MSHSML Meet 3, Event D
Study Guide

3D Exponents and Logarithms

Use of fractional, negative exponents
Simplifying expressions involving radicals
Solving equations involving radicals
Use of logarithms; identities involving logarithms
Solving logarithmic equations
Relationships between logarithms to different base

1 Contents

2 Use of Fractional, Negative Exponents

Rules for Exponents

(1)	$a^b \cdot a^c = a^{b+c}$	$3^2 \cdot 3^4 = 3^{2+4} = 3^6$
(2)	$(a^b)^c = a^{b \cdot c}$	$(3^2)^4 = 3^{2 \cdot 4} = 3^8$

(3)	$\frac{a^b}{a^c} = a^{b-c}$	$\frac{3^7}{3^2} = 3^{7-2} = 3^5$
(4)	$a^{-b} = \frac{1}{a^b}$	$3^{-2} = \frac{1}{3^2} = \frac{1}{9}$
(5)	$a^0 = 1$ for all values of a except $a = 0$.	$3^0 = 1$
(6)	$a^{1/n} = \sqrt[n]{a}$	$3^{1/2} = \sqrt[2]{3}$
(7)	$a^{m/n} = \left(\sqrt[n]{a}\right)^m$	$3^{5/2} = \left(\sqrt[2]{3}\right)^5$
(8)	$(a \cdot b)^c = a^c \cdot b^c$	$(3x)^2 = 3^2 \cdot x^2 = 9x^2$
(9)	$\left(\frac{a}{b}\right)^c = \frac{a^c}{b^c}$	$\left(\frac{x}{2}\right)^4 = \frac{x^4}{2^4} = \frac{x^4}{16}$
(10)	$\boldsymbol{(a+b)^c \neq a^c + b^c}$	Don't make this very common mistake.
(11)	$\boldsymbol{(a-b)^c \neq a^c - b^c}$	Don't make this very common mistake.

3 Simplifying Expressions Involving Radicals

4 Solving Equations Involving Radicals

1.	Solve for x if $7 \cdot 3^{(x+1)} - 3^{(1-x)} = 2$.

Solution

$$7(3^{x+1}) - (3^{1-x}) = 2$$

$$7(3^x)(3^1) - (3^1) \cdot (3^{-x}) = 2$$

$$21 \cdot (3^x) - 3 \cdot \left(\frac{1}{3^x}\right) = 2$$

Let $y = 3^x$.

$$21y - \frac{3}{y} = 2$$

We can multiply both sides by y and simplify to get a quadratic equation.

$$21y^2 - 2y - 3 = 0$$

$$(7y - 3)(3y + 1) = 0$$

$7y - 3 = 0$	$3y + 1 = 0$
$7y = 3 \Longrightarrow y = \frac{3}{7} \Longrightarrow 3^x = \frac{3}{7} \Longrightarrow 3^{x-1} = \frac{1}{7}$ $\Longrightarrow x - 1 = \log_3\left(\frac{1}{7}\right) = -\log_3(7)$ $\Longrightarrow x = 1 - \log_3(7).$	$3y = -1 \Longrightarrow y = -1/3 \Longrightarrow 3^x = -\frac{1}{3}$ But this is impossible because $3^x \geq 0$ for all real values of x.

So $x = 1 - \log_3(7)$ is the only solution. But we still need to check if this solution actually satisfies the original equation.

$$7\left(3^{(1-\log_3(7))+1}\right) - \left(3^{1-(1-\log_3(7))}\right) \stackrel{?}{=} 2$$

$$7\left(3^{2-\log_3(7)}\right) - 3^{\log_3(7)} \stackrel{?}{=} 2$$

$$\frac{7 \cdot 3^2}{3^{\log_3(7)}} - 3^{\log_3(7)} \stackrel{?}{=} 2$$

$$3^2 - 7 \stackrel{?}{=} 2$$

$$2 \stackrel{\checkmark}{=} 2$$

So $x = 1 - \log_3(7)$ is the unique solution.

