theorem of Algebra. These n complex roots are the n values of $z = \sqrt[n]{1}$.

We will start by finding the rectangular form $a + bi$ of the number 1. As this is a real number it's imaginary part b equals 0 and we simply have $1 = a + bi = 1 + 0i$.

Now we will convert this to the polar form $r\left(\cos(\theta) + i\sin(\theta)\right)$. We know that $r = \sqrt{a^2 + b^2} = \sqrt{1^2 + 0^2} = 1$ and $\theta = \tan^{-1}(b/a) = \tan^{-1}(0/1) = \tan^{-1}(0) = 0$.

That is,

$$1 = 1 + 0i = 1\left(\cos(0) + i\sin(0)\right).$$

Now we solve for all the n^{th} roots of unit using the formula

$$\sqrt[n]{r\big(\cos(\theta)+i\sin(\theta)\big)} = r^{(1/n)}\left(\cos\left(\frac{\theta+2k\pi}{n}\right)+i\sin\left(\frac{\theta+2k\pi}{n}\right)\right), \;\; k = 0,1,\dots,n-1.$$

So the n^{th} roots of unity are

$$\sqrt[n]{1} = \sqrt[n]{1\left(\cos(0)+i\sin(0)\right)} = 1^{1/n}\left(\cos\left(\frac{0+2k\pi}{n}\right)+i\sin\left(\frac{0+2k\pi}{n}\right)\right), \;\; k = 0,1,\dots,n-1$$

$$= \cos\left(\frac{2k\pi}{n}\right)+i\sin\left(\frac{2k\pi}{n}\right), \;\; k = 0,1,\dots,n-1.$$

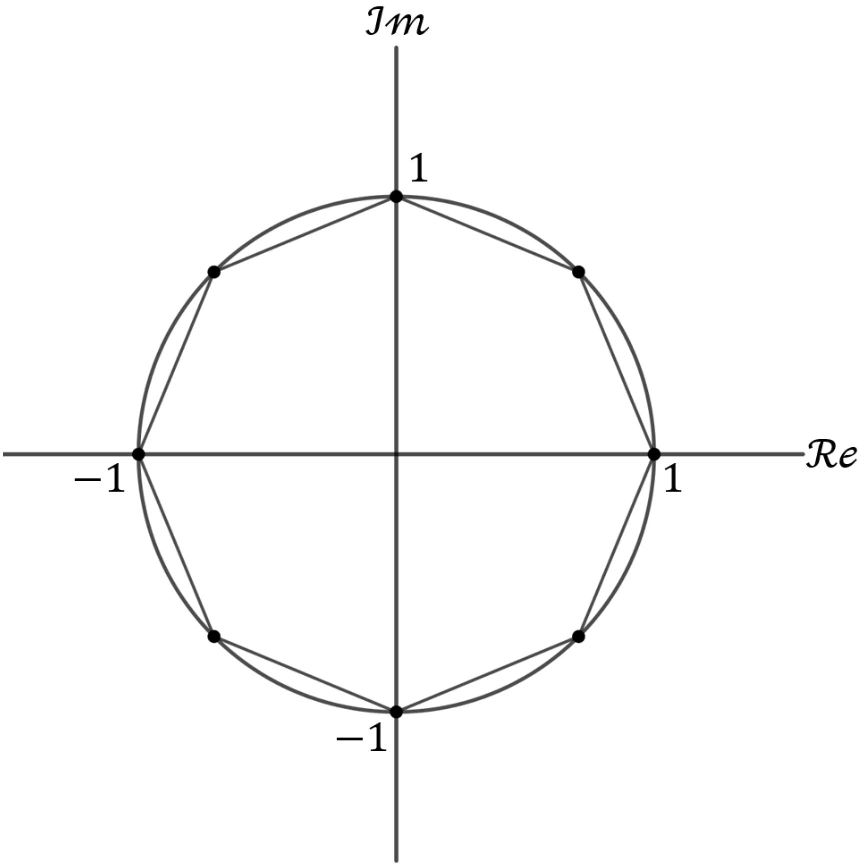
Using Euler's formula

$$e^{i\theta} = \cos(\theta)+i\sin(\theta)$$

we can also write the n^{th} roots of unity as

$$e^{i(2\pi k/n)}, \;\; k = 0,1,\dots,n-1.$$

When the n^{th} roots of unity are plotted on the complex plane (with the real part [Re] on the horizontal axis and the imaginary part [Im] on the vertical axis), we can see that they all lie on the unit circle and form the vertices of a regular polygon with n sides and a circumradius of 1.

