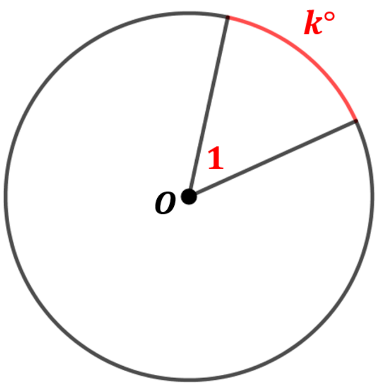
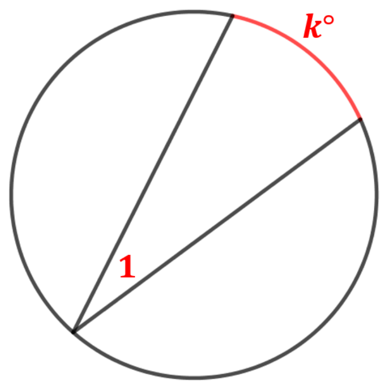
# MSHSML Meet 4, Event B
# Study Guide

## 4B Circular Figures and Solids

Central, inscribed, tangential, and exterior angles
Power of a point (chords, secants, tangents)
Interior and exterior tangents of two circles
Intercepted arcs
Area of circles, sectors, circular segments
Cylinders, cones, & spheres (including volume and surface area)

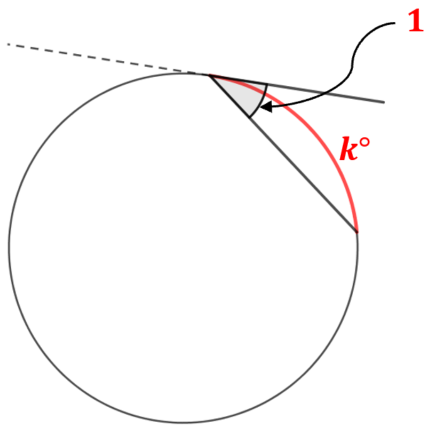
## Table of Contents

## 1. Central, Inscribed, Tangential, Interior and Exterior Angles

<table>
<tr>
<td>If an arc measures $k°$ then the associated central angle also measures $k°$.<br><br><br>$m\angle 1 = k$<br>$\angle 1$ is called a central angle<br>$O$ is the center of the circle</td>
<td>If an arc measures $k°$ then the associated inscribed angle measures $(k/2)°$.<br><br><br>$m\angle 1 = k/2$<br>$\angle 1$ is called an inscribed angle</td>
</tr>
</table>

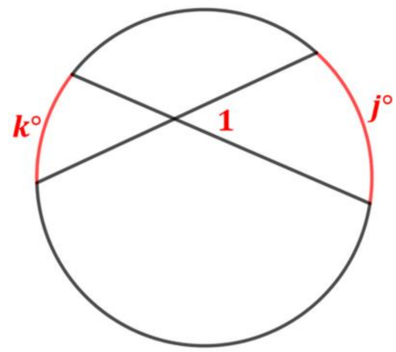
* "$m\angle 1$" is standard mathematics notation for the *measure* or size of angle #1

If an arc measures $k°$ then the associated tangent angle measures $(k/2)°$.



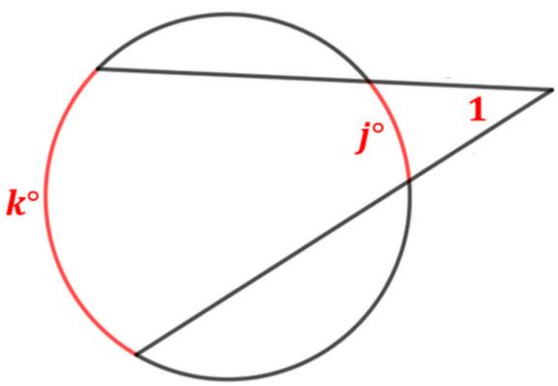

$$m\angle 1 = k/2$$

∠1 is called a **tangent angle**



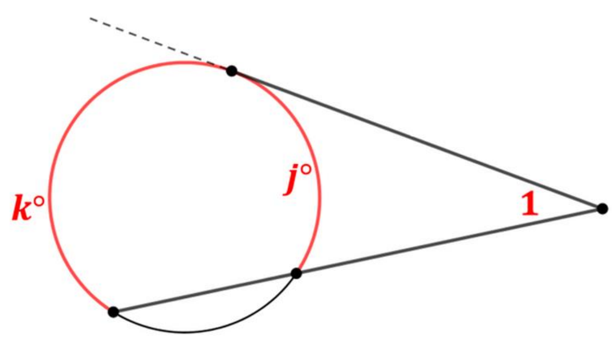

$$m\angle 1 = \frac{1}{2}(k + j)$$

∠1 is called an **interior angle**




$$m\angle 1 = \frac{1}{2}(k - j)$$

∠1 is called an **exterior angle**




$$m\angle 1 = \frac{1}{2}(k - j)$$

∠1 is called an **exterior angle**

## 2. Subtended Angle (Definition)

The angle subtended by an arc, line segment or some other object (for example, a tree) at a given point is the angle formed by drawing the line segments from that given point to the endpoints of that arc, line segment or some other object.

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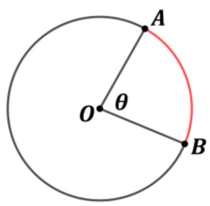
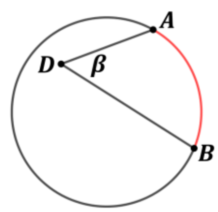
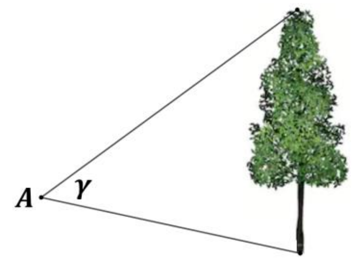
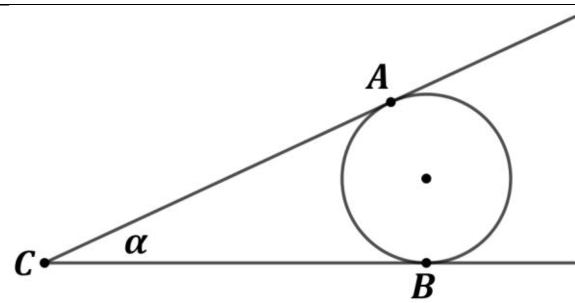
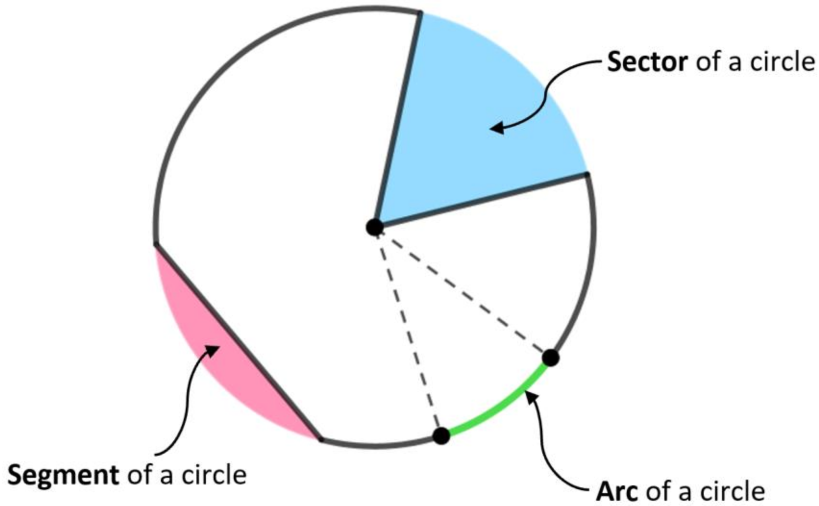
|---|---|---|
|  |  |  |
| $\theta$ is the angle subtended by arc $\widehat{AB}$ at the origin $O$ | $\beta$ is the angle subtended by arc $\widehat{AB}$ at the point $D$ | $\gamma$ is the angle subtended by this tree from the vantage point of $A$ |

In a situation where the object does not have endpoints (e.g. a circle), then the angle subtended by that object at a given point is the smallest angle from that point that fully contains that object. In the case of the circle shown below this would be the angle formed by the two tangent lines to the circle from the given point. The circle is fully contained between these two tangent lines.




$\alpha$ is the angle subtended by the above circle at the point $C$

## 3. Arcs, Sectors and Segments of a Circle



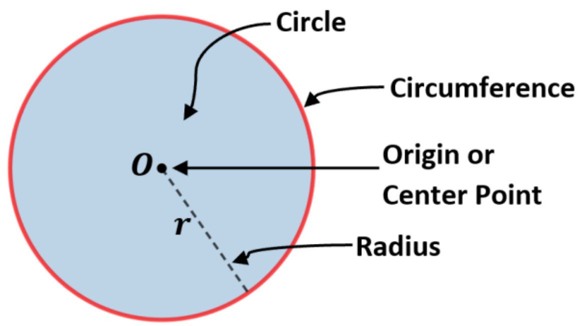

Be careful to distinguish between a "segment of a circle" and a "line segment". It is unfortunate that the same word is used to name two different geometric objects.

**Circumference and Area of a Circle**

If a circle has radius of length $r$ then

Length of Circumference $= 2\pi r$

Area of Circle $= \pi r^2$.

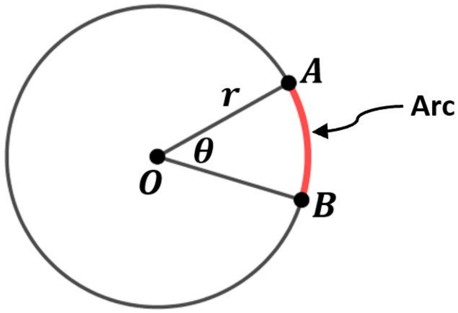


**Arc Length**

If central angle $\angle AOB$ has measure $\theta$ radians and if the circle has a radius of $r$ then

$$\text{Arc Length } \widehat{AB} = r\theta.$$

Be careful to convert $\theta$ from degrees to radians (if necessary) before using this formula.

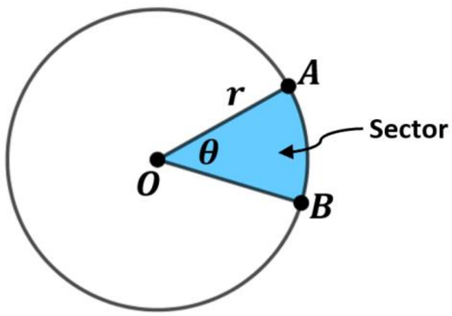



**Area of Sector**

If central angle $\angle AOB$ has measure $\theta$ radians and if the circle has a radius of $r$ then

$$\text{Area of Sector } AOB = \frac{r^2\theta}{2}.$$

Be careful to convert $\theta$ from degrees to radians (if necessary) before using this formula.

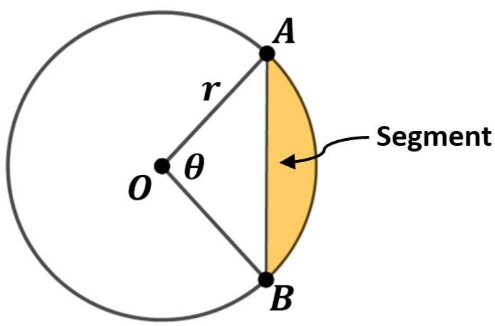



**Area of Segment**

If central angle $\angle AOB$ has measure $\theta$ radians and if the circle has a radius of $r$ then

$$\text{Area of Segment } AB = \frac{r^2}{2}\left(\theta - \sin(\theta)\right).$$

Be careful to convert $\theta$ from degrees to radians (if necessary) before using this formula.



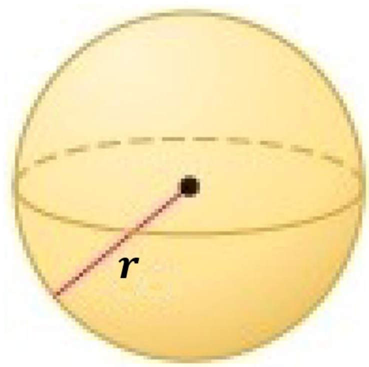
## 4. Cylinders, Cones, & Spheres (Volumes and Surface Areas)

**<u>Sphere</u>**

If a sphere has radius of length $r$ then

Surface Area of Sphere $= 4\pi r^2$

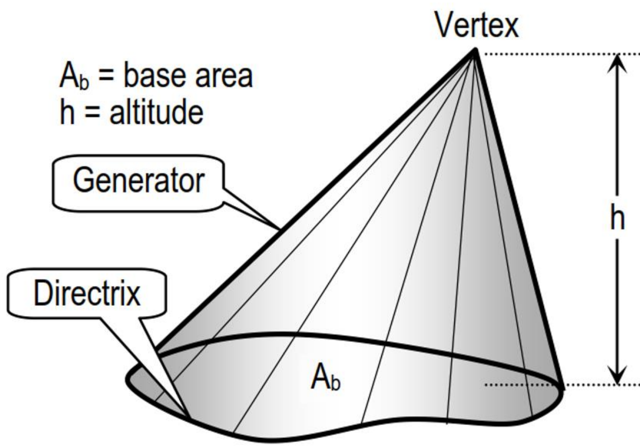
Volume of Sphere $= \frac{4}{3}\pi r^3$

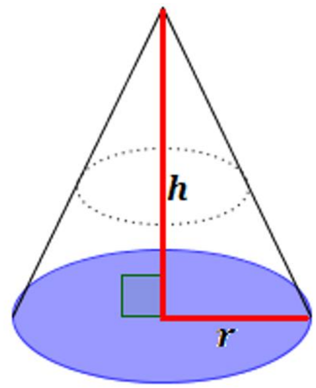



**<u>Cone</u> (General Case)**

Volume of a Cone $= \frac{1}{3}A_b h$

Remember: This formula is valid for all cones, not just circular base cones.