■

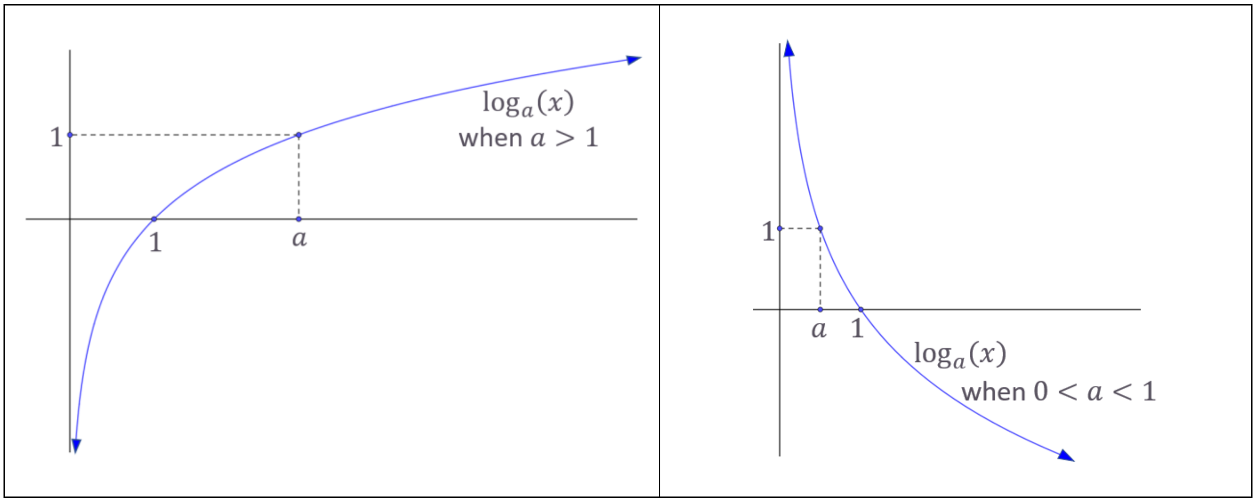
5 Use of Logarithms; Identities Involving Logarithms

Definition

$\log_a(x) = y \Leftrightarrow a^y = x,$ for all $a > 0, a \neq 1$	D1

Properties

$\log_a(1) = 0$	P1
$\log_a(a) = 1$	P2
$\log_a(xy) = \log_a(x) + \log_a(y)$	P3
$\log_a(x/y) = \log_a(x) - \log_a(y)$	P4
$\log_a(x^b) = b\log_a(x)$	P5
$\log_a(\sqrt[b]{x}) = \log_a(x)\,/b$	P6
$\log_a(x) = \log_c(x)\,/\log_c(a)$ for all $a > 0, a \neq 1, c > 0, c \neq 1$	P7
$\log_a(b) = 1/\log_b(a)$	P8
$\log_{(a^b)}(c) = \frac{1}{b}\log_a(c)$	P9
$x = \log_a(a^x)$	P10
$x = a^{\log_a(x)}$	P11
$a^{\log_c(b)} = b^{\log_c(a)}$	P12



6 Solving Logarithmic Equations

7 Relationships Between Logarithms to Different Bases

2.	Express $\log_{(8\sqrt{2})}(1024)$ as a rational number.

<u>Solution</u>

First note that $8\sqrt{2} = 2^3 \cdot 2^{1/2} = 2^{7/2}$ and $1024 = 2^{10}$.

By D1, $\log_{(8\sqrt{2})}(1024) = y$ means that $\left(2^{7/2}\right)^y = \left(8\sqrt{2}\right)^y = 1024 = 2^{10}$. That is,

$$\left(2^{7/2}\right)^y = 2^{7y/2} = 2^{10} \Longrightarrow \frac{7y}{2} = 10 \Longrightarrow y = \frac{20}{7}.$$

Alternatively, by P9

$$\log_{(8\sqrt{2})}(1024) = \log_{(2^{7/2})}(1024) = \frac{2}{7}\log_2(1024)$$

and then by P19

$$\frac{2}{7}\log_2(1024) = \frac{2}{7}\log_2(2^{10}) = \frac{2(10)}{7} = \frac{20}{7}.$$