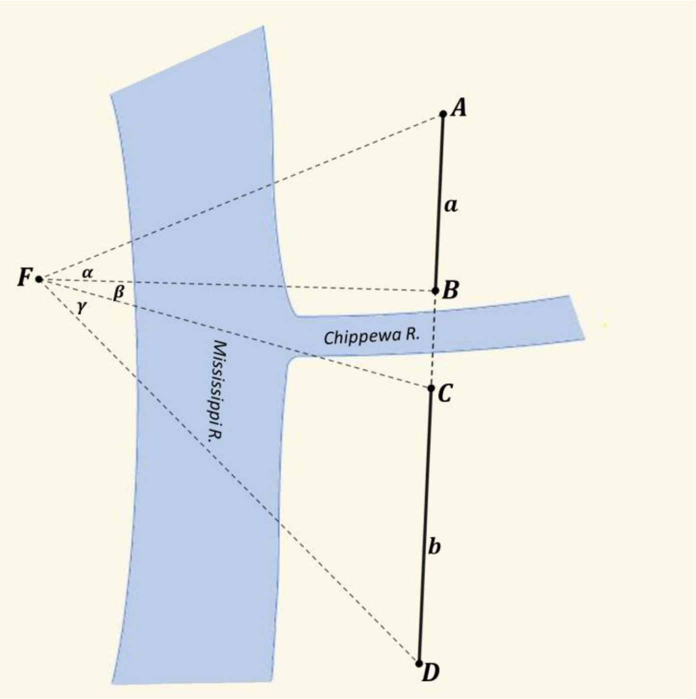


Here we are showing the 8 values of $\sqrt[8]{1}$ on the complex plane. That is, the 8^{th} roots of unity. You might be wondering why the focus on the n^{th} roots of unity and not the n^{th} roots of 7 or any other number? It turns out that the roots of unity in particular play a big role in several fields of mathematics including number theory and combinatorics.

5 Extra Problems

Problem 1.

A map maker wants to determine the distance between boundary markers A and D which are on opposite sides of the Chippewa River at a location near the confluence of the Chippewa and the Mississippi.



The map maker establishes landmarks at points B and C on the same line connecting boundary markers A and D and finds the distance a between points A and B and the distance b between points C and D. However, the map maker cannot directly measure the distance between points B and C.

The map maker uses a landmark F on the far side of the Mississippi to measure the angles α, β and γ as shown on the map above. Using this information find AD, the distance between boundary markers A and D.

The data for this problem is $a = 2320$ ft, $b = 3580$ ft, $\alpha = 24°$, $\beta = 14°$ and $\gamma = 30°$.

Solution

Let $x = BC$. Using the Law of Sines we can see that

$$\frac{a}{\sin\alpha} = \frac{FB}{\sin A} \text{ and } \frac{a+x}{\sin(\alpha+\beta)} = \frac{FC}{\sin A} \Longrightarrow \frac{FB}{FC} = \frac{a}{a+x} \cdot \frac{\sin(\alpha+\beta)}{\sin\alpha}.$$

and

$$\frac{b}{\sin\gamma} = \frac{FC}{\sin D} \text{ and } \frac{b+x}{\sin(\beta+\gamma)} = \frac{FB}{\sin D} \Longrightarrow \frac{FC}{FB} = \frac{b}{b+x} \cdot \frac{\sin(\beta+\gamma)}{\sin\gamma}.$$

Hence,

$$\frac{FB}{FC}\cdot\frac{FC}{FB}=\left(\frac{a}{a+x}\cdot\frac{\sin(\alpha+\beta)}{\sin\alpha}\right)\left(\frac{b}{b+x}\cdot\frac{\sin(\beta+\gamma)}{\sin\gamma}\right)$$

or

$$(a+x)(b+x)\sin\alpha\sin\gamma=ab\sin(\alpha+\beta)\sin(\beta+\gamma).$$

The only unknown in this last equation is the distance $x=BC$. We can rewrite this last equation as the quadratic

$$(\sin\alpha\sin\gamma)x^2+(\sin\alpha\sin\gamma)(a+b)x+c=0$$

where $c=ab(\sin\alpha\sin\gamma-\sin(\alpha+\beta)\sin(\beta+\gamma))$.

