MSHSML Meet 4, Event C
Study Guide

4C Miscellaneous Topics

Sequences: patterns and recursion formulas, arithmetic and geometric sequences

Series: partial sums, formulas for sums of consecutive integers $1 + 2 + \cdots + n$, consecutive squares $1^2 + 2^2 + \cdots + n^2$, and consecutive cubes $1^3 + 2^3 + \cdots + n^3$

Function notation

Factorial notation and the Binomial Theorem

Table of Contents

1. Sum of Consecutive Integers, Squares, Cubes

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

2. Arithmetic Sequences (or arithmetic progressions)

We say $a_1, a_2, a_3, \dots, a_n$ are in arithmetic progression or form an arithmetic sequence if the difference between successive terms is a constant.

$$a_1 = 3, a_2 = 7, a_3 = 11, a_4 = 15, a_5 = 19, a_6 = 23$$

is an example of an arithmetic sequence because $a_j - a_{j-1} = 4$ for $j = 2,3,4,5,6$

2.1 Properties of Arithmetic Sequences

If $a_1, a_2, a_3, \dots$ is an arithmetic sequence where d is the constant difference between successive terms, then

1.	$$a_n = a_m + (n-m)d, \qquad m = 1,2,\dots,n$$ Note in particular the special cases $m = 1$ and $m = n-1$. $$a_n = a_1 + (n-1)d$$ $$a_n = a_{n-1} + d.$$
2.	Finding the sum if you know the first and last term. $$\sum_{i=1}^{n} a_i = \frac{n(a_1 + a_n)}{2}$$ Finding the sum if you know the first term and the common difference. $$\sum_{i=1}^{n} a_i = \frac{n(2a_1 + (n-1)d)}{2}$$

3.	If the numbers $\{a, b, c, \ldots, z\}$ are an arithmetic sequence, then n, the number of terms in this sequence equals $$n = \left(\frac{z-a}{b-a}\right) + 1.$$ That is, the number of terms in the sequence equals $$n = \frac{\text{Last term} - \text{First term}}{d} + 1.$$ For example, there are $$n = \frac{153-(-3)}{6} + 1 = 26 + 1 = 27$$ terms in the arithmetic sequence $\{-3,3,9,15,21,27, \ldots ,147,153\}$.

3. Geometric Sequence (or geometric progressions)

We say $a_1, a_2, a_3, \ldots, a_n$ are in geometric progression or form an geometric sequence if the ratio of successive terms is a constant.

$$a_1 = 3, a_2 = 15, a_3 = 75, a_4 = 375, a_5 = 1875, a_6 = 9375$$

is an example of a geometric sequence because $a_j/a_{j-1} = 5$ for $j = 2,3,4,5,6$.

3.1 Properties of Geometric Sequences

If $a_1, a_2, a_3, \ldots$ is a geometric sequence where r is the constant ratio between successive terms, then

1.	$$a_n = a_1 r^{n-1}.$$
2.	If the constant ratio $r \neq 1$ and if n is finite, then $$\sum_{i=1}^{n} a_i = \frac{a_1(1-r^n)}{1-r}, \quad \text{provided } r \neq 1.$$

3.	If the constant ratio $r = 1$, then $a_1 = a_2 = \cdots = a_n$ and in this case $$\sum_{i=1}^{n} a_i = a_1 + a_1 + \cdots + a_1 = n \cdot a_1.$$
4.	If the number of terms in the geometric sequence is infinite, then $$\sum_{i=1}^{\infty} a_i = \frac{a_1}{1-r}, \text{ provided } \lvert r\rvert < 1.$$

4. Sequences and Series Exercises

1.	Source: MSHSML 4C163 What is the smallest integer n for which $1^3 + 2^3 + 3^3 + \cdots + n^3$ is a multiple of 77.

Solution

Using the formula for the sum of cubes, the left side of the equation is $\frac{n^2(n+1)^2}{4}$. *In order to be a multiple of 77, the numerator must be a multiple of 77. To be a multiple of 77 the number must be divisible by both 7 and by 11. The values of n for which the numerator is divisible by 11 are 10, 11, 21, 22, 32, 33, … . The first number in this list for which either n or n + 1 is a multiple of 7 is n = 21.*

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2.	Source: MSHSML 4C164 Determine the smallest $n > 2017$ for which $$2 \cdot 3 \cdot 4 + 5 \cdot 6 \cdot 7 + 8 \cdot 9 \cdot 10 + \cdots + (3n-1)(3n)(3n+1)$$ is a multiple of 27.

