MSHSML Meet 4, Event A
Study Guide

4A Algebraic Manipulation Topics

Factoring (including $x^3 + y^3$, $x^3 - y^3$)

Sums, products, quotients of rational expressions

Solving equations (including radical equations) involving these skills, but ultimately solvable by factoring or the quadratic formula (but no complex roots)

Rational exponents

Simplifying radical expressions

Function notation and variational dependencies

Table of Contents

1 Ordered Rules on Simplifying

These rules are listed with the intention that they are followed in the order stated. That is, for example, I would rationalize denominators (Rule 5) before putting terms over a common denominator (Rule 6), etc.

But to be clear, at best, these rules offer a starting point. There are exceptions to every rule for the best approach to simplifying. But these are the rules I use unless some clues are offered to take a different direction.

Rule 1. Simplify from the inside and work outward.
Rule 2. Bring all functions of the variable(s) to the same side of the equation.
Rule 3. Combine terms of like powers or with like denominators.
Rule 4. Factor an expression when you can.
Rule 5. Rationalize denominators.
Rule 6. Put terms over a common denominator.
Rule 7. Put your final answer in "simplest form" as defined by the MSHSML rules which they define through the following table.

Examples of MSHSML "simplest form":

Unacceptable	Acceptable	Reason
$\frac{6}{4}$	$\frac{3}{2}$	quotient of two relatively prime integers
$5+2$	7	simple arithmetic
3^4	81	arithmetic with numerical exponents
$\sqrt[3]{8}$	2	arithmetic with numerical roots
$\sin 30°$	$\frac{1}{2}$	commonly known "unit circle" trigonometric values
$\frac{5}{\sqrt{12}}$	$\frac{5\sqrt{3}}{6}$	"rationalized" radical form

$\frac{5}{1+2i}$	$1-2i$	$a+bi$ format for complex numbers
$\frac{1}{\frac{1}{x}+x}$	$\frac{x}{1+x^2}$	complex fractions are not allowed
$\frac{x-1}{x^2-1}$	$\frac{1}{x+1}$	quotient of two polynomials should have no common factors

In cases where there is a question as to what form is "simplest", alternate answers may be accepted. For example, $\frac{3}{2}$, $1\frac{1}{2}$, 1.5, and 1.500 would all be acceptable.

Examples

1.	4A161 Simplify the expression $\left(\sqrt{6}+\sqrt{24}\right)^2$.

Solution

Simplify inside the parentheses first (if possible) before squaring. Notice that

$$\sqrt{24}=\sqrt{4\cdot 6}=2\sqrt{6}.$$

Then

$$\left(\sqrt{6}+\sqrt{24}\right)^2=\left(\sqrt{6}+2\sqrt{6}\right)^2=\left(3\sqrt{6}\right)^2=3^2\left(\sqrt{6}\right)^2=9\cdot 6=54.$$

■

2.	4A162 The solution set of $$\frac{1}{\sqrt{x}}-\frac{1}{\sqrt{2}}<\sqrt{8}$$ can be written in the form $x>a$. Determine a exactly.

Solution

The constants $\frac{1}{\sqrt{2}}$ and $\sqrt{8}$ are "like power terms". (If we express them in the form then $\frac{1}{\sqrt{2}}x^0$ and $\sqrt{8}x^0$ we can see they are both zeroth power terms of x.) So, I would start by combining them.

$$\frac{1}{\sqrt{2}} + \sqrt{8} = \frac{\sqrt{2}}{2} + 2\sqrt{2} = \frac{5\sqrt{2}}{2}.$$

Therefore,

$$\begin{aligned}\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{2}} < \sqrt{8} &\Leftrightarrow \frac{1}{\sqrt{x}} < \frac{5\sqrt{2}}{2} \\ &\Leftrightarrow \sqrt{x} > \frac{2}{5\sqrt{2}} \\ &\Leftrightarrow x > \left(\frac{2}{5\sqrt{2}}\right)^2 = \frac{4}{50} = \frac{2}{25}.\end{aligned}$$

So $a = 2/25$.

■

3.	4A154 Determine exactly the value of $$\frac{1}{\sqrt{2}+\sqrt{1}} + \frac{1}{\sqrt{3}+\sqrt{2}} + \cdots + \frac{1}{\sqrt{400}+\sqrt{399}}.$$

Solution

Per my suggested "rules for simplifying", I would rationalize the denominators before getting a common denominator.

$$\frac{1}{\sqrt{a+1}+\sqrt{a}} = \left(\frac{1}{\sqrt{a+1}+\sqrt{a}}\right)\left(\frac{\sqrt{a+1}-\sqrt{a}}{\sqrt{a+1}-\sqrt{a}}\right) = \frac{\sqrt{a+1}-\sqrt{a}}{\left(\sqrt{a+1}\right)^2 - \left(\sqrt{a}\right)^2}$$

$$= \frac{\sqrt{a+1}-\sqrt{a}}{(a+1)-a} = \sqrt{a+1}-\sqrt{a}.$$

So,

$$\frac{1}{\sqrt{2}+\sqrt{1}} + \frac{1}{\sqrt{3}+\sqrt{2}} + \cdots + \frac{1}{\sqrt{400}+\sqrt{399}}$$
$$= \left(\sqrt{2}-\sqrt{1}\right) + \left(\sqrt{3}-\sqrt{2}\right) + \left(\sqrt{4}-\sqrt{3}\right) + \cdots + \left(\sqrt{399}-\sqrt{398}\right) + \left(\sqrt{400}-\sqrt{399}\right)$$

$$= -\sqrt{1} + (\sqrt{2} - \sqrt{2}) + (\sqrt{3} - \sqrt{3}) + (\sqrt{4} - 4) + \cdots + (\sqrt{399} - \sqrt{399}) + \sqrt{400}$$

$$= -\sqrt{1} + \sqrt{400} = -1 + 20 = 19.$$

■

4.	4A132 Simplify the expression $(x+y)^{-1}(x^{-1}+y^{-1})$ so that it no longer involves any addition or negative exponents.

Solution

First recognize that $a^{-1} = \frac{1}{a}$ so that

$$(x+y)^{-1}(x^{-1}+y^{-1}) = \left(\frac{1}{x+y}\right)\left(\frac{1}{x}+\frac{1}{y}\right).$$

Now establish a common denominator for the two terms in the second set of parentheses.

$$\frac{1}{x}+\frac{1}{y} = \frac{x+y}{xy}.$$

Therefore,

$$\left(\frac{1}{x+y}\right)\left(\frac{1}{x}+\frac{1}{y}\right) = \left(\frac{1}{x+y}\right)\left(\frac{1}{x}+\frac{1}{y}\right) = \left(\frac{1}{x+y}\right)\left(\frac{x+y}{xy}\right) = \frac{1}{xy}.$$

■

5.	4A163 Given that $$\frac{6}{x^2-1} + \frac{b}{x-1} + \frac{c}{x+1} = \frac{4}{x+1}$$ determine exactly the values of b and c.

Solution

First be sure to note that you are not being asked to solve for x where this equation is true. You are being asked to find b and c so that this equation is true for *all* x. That is, to find b and c that makes this an identity.

I would start by bringing $4/(x+1)$ over to the left-hand side of the equation. Then I would combine the terms $c/(x+1)$ and $4/(x+1)$. We then have

$$\frac{6}{x^2-1}+\frac{b}{x-1}+\frac{c-4}{x+1}=0.$$

Next factor, followed by establishing a common denominator. We see $x^2-1=(x-1)(x+1)$ and so rewriting the problem over a common denominator we have

$$\frac{6+b(x+1)+(c-4)(x-1)}{x^2-1}=0.$$

So the problem has reduced to finding x where the numerator equals 0.

$$6+bx+b+cx-4x-c+4=0$$

or

$$(b+c-4)x+(10+b-c)=0.$$

For this to hold for *all* x we need to "match coefficients" of like powers. That is, for

$$(b+c-4)x+(10+b-c)=0=0x+0$$

we must have $b+c-4=0$ and $10+b-c=0$. Solving for c in the second equation and plugging this into the first equation we have

$$b+(10+b)-4=0 \Leftrightarrow 2b=-6 \Leftrightarrow b=-3.$$

Hence, $c=10+b=10-3=7$. So $(b,c)=(-3,7)$.

■

6.	4A122 Simplify $$\frac{(x^3-8)(x^2-x-2)}{(x-2)(x^2-4)(x^3+2x^2+4x)}.$$

Solution

Remember, as per MSHSML rules, quotient of two polynomials should have no common factors when putting your answer in "simplified" form.

$$\frac{(x^3-8)(x^2-x-2)}{(x-2)(x^2-4)(x^3+2x^2+4x)}=\frac{(x-2)(x^2+2x+4)(x-2)(x+1)}{(x-2)(x-2)(x+2)x(x^2+2x+4)}=\frac{x+1}{x^2+2x}.$$

■

7.	4A102 Determine exactly all solutions to $\frac{2x+1}{x+3} - \frac{5x+4}{4x+12} = 1.$

Solution

Factor out the 4 in $4x + 12$ to get $4(x + 3)$. Now get a common denominator on the left-hand side.

$$\frac{2x+1}{x+3} - \frac{5x+4}{4x+12} = \frac{2x+1}{x+3} - \frac{5x+4}{4(x+3)}$$

$$= \frac{4(2x+1) - 5x - 4}{4(x+3)} = \frac{3x}{4(x+3)}.$$

$$\frac{3x+8}{4(x+3)} = 1 \Leftrightarrow 3x = 4x + 12 \Leftrightarrow x = -12.$$

■

2 Equating Coefficients Term by Term

If two polynomials are equal **for all x** then the two polynomials must have exactly the same coefficients *term by term*. This is called "*equating coefficients*". For example, if we are told that

$$a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = b_5x^5 + b_4x^4 + b_3x^3 + b_2x^2 + b_1x + b_0$$

for all x then it must be true that $a_5 = b_5, a_4 = b_4, a_3 = b_3,$ etc.