Using the quadratic formula to solve for x we find

$$x=\frac{-(a+b)(\sin\alpha\sin\gamma)\pm\sqrt{\left((a+b)(\sin\alpha\sin\gamma)\right)^2-4(\sin\alpha\sin\gamma)(c)}}{2\sin\alpha\sin\gamma}.$$

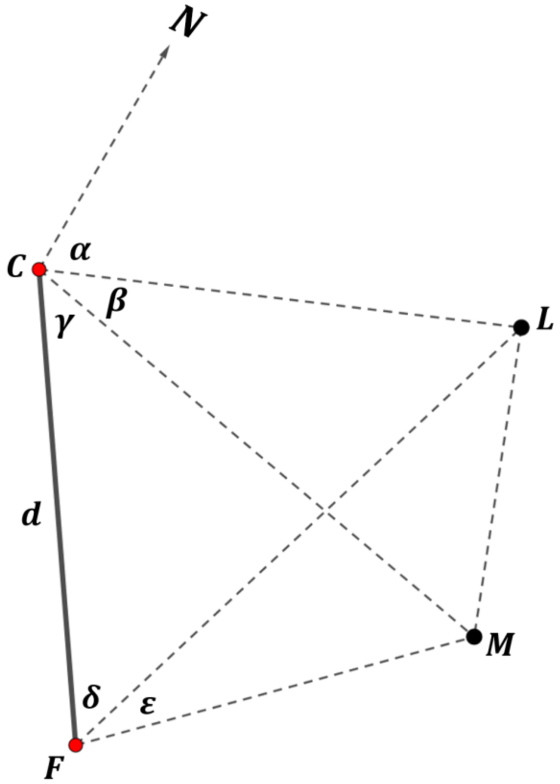
Evaluating this expression with the given data and knowing that x is necessarily positive, we find after simplification that $x\approx 1276$ ft.

Hence, $AD\approx 2320+1276+3580=7176$ ft.

■

Problem 2.

A ship at sea located at point M radios the Coast Guard Station at point F and reports they are having engine trouble and have dropped anchor at their current location. A Coast Guard rescue ship is currently at sea at location L. The Coast Guard has observation towers at points C and F which are known to be d miles apart. Assume both ships are visible from both towers.



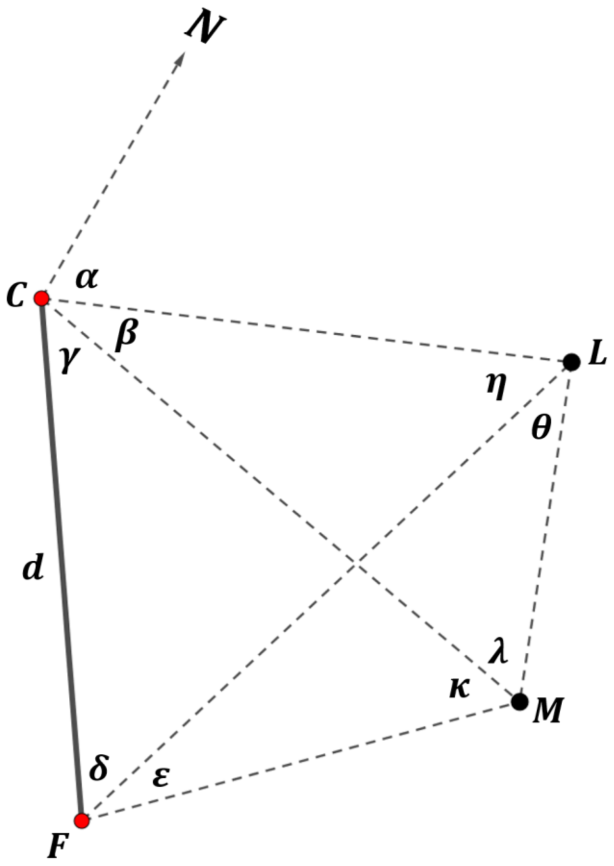
The angles β, γ, δ and ϵ are determined by officers at the two towers. Additionally, officers at C determine that the bearing (measured in degrees clockwise from North) from point C to point L is α.

From this information determine the bearing from point L to point M and find LM, the distance between the two ships.