Solution

The sum can be written as

$$2\cdot3\cdot4+5\cdot6\cdot7+8\cdot9\cdot10+\ldots+(3n-1)(3n)(3n+1)=\sum_{i=1}^{n}(3i-1)(3i)(3i+1)=\sum_{i=1}^{n}27i^3-3i=27\sum_{i=1}^{n}i^3-3\sum_{i=1}^{n}i.$$

Substituting the sum of cubes and sum of integers formulas in we obtain

$$27\sum_{i=1}^{n}i^3-3\sum_{i=1}^{n}i=\frac{27n^2(n+1)^2}{4}-\frac{3n(n+1)}{2}=\frac{3n(n+1)(9n^2+9n-2)}{4}.$$ *For this to be a multiple of 27, we need either n or n + 1 to be a multiple of 9. The quadratic term will never be divisible by 9. The smallest n > 2017 for which this occurs is 2024.*

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3.	Source: MSHSML 4C141 What is the 54th term of the sequence 3,5,7,9, … ?

Solution

The explicit formula for an arithmetic sequence is $a_n=a_1+d(n-1)$.

Here, the common difference is two; therefore, $a_{54}=3+2(54-1)=109$.

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4.	Source: MSHSML 4C143 Determine exactly the infinite sum $8-4\sqrt{2}+4-2\sqrt{2}+2-\cdots$

Solution

This is a geometric series with common ratio $\frac{-4\sqrt{2}}{8}=\frac{-\sqrt{2}}{2}$. *Its infinite sum is given by* $\frac{a_1}{1-r}=\frac{8}{1+\frac{\sqrt{2}}{2}}$. *Multiplying the numerator and denominator of this expression by 2 yields* $\frac{16}{2+\sqrt{2}}$*, and rationalizing using* $2-\sqrt{2}$ *, we have* $\frac{16(2-\sqrt{2})}{(2+\sqrt{2})(2-\sqrt{2})}=\frac{16(2-\sqrt{2})}{4-2}=16-8\sqrt{2}$.

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5.	Source: MSHSML 4C131 Determine exactly the sum of the infinite series $$5+\frac{5}{3}+\frac{5}{9}+\cdots$$

Solution

This is an infinite geometric series with $a_1=5$ *and* $r=\frac{1}{3}$. *Its sum is* $\frac{a_1}{1-r}=\frac{5}{1-\frac{1}{3}}=\frac{5}{\left(\frac{2}{3}\right)}=\frac{15}{2}$.

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6.	Source: MSHSML 4C113 If $t_1=1$, $t_2=-1$, and $t_n=\left(\frac{n-3}{n-1}\right)\cdot t_{n-2}$ for $n\geq 3$, determine exactly the value of t_{2012}.

Solution

Start computing some terms: $t_3=\left(\frac{3-3}{3-1}\right)\cdot t_1=\left(\frac{0}{2}\right)\cdot 1=0$ $\quad t_4=\left(\frac{4-3}{4-1}\right)\cdot t_2=\left(\frac{1}{3}\right)\cdot -1=-\frac{1}{3}$

$t_5=\left(\frac{5-3}{5-1}\right)\cdot t_3=\left(\frac{2}{4}\right)\cdot 0=0$ $\quad t_6=\left(\frac{6-3}{6-1}\right)\cdot t_4=\left(\frac{3}{5}\right)\cdot\left(-\frac{1}{3}\right)=-\frac{1}{5}$ $\quad t_7=\left(\frac{7-3}{7-1}\right)\cdot t_5=\left(\frac{4}{6}\right)\cdot 0=0$

Every odd-indexed term is 0, while every even-indexed term equals $-\frac{1}{n-1}$. *So* $t_{2012}=-\frac{1}{2011}$.