Examples

8.	Given that $$\frac{2x^3+14x^2+12x}{4x+4} \div (mx+n) = x$$ determine exactly the values of m and n.

Solution

Rearranging the factors this problem can be written as

$$2x^3 + 14x^2 + 12x = x(4x+4)(mx+n).$$

Now we could multiply the righthand side out then equate the coefficients of the two polynomials. This will give us two equations in two unknowns.

Two equations in two unknowns is easy enough but you could save some time by simplifying first. ***Always be on the lookout to simplify before moving on to a more time-consuming step.***

Clearly

$$2x^3 + 14x^2 + 12x = 2x(x^2 + 7x + 6) = 2x(x+1)(x+6)$$

and so both we can cancel the common factors $2, x$ and $x+1$ from both sides of the above equation ***before*** multiplying it out.

$$2x(x+1)(x+6) = 4x(x+1)(mx+n) \Longrightarrow x + 6 = 2mx + 2n.$$

Now if we equate coefficients we can immediately write down that $1x = 2mx$ and $6 = 2n$ or $m = 1/2$ and $n = 3$.

■

9.	Given that $$\frac{x^3-8}{x-2} + \frac{3x^3-9x^2+6x}{x^2-3x+2} = x^2 + bx + c$$ determine exactly the values of b and c.

Solution

We *could* blindly approach this by multiplying both sides by the common denominator on the left. We would end up, *eventually*, with a fourth-degree equation on the left and right. Equating coefficients of these two fourth degree polynomials would *eventually* allow us to solve for b and c.

But remember to simplify before doing anything else. Save time!

$$\frac{x^3-8}{x-2}=\frac{(x-2)(x^2+2x+4)}{x-2}=x^2+2x+4$$

and

$$\frac{3x^3-9x^2+6x}{x^2-3x+2}=\frac{3x(x-1)(x-2)}{(x-1)(x-2)}=3x.$$

So, this "equating coefficients" problem has simplified to

$$\left(\frac{x^3-8}{x-2}\right)+\left(\frac{3x^3-9x^2+6x}{x^2-3x+2}\right)=x^2+bx+c$$

$$(x^2+2x+4)+(3x)=x^2+bx+c$$

$$x^2+5x+4=x^2+bx+c$$

Now we can immediately see that

$$5x=bx\Longrightarrow b=5$$

$$4=c.$$

So $(b,c)=(5,4)$.

■

10.	Given that $$\frac{6}{x^2-1}+\frac{b}{x-1}+\frac{c}{x+1}=\frac{4}{x+1}$$ determine exactly the values of b and c.

Solution

By getting a common denominator, the left-hand side becomes

$$\frac{6}{x^2-1}+\frac{b}{x-1}+\frac{c}{x+1}=\frac{6+b(x+1)+c(x-1)}{(x-1)(x+1)}.$$

Now multiplying both the left side and right side by this common denominator we get

$$6 + b(x+1) + c(x-1) = \frac{4(x-1)(x+1)}{x+1} = 4(x-1)$$

or

$$(b+c)x + (6+b-c) = 4x - 4.$$

By equating coefficients, we get two equations in two unknowns which we have to solve simultaneously.

$$b + c = 4$$

$$6 + b - c = -4.$$

The c's will cancel if we add these two equations. By adding we see

$$6 + 2b = 0 \Longrightarrow b = -3.$$

Now from $b + c = 4$ we can see that $c = 7$.

$$(b, c) = (-3,7).$$

■

11.	If $x^4 + 2x^3 + ax^2 + bx + c$ is exactly divisible by $x^3 + 3x^2 - 2x + 4$, find $a + b + c$.

Solution

$$\begin{array}{r|l} & x \quad -1 \\ \hline x^3 + 3x^2 - 2x + 4 & x^4 + 2x^3 + ax^2 + bx + c \\ & \underline{x^4 + 3x^3 - 2x^2 + 4x} \\ & \quad -x^3 + (a+2)x^2 + (b-4)x + c \\ & \quad \underline{-x^3 - \quad 3x^2 + \quad 2x - 4} \\ & \qquad -(a+5)x^2 + (b-6)x + (c+4) \end{array}$$

Therefore,

$$\begin{aligned} x^4 + 2x^3 + ax^2 + bx + c &= (x^3 + 3x^2 - 2x + 4)(x-1) + (\textbf{Remainder}) \\ &= (x^3 + 3x^2 - 2x + 4)(x-1) \\ &\quad + \left(-(\boldsymbol{a}+\mathbf{5})\boldsymbol{x}^{\mathbf{2}} + (\boldsymbol{b}-\mathbf{6})\boldsymbol{x} + (\boldsymbol{c}+\mathbf{4})\right) \end{aligned}$$

But the requirement that $x^4 + 2x^3 + ax^2 + bx + c$ is exactly divisible by $x^3 + 3x^2 - 2x + 4$ is a requirement that the **remainder** equals 0 for all x.

That is, for all x

$$-(a+5)x^2+(b-6)x+(c+4)=0x^2+0x+0.$$

Equating coefficients of these two polynomials yields $-(a+5)=0$ and $b-6=0$ and $c+4=0$. Thus, $(a,b,c)=(-5,6,-4)$ and $a+b+c=-5+6-4=-3$.

■

3 Change of Variables

Broadly put, "change of variable" is the method of making a substitution that allows for a simpler approach to solving a problem which is followed by a translation of this solution in terms of the original terms of the problem.

12.	If $$f\left(\frac{1}{x+3}\right)=\frac{1}{2-5x}$$ for all $x>1$, write $f(x)$ as a rational function with no common factors shared by the numerator and denominator.

Solution

Make the change of variable $y=\frac{1}{x+3}$. We note that $x>1 \Leftrightarrow 0<y<\frac{1}{4}$.

$$y=\frac{1}{x+3}\Rightarrow 1=y(x+3)=xy+3y\Rightarrow xy=1-3y\Rightarrow x=\frac{1-3y}{y}.$$

Therefore,

$$f\left(\frac{1}{x+3}\right)=f(y)=\frac{1}{2-5\left(\frac{1-3y}{y}\right)}=\frac{y}{2y-5+15y}=\frac{y}{17y-5}$$

and

$$f\left(\frac{1}{x+3}\right)=\frac{1}{2-5x}\text{ for all }x>1\Leftrightarrow f(y)=\frac{y}{17y-5}\text{ for all }0<y<\frac{1}{4}.$$

But the graph of

$$\frac{y}{17y-5} \text{ for all } 0 < y < \frac{1}{4}$$

is the same as the graph of

$$\frac{x}{17x-5} \text{ for all } 0 < x < \frac{1}{4}.$$

That is,

$$f(x) = \frac{x}{17x-5} \text{ for all } 0 < x < \frac{1}{4}.$$

■

13.	Determine exactly all solutions to the equation $$6\left(\frac{1}{x}\right)^2 - 29\left(\frac{1}{x}\right) + 35 = 0.$$

Solution

Make the change of variable $y = \frac{1}{x}$. Then

$$0 = 6\left(\frac{1}{x}\right)^2 - 29\left(\frac{1}{x}\right) + 35 = 6y^2 - 29y + 35 = (2y-5)(3y-7)$$

$$\Rightarrow y = \frac{5}{2} \text{ or } y = \frac{7}{3}$$

$$\Rightarrow x = \frac{1}{y} = \frac{2}{5} \text{ or } x = \frac{1}{y} = \frac{3}{7}.$$

■

14.	Determine exactly all the values of a, b and c in the following system: $$\frac{ab}{a+b} = 3 \qquad \frac{ac}{a+c} = 4 \qquad \frac{bc}{b+c} = 6.$$

Solution

Make the change of variables $x = \frac{1}{a}, y = \frac{1}{b}$ and $z = \frac{1}{c}$.