The data for this problem is $\alpha = 67°$, $\beta = 33°$, $\gamma = 45°$, $\delta = 51°$, $\varepsilon = 28°$ and $d = 3$ miles.

Solution

We will define the additional angles as shown below.



From ΔCLF we can determine that $\eta = 180° - \delta - \gamma - \beta$ and from the Law of Sines,

$$\frac{\sin\eta}{d} = \frac{\sin(\gamma+\beta)}{FL} \Longrightarrow FL = \frac{d\sin(\gamma+\beta)}{\sin\eta} = \frac{d\sin(\gamma+\beta)}{\sin(180° - \delta - \gamma - \beta)}$$

$$= d \cdot \frac{\sin(\gamma+\beta)}{\sin(\delta+\gamma+\beta)}$$

From ΔCLM we can determine that $\kappa = 180° - \gamma - \delta - \varepsilon$ and from the Law of Sines,

$$\frac{\sin\kappa}{d} = \frac{\sin\gamma}{FM} \Longrightarrow FM = \frac{d\sin\gamma}{\sin\kappa} = \frac{d\sin\gamma}{\sin(180° - \gamma - \delta - \varepsilon)}$$

$$= d \cdot \frac{\sin\gamma}{\sin(\gamma+\delta+\varepsilon)}.$$

We can use the Law of Cosines with ΔFLM to solve for LM.

$$LM^2 = FL^2 + FM^2 - 2 \cdot FL \cdot FM \cdot \cos\varepsilon.$$

Thus,

$$LM = \sqrt{FL^2 + FM^2 - 2 \cdot FL \cdot FM \cdot \cos\varepsilon}$$

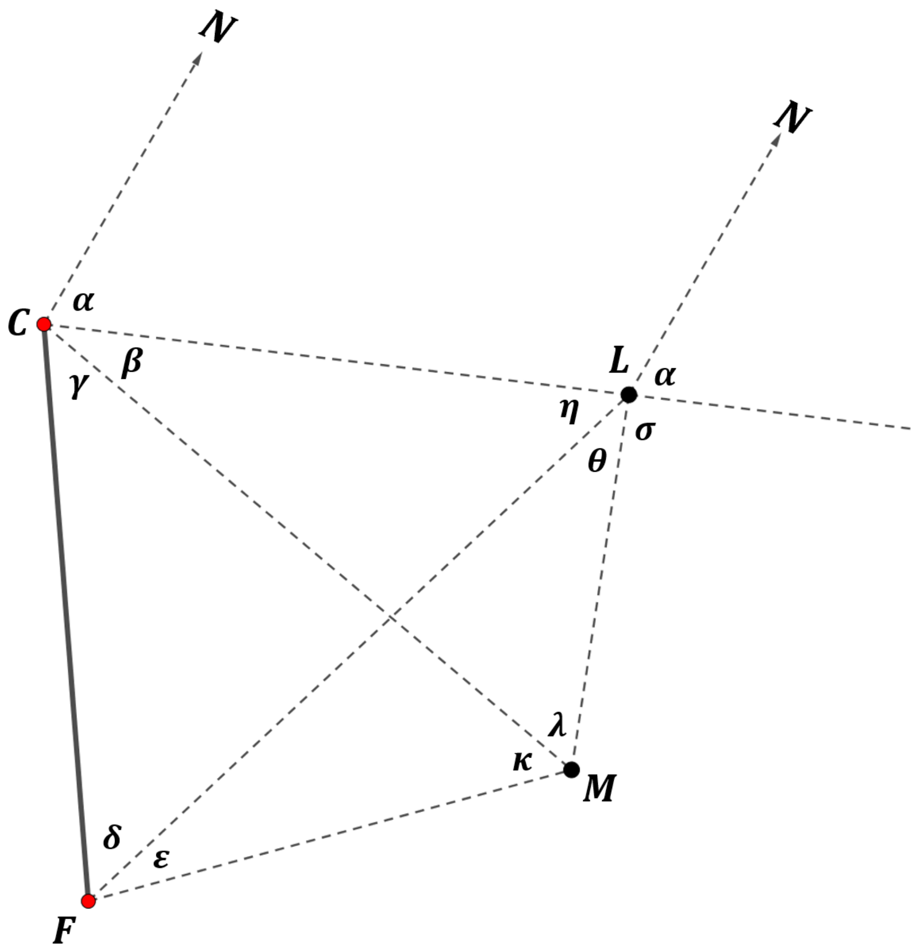
$$= \sqrt{\left(\frac{d\sin(\gamma+\beta)}{\sin(\delta+\gamma+\beta)}\right)^2 + \left(\frac{d\sin\gamma}{\sin(\gamma+\delta+\varepsilon)}\right)^2 - 2\left(\frac{d\sin(\gamma+\beta)}{\sin(\delta+\gamma+\beta)}\right)\left(\frac{d\sin\gamma}{\sin(\gamma+\delta+\varepsilon)}\right)\cos\varepsilon}$$