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7.	Source: MSHSML 4C104 If $P(n)=\frac{1}{n^2-1}-\frac{1}{n^3-n}$, determine exactly the value of $$P(2)+P(3)+P(4)+\cdots+P(2011).$$

Solution

$$P(n)=\frac{1}{n^2-1}-\frac{1}{n^3-n}=\frac{1}{n^2-1}-\frac{1}{n(n^2-1)}=\frac{n-1}{n(n-1)(n+1)}=\frac{1}{n(n+1)}, \textit{ so } P(2)=\frac{1}{6}=\frac{1}{2}-\frac{1}{3},$$

$$P(3)=\frac{1}{12}=\frac{1}{3}-\frac{1}{4},\ P(4)=\frac{1}{20}=\frac{1}{4}-\frac{1}{5}, \textit{ etc. } \textit{ Thus } \mathrm{P}(2)+\mathrm{P}(3)+\mathrm{P}(4)+\ldots+\mathrm{P}(2011)=$$

$$\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\ldots+\left(\frac{1}{2011}-\frac{1}{2012}\right)=\frac{1}{2}-\frac{1}{2012}=\frac{1005}{2012}.$$

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8.	Source: MSHSML 4C064 It is well known that the sum $1+2+3+\cdots+n$ can be expressed in the closed form $$\frac{n(n+1)}{2}.$$ Find a similar closed form for the sums (a) $1^2-2^2+3^2-4^2+\cdots-n^2$ when n is even (b) $1^2-2^2+3^2-4^2+\cdots+n^2$ when n is odd.

Solution

(a) n even

$$[1^2-2^2]+[3^2-4^2]+\ldots+[(n-1)^2-n^2]$$
$$=[(1-2)(1+2)]+[(3-4)(3+4)]+\ldots+[((n-1)-n)((n-1)+n)]$$
$$=-1\{(1+2)+(3+4)+\ldots+(n-1)+n\}=-\frac{(n+1)n}{2}$$

(b) n odd

$$1^2-[2^2-3^2]-[4^2-5^2]-\ldots-[(n-1)^2-n^2]$$
$$=1-[(2-3)(2+3)]-[(4-5)(4+5)]-\ldots-[((n-1)-n)((n-1)+n)]$$
$$=1+(2+3)+(4+5)+\ldots+(n-1)+n=\frac{n(n+1)}{2}$$

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9.	Source: MSHSML 4C031 What integer is equal to $3 + 7 + 11 + 15 + \cdots + 99$?

Solution

$$\text{Sum} = \frac{n[2a + (n-1)d]}{2}$$

$$= \frac{25[6 + 24 \cdot 4]}{2}$$

$$= 1275$$

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10.	Source: MSHSML 4C014 Let $a_0 = 2$ and let $a_k = a_{k-1} + \frac{1}{2^k}$ for all $k \geq 1$. Give a formula that expresses a_n as the difference between an integer and a power of 2.

Solution

$$a_1 = 2 + \frac{1}{2} = \frac{2^2+1}{2}$$

$$a_2 = \frac{2^2+1}{2} + \frac{1}{2^2} = \frac{2^3+2+1}{2^2}$$

$$a_3 = \frac{2^3+2+1}{2^2} + \frac{1}{2^3} = \frac{2^4+2^2+2+1}{2^3}$$

$$a_4 = \frac{2^4+2^2+2+1}{2^3} + \frac{1}{2^4} = \frac{2^5+2^3+2^2+2+1}{2^4}$$

$$\vdots$$

$$a_n = \frac{2^{n+1} + (2^{n-1} + 2^{n-2} + \cdots + 2 + 1)}{2^n}$$

The term in parentheses is the sum of a geometric progression with common ratio 2 and first term 1. It is equal to $2^n - 1$.

$$a_n = \frac{2^{n+1} + 2^n - 1}{2^n} = 2 + 1 - \frac{1}{2^n}$$

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11. Source: MSHSML 4C994

Find an exact formula for the sum of the first 100 terms of the series

$$1 + 3 + 7 + 15 + 31 + 63 + 127 + 255 + \cdots$$

Express your answer in the form $2^n - m$ for positive integers n and m.