That this will simplify the problem is *not* obvious and requires the insight (or prior experience) to notice that

$$\frac{a+b}{ab} = \frac{1}{a} + \frac{1}{b}.$$

But having noticed this, these change of variables recasts the problem as

$$x + y = \frac{1}{3}, x + z = \frac{1}{4}, \text{ and } y + z = \frac{1}{6}$$

which is much easier to work with than the problem in its original form.

$$x + y = 3 \Longrightarrow x = \frac{1}{3} - y$$

$$y + z = 6 \Longrightarrow z = \frac{1}{6} - y$$

$$\frac{1}{4} = x + z = \left(\frac{1}{3} - y\right) + \left(\frac{1}{6} - y\right) = \frac{1}{2} - 2y \Longrightarrow 2y = \frac{1}{4} \Longrightarrow y = \frac{1}{8}.$$

Therefore,

$$x = \frac{1}{3} - \frac{1}{8} = \frac{5}{24} \text{ and } z = \frac{1}{6} - \frac{1}{8} = \frac{1}{24}.$$

Therefore,

$$a = \frac{1}{x} = \frac{24}{5}, \ b = \frac{1}{y} = 8 \text{ and } c = \frac{1}{z} = 24.$$

■

15.	Source: MSHSML 4A994 Find the exact value of x which satisfies the equation $$\sqrt{\frac{x}{2}} + \sqrt{\frac{2x}{9}} + \sqrt{\frac{x}{8}} = \frac{1}{12}.$$

Solution

Let $y = \sqrt{2x}$. Then we can rewrite the left-hand side of the problem as

$$\sqrt{\frac{x}{2}}+\sqrt{\frac{2x}{9}}+\sqrt{\frac{x}{8}}=\sqrt{\frac{2x}{4}}+\sqrt{\frac{2x}{9}}+\sqrt{\frac{2x}{16}}$$

$$=\frac{y}{2}+\frac{y}{3}+\frac{y}{4}$$

$$=\left(\frac{6+4+3}{12}\right)y=\frac{13}{12}y$$

So,

$$\sqrt{\frac{x}{2}}+\sqrt{\frac{2x}{9}}+\sqrt{\frac{x}{8}}=\frac{13}{12}y=\frac{1}{12}$$

So,

$$\sqrt{2x}=y=\frac{1}{13}\Longrightarrow x=\frac{1}{2\cdot 13^2}=\frac{1}{338}.$$

■

4 Rationalizing the Denominator

The goal of "rationalizing the denominator" is to put a fraction into a mathematical equivalent form but a form which has not roots (square roots, cube roots, etc.) in the denominator. The most common application is to remove square roots from the denominator.

The "Uniform Grading Procedures" for this math contest *requires* that final answers cannot leave roots in the denominator. So be sure to rationalize *all* of your answers (or your answer will be marked wrong).

As one particular example, $\frac{5}{\sqrt{12}}=\frac{5\sqrt{3}}{6}$, but only the form $\frac{5\sqrt{3}}{6}$ will be marked correct.

Removing a Single Square-Root

$$\frac{1}{\sqrt{a}}=\left(\frac{1}{\sqrt{a}}\right)\cdot\left(\frac{\sqrt{a}}{\sqrt{a}}\right)=\frac{\sqrt{a}}{a}$$

Removing the Difference of Two-Square Roots

$$\frac{1}{\sqrt{a}-\sqrt{b}} = \left(\frac{1}{\sqrt{a}-\sqrt{b}}\right)\cdot\left(\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}+\sqrt{b}}\right) = \frac{\sqrt{a}+\sqrt{b}}{a-b}$$

Removing the Sum of Two-Square Roots

$$\frac{1}{\sqrt{a}+\sqrt{b}} = \left(\frac{1}{\sqrt{a}+\sqrt{b}}\right)\cdot\left(\frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}-\sqrt{b}}\right) = \frac{\sqrt{a}-\sqrt{b}}{a-b}$$

Removing the Sum of Three-Square Roots (requires two rationalizations)

$$\frac{1}{\sqrt{a}+\sqrt{b}+\sqrt{c}}$$

$$= \left(\frac{1}{\left(\sqrt{a}+\sqrt{b}\right)+\sqrt{c}}\right)\cdot\left(\frac{\left(\sqrt{a}+\sqrt{b}\right)-\sqrt{c}}{\left(\sqrt{a}+\sqrt{b}\right)-\sqrt{c}}\right)$$ (Rationalize once.)

$$= \frac{\sqrt{a}+\sqrt{b}-\sqrt{c}}{\left(\sqrt{a}+\sqrt{b}\right)^2-c}$$

$$= \frac{\sqrt{a}+\sqrt{b}-\sqrt{c}}{a+b-c+2\sqrt{ab}}$$

$$= \left(\frac{\sqrt{a}+\sqrt{b}-\sqrt{c}}{(a+b-c)+2\sqrt{ab}}\right)\cdot\left(\frac{a+b-c-2\sqrt{ab}}{(a+b-c)-2\sqrt{ab}}\right)$$ (Rationalize a second time.)

$$= \frac{\left(\sqrt{a}+\sqrt{b}-\sqrt{c}\right)\left(a+b-c-2\sqrt{ab}\right)}{(a+b-c)^2-4ab}$$

Removing Mixed Sum and Difference of Three-Square Roots (requires two rationalizations)

$$\frac{1}{\sqrt{a}+\sqrt{b}-\sqrt{c}}$$

$$= \left(\frac{1}{\left(\sqrt{a}+\sqrt{b}\right)-\sqrt{c}}\right)\cdot\left(\frac{\left(\sqrt{a}+\sqrt{b}\right)+\sqrt{c}}{\left(\sqrt{a}+\sqrt{b}\right)+\sqrt{c}}\right)$$ (Rationalize once.)

$$= \frac{\sqrt{a}+\sqrt{b}+\sqrt{c}}{\left(\sqrt{a}+\sqrt{b}\right)^2+c}$$

$$= \frac{\sqrt{a}+\sqrt{b}+\sqrt{c}}{a+b+c+2\sqrt{ab}}$$

$$= \left(\frac{\sqrt{a}+\sqrt{b}+\sqrt{c}}{(a+b+c)+2\sqrt{ab}}\right)\cdot\left(\frac{a+b+c-2\sqrt{ab}}{(a+b+c)-2\sqrt{ab}}\right)$$ (Rationalize a second time.)

$$= \frac{\left(\sqrt{a}+\sqrt{b}-\sqrt{c}\right)\left(a+b-c-2\sqrt{ab}\right)}{(a+b-c)^2-4ab}$$

Removing the Difference of Two Cube Roots

$$\frac{1}{\sqrt[3]{a}-\sqrt[3]{b}} = \left(\frac{1}{\sqrt[3]{a}-\sqrt[3]{b}}\right)\cdot\left(\frac{a^{2/3}+a^{1/3}\cdot b^{1/3}+b^{2/3}}{a^{2/3}+a^{1/3}\cdot b^{1/3}+b^{2/3}}\right) = \frac{a^{2/3}+a^{1/3}\cdot b^{1/3}+b^{2/3}}{a-b}$$

Removing the Sum of Two Cube Roots

$$\frac{1}{\sqrt[3]{a}+\sqrt[3]{b}} = \left(\frac{1}{\sqrt[3]{a}+\sqrt[3]{b}}\right)\cdot\left(\frac{a^{2/3}-a^{1/3}\cdot b^{1/3}+b^{2/3}}{a^{2/3}-a^{1/3}\cdot b^{1/3}+b^{2/3}}\right) = \frac{a^{2/3}-a^{1/3}\cdot b^{1/3}+b^{2/3}}{a+b}$$

Removing the Difference of a Cube Root and a Square Root

$$\frac{1}{\sqrt[3]{a}-\sqrt{b}}$$

$$= \left(\frac{1}{\sqrt[3]{a}-\sqrt{b}}\right)\cdot\left(\frac{\sqrt[3]{a}+\sqrt{b}}{\sqrt[3]{a}+\sqrt{b}}\right)$$ (Rationalize once.)

$$= \frac{\sqrt[3]{a}+\sqrt{b}}{\sqrt[3]{a^2}-b}$$

$$= \left(\frac{\sqrt[3]{a}+\sqrt{b}}{\sqrt[3]{a^2}-b}\right)\left(\frac{(a^4)^{1/3}+(a^2)^{1/3}\cdot b+b^2}{(a^4)^{1/3}+(a^2)^{1/3}\cdot b+b^2}\right)$$ (Rationalize a second time.)

$$= \frac{\left(\sqrt[3]{a}+\sqrt{b}\right)\left((a^4)^{1/3}+(a^2)^{1/3}\cdot b+b^2\right)}{a^2-b\;a^{4/3}+ba^{4/3}-b^2a^{2/3}+b^2a^{2/3}-b^3}$$

$$= \frac{\left(\sqrt[3]{a}+\sqrt{b}\right)\left((a^4)^{1/3}+(a^2)^{1/3}\cdot b+b^2\right)}{a^2-b^3}.$$

16.	Source: MSHSML 4A034 Rewrite $\frac{1}{\sqrt[3]{5}+2}$ as a fraction with an integer denominator and a numerator consisting of terms involving only integers and cube roots of integers.

Solution

$$\frac{1}{\sqrt[3]{5}+2} = \frac{1}{\sqrt[3]{5}+\sqrt[3]{8}} = \left(\frac{1}{\sqrt[3]{5}+\sqrt[3]{8}}\right)\cdot\left(\frac{5^{2/3}-5^{1/3}\cdot 8^{1/3}+8^{2/3}}{5^{2/3}-5^{1/3}\cdot 8^{1/3}+8^{2/3}}\right)$$

$$= \left(\frac{1}{\sqrt[3]{5}+2}\right)\cdot\left(\frac{\left(\sqrt[3]{5}\right)^2-2\sqrt[3]{5}+4}{\left(\sqrt[3]{5}\right)^2-2\sqrt[3]{5}+4}\right)$$

$$= \frac{\left(\sqrt[3]{5}\right)^2-2\sqrt[3]{5}+4}{\sqrt[3]{5}\left(\left(\sqrt[3]{5}\right)^2-2\sqrt[3]{5}+4\right)+2\left(\left(\sqrt[3]{5}\right)^2-2\sqrt[3]{5}+4\right)}$$

$$= \frac{\left(\sqrt[3]{5}\right)^2-2\sqrt[3]{5}+4}{\left(\left(\sqrt[3]{5}\right)^3-2\left(\sqrt[3]{5}\right)^2+4\sqrt[3]{5}\right)+\left(2\left(\sqrt[3]{5}\right)^2-4\sqrt[3]{5}+8\right)}$$

$$= \frac{\left(\sqrt[3]{5}\right)^2-2\sqrt[3]{5}+4}{5+8} = \frac{\sqrt[3]{25}-2\sqrt[3]{5}+4}{13}.$$

■

5 Absolute Values

By the definition of absolute value, we know that

$$|f(x)| = \begin{cases} f(x) & f(x) > 0 \\ 0 & f(x) = 0 \\ -f(x) & f(x) < 0. \end{cases}$$

Problems involving $|f(x)|$ are usually best handled by splitting the problem into three parts according to whether $f(x)$ is positive, zero or negative. We illustrate this idea in the following examples.