$$= d\sqrt{\left(\frac{\sin(\gamma+\beta)}{\sin(\delta+\gamma+\beta)}\right)^2 + \left(\frac{\sin\gamma}{\sin(\gamma+\delta+\varepsilon)}\right)^2 - 2\left(\frac{\sin(\gamma+\beta)}{\sin(\delta+\gamma+\beta)}\right)\left(\frac{\sin\gamma}{\sin(\gamma+\delta+\varepsilon)}\right)\cos\varepsilon}\,.$$

The data for this problem is $\alpha = 67°$, $\beta = 33°$, $\gamma = 45°$, $\delta = 51°$, $\varepsilon = 28°$ and $d = 3$ miles.

With this data entered into the formula for LM we get that $LM \approx 1.94$ miles.

Given the proximity of points C and L we can assume that the north pointing vectors from both locations are (essentially) parallel. Therefore, the bearing that the ship at L should take to reach point M is $\alpha + \sigma = \alpha + (180° - \eta - \theta)$.



We have previously determined that

$$\eta = 180° - 51° - 45° - 33° = 51°$$

and from the Law of Sines we have

$$\frac{\sin(\theta)}{FM} = \frac{\sin(\varepsilon)}{LM}.$$

Therefore

$$\theta = \sin^{-1}\left(\frac{\sin(\varepsilon) \cdot FM}{LM}\right).$$

$$= \sin^{-1}\left(\frac{\sin(28^\circ)\cdot\left(\frac{3\cdot\sin 45^\circ}{\sin(45^\circ+51^\circ+28^\circ)}\right)}{1.94}\right) \approx 38^\circ.$$

Hence the boat at L should take a bearing of

$$\alpha + (180^\circ - \eta - \theta) \approx 67^\circ + (180^\circ - 51^\circ - 38^\circ) = 158^\circ$$

clockwise from North.

■

Source: MSHSML 3C013

The expression $\frac{2\tan\left(\frac{\pi}{24}\right)}{1-\tan^2\left(\frac{\pi}{24}\right)}$ can be written in the form $a + b\sqrt{c}$ where a, b and c are integers.

Find $a + b + c$.

Solution

This problem tests whether or not you have memorized the necessary trigonometry formulas. What formulas should you memorize? At a minimum, the various sum and difference formulas and the exact results for trigonometric functions building on the 30-60-90 and the 45-45-90 triangles. You can derive other needed formulas from this information – but it can be time consuming. I recommend you memorize more formulas by building yourself a set of flashcards.

In particular, you need the following identities and results in this problem:

Tangent Identities

$$\tan(2\theta) = \frac{2\tan(\theta)}{1-\tan^2(\theta)}$$

$$\tan(\theta-\beta) = \frac{\tan(\theta)-\tan(\beta)}{1+\tan(\theta)\tan(\beta)}$$

$$\tan(60°) = \tan\left(\frac{\pi}{3}\right) = \sqrt{3}$$

$$\tan(45°) = \tan\left(\frac{\pi}{4}\right) = 1.$$

$$\frac{2\tan\left(\frac{\pi}{24}\right)}{1-\tan^2\left(\frac{\pi}{24}\right)} = \tan\left(2\left(\frac{\pi}{24}\right)\right) = \tan\left(\frac{\pi}{12}\right)$$

$$= \tan\left(\frac{\pi}{3}-\frac{\pi}{4}\right) = \frac{\tan\left(\frac{\pi}{3}\right)-\tan\left(\frac{\pi}{4}\right)}{1+\tan\left(\frac{\pi}{3}\right)\tan\left(\frac{\pi}{4}\right)}$$

$$= \frac{\sqrt{3}-1}{1+\left(\sqrt{3}\right)(1)} = \frac{\sqrt{3}-1}{1+\sqrt{3}}$$

$$= \frac{\sqrt{3}-1}{1+\sqrt{3}}\cdot\left(\frac{1-\sqrt{3}}{1-\sqrt{3}}\right)$$

$$= \frac{\sqrt{3}-1-3+\sqrt{3}}{1-3} = \frac{2\sqrt{3}-4}{-2} = 2-\sqrt{3}.$$

Finally, we recognize this answer is of the required form $a+b\sqrt{c}$ with $a=2, b=-1$ and $c=3$. Therefore, $\boxed{a+b+c=2-1+3=4.}$