<u>Solution</u>

$$(2-1)+(2^2-1)+(2^3-1)+(2^4-1)+\cdots+(2^{100}-1)=$$

$$2\left[1+2+2^2+\cdots+2^{99}\right]-100 =$$

$$2\left[\frac{2^{100}-1}{2-1}\right]-100 = 2^{101}-102$$

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5. Factorial Notation

$$k! = \begin{cases} k(k-1)(k-2)\cdots 3(2)(1) & k = 1,2,3,\ldots \\ 1 & k = 0 \end{cases}$$

12. Source: MSHSML 4C993

Express

$$\frac{1\cdot 3\cdot 5\cdots(2n-1)}{4\cdot 8\cdot 12\cdots(4n)}$$

in the form

$$\frac{A!}{B^n(C!)^2}.$$

<u>Solution</u>

$$\frac{[1\cdot 3\cdot 5\cdots(2n-1)][2\cdot 4\cdots(2n)]}{(4\cdot 1)(4\cdot 2)\cdots(4\cdot n)[(2\cdot 1)(2\cdot 2)\cdots(2\cdot n)]}$$

$$= \frac{(2n)!}{4^n\cdot n!\, 2^n\cdot n!}$$

$$= \frac{(2n)!}{8^n\,(n!)^2}$$

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6. Binomial Coefficients

$$\binom{n}{i} = \frac{n!}{i!\,(n-i)!} \text{ provided } i \le n \text{ and both are integers.}$$

Note that $\binom{n}{i} = \binom{n}{n-i}$. Also note that $\binom{n}{i}$ can also be written as ${}_nC_i$.

7. Binomial Theorem

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i} \quad n = 0,1,2,3\ \ldots$$

7.1 Binomial Theorem Exercises

13.	Source: MSHSML 4C184 When $\left(\sqrt{3}+\sqrt[3]{2}\right)^9$ is expanded, what is the sum of all the terms that are integers?

Solution

All terms of the expansion are of the form: $\binom{9}{k}\left(\sqrt{3}\right)^{9-k}\left(\sqrt[3]{2}\right)^k=\binom{9}{k}(3)^{\frac{9-k}{2}}(2)^{\frac{k}{3}}$. *For this to be an integer, both* $\frac{9-k}{2}$ *and* $\frac{k}{3}$ *must be integers. This only occurs when* $k=3$ *or* $k=9$.

Let $k=3:\binom{9}{3}(3)^{\frac{9-3}{2}}(2)^{\frac{3}{3}}=84\cdot27\cdot2=4536.$ Let $k=9:\binom{9}{9}(3)^{\frac{9-9}{2}}(2)^{\frac{9}{3}}=1\cdot1\cdot8=8.$

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14.	Source: MSHSML 4C172 What is the integer coefficient of the x^8 term in the expansion of $(2x^2-5)^7$?

Solution

Each tern of the expansion is of the form: $\binom{7}{k}\left(2x^2\right)^{7-k}(-5)^k$. *Expanding,*

$\binom{7}{k}(2)^{7-k}\left(x^2\right)^{7-k}(-5)^k=\binom{7}{k}(2)^{7-k}(-5)^k(x)^{14-2k}$. *Therefore,* $14-2k=8\rightarrow k=3$.

So the coefficient of x^8 *is* $\binom{7}{3}(2)^4(-5)^3=35\cdot16\cdot(-125)=-70000$.

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15.	Source: MSHSML 4C142 What is the coefficient of the x^2 term in the expansion of $(2x-3)^5$?

Solution

By the Binomial Theorem, the x^2 *term will be* ${}_5C_3(2x)^2(-3)^3 = 10\cdot 4x^2(-27)$,

so the coefficient is $10\cdot 4\cdot(-27) = -1080$.

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16.	Source: MSHSML 4T132 Calculate the value of $$2^{99}\binom{100}{1} - 2^{98}\binom{100}{2} + 2^{97}\binom{100}{3} + \cdots - 2^{2}\binom{100}{98} + 2^{1}\binom{100}{99}.$$ You may express your answer as a power or binomial coefficient if necessary.