■

3.	Express $\log_2(3) \cdot \log_3(4) \cdot \log_4(5) \cdots \log_{511}(512)$ as an integer.

Solution

First use P7 to express each of these log factors in terms of a common base. It does not matter what common base you pick to work with. I'll pick base 2. Then

$$\log_2(3) \cdot \log_3(4) \cdot \log_4(5) \cdot \log_5(6) \cdots \log_{510}(511) \cdot \log_{511}(512)$$

$$= \log_2(3) \cdot \frac{\log_2(4)}{\log_2(3)} \cdot \frac{\log_2(5)}{\log_2(4)} \cdot \frac{\log_2(6)}{\log_2(5)} \cdots \frac{\log_2(511)}{\log_2(510)} \cdot \frac{\log_2(512)}{\log_2(511)}$$

which simplifies to just $\log_2(512)$ after all the cancelling of like factors in the numerator and denominator. But $512 = 2^9$ and by P10, $\log_2(512) = \log_2(2^9) = 9$.

■

4.	Simplify $$\log_b(\tan(1°)) + \log_b(\tan(2°)) + \log_b(\tan(3°)) + \cdots + \log_b(\tan(89°))$$ where $b > 0, b \neq 1$.

Solution

First, remember the trig identities

$$\tan(k°) = \cot(90° - k°) = \frac{1}{\tan(90° - k°)}.$$

Then

$$\log_b(\tan(1°)) + \log_b(\tan(2°)) + \log_b(\tan(3°)) + \cdots + \log_b(\tan(89°))$$

$$= \left(\log_b(\tan(1°)) + \cdots + \log_b(\tan(44°))\right) + \log_b(\tan(45°))$$

$$+ \left(\log_b\big(\tan(46°)\big) + \cdots + \log_b(\tan(89°))\right)$$

$$= \left(\log_b(\tan(1°)) + \cdots + \log_b(\tan(44°)) \right) + \log_b(\tan(45°))$$

$$+ \left(\log_b\left(\frac{1}{\tan(90° - 46°)}\right) + \cdots + \log_b\left(\frac{1}{\tan(90° - 89°)}\right) \right).$$

But by P4 and then P1

$$\log_b\left(\frac{1}{\tan(90° - k°)}\right) = \log_b(1) - \log_b\left(\tan(90° - k°)\right) = \log_b\left(\tan(90° - k°)\right).$$

So, for example,

$$\log_b\left(\frac{1}{\tan(90° - 46°)}\right) = -\log_b\left(\tan(90° - 46°)\right) = -\log_b\left(\tan(44°)\right)$$

and

$$\log_b\left(\frac{1}{\tan(90° - 89°)}\right) = -\log_b\left(\tan(90° - 89°)\right) = -\log_b\left(\tan(1°)\right).$$

Thus,

$$\left(\log_b(\tan(1°)) + \cdots + \log_b(\tan(44°)) \right) + \log_b(\tan(45°))$$

$$+ \left(\log_b\left(\frac{1}{\tan(90° - 46°)}\right) + \cdots + \log_b\left(\frac{1}{\tan(90° - 89°)}\right) \right)$$

$$= \left(\log_b(\tan(1°)) + \cdots + \log_b(\tan(44°)) \right) + \log_b(\tan(45°))$$

$$+ \left(-\log_b(\tan(44°)) - \log_b(\tan(43°)) - \cdots - \log_b(\tan(1°)) \right)$$

$$= \log_b(\tan(45°))$$

$$= \log_b(1) = 0.$$

■

5.	Solve the system $$5\left(\log_x(y)+\log_y(x)\right)=26$$ $$xy=64$$

Solution

First, we need to get both logs in terms of the same base. By P8,

$$\log_y(x)=\frac{1}{\log_x(y)}.$$

Thus,

$$26=5\left(\log_x(y)+\log_y(x)\right)=5\log_x(y)+\frac{5}{\log_x(y)}=\frac{5(\log_x(y))^2+5}{\log_x(y)}$$

or

$$5(\log_x(y))^2-26(\log_x(y))+5=0.$$

Factoring the left-hand side we have

$$(5\log_x(y)-1)(\log_x(y)-5)=0.$$

Now

$$5\log_x(y)-1=0\Longrightarrow\log_x(y)=\frac{1}{5}\Longrightarrow y=x^{1/5}$$

and

$$\log_x(y)-5=0\Longrightarrow\log_x(y)=5\Longrightarrow y=x^5.$$

So, there are actually two systems to solve.

$$y=x^{1/5}\text{ and }xy=64\Longrightarrow x^{6/5}=64\Longrightarrow x=\left(64^{1/6}\right)^5=32, y=2$$

and

$$y=x^5\text{ and }xy=64\Longrightarrow x^6=64\Longrightarrow x=2, y=32.$$

So, the two solutions are $(x,y)=(32,2)$ and $(x,y)=(2,32)$.