17.	Source: MSHSML 4A072 Find all x that satisfy $\sqrt{(x-3)^2} - 2x + 1 = 0$.

Solution

A key issue here is recognize that $\sqrt{(x-3)^2} = |x-3|$ and not just $(x-3)$.

To be clear on this point, when we plug in $x = 0$ into $\sqrt{(x-3)^2}$ we get $\sqrt{(0-3)^2} = \sqrt{9} = 3$ but when we plug $x = 3$ int $(x-3)$ we get $(-0-3) = -3$.

So, the problem is actually asking us to solve $|x-3| - 2x + 1 = 0$.

The presence of this absolute value should automatically get you thinking "cases". Remember that

$$|x-3| = \begin{cases} x-3 & x-3 > 0 \\ 0 & x-3 = 0 \\ -(x-3) & x-3 < 0. \end{cases}$$

That is,

$$|x-3| = \begin{cases} x-3 & x > 3 \\ 0 & x = 3 \\ 3-x & x < 3. \end{cases}$$

So

$$|x-3| - 2x + 1 = \begin{cases} (x-3) - 2x + 1 & x > 3 \\ (0) - 2x + 1 & x = 3 \\ (3-x) - 2x + 1 & x < 3. \end{cases}$$

We must handle each of these three cases as a separate problem.

Case $x > 3$	Case $x = 3$	Case $x < 3$
$(x-3) - 2x + 1 = 0$ $-x - 2 = 0$ $x = 2$	$-2x + 1 = 0$ $x = 1/2$	$(3-x) - 2x + 1 = 0$ $-3x + 4 = 0$ $x = 4/3$

So, our solution set consists of the set of all x such that

$$\left\{(x \geq 3 \text{ and } x = 2) \text{ or } (x = 3 \text{ and } x = 1/2) \text{ or } \left(x < 3 \text{ and } x = \frac{4}{3}\right)\right\}.$$

Clearly the only x that belongs to this set is $x = 4/3$.

■

18.	Find all pairs (x, y) that simultaneously satisfy $$\lvert x \rvert + y = 5$$ $$\lvert x \rvert y - x^2 = 0.$$

Solution

Remember that

$$\lvert x \rvert = \begin{cases} x & x > 0 \\ 0 & x = 0 \\ -x & x < 0. \end{cases}$$

Solve each case as a separate problem.

Case $x > 0$	Case $x = 0$	Case $x < 0$
$(x) + y = 5$ $(x)y - (x)^2 = 0$ The second equation simplifies to $x(y - x) = 0$ but we are in a case where $x \neq 0$ so $x(y - x) = 0 \Leftrightarrow y - x = 0$. Therefore in this	$(0) + y = 5$ $(0)y - (0)^2 = 0.$ The second equation simplifies to $0 = 0$. Therefore, this case reduces to	$(-x) + y = 5$ $(-x)y - (-x)^2 = 0$ The second equation simplifies to $(-x)(y + x) = 0$ but we are in a case where $x \neq 0$ so $(-x)(y + x) = 0 \Leftrightarrow y + x = 0$. Therefore in

case the problem simplifies to solving $$x+y=5$$ $$y-x=0.$$ $$y=x$$ $$2y=5 \Longrightarrow y=5/2.$$ $$(x,y)=\left(\frac{5}{2},\frac{5}{2}\right).$$	$(x,y)=(0,5)$.	this case the problem simplifies to solving $$-x+y=5$$ $$y+x=0$$ $$y=-x$$ $$-x-x=5 \Longrightarrow -2x=5$$ $$x=-5/2$$ $$(x,y)=\left(-\frac{5}{2},\frac{5}{2}\right).$$

So, there are three (x,y) pairs that solve this system. $(x,y)=(5/2,5/2)$, $(x,y)=(0,5)$ and $(x,y)=(-5/2,5/2)$.

■

6 Useful Factoring Formulas

$$(a+b)^2=a^2+2ab+b^2$$

$$(a-b)^2=a^2-2ab+b^2$$

$$(a+b)^3=a^3+3a^2b+3ab^2+b^3$$

$$(a-b)^3=a^3-3a^2b+3ab^2-b^3$$

$$a^2-b^2=(a-b)(a+b)$$

$$a^3-b^3=(a-b)(a^2+ab+b^2)$$

$$a^3+b^3=(a+b)(a^2-ab+b^2)$$

$$a^n-b^n=(a-b)(a^{n-1}+a^{n-2}b+\cdots+ab^{n-2}+b^{n-1})$$

$$a^n-b^n=(a+b)(a^{n-1}-a^{n-2}b+\cdots+ab^{n-2}-b^{n-1}) \text{ for even values of } n$$

$$a^n+b^n=(a+b)(a^{n-1}-a^{n-2}b+\cdots-ab^{n-2}+b^{n-1}) \text{ for odd values of } n.$$

19.	Source: MSHSML 4A174 Determine exactly the value of $$\frac{1}{\sqrt[3]{4}+\sqrt[3]{6}+\sqrt[3]{9}}.$$

Clarification: What this question means by "determine exactly" is to simplify this expression "as much as possible". But that is clearly too vague. So, let me add that you should simplify this expression into the form $c\sqrt[3]{d}+e\sqrt[3]{f}$ for some integers c, d, e and f.

Solution

Look for clues! Notice that

$$\sqrt[3]{4}+\sqrt[3]{6}+\sqrt[3]{9}=\sqrt[3]{2\cdot 2}+\sqrt[3]{2\cdot 3}+\sqrt[3]{3\cdot 3}.$$

Is this a way to use this symmetry? Let's take a couple more steps.

$$\begin{aligned}\sqrt[3]{2\cdot 2}+\sqrt[3]{2\cdot 3}+\sqrt[3]{3\cdot 3} &= \sqrt[3]{2}\cdot\sqrt[3]{2}+\sqrt[3]{2}\cdot\sqrt[3]{2}+\sqrt[3]{3}\cdot\sqrt[3]{3}\\ &= \left(\sqrt[3]{2}\right)^2+\left(\sqrt[3]{2}\right)\left(\sqrt[3]{3}\right)+\left(\sqrt[3]{3}\right)^2\end{aligned}$$

Look at the form of this last expression: a^2+ab+b^2, with $a=\sqrt[3]{2}$ and $b=\sqrt[3]{3}$.

Where have we seen this before? Remember that

$$a^3-b^3=(a-b)(a^2+ab+b^2).$$

which implies

$$a^2+ab+b^2=\frac{a^3-b^3}{a-b}.$$

This shows again the need to ***memorize*** the special case factoring formulas listed above. You need to recognize certain forms by inspection.

Thus,

$$\left(\sqrt[3]{2}\right)^2 + \left(\sqrt[3]{2}\right)\left(\sqrt[3]{3}\right) + \left(\sqrt[3]{3}\right)^2 = \frac{\left(\sqrt[3]{2}\right)^3 - \left(\sqrt[3]{3}\right)^3}{\left(\sqrt[3]{2}\right) - \left(\sqrt[3]{3}\right)} = \frac{2-3}{\left(\sqrt[3]{2}\right) - \left(\sqrt[3]{3}\right)}.$$

Therefore,

$$\frac{1}{\sqrt[3]{4} + \sqrt[3]{6} + \sqrt[3]{9}} = \frac{1}{\left(\frac{2-3}{\left(\sqrt[3]{2}\right) - \left(\sqrt[3]{3}\right)}\right)} = (-1)\left(\sqrt[3]{2} - \sqrt[3]{3}\right) = \sqrt[3]{3} - \sqrt[3]{2}.$$

■

7 Functional Equations

If $g(x)$ is a continuous* function, then the following results are true for all real x. These results are collectively known as **Cauchy's functional equations**.

(1)	If $g(ab) = g(a) \cdot g(b)$ for all real numbers a and b, then $g(x) = x^c$ for some c.
(2)	If $g(ab) = g(a) \cdot g(b)$ for all real numbers a and b and $g(0) \neq 0$, then $g(x) = 1$.
(3)	If $g(ab) = g(a) + g(b)$ for all real numbers a and b, then $g(x) = c\ln(x)$ for some c.
(4)	If $g(a+b) = g(a) \cdot g(b)$ for all real numbers a and b, then $g(x) = e^{cx}$ for some c.
(5)	If $g(a+b) = g(a) + g(b)$ for all real numbers a and b, then $g(x) = cx$ for some c.

*Continuity is a sufficient but not a necessary condition for these results to hold.