■

Source: MSHSML 3C014

4. In ΔABC (Figure 4), use the same inverse trigonometric function to express each of the three interior angles in terms of an integer. (Two correct answers earn 1 point)

$\angle A = Arc(\ ?\ )\underline{\quad ?}$ $\angle B = Arc(\ ?\ )\underline{\quad ?}$ $\angle C = Arc(\ ?\ )\underline{\quad ?}$

$\angle A =$ ________ $\angle B =$ ________ $\angle C =$ ________

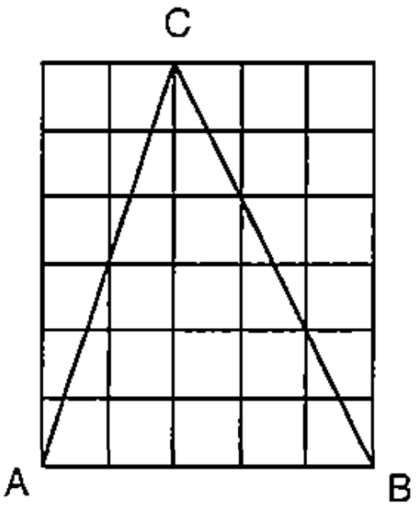
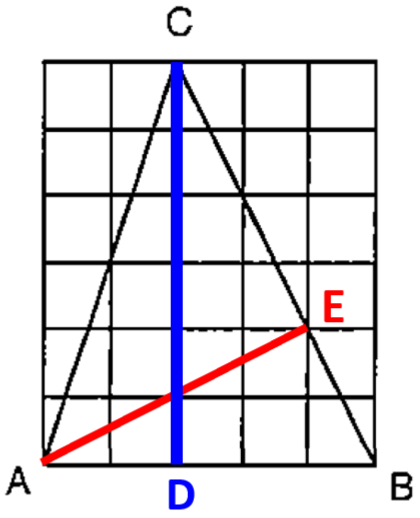


Figure 4

Solution

The key step is in recognizing the need for an **auxiliary line(s)** (an extra line(s) need to complete a proof or solve a geometry or trigonometry problem). Recognizing what extra line(s) are needed is generally not immediately obvious.



First consider the right triangle ΔACD. Using rt.ΔACD we have

$$\tan(A) = \frac{CD}{AD} = \frac{6}{2} = 3 \Longrightarrow A = \arctan(3).$$

Second consider the right triangle ΔBCD. Using rt. ΔBCD we have

$$\tan(B) = \frac{CD}{DB} = \frac{6}{3} = 2 \Longrightarrow B = \arctan(2).$$

Finally consider the triangle ΔACE. Is this a right triangle? Let's find the lengths of each side using the distance formula between two points.

$$\text{dist}\big((x_2, y_2), (x_1, y_1)\big) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

The point A has coordinates $(0,0)$

The point E has coordinates $(4,2)$

The point C has coordinates $(2,6)$.

$$AE = \text{dist}\big((4,2), (0,0)\big) = \sqrt{(0-4)^2 + (0-2)^2} = \sqrt{20} = 2\sqrt{5}$$

$$CE = \text{dist}\big((2,6), (4,2)\big) = \sqrt{(4-2)^2 + (2-6)^2} = \sqrt{20} = 2\sqrt{5}$$

$$AC = \text{dist}\big((2,6), (0,0)\big) = \sqrt{(0-2)^2 + (0-6)^2} = \sqrt{40} = 2\sqrt{10}.$$

Notice that

$$AE^2 + CE^2 = \left(2\sqrt{5}\right)^2 + \left(2\sqrt{5}\right)^2 = 40$$

$$AC^2 = \left(2\sqrt{10}\right)^2 = 40.$$

So,

$$AE^2 + CE^2 = AC^2.$$

Hence by the converse of the Pythagorean formula, ΔACE is a right triangle and because the two legs have the same length (namely, $2\sqrt{5}$), it is a 45-45-90 triangle.