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6.	Solve the equation $$\log_{1/3}\left(\cos(x)+\frac{\sqrt{5}}{6}\right)+\log_{1/3}\left(\cos(x)-\frac{\sqrt{5}}{6}\right)=2.$$

Solution

By P3 the right-hand side simplifies to

$$2=\log_{1/3}\left(\left(\cos(x)+\frac{\sqrt{5}}{6}\right)\left(\cos(x)-\frac{\sqrt{5}}{6}\right)\right)$$

$$=\log_{1/3}\left(\cos^2(x)-\frac{5}{36}\right)$$

By D1 this means

$$\left(\frac{1}{3}\right)^2=\cos^2(x)-\frac{5}{36}$$

or

$$\cos^2(x)=\frac{1}{9}+\frac{5}{36}=\frac{9}{36}\Longrightarrow\cos(x)=\pm\frac{1}{2}.$$

But $\cos(x)=-1/2$ is impossible because

$$x=-\frac{1}{2}-\frac{\sqrt{5}}{6}<0$$

and $\log_a(x)$ is undefined for all $x\le 0$. Therefore, $\cos(x)=1/2$ and

$$\cos(x)=\frac{1}{2}\Longrightarrow x=2n\pi\pm\frac{\pi}{3}\ \text{ for all integers } n.$$

■

7.	For what x value will $4\log_3(x)=4$?

Solution

$$4\log_3(x) = 4 \Longrightarrow \log_3(x) = 1 \Longrightarrow x = 3^1 = 3.$$

■

8.	If $x > 2y > 0$ and $2\log_b(x - 2y) = \log_b(x) + \log_b(y)$, determine x/y exactly.

Solution

$$2\log_b(x - 2y) = \log_b((x - 2y)^2)$$

$$\log_b(x) + \log_b(y) = \log_b(xy)$$

Therefore,

$$\log_b((x - 2y)^2) = \log_b(xy).$$

Therefore,

$$(x - 2y)^2 = xy$$

$$x^2 - 2xy + 4y^2 = xy$$

$$x^2 - 3xy + 4y^2 = 0$$

$$(x - 4y)(x + y) = 0$$

But notice that $x > 2y > 0$ means that $x + y \neq 0$. Therefore, it is necessary that $x - 4y = 0$.

Hence,

$$x = 4y \quad \text{and} \quad \frac{x}{y} = 4.$$

■

9.	The solution set of all x values for which $\log_4(x) - \log_4(2) + \log_4(x - 4) \leq 2$ can be written as $a < x \leq b$. Determine exactly the values of a and b.

Solution

First note that

$$\log_4(x) - \log_4(2) + \log_4(x - 4) = \log_4\left(\frac{x(x - 4)}{2}\right).$$

Hence,

$$\log_4(x) - \log_4(2) + \log_4(x-4) \le 2 \Leftrightarrow \log_4\left(\frac{x(x-4)}{2}\right) \le 2.$$

We know that for any increasing function $g(x)$,

$$a \le b \Leftrightarrow g(a) \le g(b).$$

Furthermore, $g(x) = 4^x$ is an increasing function for all x. Therefore,

$$\log_4\left(\frac{x(x-4)}{2}\right) \le 2$$

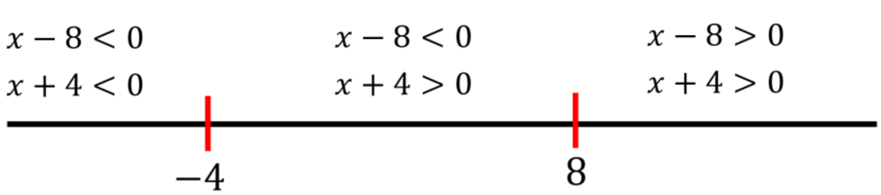
$$\Leftrightarrow 4^{\log_4\left(\frac{x(x-4)}{2}\right)} \le 4^2$$