<u>Solution</u>

$$(1-2)^{100} = \sum_{j=0}^{100}\binom{100}{j}(-1)^j 2^{100-j}$$

$$= 2^{100}\binom{100}{0}$$

$$-\left(2^{99}\binom{100}{1} - 2^{98}\binom{100}{2} + 2^{97}\binom{100}{3} + \cdots - 2^{2}\binom{100}{98} + 2^{1}\binom{100}{99}\right)$$

$$+ 2^{0}\binom{100}{100}$$

So,

$$2^{99}\binom{100}{1} - 2^{98}\binom{100}{2} + 2^{97}\binom{100}{3} + \cdots - 2^{2}\binom{100}{98} + 2^{1}\binom{100}{99}$$

$$= 2^{100}\binom{100}{0} + 2^{0}\binom{100}{100} - (1-2)^{100} = 2^{100} + 1 - 1 = 2^{100}.$$

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17.	Source: MSHSML 4C124 Find <u>all</u> ordered triples of integers (x, m, n), with $x, m, n \geq 0$, for which $$\sum_{k=0}^{m} \binom{m}{k} x^k 3^{m-k} = 1024 \quad \text{and} \quad \sum_{k=0}^{n} \binom{n}{k} x^k (-2)^{n-k} = 729.$$

Solution

The given sigma notations are representative of the binomial expansions of $(x+3)^m$ *and* $(x-2)^n$ *respectively, so we have* $(x+3)^m = 1024 = (\pm 2)^{10} = 4^5 = (\pm 32)^2 = 1024^1$ *and* $(x-2)^n = 729 = (\pm 3)^6 = 9^3 = (\pm 27)^2 = 729^1$. *The first set of equalities gives us* $x = -5, -1, 1, 29, -35,$ *and* 1021. *The second set of equalities yields* $x = -1, 5, 11, -25, 29,$ *and* 731.
Since $x \geq 0$, $x = 29$, *and the only ordered triple that works is* $(x, m, n) = (29, 2, 2)$.

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18.	Source: MSHSML 4T125 Determine exactly all real numbers c for which $$\sum_{n=0}^{6} \binom{6}{n} 3^n c^{6-n} = 1000.$$

Solution

$$\sum_{n=0}^{6} \binom{6}{n} 3^n c^{6-n} = \binom{6}{0} 3^0 c^6 + \binom{6}{1} 3^1 c^5 + \ldots + \binom{6}{5} 3^5 c^1 + \binom{6}{6} 3^6 c^0,$$

which is the binomial expansion of $(3+c)^6$.

So $(3+c)^6 = 1000 = 10^3 \Rightarrow (3+c)^2 = 10 \Rightarrow 3+c = \pm\sqrt{10} \Rightarrow c = \boxed{-3 \pm \sqrt{10}}$.

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19.	Source: MSHSML SC092 In the expansion of $$\left(\frac{2}{a^2} - \frac{a^3}{4}\right)^5$$ there is a term which can be simplified so that the variable a does not appear. Write that simplified term.

Solution

$$\left(\frac{2}{a^2} - \frac{a^3}{4}\right)^5 = \sum_{j=0}^{5} \binom{5}{j} \left(\frac{2}{a^2}\right)^j \left(-\frac{a^3}{4}\right)^{5-j}$$

$$= \sum_{j=0}^{5} \binom{5}{j} 2^j \left(-\frac{1}{4}\right)^{5-j} \cdot a^{3(5-j)-2j}$$

$$= \sum_{j=0}^{5} \binom{5}{j} 2^j \left(-\frac{1}{4}\right)^{5-j} \cdot a^{15-5j}.$$

The only term in the above with a^0 (*i.e.* variable a does not appear) is when $15 - 5j = 0$. That occurs when $j = 3$. In this case we have

$$\left(\binom{5}{3} 2^3 \left(-\frac{1}{4}\right)^{5-3}\right) \cdot a^0$$

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20.	Source: MSHSML 4C084 For certain ordered pairs of integers (x, n), with $n > 0$, $$\sum_{k=0}^{n} x^k 3^{n-k} \binom{n}{k} = 4096.$$ Compute the sum of all possible values of x from these ordered pairs.