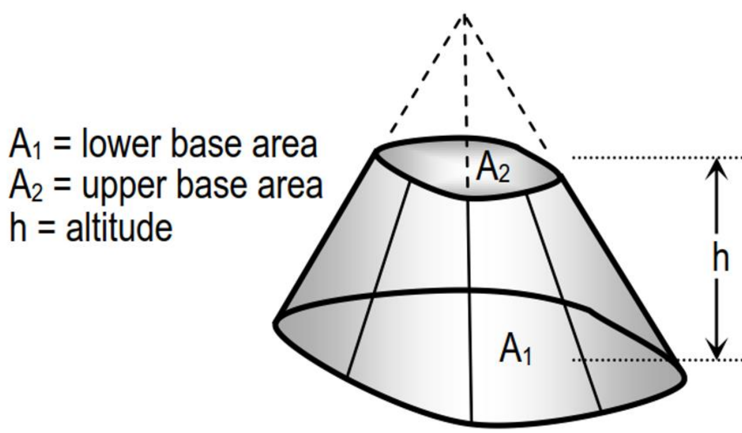


| **Right Circular Cone**<br><br>If a right circular cone of height $h$ and base radius $r$ we have<br><br>$$\text{Volume} = \frac{1}{3}\pi r^2 h$$<br><br>$$\text{Lateral Surface Area} = \pi r\sqrt{r^2 + h^2}$$<br><br>$$\text{Total Surface Area} = \pi r\left(\sqrt{r^2 + h^2} + r\right)$$<br><br>Note: Lateral means "sides only". So the lateral surface area does not include the area of the base. | <br><br>Right Circular Cone |
|---|---|

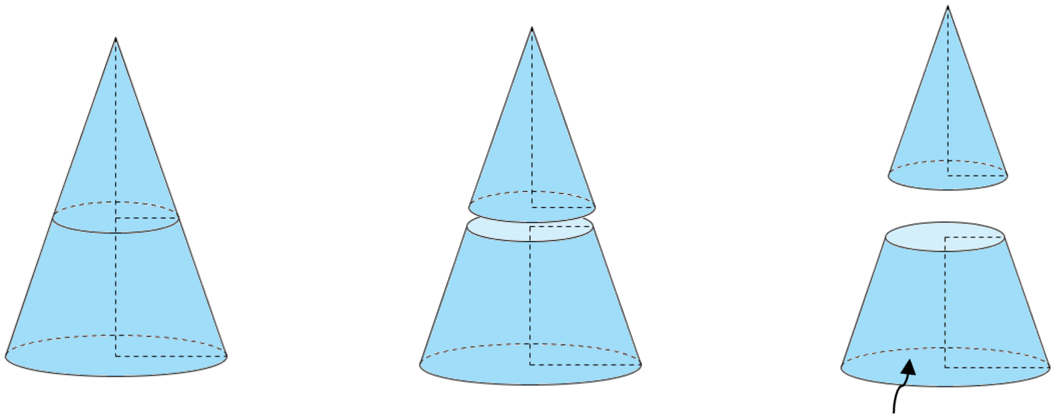
**Frustum of a Cone** (General Case)

Volume of a Frustum of a Cone

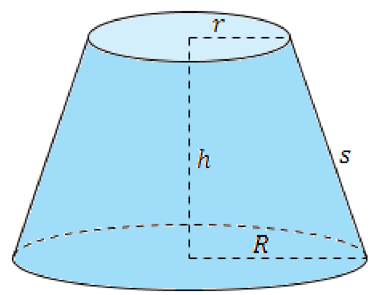
$$= \frac{h}{3}\left(A_1 + A_2 + \sqrt{A_1 A_2}\right)$$



**Frustum of a Right Circular Cone**




If a frustum of a right circular cone has a bottom radius of $R$, a top radius of $r$, a height of $h$ and a slant height of $s$, then




$$\text{Volume} = \frac{\pi h}{3}\left(R^2 + r^2 + Rr\right)$$

$$\text{Lateral Surface Area} = \pi(R + r)s$$

$$s = \text{Slant Height} = \sqrt{h^2 + (R - r)^2}$$

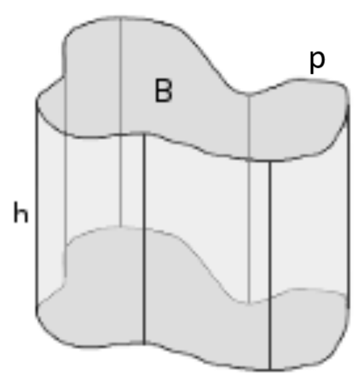
**Right Cylinders**

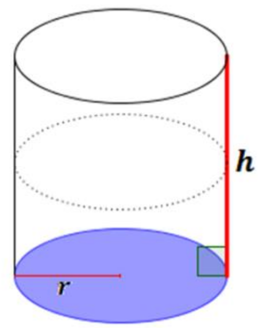
If a right cylinder has an upper and lower base with area $B$ and perimeter $p$ and a height of $h$ then

$$\text{Volume} = Bh$$

$$\text{Lateral Surface Area} = ph$$

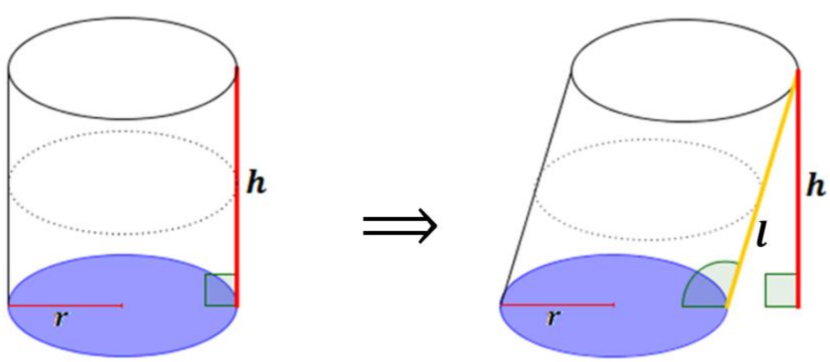
$$\text{Total Surface Area} = ph + 2B$$



<table>
<tr>
<td>Right Circular Cylinder<br><br>If a cylinder of height $h$ has a circular base with radius $r$ and if the height is perpendicular to the base then<br><br>$$\text{Volume} = \pi r^2 h$$<br>$$\text{Lateral Surface Area} = 2\pi r h$$<br>$$\text{Total Surface Area} = 2\pi r h + 2\pi r^2$$<br>Note 1: Lateral means “sides only”.  So lateral surface area does not include the area of either the upper or lower base.</td>
<td><br><br>Right Circular Cylinder</td>
</tr>
</table>

## 4.1 Oblique Circular Cylinder

The oblique circular cylinder shown below is formed by tilting a right circular cylinder of height $h$ and base radius $r$ in such a way that the base of the right cylinder does not move, the circular cross sections remain circular and the height is not altered.

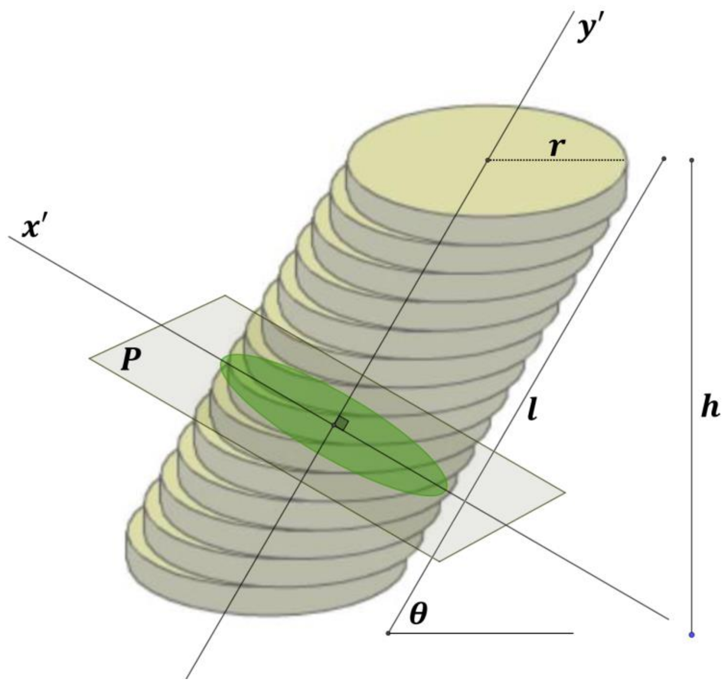



Right Circular Cylinder    Oblique Circular Cylinder

The line segment of length $l$ of the oblique cylinder is called an *element* of the cylinder.  (It can also be called the *slant height* of the cylinder.)

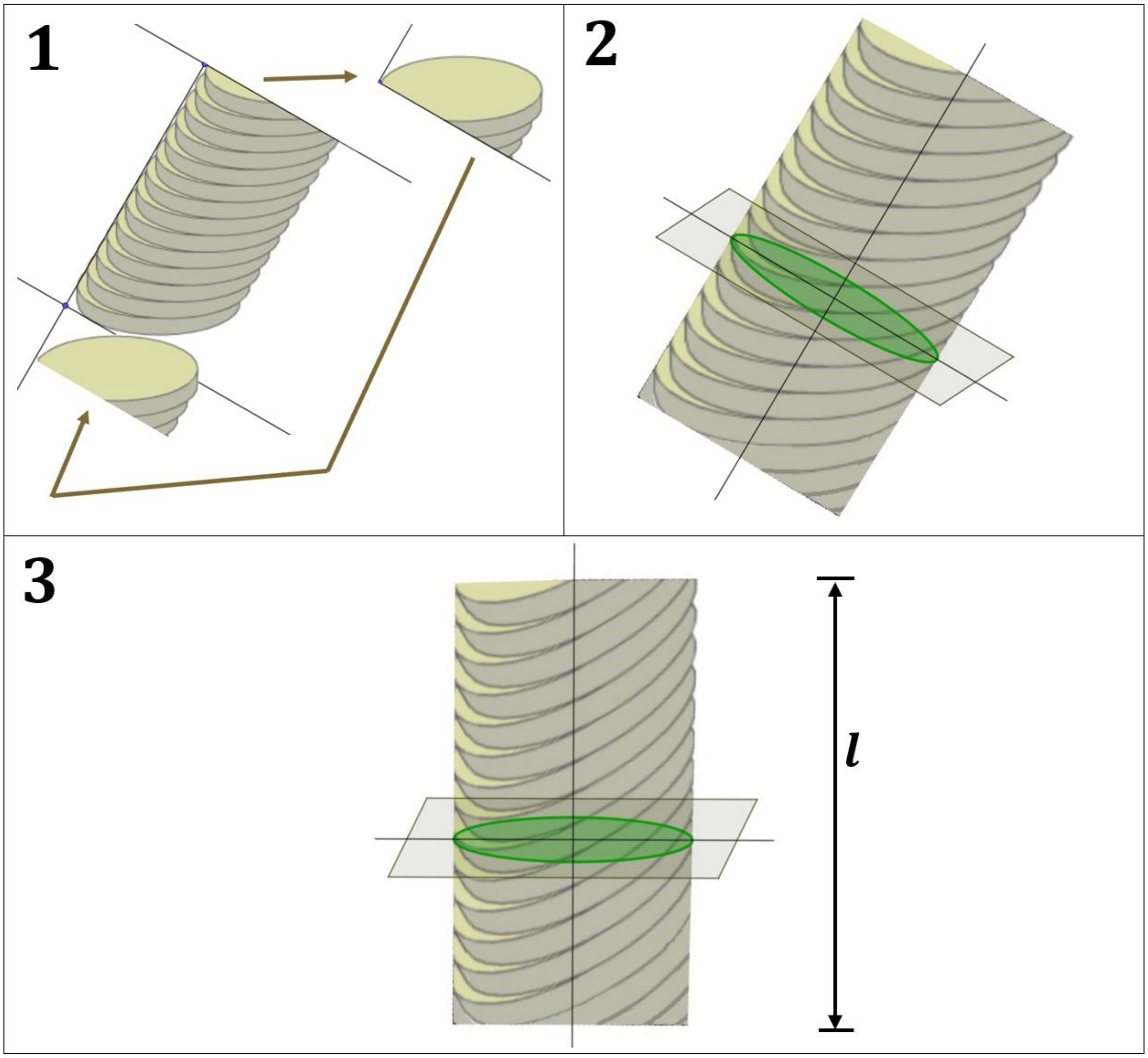
### 4.1.1 Volume of an Oblique Circular Cylinder

$(x', y')$ defines an orthogonal (at right angles) coordinate system for the oblique cylinder shown below.




Plane $P$ cuts the axis $y'$ at a right angle. The cross section, highlighted in bright green, of the cylinder on plane $P$ is an ellipse (see for example, *Projective Geometry*, C. V. Durell, pages 24-25). This ellipse is a ***right section*** because plane $P$ is orthogonal to axis $y'$.

Now imagine shearing off the top of this cylinder and sliding this top piece up under the bottom as shown in Steps 1 and 2 below and then rotating as in Step 3.




The cut and pasted cylinder shown in Step 3 is a right cylinder with elliptical cross sections. Furthermore, the cutting and pasting has not changed the volume of the cylinder. That is, the volume of the right cylinder in Step 3 is the *same as the volume of the original oblique cylinder*. But by definition, the volume of a right cylinder equals its cross-sectional area times its height. Therefore,

$$\text{Volume}(\text{oblique cylinder}) = \text{Area}(\text{ellipse}) \cdot l.$$

The equation of the ellipse in Step 3 is

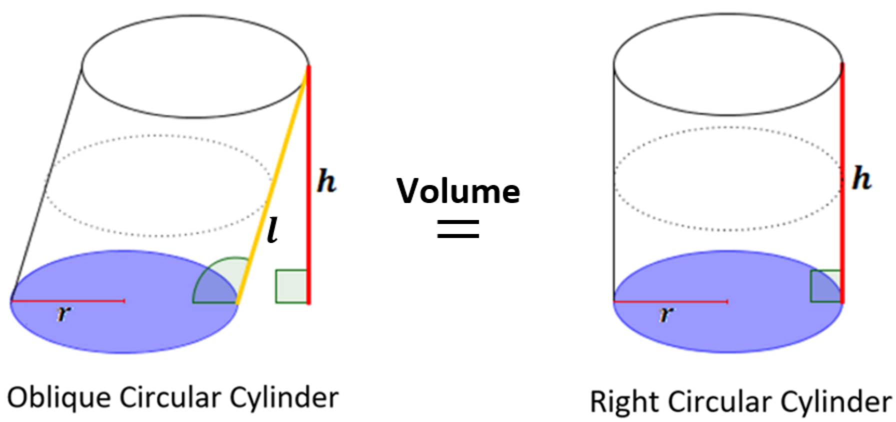
$$\frac{x^2}{r^2} + \frac{y^2}{(r\sin(\theta))^2} = 1.$$

(*Projective Geometry*, C. V. Durell, pages 24-25)

The area of this ellipse is $\pi(r)(r\sin(\theta))$. Also, we see from the initial diagram of the oblique cylinder that $\sin(\theta) = h/l$ or $l = h/\sin(\theta)$. Hence

$$\begin{aligned}\text{Volume(oblique cylinder)} &= \text{Area(ellipse)} \cdot \text{Length(element)} \\ &= (\pi r^2 \sin(\theta)) \cdot \left(\frac{h}{\sin(\theta)}\right) \\ &= \pi r^2 h\end{aligned}$$

which we recognize as the volume of a right circular cylinder with height $h$ and radius $r$.