Example

20.	4A134 For all real numbers a and b, the function g satisfies the equation $g(ab) = g(a) \cdot g(b)$. If $g(0) \neq 0$, determine exactly the value of $g(2013) + g(2014)$.

Solution

From Result (2) above, $g(x) = 1$ for all x. Therefore, $g(2013) + g(2014) = 1 + 1 = 2$.

To see why Result (2) is true, consider taking $a = 0$ and $b = x$ in the result $g(ab) = g(a)g(b)$. We find $g(0 \cdot x) = g(0) \cdot g(x)$, which implies $g(0) = g(0) \cdot g(x)$. As the problem stipulates that $g(0) \neq 0$, we can divide both sides of this last equation by $g(0)$ to see that $g(x) = 1$. As this derivation holds for all x we have established that $g(x) = 1$ for all x.

■

8 Partial Fraction Decomposition

21.	Find constants A, B and C such that $$\frac{2x^2+3}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$ for all x.

Solution

For now. let's take it for granted that there exists a *unique* set of constants A, B and C such that this expression is true for all x. Now multiply both sides of this identity by $x(x-1)^2$. After simplification, we will have the identity

$$2x^2 + 3 = A(x-1)^2 + Bx(x-1) + Cx.$$

If we plug in three different values for x, say x_0, x_1 and x_2 then we get three equations that will allow us to solve for the three unknowns A, B and C.

$$2x_0^2 + 3 = A(x_0-1)^2 + Bx_0(x_0-1) + Cx_0$$
$$2x_1^2 + 3 = A(x_1-1)^2 + Bx_1(x_1-1) + Cx_1$$
$$2x_2^2 + 3 = A(x_2-1)^2 + Bx_2(x_2-1) + Cx_2.$$

And assuming that A, B and C are *unique* we will get the same values for these three unknowns *no matter what values we picked* for x_0, x_1 and x_2.

So obviously we should pick three values x_0, x_1 and x_2 that will make the process of solving for A, B and C as simple as possible.

Notice that if you take $x_0 = 1$ the terms $A(1-1)^2$ and $B(1)(1-1)$ will both equal 0 and leaves you with the equation

$$2(1^2) + 3 = A(1-1)^2 + B(1)(1-1) + C(1) = C.$$

Therefore, $C = 5$. If you take $x_1 = 0$ the terms $B(0)(0-1)$ and $C(0)$ will both equal 0 leaves you with the equation

$$2(0^2) + 3 = A(0-1)^2 + B(0)(0-1) + C(0) = A.$$

Therefore, $A = 3$. There is no other value of x that will make several terms drop out so just pick something simple for x_2 such as $x_2 = 2$. Then

$$2(2^2) + 3 = (3)(2-1)^2 + B(2)(2-1) + 5(2) = 2B + 13.$$

Solving for B in this last equation we get $B = -1$.

■

This kind of problem is called a **partial fraction decomposition** of a rational function. The general format starts with a rational function

$$f(x) = \frac{P(x)}{Q(x)}$$

where

- $P(x)$ and $Q(x)$ are polynomials with real valued coefficients
- $P(x)$ and $Q(x)$ have no common factors (*i.e.* relatively prime)
- the degree of $P(x)$ is strictly less than the degree of $Q(x)$
- $Q(x)$ is completely factored into its linear and irreducible quadratic factors.

Remember that the general quadratic function $ax^2 + bx + c$ is *irreducible* over the reals (cannot be factored without introducing non-real complex numbers) if the discriminant $b^2 - 4ac$ is negative.

It is a corollary of Gauss's Fundamental Theorem of Algebra that every polynomial with real valued coefficients can be factored into linear and irreducible quadratic factors. This theorem does not tell us how to find such a factorization but rather it tells us that such a factorization necessarily exists.

Some examples of $f(x)$ would be:

$$f(x) = \frac{x^2 + 1}{(x-1)(x-2)(x-3)}$$

$$f(x) = \frac{-2x+4}{(x-1)(x-1)(x^2+1)} = \frac{-2x+4}{(x-1)^2(x^2+1)}$$

and

$$f(x) = \frac{3x^4+x^3+20x^2+3x+31}{(x+1)(x^2+4)(x^2+4)} = \frac{3x^4+x^3+20x^2+3x+31}{(x+1)(x^2+4)^2}.$$

Notice that in each of these examples the denominator $Q(x)$ (if we multiplied all the factors) would be a polynomial whose degree is strictly larger than the degree of its numerator polynomial $P(x)$.

Take the last example. If we multiplied out the denominator $(x+1)(x^2+4)^2$ we would get

$$(x+1)(x^2+4)^2 = x^5+x^4+8x^3+8x^2+16x+16$$

which is a fifth-degree polynomial which is strictly larger than its fourth-degree numerator.

Notice that the problem we just finished solving

$$f(x) = \frac{2x^2+3}{x(x-1)(x-1)} = \frac{2x^2+3}{x(x-1)^2}$$

has this form. In the language of partial fractions a denominator $Q(x)$ of the form

$$Q(x) = (2x-3)(x-2)^3(2x^2+5)^2$$

is said to have three *distinct* factors with the linear factor $(x-2)$ repeated three times and the irreducible quadratic factor repeated twice.

Look back at the problem we just solved where we showed

$$\frac{2x^2+3}{x(x-1)^2} = \frac{3}{x} + \frac{-1}{x-1} + \frac{5}{(x-1)^2}$$

for all x.

We broke up (decomposed) a complicated fraction (rational function) on the left-hand side of the equation into several smaller (partial) fractions on the right-hand side.

A key step in partial fraction problems is knowing how to set up the denominators on the right-hand side of the equation in a way where there will necessarily exist constants that make this equation an identity (true for *all* x.)

Partial Fraction Decomposition Theorem

Let the rational function $f(x) = P(x)/Q(x)$ be such that P and Q meet the four conditions set our above.

Suppose that for every distinct factor $(ax+b)^k$ in $Q(x)$ we put the terms

$$\frac{C_1}{ax+b}+\frac{C_2}{(ax+b)^2}+\cdots+\frac{C_k}{(ax+b)^k}$$

in the decomposition where $C_1, C_2, \dots, C_k$ are a set of unknown constants and for every distinct factor $(ax^2+bx+c)^k$ in $Q(x)$ we put the terms

$$\frac{D_1x+E_1}{ax^2+bx+c}+\frac{D_2x+E_2}{(ax^2+bx+c)^2}+\cdots+\frac{D_kx+E_k}{(ax^2+bx+c)^k}$$

in the decomposition where $D_1, D_2, \dots, D_k$ and $E_1, E_2, \dots, E_k$ are a set of unknown constants.

Then there exists *unique* values for all these unspecified constants that will make this decomposition an identity.

Look back at the three examples we listed for $f(x)$. The Partial Fraction Decomposition Theorem tells us there will exists constants to make each of the following an equations an identity:

$$\frac{x^2+1}{(x-1)(x-2)(x-3)}=\frac{A}{x-1}+\frac{B}{x-2}+\frac{C}{x-3}$$

$$\frac{-2x+4}{(x-1)^2(x^2+1)}=\left(\frac{A}{x-1}+\frac{B}{(x-1)^2}\right)+\frac{Cx+D}{x^2+1}$$

and

$$\frac{3x^4+x^3+20x^2+3x+31}{(x+1)(x^2+4)^2}=\frac{A}{x+1}+\left(\frac{Bx+C}{x^2+4}+\frac{Dx+E}{(x^2+4)^2}\right).$$

22.	Source: MSHSML 4A124 Express $$\frac{2x-1}{(x^2+x+1)(x+1)}$$ as the sum of two **simplified** rational expressions.

Solution

According to the Partial Fraction Decomposition Theorem there will exist constants A, B and C such that for all x

$$\frac{P(x)}{Q(x)} = \frac{2x-1}{(x^2+x+1)(x+1)} = \frac{Ax+B}{x^2+x+1} + \frac{C}{x+1}.$$

Multiplying both sides by $Q(x)$ and simplifying we have

$$2x-1 = (x+1)(Ax+B) + (x^2+x+1)C.$$

Because we have three unknown constants we need to form three equations by picking (any) three values of x and plugging them into this equation. Plugging in $x=-1$ is certainly a good choice because it will "zero out" the term $(-1+1)(A(-1)+B)$. Plugging in $x=0$ is also a good choice because that will eliminate the unknown A in the term $(0+1)(A(0)+B)$.

After that just pick any other values of x. Picking the x values $-1, 0$ and 1 we have the three equations

$$2(-1)-1 = (-1+1)(A(-1)+B) + ((-1)^2+(-1)+1)C$$

$$2(0)-1 = (0+1)(A(0)+B) + (0^2+0+1)C$$

$$2(1)-1 = (1+1)(A(1)+B) + (1^2+1+1)C.$$

From the top equation we have $-3 = 0 + C$ which immediately gives us $C=-3$. From the second and third equations we have

$$-1 = B + C = B - 3 \Rightarrow B = 2.$$

$$1 = 2(A+B) + 3C = 2A + 2B + 3C = 2A + 2(2) + 3(-3) \Rightarrow 2A = 6 \Rightarrow A = 3.$$

Therefore,

$$\frac{2x-1}{(x^2+x+1)(x+1)} = \frac{3x+2}{x^2+x+1} + \frac{-3}{x+1}.$$

■

The Partial Fraction Decomposition Theorem states there will always exist constants that will decompose the rational function $P(x)/Q(x)$ into the partial fractions of the exact form as indicated by the theorem.