Solution

The left side of the equation represents the binomial expansion of $(x+3)^n$. *Thus,* $(x+3)^n = 4096 = (\pm 2)^{12} = (\pm 4)^6 = (\pm 8)^4 = 16^3 = (\pm 64)^2 = 4096^1$. *These yield:*

$$x = -5 \textit{ or } -1,\quad -7 \textit{ or } 1,\quad -11 \textit{ or } 5,\quad 13,\quad -67 \textit{ or } 61,\quad 4093.$$

The sum of these x-values is $4(-6)+13+4093 = 4082$.

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21.	Source: MSHSML 4C073 Find the sum of all positive integers n for which $$\left(4x^n + \frac{x^{-3}}{2}\right)^{10}$$ will have a term that is x-free.

Solution

Powers of x will be $(x^n)^{10-r}(x^{-3})^r$ for $r = 0, 1, \ldots, 10$. When will $10n - rn - 3r = 0$? $n = \dfrac{3r}{10-r}$

See table in right margin.

r	$\frac{3r}{10-r}$
0	0
1	$3/9$
2	$6/8$
3	$9/7$
4	$12/6 = 2$
5	$15/5 = 3$
6	$18/4$
7	$21/3 = 7$
8	$24/2 = 12$
9	$27/1 = 27$
10	–
	$\overline{51}$

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22.	Source: MSHSML 4C062 The expansion of $\left(\frac{2}{a} + \frac{a^2}{4}\right)^8$ includes a term of the form ra where r is an integer. What is r?

<u>Solution</u>

term of interest $= 8 \cdot 7 \frac{2^5}{2^6} a$

$$\left(\frac{2}{a}\right)^8 + 8\left(\frac{2}{a}\right)^7\left(\frac{a^2}{4}\right) + \frac{8\cdot 7}{2}\left(\frac{2}{a}\right)^6\left(\frac{a^2}{4}\right)^2 + \underbrace{\frac{8\cdot 7\cdot 6}{3\cdot 2}\left(\frac{2}{a}\right)^5\left(\frac{a^2}{4}\right)^3}_{\text{term of interest}} + \ldots$$

$$r = 8 \cdot 7 \cdot \frac{2^5}{2^6} = 28$$

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23.	Source: MSHSML 4C004 The coefficient of $a^k b^{41-k}$ in the expansion of $$\left(\frac{a}{2} + \frac{b}{2}\right)^{41}$$ Is the same as the coefficient of $a^{k+1} b^{41-k}$ in the expansion of $$\left(\frac{a}{2} + \frac{b}{2}\right)^{42}.$$ What is k?

<u>Solution</u>

Compare

$$\binom{41}{k}\left(\frac{a}{2}\right)^{k}\left(\frac{b}{2}\right)^{41-k} \text{ and } \binom{42}{k+1}\left(\frac{a}{2}\right)^{k+1}\left(\frac{b}{2}\right)^{42-(k+1)}$$

$$\frac{41!}{(41-k)!\,k!}\,\frac{1}{2^{41}} = \frac{42 \cdot 41!}{(41-k)!\,(k+1)k!} \cdot \frac{1}{2 \cdot 2^{41}}$$

$$1 = \frac{42}{(k+1)(2)}$$

$$2k + 2 = 42$$

$$\boxed{k = 20}$$

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24.	Source: MSHSML SC152 Calculate the coefficient of the x^2 term in $(5x - 1)^7$.

<u>Solution</u>

$$(5x-1)^7 = \sum_{j=0}^{7} \binom{7}{j} (5x)^j (-1)^{7-j}$$

The only term in the above with x^2 is when $j = 2$. In this case we have

$$\left(\binom{7}{2} 5^2 (-1)^5\right) \cdot x^2.$$

So, the coefficient of x^2 in $(5x-1)^7$ is

$$\binom{7}{2} 5^2 (-1)^5 = 21 \cdot 25 \cdot (-1) = -525.$$

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25.	Source: MSHSML 5C061 In the expansion of $(x+y)^{15}$, what is the coefficient of $x^9 y^6$?