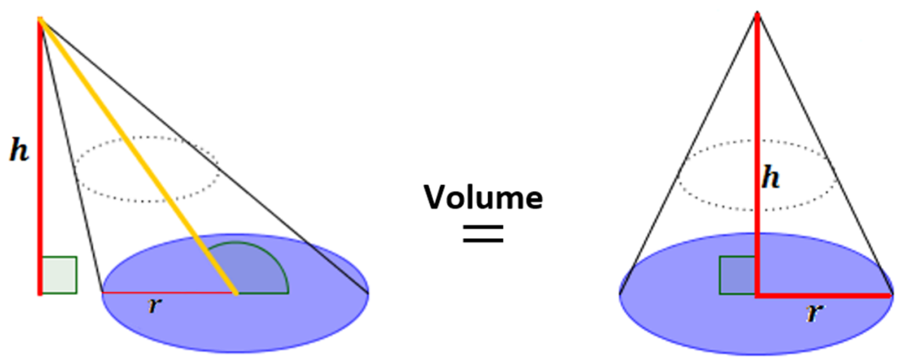



It is <u>not</u> just a coincidence that the volume of the oblique circular cylinder equals the volume of a right circular cylinder.

In fact, this equivalence (which extends beyond just cylinders) is known as *Cavalieri's principle*, which we first introduced in the Study Guide for Meet 2, Event B. We will repeat the definition here.

> **Cavalieri's Principle**
>
> Two solids with the same height and equal cross-sectional area at all values of that height will have equal volumes.

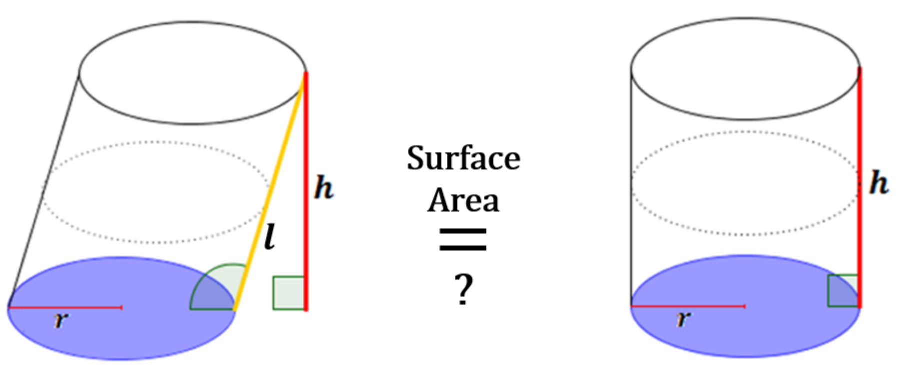
For example, it follows from Cavalieri's principle that the volume of an oblique cone can be found by comparing it to a right cone.




Oblique Circular Cone Right Circular Cone

### 4.1.2 Surface Area of an Oblique Cylinder

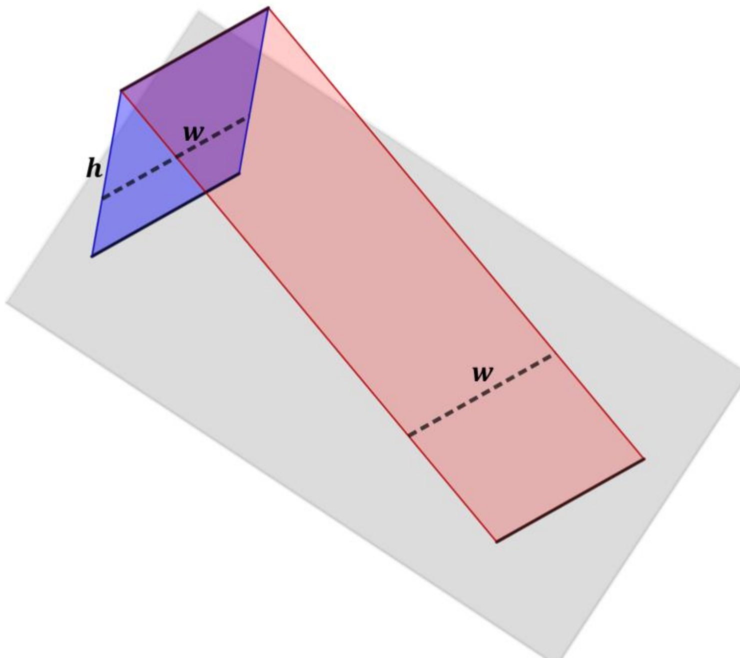
Can we use Cavalieri's Principle to find the surface area of an oblique cylinder?




Oblique Circular Cylinder Right Circular Cylinder

Unfortunately, this the answer is **no**. It is a common misconception that these two cylinders will have same surface area. It is a natural mistake to make, especially if you have only seen Cavalieri's Principle introduced as an "intuitive" result.

The following simple example of two boards stacked to form a lean-to shows how blindly applying Cavalieri's Principle can go badly wrong. Obviously, the cross-sectional width equals $w$ for both the blue and red rectangular boards at all points along the common height $h$. But it is just as obvious that these two boards do not have the same area or the same perimeter. (This example is taken from https://sites.math.washington.edu/~nwmi/materials/Cavalieri.pptx.pdf).

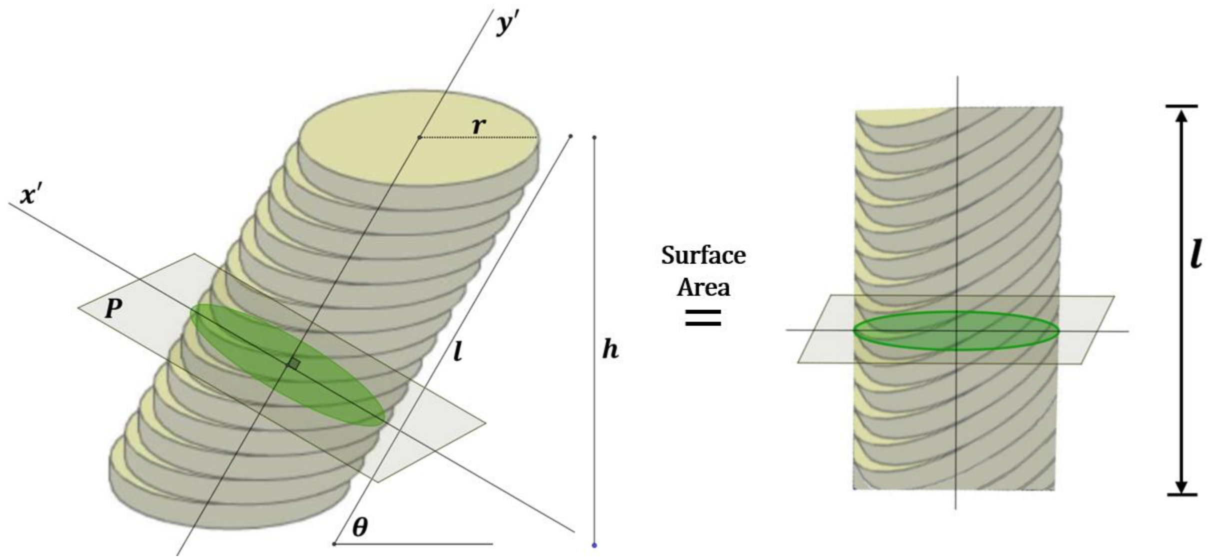



The bottom line is that Cavalieri's Principle applies to **solids** with the same height and equal cross-sectional at all values of that height.

In our oblique circular cylinder situation, we can (correctly) argue that the surface area of the original oblique cylinder equals the surface area of the right cylinder whose cross section is given by the ellipse

$$\frac{x^2}{r^2} + \frac{y^2}{(r\sin(\theta))^2} = 1.$$

(see diagrams below)




It follows that

$$\text{Surface Area(oblique cylinder)} = \text{Perimeter(ellipse)} \cdot \text{Length(element)}.$$

Unfortunately, it is well known that the perimeter of an ellipse does not have a *closed form* (this is the term used in mathematics to mean "no simple expression exists").

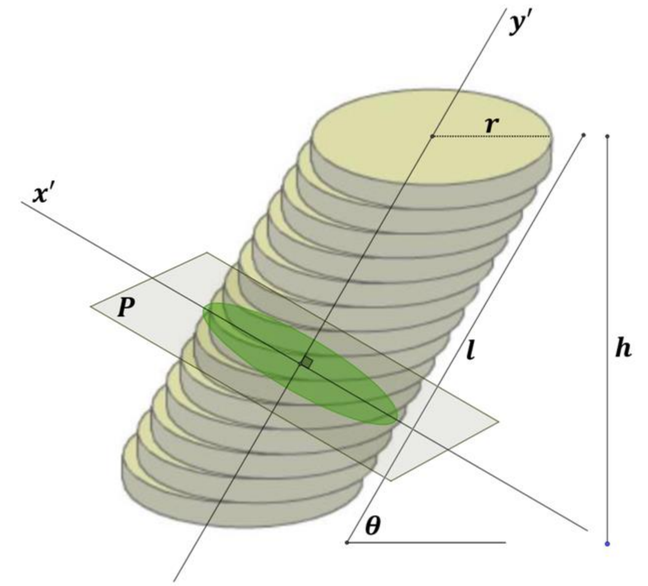
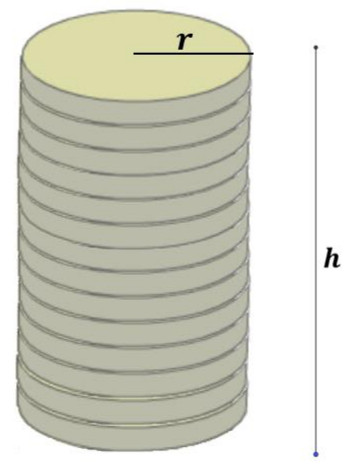
When you take calculus 2 you will learn a general formula for arc length which will allow you to write

$$\text{Perimeter(ellipse)} = r\int_0^{2\pi} \sqrt{\cos^2(x) + \sin^2(\theta)\sin^2(x)}\, dx$$

for the ellipse

$$\frac{x^2}{r^2} + \frac{y^2}{(r\sin(\theta))^2} = 1.$$

And you can use a computer package (such as Wolfram Alpha) to approximate this integral with as much precision as you like.

| | |
|---|---|
|  |  |
| Surface Area $= \text{Perimeter}(\text{Ellipse}) \cdot l$ $= \text{Perimeter}(\text{Ellipse}) \cdot h/\sin(\theta)$ | Surface Area $= \text{Perimeter}(\text{Circle}) \cdot h$ $= 2\pi r \cdot h$ |

How substantial is the difference between these two formulas?

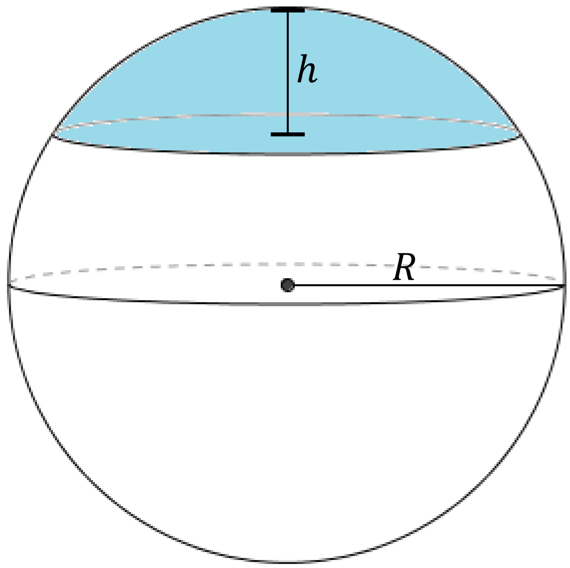
We used the online (free) program *Wolfram Alpha* to evaluate the integral for the perimeter of an ellipse to construct the following table.

| | **Surface Area Oblique Cylinder with $r = 1, h = 3$** | **Surface Area Right Cylinder with $r = 1, h = 3$** |
|---|---|---|
| $\theta = 90°$ | $6\pi$ | $6\pi \approx 18.8496$ |
| $\theta = 60°$ | 20.3338 | 18.8496 |
| $\theta = 45°$ | 22.9212 | 18.8496 |
| $\theta = 30°$ | 29.0653 | 18.8496 |
| $\theta = 15°$ | 49.9069 | 18.8496 |

Obviously, the error grows quickly as $\theta$ moves away from 90° (*i.e.* the amount of "tilt" increases).