However, it does *not* say there cannot be other ways to decompose $P(x)/Q(x)$. If you come to a problem where you are asked to decompose $P(x)/Q(x)$ in a way that isn't the form specified in the PFDT, then just assume that it will be possible (or they would not have asked the questions) and take the same approach to find it.

23.	Source: MSHSML 4T121 Find integers A, B, C and D such that $$\frac{P(x)}{Q(x)} = \frac{x^3+8x+32}{(x^2+4)(x+2)^2} = \frac{Ax+B}{x^2+4} + \frac{Cx+D}{(x+2)^2}$$ for all x.

Solution

The Partial Fraction Decomposition Theorem guarantees there will exist a unique set of constants A, B, C and D such that for all x

$$\frac{x^3+8x+32}{(x^2+4)(x+2)^2} = \frac{Ax+B}{x^2+4} + \frac{C}{x+2} + \frac{D}{(x+2)^2}.$$

The left-hand side of the identity in the problem statement isn't in this exact form. But just assume that constants will exist to solve the problem or else they wouldn't have asked the question). So just move forward on faith and use the same methods to find the unknown constants.

Multiplying both sides of the given identity by $Q(x)$ and simplifying we have the new identity

$$x^3+8x+32 = (x+2)^2(Ax+B) + (x^2+4)(Cx+D).$$

This equation has four unknowns so we will need to pick four values for x to form four equations so we can solve for all the unknowns. Below, I've picked the four x values $-2, -1, 0$ and 1.

$$(-2)^3 + 8(-2) + 32 = (-2+2)^2(A(-2)+B) + ((-2)^2+4)(C(-2)+D)$$

$$(-1)^3 + 8(-1) + 32 = (-1+2)^2(A(-1)+B) + ((-1)^2+4)(C(-1)+D)$$

$$0^3 + 8(0) + 32 = (0+2)^2(A(0)+B) + (0^2+4)(C(0)+D)$$

$$1^3 + 8(1) + 32 = (1+2)^2(A(1)+B) + (1^2+4)(C(1)+D)$$

Remember there are no "wrong" values of x to pick. If you and I pick different values of x to work with we will necessarily end up with the *same* values for A, B, C and D.

These equations simplify to

$$8 = 0^2(-2A+B) + 8(-2C+D) \Longrightarrow D - 2C = 1$$

$$23 = 1^2(-A+B) + 5(-C+D) \Longrightarrow -A + B - 5C + 5D = 23$$

$$32 = 4B + 4D \Longrightarrow B + D = 8$$

$$41 = 9(A+B) + 5(C+D) \Longrightarrow 9A + 9B + 5C + 5D = 41.$$

From the first equation we get $D = 2C + 1$. From the third equation we get

$$B = 8 - D = 8 - (2C+1) = 7 - 2C.$$

Therefore, we can simplify the second equation to

$$-A + (7 - 2C) - 5C + 5(2C+1) = 23$$

or

$$A = 7 - 2C - 5C + 10C + 5 - 23 = -11 + 3C.$$

At this point we have determined that

$$A = -11 + 3C$$

$$B = 7 - 2C$$

$$D = 2C + 1.$$

Substituting these results into the fourth equation we have

$$9(-11+3C) + 9(7-2C) + 5C + 5(2C+1) = 41$$

or

$$24C - 31 = 41.$$

Therefore, $C = 3$. Hence $A = -2, B = 1$ and $D = 7$. This establishes the identity

$$\frac{x^3 + 8x + 32}{(x^2+4)(x+2)^2} = \frac{-2x+1}{x^2+4} + \frac{3x+7}{(x+2)^2}.$$

■

8.1 Telescoping Sums

Partial fraction decompositions are often useful in converting a regular sum into a "telescoping" sum (one where adjacent terms can cancel each other). Consider the following problems.

24.	Source: MSHSML 4A044 Express $$\sum_{n=1}^{100}\frac{1}{n(n+1)}=\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\cdots+\frac{1}{10100}$$ as the quotient of two relatively prime integers.

Solution

Start by finding the partial fraction decomposition of

$$\frac{1}{n(n+1)}=\frac{A}{n}+\frac{B}{n+1}.$$

Multiplying both sides by $n(n+1)$ we have the identity

$$1=A(n+1)+Bn$$

involving two unknowns. Taking $n=-1$ gives $1=A(0)+B(-1)=-B$. So $B=-1$. Taking $x=0$ gives $1=A(0+1)+B(0)=A$. So $A=1$.

That is, we have by partial fractions that for all x

$$\frac{1}{n(n+1)}=\frac{1}{n}-\frac{1}{n+1}.$$

Therefore,

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{100\cdot 101}$$

$$=\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{99}-\frac{1}{100}\right)+\left(\frac{1}{100}-\frac{1}{101}\right)$$

$$=\frac{1}{1}+\left(-\frac{1}{2}+\frac{1}{2}\right)+\left(-\frac{1}{3}+\frac{1}{3}\right)+\left(-\frac{1}{4}+\frac{1}{4}\right)+\cdots+\left(-\frac{1}{100}+\frac{1}{100}\right)-\frac{1}{101}$$

$$= \frac{1}{1} - \frac{1}{101} = \frac{101-1}{1(101)} = \frac{100}{101}.$$

■

25.	Source: MSHSML SI1211 Determine exactly the sum $$\frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{2012\cdot 2013}.$$

Solution

Start by finding the partial fraction decomposition of

$$\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}.$$

Multiplying both sides by $x(x+1)$ we have the identity

$$1 = A(x+1) + Bx$$

involving two unknowns. Taking $x = -1$ gives $1 = A(0) + B(-1) = -B$. So $B = -1$. Taking $x = 0$ gives $1 = A(0+1) + B(0) = A$. So $A = 1$.

That is, we have by partial fractions that for all x

$$\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}.$$

Therefore,

$$\frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{2012\cdot 2013}$$

$$= \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{2011} - \frac{1}{2012}\right) + \left(\frac{1}{2012} - \frac{1}{2013}\right)$$

$$= \frac{1}{2} + \left(\frac{1}{3} - \frac{1}{3}\right) + \left(\frac{1}{4} - \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{2012} - \frac{1}{2012}\right) - \frac{1}{2013}$$

$$= \frac{1}{2} - \frac{1}{2013} = \frac{2013-2}{2(2013)} = \frac{2011}{4026}.$$

■

9 Nested Roots

The expression $\sqrt{a + b\sqrt{c}}$ is called a "nested" square root because it involves a square root inside another square root.

The identity

$$\sqrt{a \pm b\sqrt{c}} = \sqrt{\frac{a + \sqrt{a^2 - b^2c}}{2}} \pm \sqrt{\frac{a - \sqrt{a^2 - b^2c}}{2}} \qquad (*)$$

is valid for all nonnegative c and if $a^2 - b^2c$ is a perfect square then we will be able to use this identity to "denest" these radicals.

It can be shown that the case of $a^2 - b^2c$ a perfect square is the *only* situation when we can "denest" an expression of this form.

26.	Source: MSHSML 4A173 $\sqrt{6 - 2\sqrt{5}}$ can be written in the form $a + b\sqrt{c}$, where a, b, and c are integers and c has no square factors. Determine the ordered triple (a, b, c).

<u>Solution</u>

Use the above identity $(*)$ to denest this nested root.

$$\sqrt{6 - 2\sqrt{5}} = \sqrt{a \pm b\sqrt{c}}$$

for $a = 6, b = 2$ and $c = 5$. Notice that $a^2 - b^2c = 6^2 - 2^2 \cdot 5 = 2^2$, which is a perfect square. So, the above identity will work to denest the left-hand side. Applying this identity we see that

$$\sqrt{6 - 2\sqrt{5}} = \sqrt{\frac{6 + \sqrt{6^2 - 2^2 \cdot 5}}{2}} - \sqrt{\frac{6 - \sqrt{6^2 - 2^2 \cdot 5}}{2}}$$

$$= \sqrt{\frac{6+4}{2}} - \sqrt{\frac{6-4}{2}}$$

$$= \sqrt{5} - 1$$

$$= -1 + 1 \cdot \sqrt{5}.$$

That is, $\sqrt{6 - 2\sqrt{5}} = a + b\sqrt{c} = -1 + 1 \cdot \sqrt{5}$. So $(a, b, c) = (-1,1,5)$.