$$\Leftrightarrow \frac{x(x-4)}{2} \le 16$$

$$\Leftrightarrow x^2 - 4x - 32 \le 0$$

$$\Leftrightarrow (x-8)(x+4) \le 0.$$

Now looking at the number line shown below we can see that $(x-8)(x+4) < 0$ only for $-4 < x < 8$ and $(x-8)(x+4) = 0$ only for $x = -4$ and $x = 8$.



So, $(x-8)(x+4) \le 0 \Leftrightarrow -4 \le x \le 8$.

But there is a problem if $x = -4$ because in order for $\log_4(x) - \log_4(2) + \log_4(x-4)$ to be defined we need $x > 4$. (Otherwise $\log_4(x-4)$ is undefined.). Therefore,

$$\log_4(x) - \log_4(2) + \log_4(x-4) \le 2 \Leftrightarrow -4 < x \le 8.$$

Hence, $a = 4$ and $b = 8$.

■

10.	Determine exactly all values x for which $\log_2(x) + \log_2(x-7) = 3$.

Solution

$$3 = \log_2(x) + \log_2(x-7) = \log_2\big(x(x-7)\big)$$

$$2^3 = x(x-7)$$

$$x^2 - 7x - 8 = 0$$

$$(x-8)(x+1) = 0$$

$$x = 8, x = -1.$$

But $\log_2(x) + \log_2(x-7)$ is not defined at $x = -1$ so this is an extraneous root. So $x = 8$ is the only solution.

■

11.	Determine exactly the x value where graphs of $y = \log_3(x)$ and $y = \log_{1/9}(x) + 12$ intersect.

Solution

First note that by the change of base formula we have

$$\log_{1/9}(x) = \frac{\log_3(x)}{\log_3(9^{-1})} = \frac{\log_3(x)}{-\log_3(9)} = \frac{\log_3(x)}{-2}.$$

These graphs intersect when

$$\log_3(x) = \log_{1/9}(x) + 12 = -\frac{\log_3(x)}{2} + 12.$$

Simplifying we have

$$\log_3(x)\left(1 + \frac{1}{2}\right) = 12$$

or

$$\log_3(x) = 12\left(\frac{2}{3}\right) = 8.$$

Therefore,

$$x = 3^8 = 6561.$$

■

12.	How many digits does 8^{330} have? Hint: It might be useful to know that $0.90308 < \log_{10}(8) < 0.90309$.

Solution

Let n be the integer such that $10^n < 8^{330} < 10^{n+1}$. Then 8^{330} would have $n+1$ digits by the definition of logarithms (base 10). We know that $\log_a(x)$ is an increasing function for all $a > 1$. Hence $\log_{10}(x)$ is an increasing function. And we also know that for any increasing function $g(x)$,

$$a < b \Leftrightarrow g(a) < g(b).$$

So,

$$10^n < 8^{330} < 10^{n+1} \Leftrightarrow \log_{10}(10^n) < \log_{10}(8^{330}) < \log_{10}(10^{n+1})$$

$$\Leftrightarrow n < 330\log_{10}(8) < n+1.$$

Furthermore,

$$0.90308 < \log_{10}(8) < 0.90309$$

$$\Leftrightarrow 330(0.90308) < 330\log_{10}(8) < 330(0.90309)$$

$$\Leftrightarrow 298.0164 < 330\log_{10}(8) < 298.0197.$$

Therefore, 8^{330} has 299 digits.