## 5. Spherical Caps and Sectors

**Surface Area of Spherical Cap**




$h =$ height of spherical cap

$R =$ radius of sphere

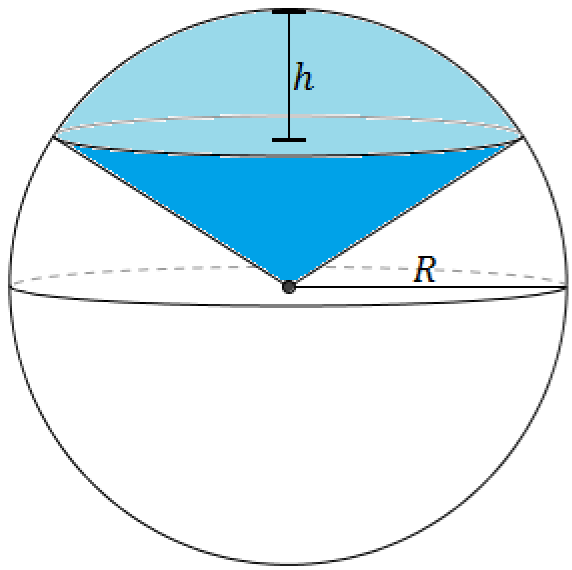
Surface Area of Hemisphere $= 2\pi R^2$

$$\frac{h}{R} = \frac{\text{Surface Area Cap}}{\text{Surface Area Hemisphere}}$$

Therefore,

$$\text{Surface Area of Spherical Cap with height } h = \left(\frac{h}{R}\right)(2\pi R^2) = 2\pi Rh$$

**Volume of Spherical Sector**




$h$ = height of spherical sector

$R$ = radius of sphere

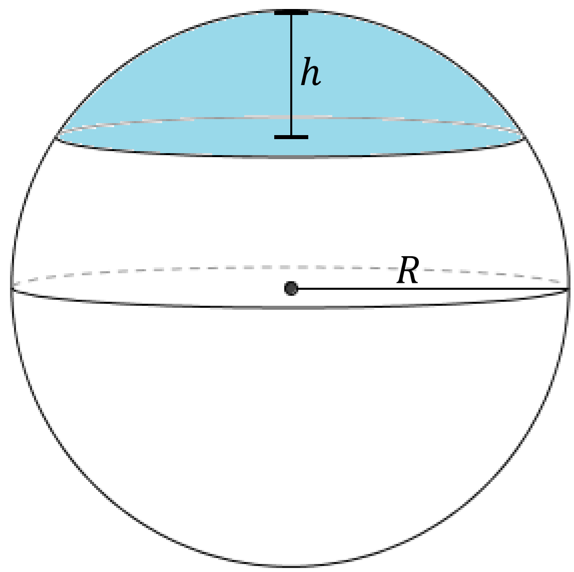
$$\text{Volume of Hemisphere} = \frac{2}{3}\pi R^3$$

$$\frac{h}{R} = \frac{\text{Volume Spherical Sector}}{\text{Volume Hemisphere}}$$

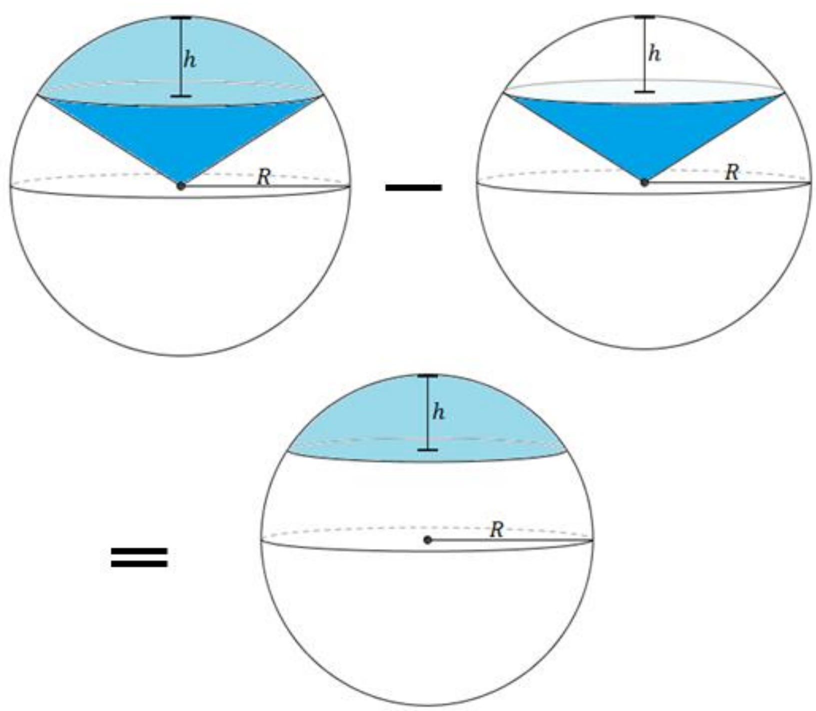
Therefore,

$$\text{Volume of Spherical Sector with height } h = \left(\frac{h}{R}\right)\left(\frac{2}{3}\pi R^3\right) = \left(\frac{2}{3}\right)\pi R^2 h$$

## Volume of Spherical Cap




The key step is to recognize that




We have already found the volume of the spherical sector. Now we need to find the volume of the cone. This cone has height $R - h$ and slant height $R$. If we let $r$ equal the radius of this cone, then it follows from the Pythagorean Theorem that

$$r^2 + (R - h)^2 = R^2.$$

This means that

$$r^2 = R^2 - (R-h)^2 = 2Rh - h^2.$$

The formula for the volume of a cone with height $R - h$ and radius $r$ is

$$\frac{1}{3}\pi \cdot r^2 \cdot (R-h) = \frac{1}{3}\pi \cdot (2Rh - h^2) \cdot (R-h)$$

Therefore, by subtraction, the volume of the spherical cap is

$$\frac{2}{3}\pi R^2 h - \frac{1}{3}\pi \cdot (2Rh - h^2) \cdot (R-h)$$

$$= \frac{1}{3}\pi h^2(3R - h)$$

after simplification.

Sometimes it is convenient to express this formula in terms of $r$ (the radius of the base circle of the spherical cap) and $h$.

Recall that we just showed that

$$r^2 = R^2 - (R-h)^2 = 2Rh - h^2.$$

This means that

$$Rh = \left(\frac{1}{2}\right)(r^2 + h^2).$$

So,

$$\frac{1}{3}\pi h^2(3R - h) = \frac{1}{3}\pi h(3Rh - h^2)$$

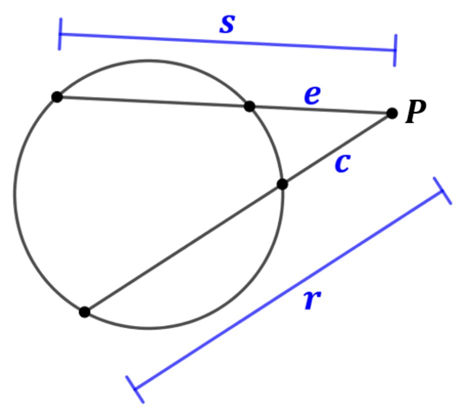
$$= \frac{1}{6}\pi h(3(r^2 + h^2) - 2h^2)$$

$$= \frac{1}{6}\pi h(3r^2 + h^2).$$

# 6. Power of a Point Theorems

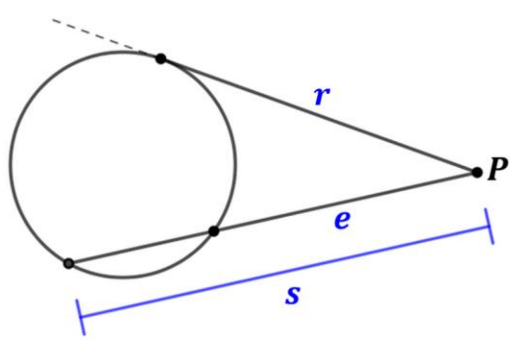
## 6.1 Secant-Secant Theorem

If two secant lines are drawn through the point $P$ taken outside the circle, then $\boxed{s \cdot e = r \cdot c}$ as illustrated in the figure below.



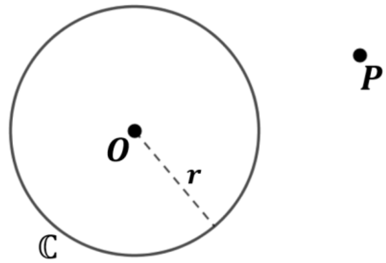

## 6.2 Secant-Tangent Theorem

If one secant line and one tangent line are drawn through the point $P$ taken outside the circle, then $\boxed{s \cdot e = r^2}$ as illustrated in the figure below.

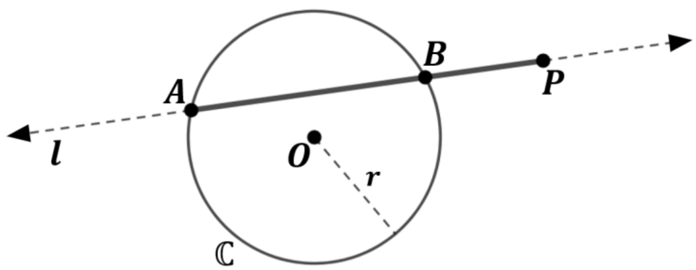



**How do these two theorems help us?**

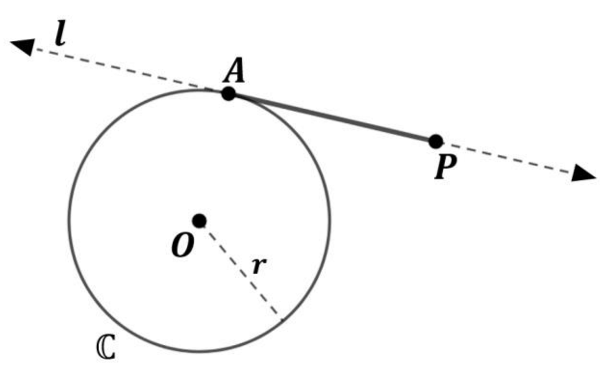
What the Secant-Secant Theorem tells us is that for the circle $\mathbb{C}$ with center $O$ and radius $r$ and for any fixed point $P$ taken outside circle $\mathbb{C}$,



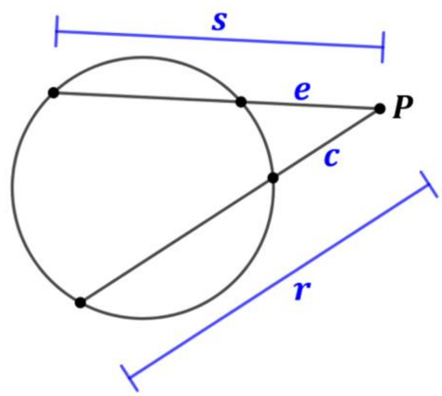

the product $PA \cdot PB$ is the same (a constant) for *<u>every secant line</u>* $l$ of circle $\mathbb{C}$ going through the fixed point $P$. The constant value of $PA \cdot PB$ is called the **power of the point** $P$ taken *outside* the circle $\mathbb{C}$.




What the Secant-Tangent Theorem tells us is that if we consider *any* tangent line $l$ to this same circle $\mathbb{C}$ that goes through the same fixed point $P$ (obviously there are exactly two such tangent lines $l$) then $PA \cdot PA = (PA)^2$ will again equal the same constant "power of the point" that we found for point $P$ in the Secant-Secant Theorem.



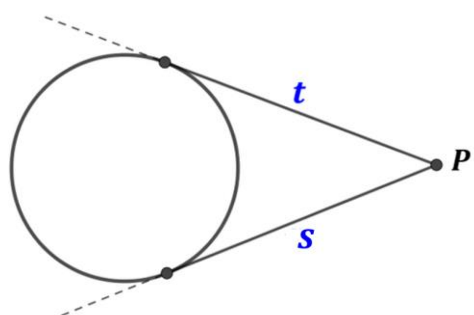
Perhaps the reason the Secant-Secant and the Secant-Tangent Theorem are (almost) always stated in the form we have given them is that this form makes it clear how to use these theorems to solve a typical contest problem such as "find $c$ in the figure below"




for some given values of $s, e$ and $r$.

### 6.2.1 Tangent-Tangent Theorem

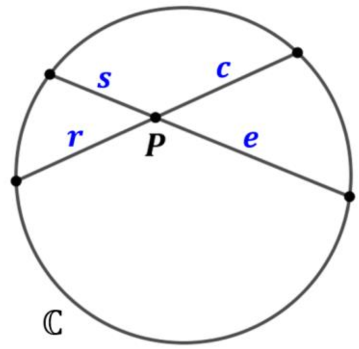
The Tangent-Tangent Theorem which tells us that $s = t$ in the figure below where we have taken the two tangent lines to the circle $\mathbb{C}$ going through the fixed point $P$ taken outside the circle $\mathbb{C}$, follows as a corollary to the Tangent-Secant Theorem.



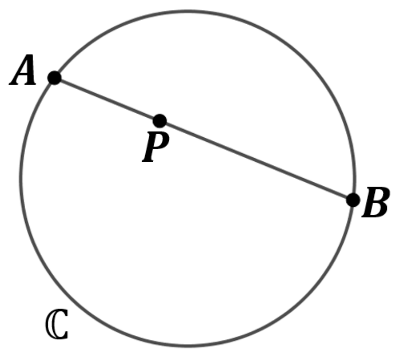

## 6.3 Intersecting Chords (or the Chord-Chord) Theorem

For *any* two chords of the circle $\mathbb{C}$ going through a fixed point $P$ taken *inside* the circle $\mathbb{C}$,

$$s \cdot e = r \cdot c.$$




Another way of stating this result is that $PA \cdot PB$ is constant for every chord $\overline{AB}$ of the circle $\mathbb{C}$ going through the point $P$ taken *inside* the circle $\mathbb{C}$.

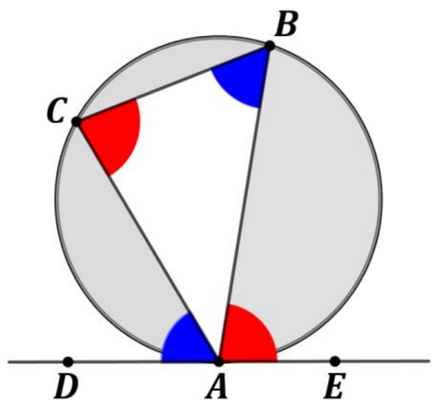



The constant product $PA \cdot PB$ is called the power of the point $P$ taken *inside* circle $\mathbb{C}$.