■

27.	Source: MSHSML 4T852 Write $\sqrt{6 - 3\sqrt{3}}$ as an expression without nested square roots.

Solution

Again use the above identity $(*)$ to denest this nested root.

$$\sqrt{a \pm b\sqrt{c}} = \sqrt{\frac{a + \sqrt{a^2 - b^2c}}{2}} \pm \sqrt{\frac{a - \sqrt{a^2 - b^2c}}{2}}$$

$$a = 6, b = 3, c = 3$$

$$\sqrt{\frac{6 + \sqrt{6^2 - 3^2 \cdot 3}}{2}} - \sqrt{\frac{6 - \sqrt{6^2 - 3^2 \cdot 3}}{2}} = \sqrt{\frac{6 + \sqrt{9}}{2}} - \sqrt{\frac{6 - \sqrt{9}}{2}}$$

$$= \sqrt{\frac{6+3}{2}} - \sqrt{\frac{6-3}{2}} = \frac{3}{\sqrt{2}} - \frac{\sqrt{3}}{\sqrt{2}} = \frac{3\sqrt{2} - \sqrt{6}}{2}$$

■

10 Extra Practice Problems with Solutions for Test 4A

28.	Source: MSHSML 4A071 Find all x that satisfy $\sqrt{2x-7} = \sqrt{x-1}$.

Solution

$$\begin{aligned} \sqrt{2x-7} &= \sqrt{x-1} \\ \Longrightarrow \left(\sqrt{2x-7}\right)^2 &= \left(\sqrt{x-1}\right)^2 \\ \Longrightarrow 2x-7 &= x-1 \\ \Longrightarrow x &= 6. \end{aligned}$$

But don't forget that squaring both sides in the first step above can produce ***extraneous*** solutions. Is the above solution $x = 6$ extraneous? You have to check! Does $x = 6$ satisfy the original equation $\sqrt{2x-7} = \sqrt{x-1}$? Yes. That is,

$$\sqrt{2(6)-7} = \sqrt{5} \text{ and } \sqrt{6-1} = \sqrt{5}.$$

■

29.	Source: MSHSML 4A073 The roots of $ax^2 + bx + 1 = 0$ are -2 and 3. What are the roots of $bx^2 + ax - 1 = 0$?

Solution

If the roots of $ax^2 + bx + 1 = 0$ are -2 and 3 then $ax^2 + bx + 1 = k\big(x-(-2)\big)(x-3)$ for some constant k.

But

$$k\big(x-(-2)\big)(x-3) = k(x+2)(x-3) = k(x^2+2x-3x-6) = kx^2 - kx - 6k.$$

So,

$$ax^2 + bx + 1 = kx^2 - kx - 6k \text{ for } \textbf{all } x$$

But remember that the only way that two polynomials can be equal for **all** x is if those two polynomials agree coefficient by coefficient.

In our above situation, this means that a, coefficient of x^2 in $ax^2 + bx + 1$ must equal k, the coefficient of x^2 in $kx^2 - kx - 6k$.

And b must equal $-k$. And 1 must equal $-6k$.

But $1 = -6k \Leftrightarrow k = -1/6$. So $a = k = -1/6$ and $b = -k = -(-1/6) = 1/6$.

Hence,

$$bx^2 + ax - 1 = \frac{1}{6}x^2 - \frac{1}{6}x - 1 = \left(\frac{1}{6}\right)(x^2 - x - 6) = \left(\frac{1}{6}\right)(x-3)(x+2).$$

Therefore, the roots of $bx^2 + ax - 1$ are $x = 3$ and $x = -2$.

■

30.	Source: MSHSML 4A063 Given that $f(x) = \frac{x - \frac{1}{x}}{x+1}$, find $[f(x^{-1})][f(x)]^{-1}$.

Solution

$$f(x^{-1}) = f\left(\frac{1}{x}\right) = \frac{\left(\frac{1}{x}\right) - \frac{1}{\left(\frac{1}{x}\right)}}{\left(\frac{1}{x}\right) + 1} = \frac{\left(\frac{1}{x}\right) - x}{\left(\frac{1}{x}\right) + 1} = \frac{\frac{1-x^2}{x}}{\frac{1+x}{x}} = \frac{1-x^2}{1+x} = \frac{(1-x)(1+x)}{1+x} = 1 - x$$

provided $x \neq 0$ and $x \neq -1$. (We would be dividing by 0 when $x = 0$ and/or $x = -1$.)

$$\left(f(x)\right)^{-1} = \frac{1}{f(x)} = \frac{1}{\left(\frac{x - \frac{1}{x}}{x+1}\right)} = \frac{1}{\frac{\left(\frac{x^2-1}{x}\right)}{x+1}} = \frac{1}{\left(\frac{x^2-1}{x(x+1)}\right)}$$

$$= \frac{x(x+1)}{x^2-1} = \frac{x(x+1)}{(x+1)(x-1)} = \frac{x}{x-1}$$

provided $x \neq -1$. (We would be dividing by 0 when $x = -1$.)

Therefore,

$$f(x^{-1})\cdot\left(f(x)\right)^{-1} = (1-x)\cdot\left(\frac{x}{x-1}\right) = (-1)(x-1)\cdot\left(\frac{x}{x-1}\right) = -x$$

provided $x \neq 1$. (We would be dividing by 0 when $x = 1$.)

So,

$$f(x^{-1})\cdot\left(f(x)\right)^{-1} = x$$

except for $x = -1, x = 0$ or $x = 1$ where this product is undefined.

■

31.	Source: MSHSML 4A064 Find all lattice points interior to the first quadrant, that is all points (m, n) where m and n are positive integers, that lie of the graph of $x^2 + y^2 + 2xy - 4x - 4y - 5 = 0$.

Solution

General Rule of Thumb.

If you are trying to find solutions to $f(x) = 0$ or to $f(x, y) = 0$ as we have above, always look to see if you can ***factor*** $f(\,)$. If, in the problem above, we can write $f(x, y) = g(x, y)\cdot h(x, y)$ then solving $f(x, y) = 0$ is equivalent to solving the two simpler problems $g(x, y) = 0$ and $h(x, y) = 0$.

Unfortunately, there is no formula or algorithm that shows how to factor every expression. But a good starting point for factoring is to look for any clues that the form of your expression might offer.

For example, does $x^2 + y^2 + 2xy$ look familiar? Of course,

$$x^2 + 2xy + y^2 = (x + y)^2.$$

Now notice that $-4x - 4y = (-4)(x + y)$. So,

$$x^2 + y^2 + 2xy - 4x - 4y - 5 = (x + y)^2 - 4(x + y) - 5 = 0.$$

Does this help? YES! We can now see that our expression is a quadratic function in the expression $x + y$. That is, the above expression has the form $z^2 - 4z - 5$ with $z = x + y$.

But $z^2 - 4z - 5 = (z - 5)(z + 1)$. So

$$\begin{aligned} x^2 + y^2 + 2xy - 4x - 4y - 5 &= (x + y)^2 - 4(x + y) - 5 \\ &= \big((x + y) - 5\big)\big((x + y) + 1\big). \end{aligned}$$

Aha! We have a factorization.

Is this helpful? For sure.

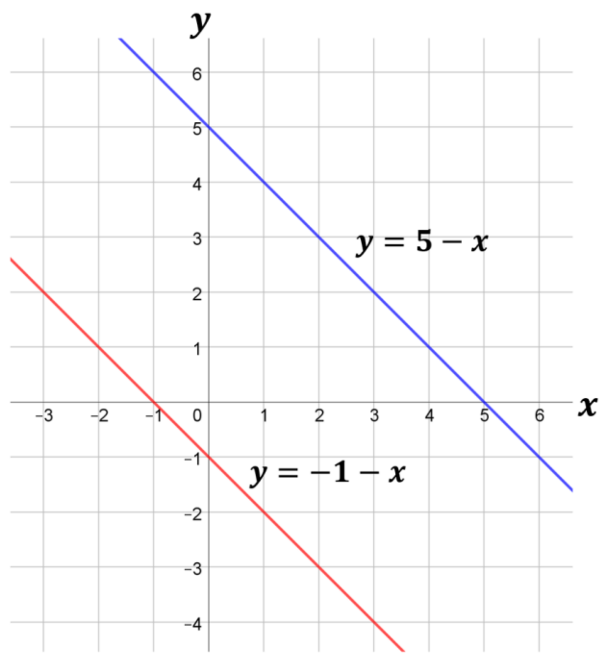
Finding (x, y) pairs where $x^2 + y^2 + 2xy - 4x - 4y - 5 = 0$ is much, much harder than finding (x, y) pairs where $(x + y) - 5 = 0$ ***or*** where $(x + y) + 1 = 0$.

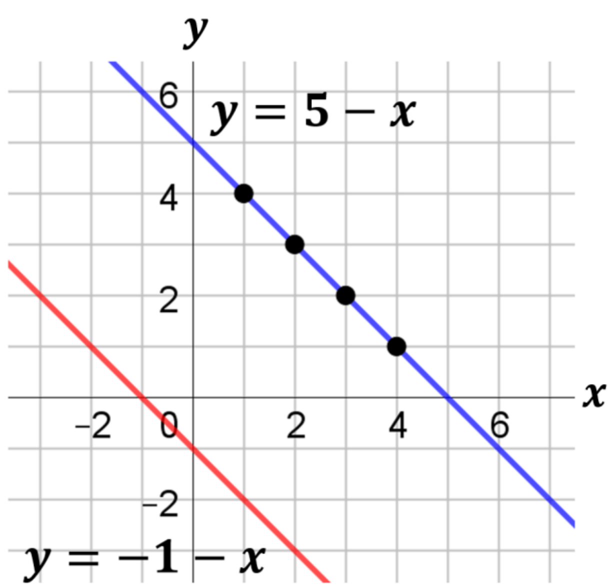
In fact, $(x + y) - 5 = 0$ is equivalent to $y = 5 - x$. That is, $(x + y) - 5 = 0$ for all (x, y) on the line $y = 5 - x$.

And $(x + y) + 1 = 0$ is equivalent to $y = -1 - x$. But we can easily see that $y < 0$ whenever $x > 0$ on the line $y = -1 - x$. So there are no points interior to the first quadrant (where $x > 0$ and $y > 0$) where $y = -1 - x$.

So, the problem has reduced to finding all (x, y) with x and y both integers (we are instructed to find lattice point solutions) interior the first quadrant (*i.e.* where x and y are both positive) such that $y = 5 - x$.