Consider the right triangle ΔACE. Using rt. ΔACE we have

$$\tan(C) = \frac{AE}{CE} = \frac{2\sqrt{5}}{2\sqrt{5}} = 1 \Longrightarrow C = \arctan(1).$$

So, our final conclusion is that

$$\boxed{\begin{gathered} A = \arctan(3) \\ B = \arctan(2) \\ C = \arctan(1). \end{gathered}}$$

Incidentally, we know that

$$A + B + C = \pi$$

because these are the three angles of a triangle. So, we can also note that we have just proven that

$$\boxed{\arctan(3) + \arctan(2) + \arctan(1) = \pi.}$$

■

Source: MSHSML 3C002

2. What is the radian measure of the angle α in the third quadrant that satisfies

$$\sin^2 \alpha + \cos\alpha = \frac{1}{4}?$$

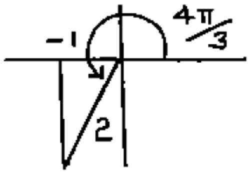
Solution

$$1 - \cos^2\alpha + \cos\alpha = \frac{1}{4}$$

$$4\cos^2\alpha - 4\cos\alpha - 3 = 0$$

$$(2\cos\alpha + 1)(2\cos\alpha - 3) = 0$$

$$\cos\alpha = -\frac{1}{2}\,, \quad \xcancel{\cos\alpha = \frac{3}{2}}$$



■

Source: MSHSML 3C003

3. Given the quadrilateral ABCD in Figure 3 with sides of lengths 3, 4, 5, and 6 and $\angle C = \arccos\frac{1}{8}$, find the length of BD.

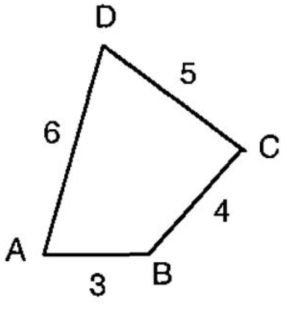


Figure 3

Solution

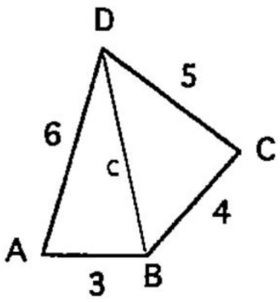


Figure 3

$$c^2 = 16 + 25 - 2(4)(5)\cos C$$
$$= 41 - 40\left(\tfrac{1}{8}\right) = 36$$
$$c = 6$$

■

Source: MSHSML 3C004

4. The ice-cream-cone shaped region in Figure 4 is formed by using the base PQ of the isosceles ΔPQR as the diameter of a semicircle. Let $\theta = \angle PRQ$. If $S(\theta)$ is the area of the semi-circle and $T(\theta)$ is the area of the triangle, find constants a and b such that $\frac{S(\theta)}{T(\theta)} = a\tan b\theta$

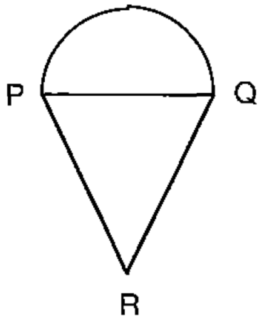
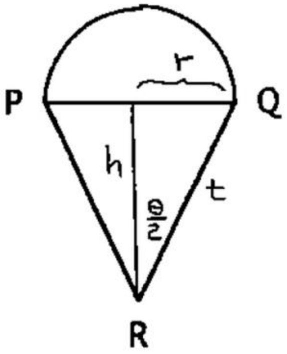


Figure 4

Solution



$$\sin\frac{\theta}{2} = \frac{r}{t} \qquad \cos\frac{\theta}{2} = \frac{h}{t}$$

$$\frac{S(\theta)}{T(\theta)} = \frac{\pi\left[t\ \sin\frac{\theta}{2}\right]^2\left(\frac{1}{2}\right)}{\frac{1}{2}\left(2t\sin\frac{\theta}{2}\right)\left(t\cos\frac{\theta}{2}\right)} = \frac{\pi\ \sin\frac{\theta}{2}}{2\cos\frac{\theta}{2}}$$

$$= \frac{\pi}{2}\tan\left(\frac{1}{2}\right)\theta$$

■