## 7. Alternate Segments Theorem (*aka* Tangent-Chord Theorem)

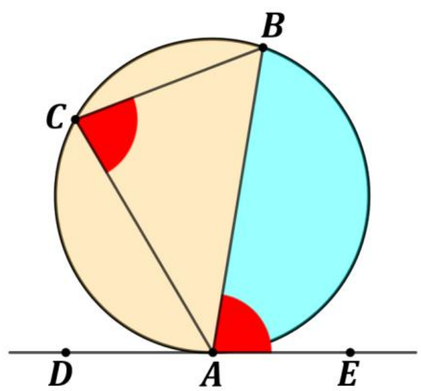
The **alternate segment theorem** (less commonly known as the **tangent-chord theorem**) states that “in any circle, the angle between a chord and a tangent through one of the end points of the chord is equal to the angle subtended by the chord in the alternate segment”.

What does this mean?  It translates to saying that the angles of the same color in the diagram below are equal to each other.



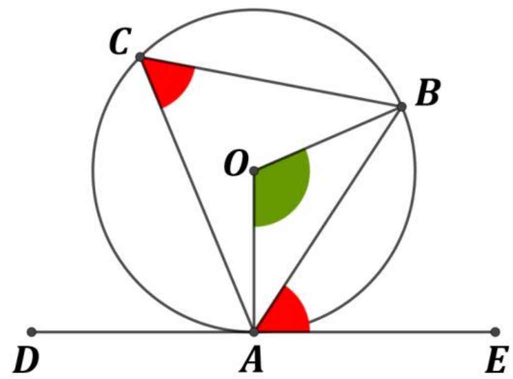

That is, $m\angle CBA = m\angle DAC$ and $m\angle ACB = m\angle BAE$.

Why the name "alternate segment" theorem? Chord $\overline{AB}$ in the diagram below divides the circle into two segments - the tan colored segment and the cyan colored segment.




$\angle ACB$ is the angle subtended by chord $\overline{AB}$ and $\angle ACB$ is in the (tan colored) segment, the segment which is on the alternate side (*i.e.* other side) of chord $\overline{AB}$ than tangent angle $\angle BAE$ is on.

**Corollary**



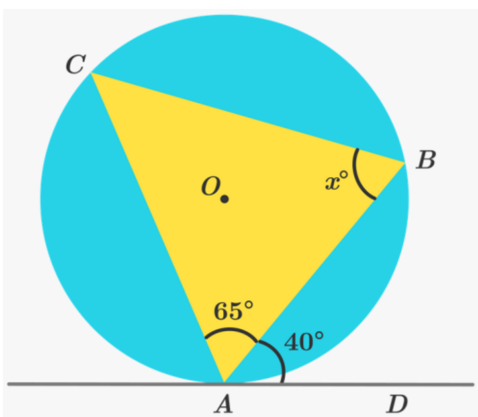

The measure of the central angle $AOB$ has twice the measure of tangent angle $BAE$.

Why? We know that $m\angle BAE = m\angle ACB$ by the Alternate Segment Theorem.

But the inscribed angle $ACB$ has twice the measure of the central angle $AOB$ because both angles are subtended by arc $BA$. Therefore,

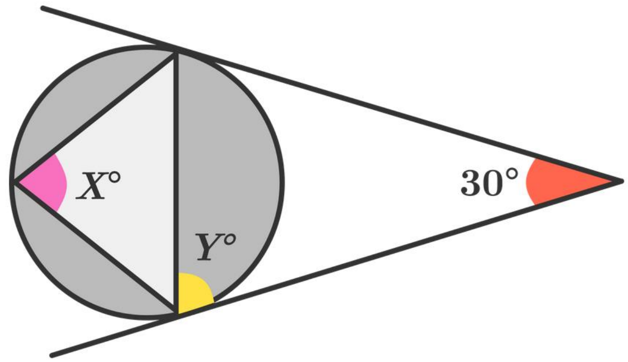
$$m\angle AOB = 2 \cdot m\angle BAE.$$

Example 2.




Find the value of $x$ in the above diagram.

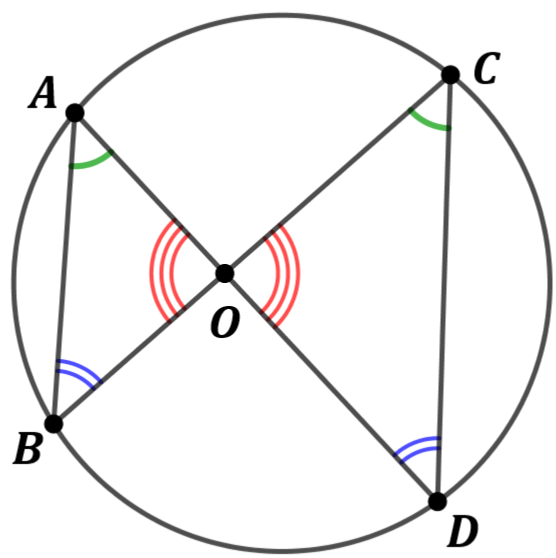
Example 3.



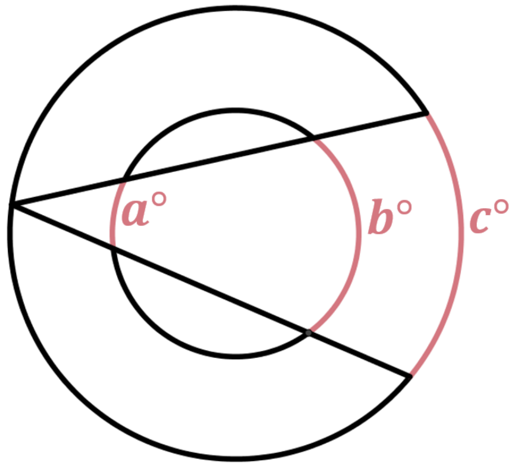

Find $X + Y$ in the above diagram.

## 8. An Assortment of Circle Theorems

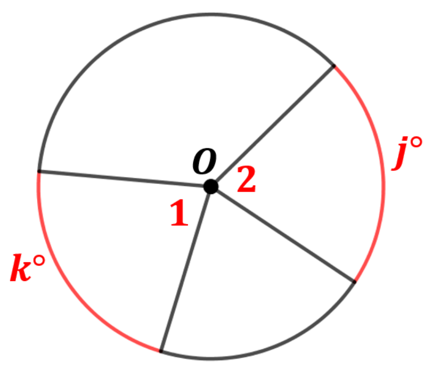
**Bow Tie Theorem**



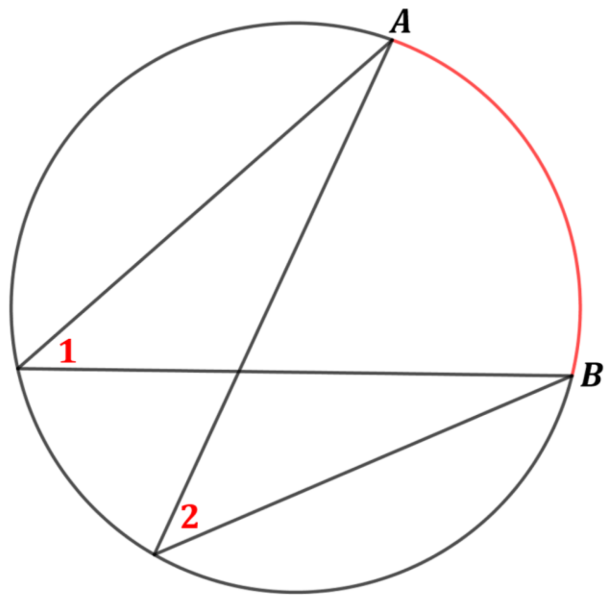

| | |
|---|---|
| $m\angle BAD = m\angle DCB$ | (Both angles intercept $\widehat{BD}$) |
| $m\angle BOA = m\angle DOC$ | (Vertical angles) |
| $m\angle ABC = m\angle CDA$ | (Both angles intercept $\widehat{AC}$) |
| $\therefore \quad \Delta AOB \sim \Delta COD$ | (AAA criteria) |




$c = b - a$

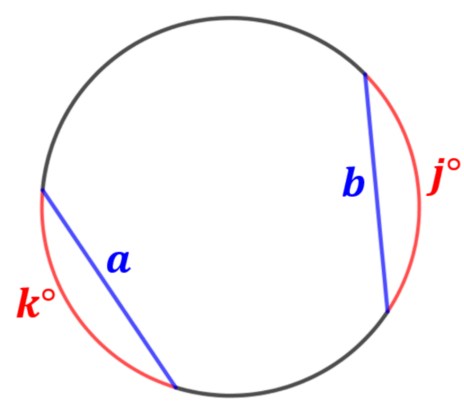



In the same circle or in congruent circles, congruent minor arcs have congruent central angles and congruent central angles have congruent arcs. In reference to the above figure, if $k = j$, then $m\angle 1 = m\angle 2$ and if $m\angle 1 = m\angle 2$, then $k = j$.

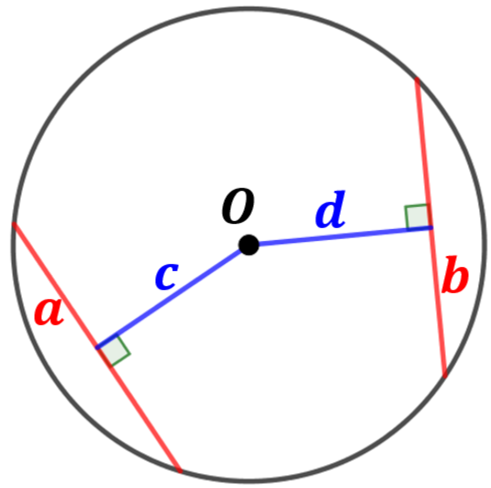



If two inscribed angles intercept the same arc, then the angles are congruent. In reference to the above figure, inscribed angles $\angle 1$ and $\angle 2$ both intercept the same arc $\widehat{AB}$, therefore, $m\angle 1 = m\angle 2$.

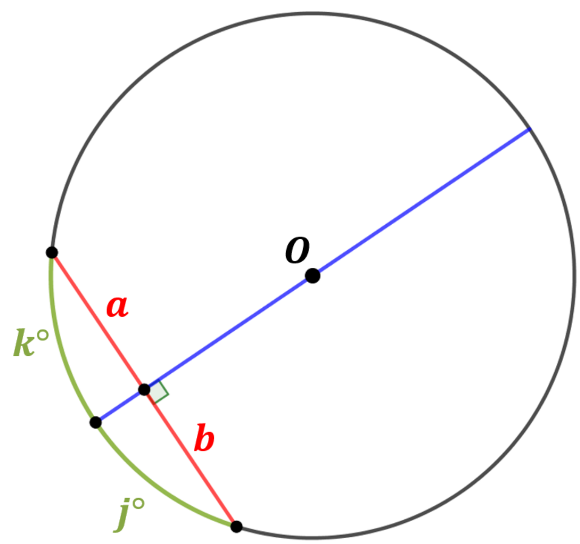
Word Usage Note: We can say $\angle 1$ ***intercepts*** arc $\widehat{AB}$ and we can say arc $\widehat{AB}$ ***subtends*** $\angle 1$.



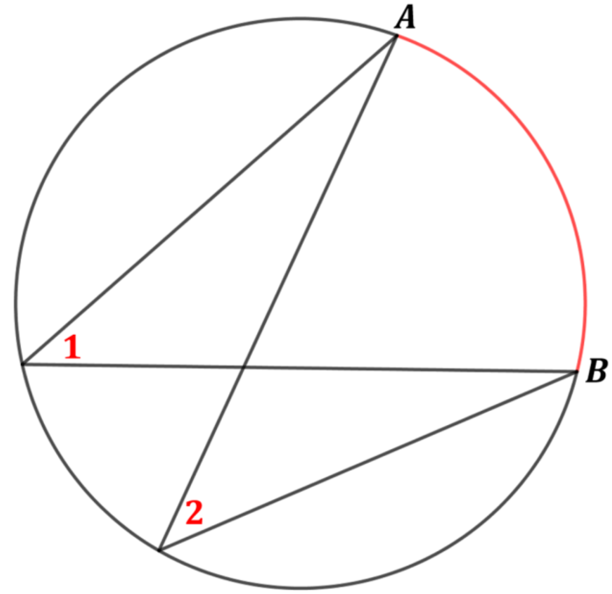

In the same circle or in congruent circles, congruent arcs have congruent chords and congruent chords have congruent arcs. In reference to the above figure, if $k = j$, then $a = b$ and if $a = b$, then $k = j$.




In the same circle or in congruent circles, chords equally distant from the center are congruent and congruent chords are equally distant from the center. In reference to the above figure, if $c = d$, then a$= b$ and if $a = b$, then $c = d$.

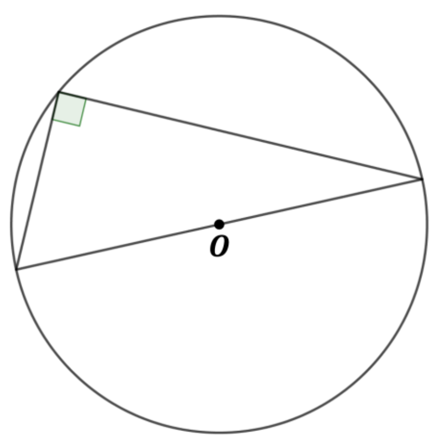



A diameter that is perpendicular to a chord bisects the chord and its arc. In reference to the above figure, $a = b$ and $k = j$.

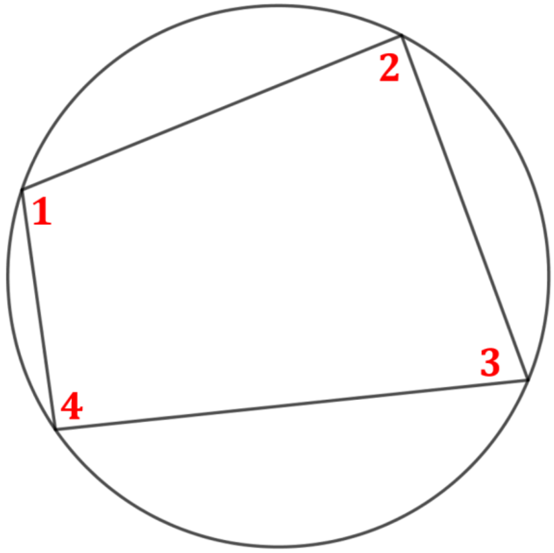



If two inscribed angles intercept the same arc, then the angles are congruent. In reference to the above figure, inscribed angles $\angle 1$ and $\angle 2$ both intercept the same arc $\widehat{AB}$, therefore, $m\angle 1 = m\angle 2$.