■

13.	Express $\log_6(12)$ as a function of a where $a = \log_{12}(54)$.

Solution

Let $x = \log_2(3)$. Then

$$
\begin{aligned}
a &= \log_{12}(54) = \log_{12}(2^1 \cdot 3^3) \\
&= \log_{12}(2) + 3\log_{12}(3) \\
&= \frac{1}{\log_2(12)} + \frac{3}{\log_3(12)} \\
&= \frac{1}{\log_2(2^2 \cdot 3^1)} + \frac{3}{\log_3(2^2 \cdot 3^1)} \\
&= \frac{1}{2 + \log_2(3)} + \frac{3}{1 + 2\log_3(2)} \\
&= \frac{1}{2 + \log_2(3)} + \frac{3}{1 + \frac{2}{\log_2(3)}} \\
&= \frac{1}{2 + x} + \frac{3}{1 + \frac{2}{x}} \\
&= \frac{1}{2 + x} + \frac{3x}{2 + x} = \frac{1 + 3x}{2 + x}.
\end{aligned}
$$

Solving for x we have

$$
\begin{aligned}
a = \frac{1 + 3x}{2 + x} &\Leftrightarrow 2a + ax = 1 + 3x \\
&\Leftrightarrow x(3 - a) = 2a - 1 \\
&\Leftrightarrow x = \frac{2a - 1}{3 - a}.
\end{aligned}
$$

But

$$
\begin{aligned}
\log_6(12) = \log_6(2 \cdot 6) &= 1 + \log_6(2) \\
&= 1 + \frac{1}{\log_2(6)} \\
&= 1 + \frac{1}{1 + \log_2(3)} \\
&= 1 + \frac{1}{1 + x} = \frac{2 + x}{1 + x}.
\end{aligned}
$$

Therefore,

$$\log_6(12) = \frac{2+x}{1+x} = \frac{2+\left(\frac{2a-1}{3-a}\right)}{1+\left(\frac{2a-1}{3-a}\right)}$$

$$= \frac{\frac{2(3-a)+(2a-1)}{3-a}}{\frac{(3-a)+(2a-1)}{3-a}}$$

$$= \frac{6-2a+2a-1}{3-a+2a-1} = \frac{5}{2+a}.$$

■

14.	If $\log_b(a) = c$ and $\log_x(b) = c$, then find $\log_a(x)$ in terms of c.

Solution

$$\log_b(a) = c \Leftrightarrow a = b^c$$

$$\log_x(b) = c \Leftrightarrow b = x^c$$

Therefore,

$$a = b^c = (x^c)^c = x^{c^2}.$$

Therefore,

$$\log_a(a) = \log_a\left(x^{c^2}\right)$$

$$\Leftrightarrow 1 = c^2 \log_a(x)$$

$$\Leftrightarrow \log_a(x) = \frac{1}{c^2}.$$

■

15.	If $\log_b N = r$, what is $\log_{b^2} N$? (Source: MSHSML 3D051)

Solution

$$N = b^r = (b^2)^{\frac{r}{2}}$$

$$\log_{b^2} N = \frac{r}{2}$$

■

16.	Express $\log_8 40$ in the form $\frac{a+\log 4}{b \log 2}$, understanding $\log 4$ and $\log 2$ are written to base 10. (Source: MSHSML 3D053)

Solution

$$\text{Let } \log_8 (40) = r$$

$$\text{Then } (2^3)^r = 40$$

$$r \log 2^3 = \log 4 + \log 10$$

$$3r \log 2 = \log 4 + 1$$

$$r = \frac{1 + \log 4}{3 \log 2}$$

■

17.	Given that $0 < M < 1 < N$ and that $4 \log_M N = \log_N M$, what is $\log_N MN$? (Source: MSHSML 3D054)

Solution

Let $\log_N M = r$ so $N^r = M$

$N = M^{1/r}$ so $\log_M N = \frac{1}{r}$

$\therefore\ 4 \cdot \frac{1}{r} = r$; $r^2 = 4$

Case 1 $r = 2$.

Then $N^2 = M$, contradicting $M < 1 < N$.