While not necessary, it might help to graph these two lines.





Clearly the only (x, y) pairs which are

- interior to the first quadrant (*i.e.* $x > 0$ and $y > 0$)
- where x and y are both integers (*i.e.* lattice points)
- where x and y are on the red line ***<u>or</u>*** on the blue line

are the pairs $(x, y) = (1,4), (2,3), (3,2)$ and $(4,1)$.

■

32.	Source: MSHSML 4A023 Find all the ordered pairs (x, y) that simultaneously satisfy the two equations $$x^3 - y^3 = 35$$ $$x^2 + y^2 = 7 - xy.$$

Solution

Except for the simplest situation of solving a system of ***linear*** equations, there is no "one method fits all" approach to solving a system of equations. And clearly the above has quadratic and cubic terms, so it is not linear.

Remember the "Rule of Thumb" I mentioned in a previous problem about the usefulness of factoring when trying to solving an equation? Well the same rule of thumb applies to solving a system of equations.

Do you see anything in this system that looks like it has been set up for factoring? How about $x^3 - y^3$? Look back in the Test 4A Study Guide and turn to the section on "Useful Factoring Formulas". You should memorize these formulas. They arise frequently. We see that

$$x^3 - y^3 = (x - y)(x^2 + xy + y^2).$$

Aha! Notice that the second equation in our system can be written as $x^2 + xy + y^2 = 7$. And we notice that conveniently the expression $x^2 + xy + y^2$ is part of the factorization of $x^3 - y^3$.

So,

$$\begin{Bmatrix} x^3 - y^3 = 35 \\ x^2 + y^2 = 7 - xy \end{Bmatrix} \Leftrightarrow \begin{Bmatrix} (x - y)(x^2 + xy + y^2) = 35 \\ x^2 + y^2 = 7 - xy \end{Bmatrix}$$

$$\Leftrightarrow \begin{Bmatrix} (x - y)(x^2 + xy + y^2) = 35 \\ x^2 + xy + y^2 = 7 \end{Bmatrix}$$

$$\Leftrightarrow \begin{Bmatrix} (x - y)(7) = 35 \\ x^2 + xy + y^2 \end{Bmatrix}$$

$$\Leftrightarrow \begin{Bmatrix} x - y = 5 \\ x^2 + xy + y^2 = 7 \end{Bmatrix}$$

$$\Leftrightarrow \begin{Bmatrix} x = y + 5 \\ x^2 + xy + y^2 = 7 \end{Bmatrix}$$

$$\Leftrightarrow \begin{Bmatrix} x = y + 5 \\ (y+5)^2 + (y+5)y + y^2 - 7 = 0 \end{Bmatrix}.$$

Notice that the second equation in our system *only* involves y. So, we can use this second equation to solve for y.

Furthermore,

$$\begin{aligned}(y+5)^2 + (y+5)y + y^2 - 7 &= (y^2 + 10y + 25) + (y^2 + 5y) + y^2 - 7 \\ &= 3y^2 + 15y + 18 \\ &= 3(y+2)(y+3).\end{aligned}$$

So,

$$\begin{aligned}(y+5)^2 + (y+5)y + y^2 - 7 = 0 &\Leftrightarrow 3(y+2)(y+3) = 0 \\ &\Leftrightarrow y = -2 \text{ or } y = -3.\end{aligned}$$

But remember, $x = y + 5$. So the set of all (x, y) pairs that satisfy the given system of equations would be $(3, -2)$ and $(2, -3)$.

■

33.	Source: MSHSML 4A024 Factor $6x^2 + x - 2y^2 + 10y + xy - 12$ into the product of two first degree terms in x and y.

Solution

It is often a good idea to group together all the equal power terms and see if the groupings factor individually in some way.

By "equal power terms" I mean to say $x^{a_1}y^{a_2}$ and $x^{b_1}y^{b_2}$ are "equal power terms" if $a_1 + a_2 = b_1 + b_2$.

By this understanding, x^2, xy and y^2 are "equal power terms". So grouping like this we have

$$6x^2 + x - 2y^2 + 10y + xy - 12 = (6x^2 + xy - 2y^2) + (x + 10y) - 12.$$

Notice that

$$6x^2 + xy - 2y^2 = (3x + 2y)(2x - y).$$

What would be *really* convenient is if the grouping $(x + 10y)$ could somehow be expressed in terms of the factors $(3x + 2y)$ and $(2x - y)$ that occur in the first grouping.

Is it possible to write

$$x + 10y = a(3x + 2y) + b(2x - y)$$

for some constants a and b? Now remember that two polynomials can only equal each other for all x and y if the coefficients match up term by term.

$$x + 10y = a(3x + 2y) + b(2x - y) = (3a + 2b)x + (2a - b)y.$$

So if this is going to work (nothing guarantees it will), then we must have

$$3a + 2b = 1 \quad \text{and} \quad 2a - b = 10.$$

From the second equation, $b = 2a - 10$. Substituting this into the first equation, we have

$$3a + 2(2a - 10) = 1$$
$$7a = 21$$
$$a = 3.$$

But $b = 2a - 10$, so $b = 2(3) - 10 = -4$. That is, $(a, b) = (3, -4)$ will work!

$$x + 10y = 3(3x + 2y) - 4(2x - y).$$

Plugging this back into our problem, we have

$$\begin{aligned} &6x^2 + x - 2y^2 + 10y + xy - 12 \\ &\qquad = (6x^2 + xy - 2y^2) + (x + 10y) - 12 \\ &\qquad = (3x + 2y)(2x - y) + 3(3x + 2y) - 4(2x - y) - 12 \end{aligned}$$

Now think about what this looks like. What do you get when you "foil" $(s + 3)(t - 4)$?

$$(s + 3)(t - 4) = st + 3t - 4s - 12.$$

How does this fit into our problem? Let $s = 2x - y$ and let $t = 3x + 2y$. Then

$$st + 3t - 4s - 12 = (2x - y)(3x + 2y) + 3(3x + 2y) - 4(2x - y) - 12.$$

Another Aha! So,

$$\begin{aligned}
&(2x - y)(3x + 2y) + 3(3x + 2y) - 4(2x - y) - 12 \\
&\qquad = st + 3t - 4s - 12 \\
&\qquad = (s + 3)(t - 4) \\
&\qquad = (2x - y + 3)(3x + 2y - 4).
\end{aligned}$$

And there we have it.

$$6x^2 + x - 2y^2 + 10y + xy - 12 = (2x - y + 3)(3x + 2y - 4).$$

■

34.	Source: MSHSML 4A993 Find the exact value of x, different from 1, such that $1 + 2\sqrt{x} - \sqrt[3]{x} - 2\sqrt[6]{x} = 0$.

Solution

Let $y = \sqrt[6]{x}$. Then $y^3 = \left(\sqrt[6]{x}\right)^3 = x^{3/6} = \sqrt{x}$ and $y^2 = \left(\sqrt[6]{x}\right)^2 = x^{2/6} = \sqrt[3]{x}$. Therefore, we can rewrite the entire left-hand side as

$$1 + 2\sqrt{x} - \sqrt[3]{x} - 2\sqrt[6]{x} = 1 + 2y^3 - y^2 - 2y.$$

So in terms of y the problem becomes, solve

$$1 + 2y^3 - y^2 - 2y = 0.$$

Now what? You guessed it, FACTOR! How? Look for clues in the form of the problem. Do you see any way to group the four terms into two groups such that each group has something in common?

Notice that $2y^3 - 2y = 2y(y^2 - 1)$ and $-y^2 + 1 = (-1)(y^2 - 1)$.

Aha! Both of these groupings contain the factor $y^2 - 1$.

$$\begin{aligned} 0 &= 1 + 2y^3 - y^2 - 2y \\ &= (2y^3 - 2y) + (-y^2 + 1) \\ &= 2y(y^2 - 1) + (-1)(y^2 - 1) \\ &= (y^2 - 1)(2y - 1) \\ &= (y - 1)(y + 1)(2y - 1) \end{aligned}$$

We know that

$$1 + 2y^3 - y^2 - 2y = (y - 1)(y + 1)(2y - 1) = 0$$

$\Leftrightarrow y - 1 = 0$ or $y + 1 = 0$ or $2y - 1 = 0$

$\Leftrightarrow y^2 = 1$ or $y = \dfrac{1}{2}$

$\Leftrightarrow y = -1$ or $y = 1$ or $y = \dfrac{1}{2}$

$\Leftrightarrow \sqrt[6]{x} = -1$ or $\sqrt[6]{x} = 1$ or $\sqrt[6]{x} = \dfrac{1}{2}$

$\Leftrightarrow \sqrt[6]{x} = -1$ or $\sqrt[6]{x} = 1$ or $\sqrt[6]{x} = \dfrac{1}{2}$

Now there are no real numbers such that $\sqrt[6]{x} = -1$. Furthermore

$$\sqrt[6]{x} = 1 \Leftrightarrow x = 1$$

and

$$\sqrt[6]{x} = \frac{1}{2} \Leftrightarrow x = \left(\frac{1}{2}\right)^6 = \frac{1}{64}.$$

So, the only value of x (other than $x = 1$) that satisfies the equation

$$1 + 2\sqrt{x} - \sqrt[3]{x} - 2\sqrt[6]{x} = 0$$

is $x = 1/64$.

∎