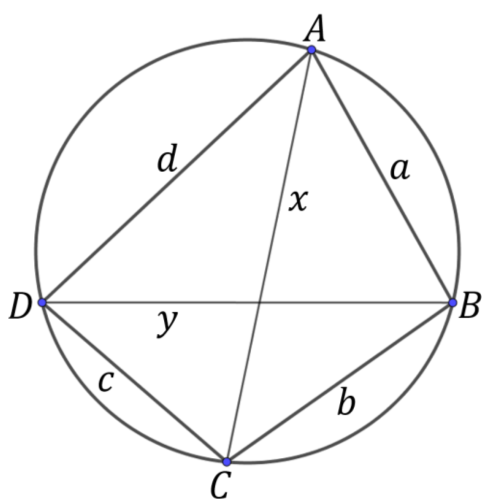
Word Usage Note: We can say $\angle 1$ ***intercepts*** arc $\widehat{AB}$ and we can say arc $\widehat{AB}$ ***subtends*** $\angle 1$.




An angle inscribed in a semicircle is a right angle.



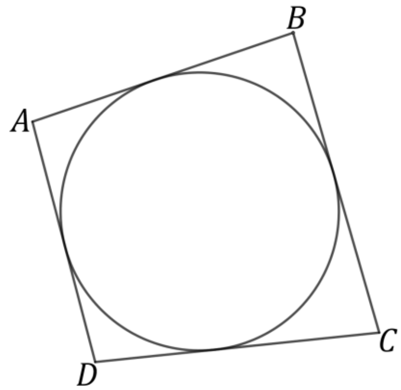

If a quadrilateral is inscribed in a circle (*i.e.* a *cyclic* quadrilateral), then its opposite angles are supplementary. In reference to the above figure, $m\angle 1 + m\angle 3 = 180°$ and $m\angle 2 + m\angle 4 = 180°$.



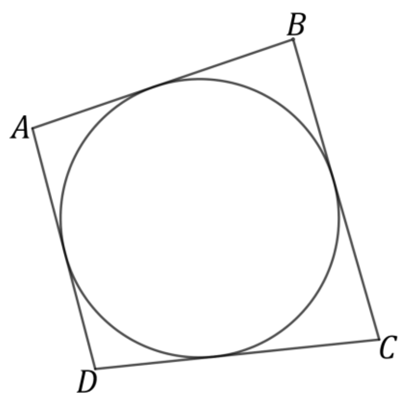

**Ptolemy's Theorem**

Let $x$ and $y$ be the lengths of the diagonals in a cyclic quadrilateral with sides $a, b, c$ and $d$. Then

$$xy = ac + bd.$$



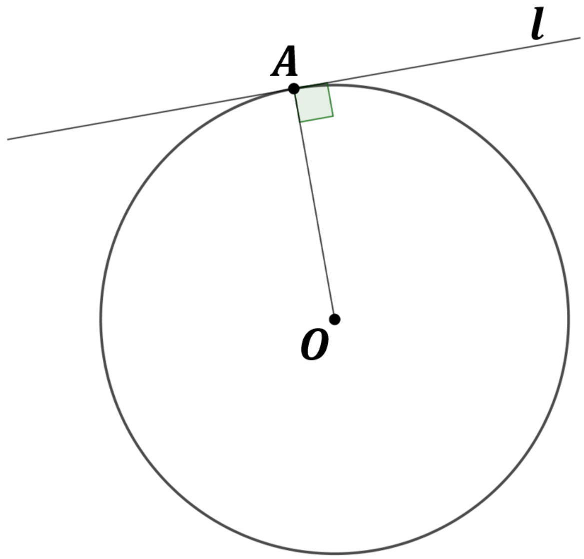

If a circle can be inscribed in quadrilateral $ABCD$, then $\overline{AB}+\overline{CD}=\overline{BC}+\overline{DA}$.
(Part 1 of Pitot's Theorem)



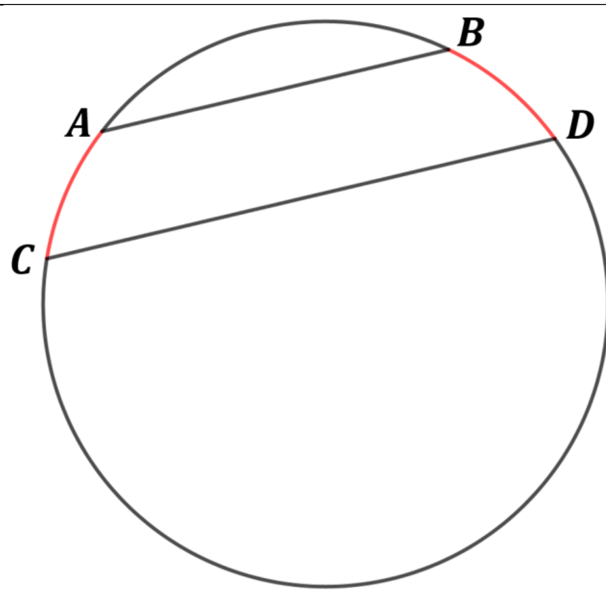

If a circle can be inscribed in quadrilateral $ABCD$, then

$$\overline{AB}+\overline{CD}=\overline{BC}+\overline{DA}=\frac{\overline{AB}+\overline{BC}+\overline{CD}+\overline{DA}}{2}.$$

(Part 2 of Pitot's Theorem)

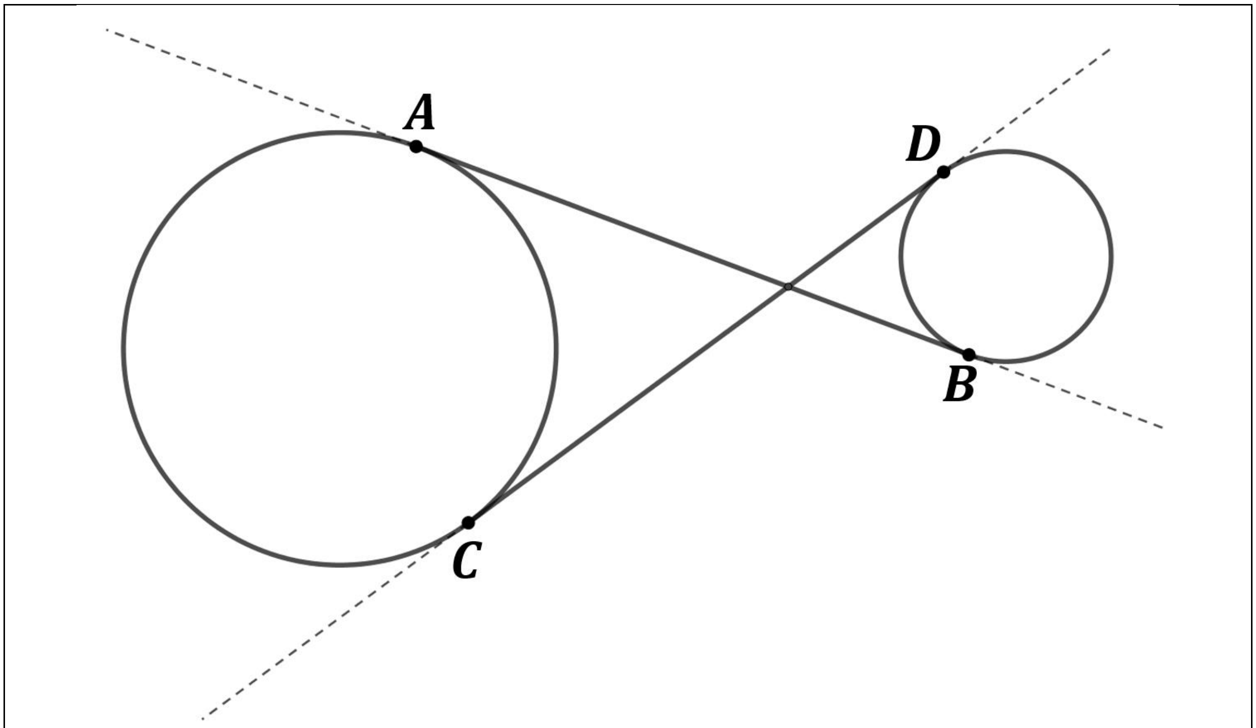



If a line is tangent to a circle, then the line is perpendicular to the radius drawn to the point of tangency and if a line going through point on a circle is perpendicular to the radius drawn to this same point, then this line is tangent to the circle at this point. In reference to the above figure, if $l$ is tangent to the circle at $A$, then $m\angle A = 90°$ and if $m\angle A = 90°$ then $l$ is tangent to the circle.

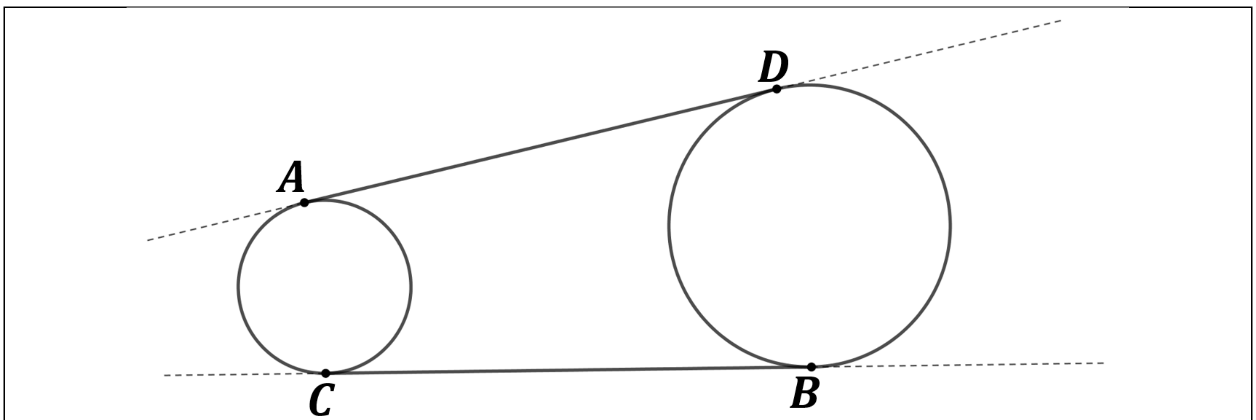



If chords $\overline{AB}$ and $\overline{CD}$ are parallel (which is denoted by $\overline{AB} \parallel \overline{CD}$), then $\widehat{AC} = \widehat{BD}$.

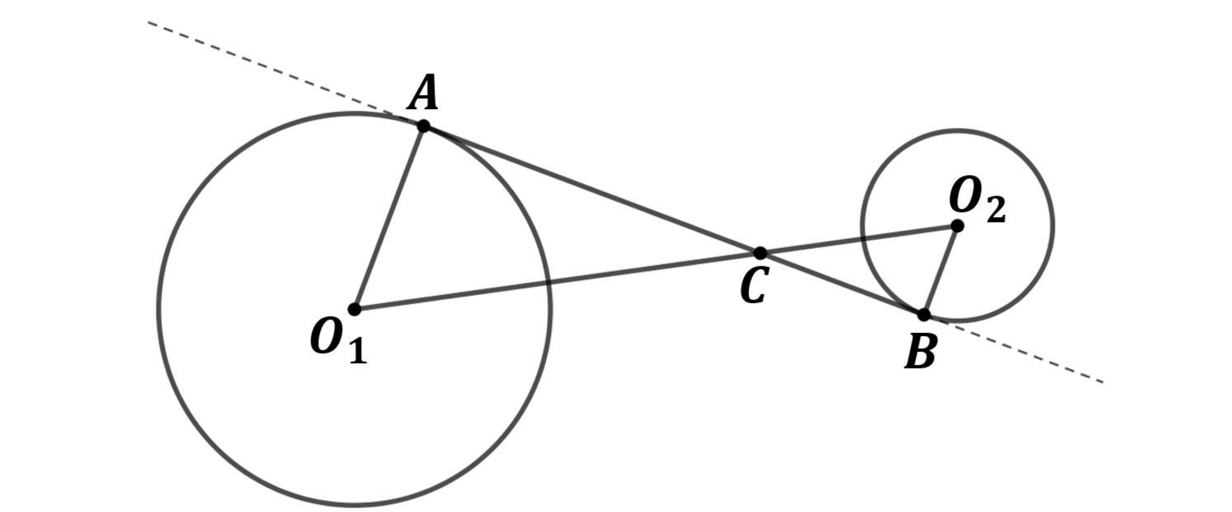
## 9. Interior and Exterior Tangents of Two Circles




Common internal tangents are equal in length. In reference to the above diagram, $\overline{AB} = \overline{CD}$.




Common external tangents are equal in length. In reference to the above diagram, $\overline{AD} = \overline{CB}$.




$\Delta AO_1C$ and $\Delta BO_2C$ are similar triangles.

Find the ***length of the common <u>internal</u> tangent*** of two circles with radii $r_1$ and $r_2$ whose centers are distance $d$ apart.