Case 2 $r = -2$

Then $M = N^{-2}$ so $MN = N^{-1}$

$\log_N MN = -1$

■

18.	Given that $\log_b 35 = r$ and $\log_b 49 = s$, express $\log_b 175$ in terms of r and s. (Source: MSHSML 3D013)

Solution

$$s = \log_b(49) = 2\log_b(7) \Longrightarrow \boxed{\log_b(7) = \frac{s}{2}}$$

$$r = \log_b(35) = \log_b(5) + \log_b(7) = \log_b(5) + \frac{s}{2} \Longrightarrow \boxed{\log_b(5) = r - \frac{s}{2}}$$

$$\log_b(175) = \log_b(5^2 \cdot 7) = 2\log_b(5) + \log_b(7) = 2\left(r - \frac{s}{2}\right) + \frac{s}{2} = 2r - \frac{s}{2}.$$

■

19.	Given that $\log_2 3 = r$ and $\log_2 b = s$, express $\log_b 48$ in terms of r and s. (Source: MSHSML 3D014)

Solution

$$s = \log_2(b) = \frac{\log_b(b)}{\log_b(2)} = \frac{1}{\log_b(2)} \Longrightarrow \boxed{\log_b(2) = \frac{1}{s}}$$

$$r = \log_2(3) = \frac{\log_b(3)}{\log_b(2)} \Longrightarrow \boxed{\log_b(3) = r \cdot \log_2(2) = \frac{r}{s}}$$

$$\log_b(48) = \log_b(2^4 \cdot 3) = 4\log_b(2) + \log_b(3) = 4\left(\frac{1}{s}\right) + \frac{r}{s} = \frac{4+r}{s}.$$

■

20.	$\log_{a^2} N = \frac{1}{2}$. Express N in terms of a, paying attention to the possibility that a might be negative. (Source: MSHSML 3D001)

Solution

$$[a^2]^{1/2} = N$$

$$N = \sqrt{a^2} = |a|$$

■

21.	Given that for some $b > 0$, $\log_b M = 1.4468$ and $\log_b N = 0.6117$, express $\frac{N^2}{\sqrt{M}}$ as a power of b. (Source: MSHSML 3D002)

Solution

$$\log_b \frac{N^2}{\sqrt{M}} = 2\log_b N - \frac{1}{2}\log_b M$$

$$\begin{array}{r} 1.2234 \\ .7234 \\ \hline .5000 \end{array}$$

$$b^{1/2} = \frac{N^2}{\sqrt{M}}$$

■

22.	For some $b > 0$, $\log_b 5x^4 = 7$ and $\log_b x^2 = 3$. Find x in exact form. (Source: MSHSML 3D003)

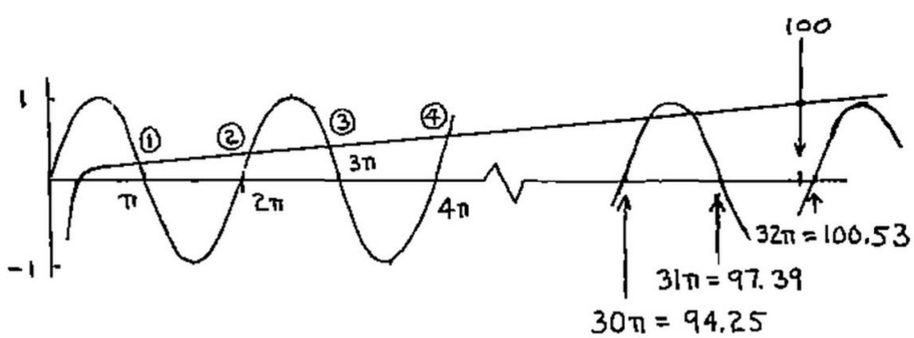
Solution

$$5x^4 = b^7 \Rightarrow 5^{1/7} x^{4/7} = b$$
$$x^2 = b^3 \Rightarrow x^{2/3} = b$$
$$x^{2/3} = 5^{1/7} x^{4/7}$$
$$x^{\frac{14-12}{21}} = 5^{1/7}$$
$$x^{2/3} = 5 \;;\quad x = 5^{3/2}$$

■

23.	For $0 < x < \infty$, find the number of intersections of the graphs of $y = \log_{100} x$ and $y = \sin x$. (Source: MSHSML 3D004)

Solution



Interval	Intersections
[0, 2π]	1
[2π, 4π]	2
[4π, 6π]	2
⋮	⋮
[30π, 32π]	2

Intervals [2π, 4π] to [30π, 32π]: $15 \cdot 2 + 1$

■