Solution

By the Binomial Theorem

$$(x+y)^{15} = \sum_{j=0}^{15} \binom{15}{j} x^j y^{15-j}.$$

So, the coefficient of $x^9 y^6$ in $(x+y)^{15}$ is

$$\binom{15}{9} = \frac{15!}{9!\,(15-9)!} = 5005.$$

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26. Source: MSHSML 5C051

In the expansion of

$$\left(\frac{1}{2}a + 4\right)^8,$$

what will be the coefficient of a^5?

Solution

$$\binom{8}{3}\left(\frac{a}{2}\right)^5 (4)^3 = \frac{8!}{5!\,3!}\,\frac{1}{2^5}\left(2^2\right)^3 a^5$$

$$= 112\,a^5$$

27. Source: MSHSML 5C031

In the binomial expansion

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

the sum of the coefficients is $1 + 4 + 6 + 4 + 1 = 16$. What is the sum of the coefficients in the expansion of $(a+b)^9$?

Solution

Set $a = b = 1$

$$(1+1)^9 = 2^9 = 512$$

28. Source: MSHSML 5C962

$(a+b)^{10} = a^{10} + 10a^9b + 45a^8b^2 + \cdots$ will be a polynomial in the two variables a and b. What will be the sum of the coefficients of this polynomial?

Solution

Set $a = b = 1$ to get the sum $= 2^{10}$

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29.	Source: MSHSML 5C921 In the expansion of $$\left(\frac{x}{2}+12\right)^8,$$ what is the coefficient of x^6?

Solution

Sixth power term is $\binom{8}{6}\left(\frac{x}{2}\right)^6\left(2^2\cdot 3\right)^2 = \frac{8\cdot 7\cdot 6!}{2\cdot 6!}\frac{1}{2^6}\, 2^4\cdot 3^2 x^6 = 63x^6$

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30.	Simplify $$\binom{11}{0}+\binom{11}{1}+\binom{11}{2}+\binom{11}{3}+\ \dots\ +\binom{11}{8}+\binom{11}{9}+\binom{11}{10}+\binom{11}{11}.$$

Solution

By the Binomial Theorem

$$(x+y)^{11} = \sum_{j=0}^{11}\binom{11}{j}x^j y^{11-j}$$

for all real numbers x and y. If we take $x = 1$ and $y = 1$ in the above sum we find

$$(1+1)^{11} = \sum_{j=0}^{11} \binom{11}{j} 1^j 1^{11-j} = \sum_{j=0}^{11} \binom{11}{j}$$

$$= \binom{11}{0} + \binom{11}{1} + \cdots + \binom{11}{10} + \binom{11}{11}.$$

So

$$\binom{11}{0} + \binom{11}{1} + \cdots + \binom{11}{10} + \binom{11}{11} = (1+1)^{11} = 2^{11}.$$

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31.	Find $$\binom{18}{0} - \binom{18}{1} + \binom{18}{2} - \binom{18}{3} + \ \dots\ + \binom{18}{14} - \binom{18}{15} + \binom{18}{16}$$ where the terms continue to alternate $+, -, +, -, \dots$ throughout the sum.

Solution

By the Binomial Theorem

$$(x+y)^{18} = \sum_{j=0}^{18} \binom{18}{j} x^j y^{18-j}$$

for all real numbers x and y. If we take $x = 1$ and $y = -1$ in the above sum we find

$$(1-1)^{18} = \sum_{j=0}^{18} \binom{18}{j} 1^j (-1)^{18-j}$$

$$= \binom{18}{0} - \binom{18}{1} + \ \dots\ + \binom{18}{14} - \binom{18}{15} + \binom{18}{16} - \binom{18}{17} + \binom{18}{18}$$

$$= \left(\binom{18}{0} - \binom{18}{1} + \ldots + \binom{18}{14} - \binom{18}{15} + \binom{18}{16} \right) - \binom{18}{17} + \binom{18}{18}.$$

So,

$$\binom{18}{0} - \binom{18}{1} + \binom{18}{2} - \binom{18}{3} + \ldots + \binom{18}{14} - \binom{18}{15} + \binom{18}{16}$$

$$= (1-1)^{18} + \binom{18}{17} - \binom{18}{18} = 0 + 18 - 1 = 17.$$

■