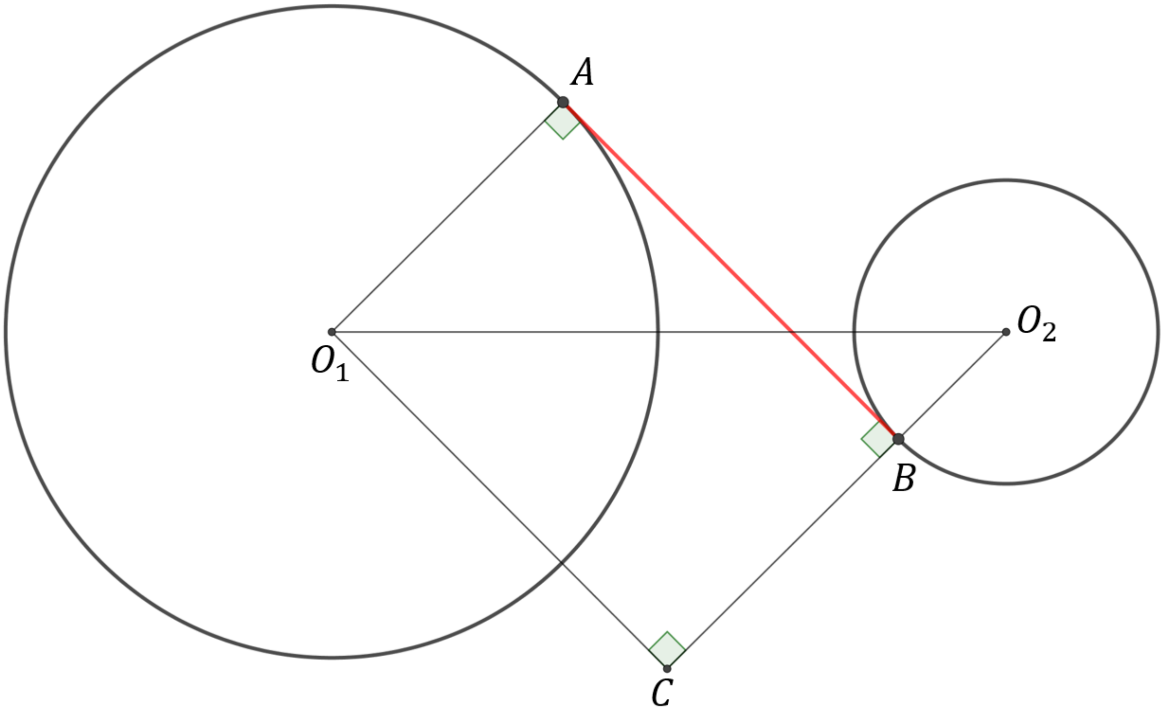
<u>Answer</u>

$$\sqrt{d^2 - (r_1 + r_2)^2}$$

<u>Solution</u>

Let $AB$ be the common internal tangent of two circles with radii $O_1A = r_1$ and $O_2B = r_2$ and where $O_1O_2$, the distance between the centers, equals $d$ as shown in the figure below.

Because radii $O_1A$ and $O_2B$ are parallel because they are both perpendicular to $AB$. Construct $BC$ by extending radius $O_2B$ by the length of radius $O_1A$ to form rectangle $AO_1CB$. It follows that $AB = O_1C$.




By the Pythagorean Theorem,

$$AB = O_1C = \sqrt{O_1O_2 - O_2C} = \sqrt{d^2 - (r_1 + r_2)^2}.$$

■

Find the ***length of the common <u>external</u> tangent*** of two circles with radii $r_1$ and $r_2$ whose centers are distance $d$ apart.

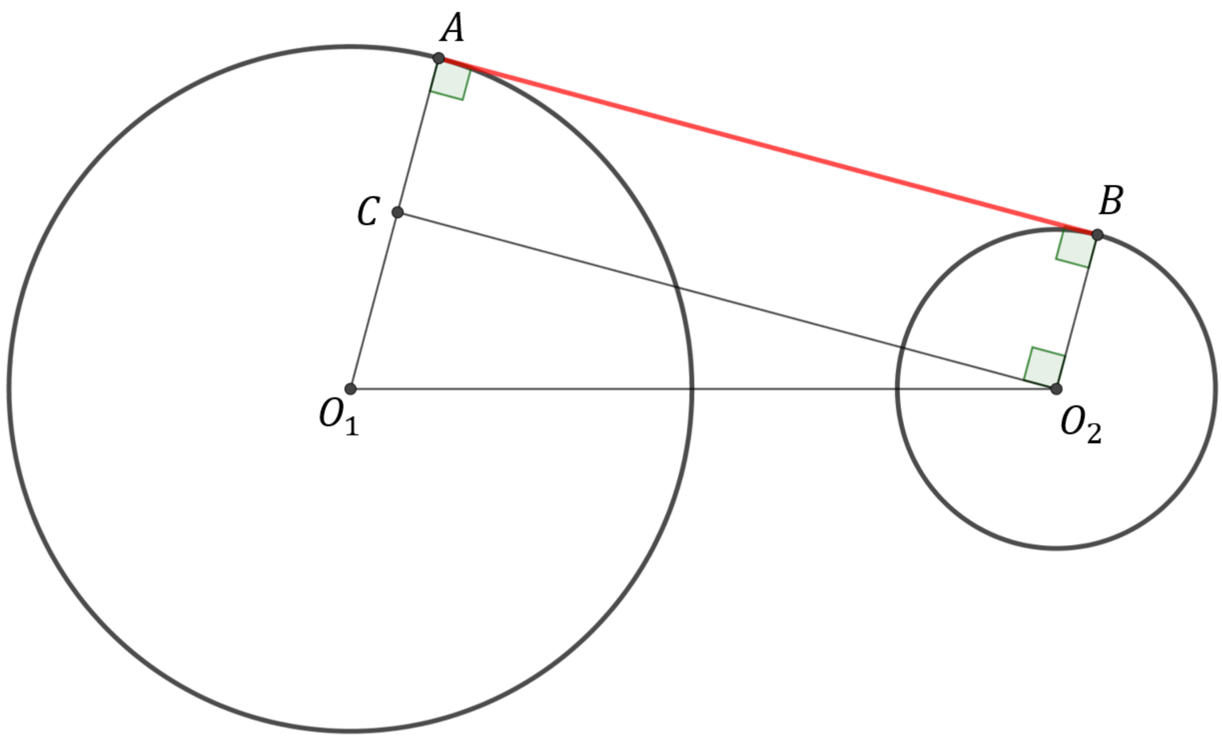
Answer

$$\sqrt{d^2 - (r_1 - r_2)^2}$$

Solution

Let $AB$ be the common external tangent of two circles with radii $O_1A = r_1$ and $O_2B = r_2$ and where $O_1O_2$, the distance between the centers, equals $d$ as shown in the figure below.

Because radii $O_1A$ and $O_2B$ are parallel because they are both perpendicular to $AB$. Construct point $C$ on $O_1A$ so that $O_2A$ is parallel to $AB$ to form rectangle $ACO_2B$. It follows that $AB = O_2C$.




By the Pythagorean Theorem,

$$AB = O_2C = \sqrt{(O_1O_2)^2 - (O_1C)^2} = \sqrt{d^2 - (r_1 - r_2)^2}.$$

■

## 10. Regiomontanus' Problem

The Regiomontanus problem is to find where a person should stand in order to maximize their visual angle when looking up at a painting hung on a wall, a statue on a pedestal or a steeple on a church.

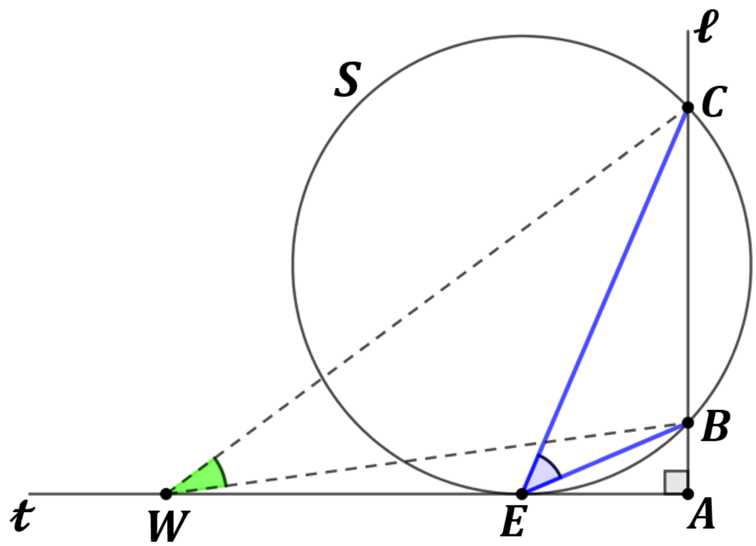
The problem was posed by Johannes Müller (also known as Regiomontanus) in 1471. Today the problem is a standard exercise for first year calculus students. However in this section we show how to find the answer *without* using calculus (which, by the way, had not been invented in Regiomontanus' time).

The geometric approach taken here dates back to a note by Ad. Lorsch, "Ueber eine Maximumaufgabe," *Zeitschrift für Mathematik und Physik* (1878).

We will break the problem into two parts. In the first part we identify conditions for a maximal angle. In the second part we identify distances and heights when the visual angle is at a maximum. These two parts are followed with some applications.

**(Part 1) Maximum Angle**

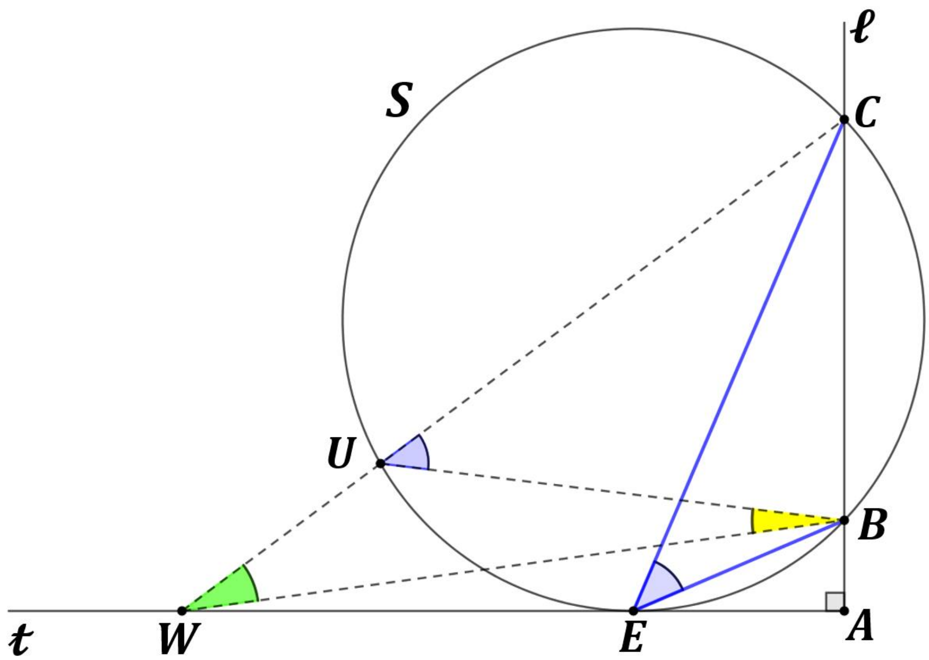
Let $S$ be that circle that goes through the points $C, B$ and $E$ and is tangent to line $t$ at the point $E$ (as illustrated in the diagram below). Assume that lines $l$ and $t$ are perpendicular. Let $W$ be any point on the line $t$ other than $E$.




Then $m\angle CWB < m\angle CEB$.

**Proof**

Let $U$ be the point of intersection of $\overline{WC}$ and circle $S$ and draw the auxiliary line $\overline{UB}$.




We note that $m\angle CEB = m\angle CUB$ because both are inscribed angles subtended by $\overline{CB}$.

Then by the Exterior Angle Theorem for a triangle, $m\angle CUB = m\angle UWB + m\angle WBU$.

Therefore,

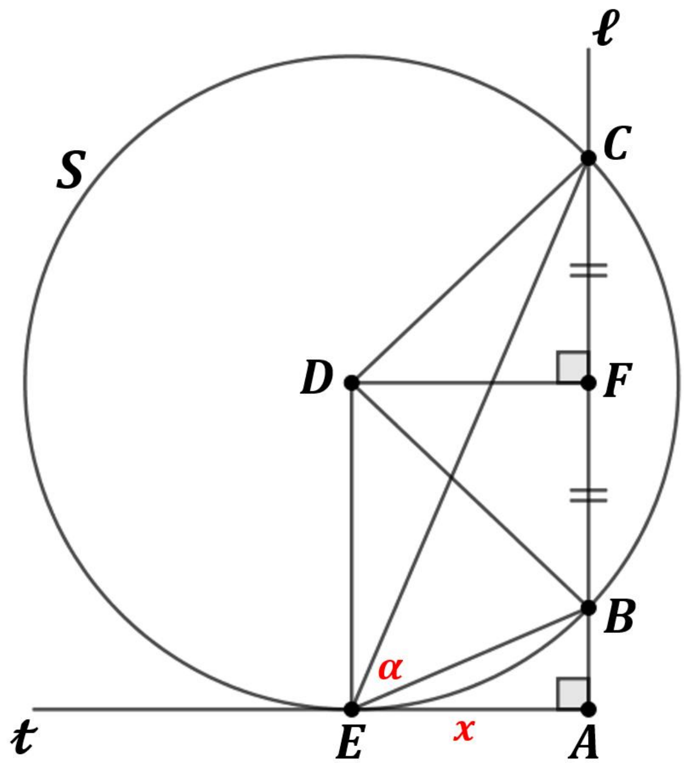
$$m\angle CEB = m\angle CUB = m\angle UWB + m\angle WBU > m\angle UWB = m\angle CWB$$

as long as $W$ and $E$ are different points on $t$.

■

**(Part 2) Measurements on a Circle Containing Two Points and Tangent to a Given Line**

Let $S$ be that circle with center point $D$ that goes through the points $C, B$ and $E$ and is tangent to line $t$ at the point $E$ (as illustrated in the diagram below). Assume that lines $l$ and $t$ are perpendicular and $\overline{DE}$ is the perpendicular bisector of $\overline{CB}$. Also assume $m\angle CEB = \alpha$ and $EA = x$.




(a) Find $m\angle CDF$ as a function of $\alpha$.

(b) Find $m\angle BEA$ as a function of $\alpha$.

(c) Find $AB$ as a function of $\alpha$ and $x$.

(d) Find $BC$ as a function of $\alpha$ and $x$.

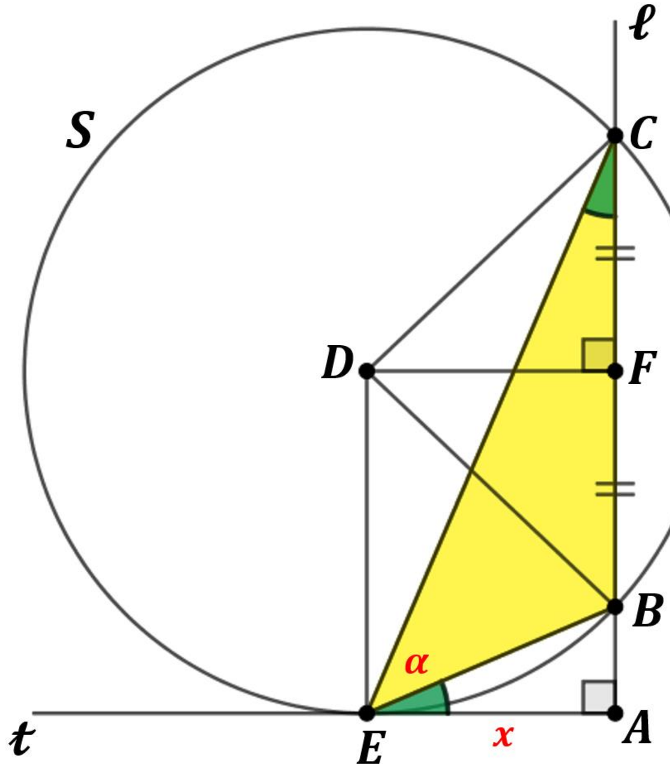
(e) Find $x$ as a function of $AB$ and $BC$.

**Solution**

(a) $m\angle CDB = 2\alpha$ because the central angle $\angle CDB$ has twice the measure of the inscribed angle $\angle CEB$ subtended by the same arc $\widehat{CB}$.

$\overline{DC}$ and $\overline{DB}$ are both radii and hence $\Delta CDB$ is an isosceles triangle. Therefore $\overline{DE}$, the perpendicular bisector of $\overline{CB}$, bisects $\angle CDB$. Hence, $m\angle CDF = \frac{1}{2}m\angle CDB = \alpha$.

(b) First recognize that $m\angle BEA = m\angle BCE$ by the Alternating Segments Theorem (see diagram below).




But $m\angle BCE = \frac{1}{2}m\angle BDE$ because the central angle $\angle BDE$ has twice the measure of the inscribed angle $\angle BCE$ subtended by the same arc $\widehat{BE}$.

Furthermore $m\angle FDE = 90°$ because $\angle DEA, \angle EAF$ and $\angle AFD$ are all right angles in rectangle $DEAF$. Therefore,

$$m\angle BEA = m\angle BCE = \frac{1}{2}m\angle BDE = \frac{1}{2}(m\angle FDE - m\angle FDB) = \frac{1}{2}(m\angle FDE - m\angle FDC)$$

But we have already shown that $m\angle FDC = \alpha$ and we can see that $m\angle FDE = 90°$ because $\angle DEA, \angle EAF$ and $\angle AFD$ are all right angles in rectangle $\square DEAF$. Therefore,

$$m\angle BEA = \frac{1}{2}(m\angle FDE - m\angle FDC) = \frac{1}{2}(90° - \alpha).$$

(c) In $\Delta CDF$ we have

$$\tan(\angle CDF) = \frac{CF}{DF}.$$

So,

$$BC = 2 \cdot CF = 2 \cdot \left(DF \cdot \tan(\angle CDF)\right) = 2x\tan(\alpha)$$

because $DF = EA = x$ (as $\overline{ED}$ and $\overline{AF}$ are parallel).

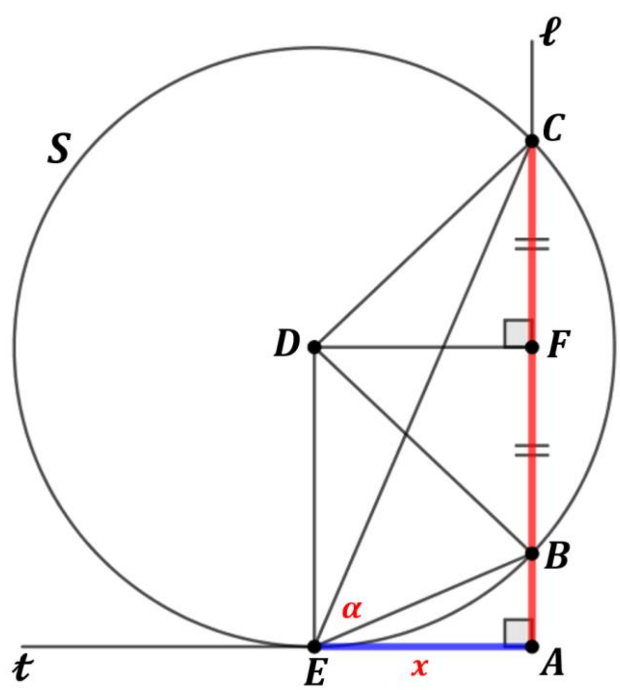
(d) In $\Delta BEA$ we have

$$\tan(\angle BEA) = \frac{AB}{EA}.$$

So,

$$AB = EA \cdot \tan(\angle BEA) = x\tan\left(45° - \frac{\alpha}{2}\right).$$

(e) The secant-tangent theorem tells us that




$$x^2 = EA^2 = AB \cdot AC = AB(AB + BC).$$

■

**Problem 1.**

The bronze statue of the Marquis de Lafayette in Lafayette Square in Washington, D.C. is 11 ft high and the marble pedestal and granite foundation combined stands 25 ft high.

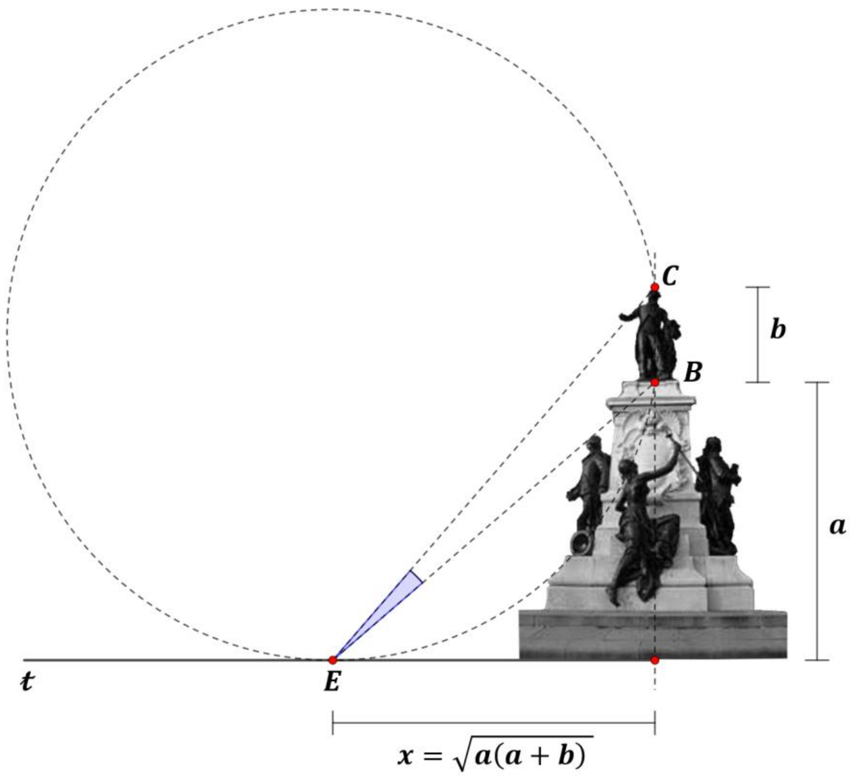
(i) How far away from the monument should a bug get in order to have the largest viewing angle of Lafayette's statue on top?

(ii) How far away from the monument should a tourist, whose eyes tourist's eyes are $5.\overline{6}$ feet above the ground, stand in order to have the largest viewing angle of Lafayette's statue on top?

**Solution**

(i)

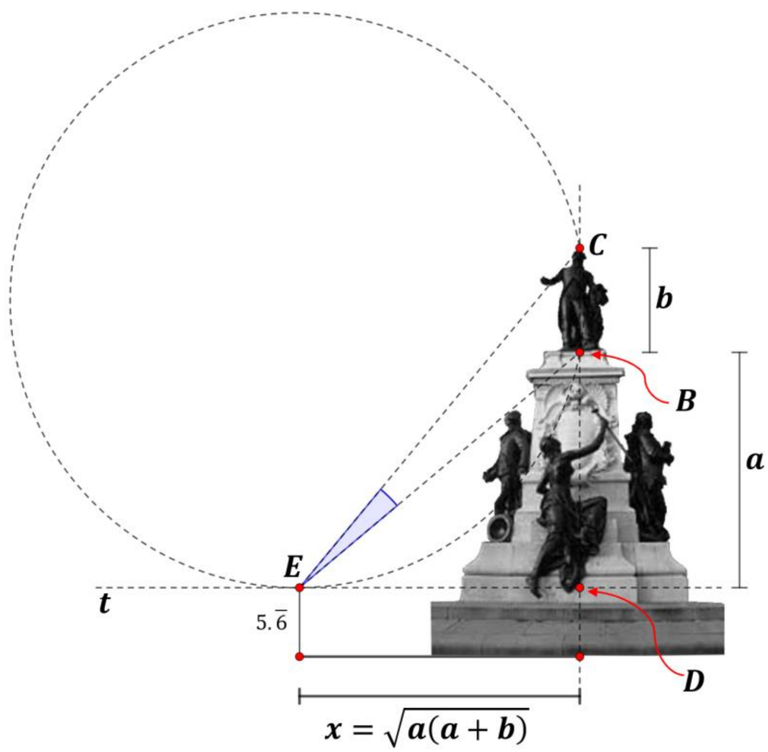
From Part (1) we know that the point of tangency of the circle which is tangent to the eye level of the bug (the ground) and contains the top (point $C$) and the bottom (point $B$) of the statue will determine the maximum viewing angle $\angle CEB$. And from Part (2) we know that the point of tangency $E$ on the tangent line $t$ (the ground) which achieves the maximum viewing angle will be a distance $x = \sqrt{a(a+b)}$ from the monument where $a$ equals the distance from the <u>bug's</u> eye level to the bottom of the statue and $b$ equals the height of the statue.




In this problem $a = 25$ feet and $b = 11$ feet. Therefore, in order to maximize the viewing angle $\angle CEB$, the bug should be a distance $x = \sqrt{25(25+11)} = 30$ feet from the monument.

(ii)

The only parameter that changes is $a$. Now $a$ equals the distance from the tourist's eye level to the bottom of the statue.



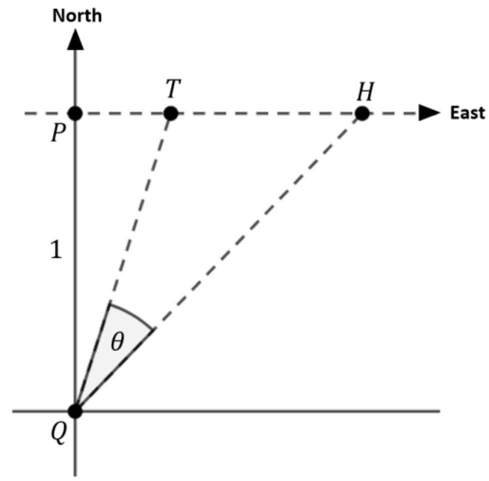

In this situation, $a = 25 - 5.\overline{6} = 19.\overline{3}$. Therefore, in order to maximize the viewing angle $\angle CEB$, the tourist should stand $x = \sqrt{19.\overline{3} \cdot (19.\overline{3} + 11)} \approx 24.2$ feet from the monument.

■

**Problem 2.**

Tugboats Theodore and Hank start off positioned together at a point $P$ a mile due north of the harbormaster's tower position at point $Q$. Then the tugboats take off together from point $P$ traveling due east from the tower. Theo goes $5$ mph and Hank goes $15$ mph.

How long will it take until Theodore and Hank to reach the position where the angle of sight $\theta$ between the tugboats is at a maximum from the harbormaster's position in the tower? What will the angle of sight $\theta$ be at that moment?




**Solution**

Let $a$ miles be the distance that Tugboat Theodore has traveled when the angle $\theta$ is maximized. It follows that Tugboat Hank will have traveled $3a$ miles and the distance between the tugs will be $b = 3a - a = 2a$ at that point. From Part 2 of this section the distance between points $P$ and $Q$ will be $x = \sqrt{a(a+b)}$ when $\theta$ is at its maximum value. So, we can work backwards to find $a$ and $b$ when $\theta$ is at a maximum and $x = 1$.

$$1 = \sqrt{a(a+b)} = \sqrt{a(a+2a)} = \sqrt{3a^2} = \sqrt{3} \cdot a$$

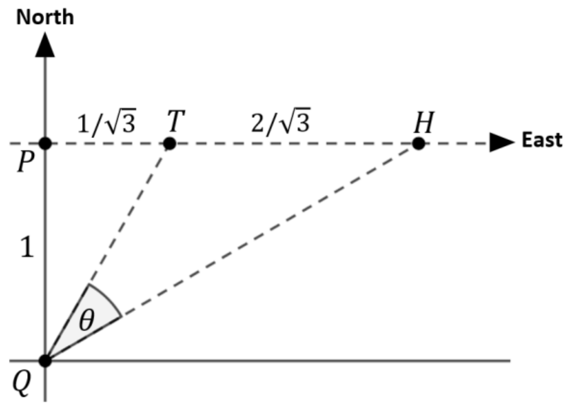
or

$$a = \frac{1}{\sqrt{3}} \text{ and } b = \frac{2}{\sqrt{3}}.$$

So, it will take the tugboats

$$\frac{1/\sqrt{3}}{5} \approx 0.115 \text{ hours} \approx 6.9 \text{ minutes}$$

for $\theta$ to reach its maximum value.




Let $\delta = \angle PQT$. Then

$$\tan(\delta) = \frac{1/\sqrt{3}}{1} \text{ and } \tan(\delta + \theta) = \frac{3/\sqrt{3}}{1} \Longrightarrow \delta = 30° \text{ and } \delta + \theta = 60°$$

at the moment when the angle of sight $\theta$ is maximized. Therefore, $\theta = 30°$ is the maximum angle of sight.