MSHSML Meet 4, Event D
Study Guide

4D Analytic Geometry of the Conic Sections

Using the standard forms of equations of the conic sections

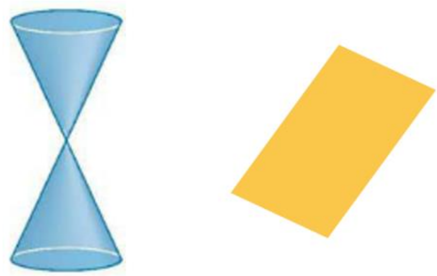
Graphs, including the location of foci, directrices, and asymptotes

Use of properties of conics to solve applied problems, including max-min for parabolas

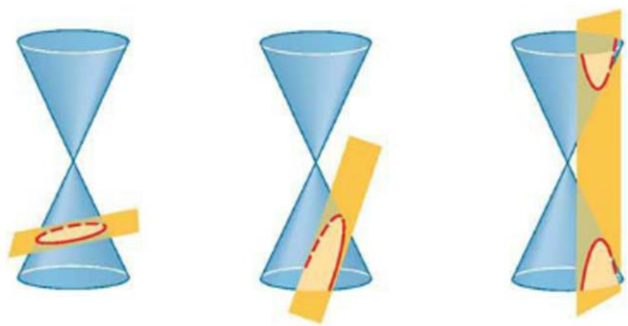
Contents

1 Conic Sections

Consider the possible shapes that you can generate if you intersect a double cone with a plane that does not go through the center point where the two cones touch. As you think about this remember that in geometry a cone only includes the points on the surface. The points in the interior are not included in the definition of a cone.



There are actually only three possible shapes of the intersection - the ellipse, the parabola and the hyperbola.



ellipse parabola hyperbola

Some books define a circle as a fourth possible shape

circle

But in the study of conic sections a circle is mostly identified as a special case of an ellipse. In any event, these shapes are called **conic sections**, or **conics**, because of their connection with a cone.

Conic sections can be described in three ways:

(*i*) as the intersection of a right circular and a plane where the plane meets certain conditions

(*ii*) as an equation of the form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ where the coefficients meet certain conditions (*iii*) as the locus of points in a given plane meeting certain conditions.

We considered method (*i*) in the above discussion. Next we consider method (*ii*). The following table gives us a simple check to tell which conic we have for any given set of coefficients A, B, C, D, E and F in the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

We can see from this table that all comes down to whether $B^2 - 4AC$ is positive (hyperbola), negative (ellipse) or zero (parabola).

Ellipse	$B^2 - 4AC < 0$
Parabola	$B^2 - 4AC = 0$
Hyperbola	$B^2 - 4AC > 0$

Because the circle is such an important special case of an ellipse we make a note that within the case where $B^2 - 4AC < 0$, if $A = C$ and $B = 0$, the resulting equation is necessarily a circle.

Circle	$B^2 - 4AC < 0$, and $A = C$ and $B = 0$

Special Case of $\boldsymbol{B = 0}$.

We begin by limiting ourselves to the case when there is no xy term. That is, the case when $B = 0$. This is an important special case because when $B = 0$ the **graph of the conic will be symmetric to a line parallel and/or perpendicular to the x-axis**.

1.1 Parabolas

Parabola with Vertex $(\boldsymbol{h}, \boldsymbol{k})$		
General Shape	∪ or ∩	⊂ or ⊃
Standard Form of the Equation	$y = a(x-h)^2 + k$ with $a \neq 0$	$x = a(y-k)^2 + h$ with $a \neq 0$
Vertex	(h, k)	(h, k)
Axis of Symmetry	$x = h$	$y = k$
Focus	$\left(h, k + \frac{1}{4a}\right)$	$\left(h + \frac{1}{4a}, k\right)$
Directrix	$y = k - \frac{1}{4a}$	$x = h - \frac{1}{4a}$
Direction of Opening	Upward (*i.e.* ∪) if $a > 0$ Downward (*i.e.* ∩) if $a < 0$	Right (*i.e.* ⊂) if $a > 0$ Left (*i.e.* ⊃) if $a < 0$
Vertices (Endpoints) of Latus Rectum [The latus rectum is the line segment with endpoints on the parabola that goes through focus and is parallel to the directrix.]	$\left(h - \left\lvert\frac{1}{2a}\right\rvert, k + \frac{1}{4a}\right)$, $\left(h + \left\lvert\frac{1}{2a}\right\rvert, k + \frac{1}{4a}\right)$	$\left(h + \frac{1}{4a}, k - \left\lvert\frac{1}{2a}\right\rvert\right)$, $\left(h + \frac{1}{4a}, k - \left\lvert\frac{1}{2a}\right\rvert\right)$
Length of Latus Rectum	$\left\lvert\frac{1}{a}\right\rvert$ units	$\left\lvert\frac{1}{a}\right\rvert$ units

Exercise 1.

Identify the type of conic, put it in standard form, and sketch the graph. Label the appropriate information for this conic:

$$\boldsymbol{x^2 - 6x + 2y + 9 = 0}$$

Solution

Identify the type of conic.

The coefficients of x^2 and y^2 in $Ax^2 + Cy^2 + Dx + Ey + F = 0$ are the key to identifying the type of conic. In this case $A = 1$ and $C = 0$. Therefore, this is a parabola. (See the above chart.)

Put it in standard form.

The key to putting any of the conics in standard form is the "***completing the square***" formula.

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right).$$

Applying this formula to

$$x^2 - 6x = x^2 - 6x + 0$$

we get

$$x^2 - 6x = \left(x + \frac{(-6)}{(2 \cdot 1)}\right)^2 + \left(0 - \frac{(-6)^2}{(4 \cdot 1)}\right) = (x-3)^2 - 9.$$

So,

$$x^2 - 6x + 2y + 9 = 0$$

$$\Longrightarrow (x-3)^2 - 9 + 2y + 9 = 0$$

$$\Longrightarrow (x-3)^2 + 2y = 0$$

$$\Longrightarrow 2y = -(x-3)^2$$

$$\Longrightarrow y = \left(-\frac{1}{2}\right)(x-3)^2$$

We recognize this from the chart above as a parabola of the type

$$y = a(x-h)^2 + k, \text{ where } a \neq 0$$

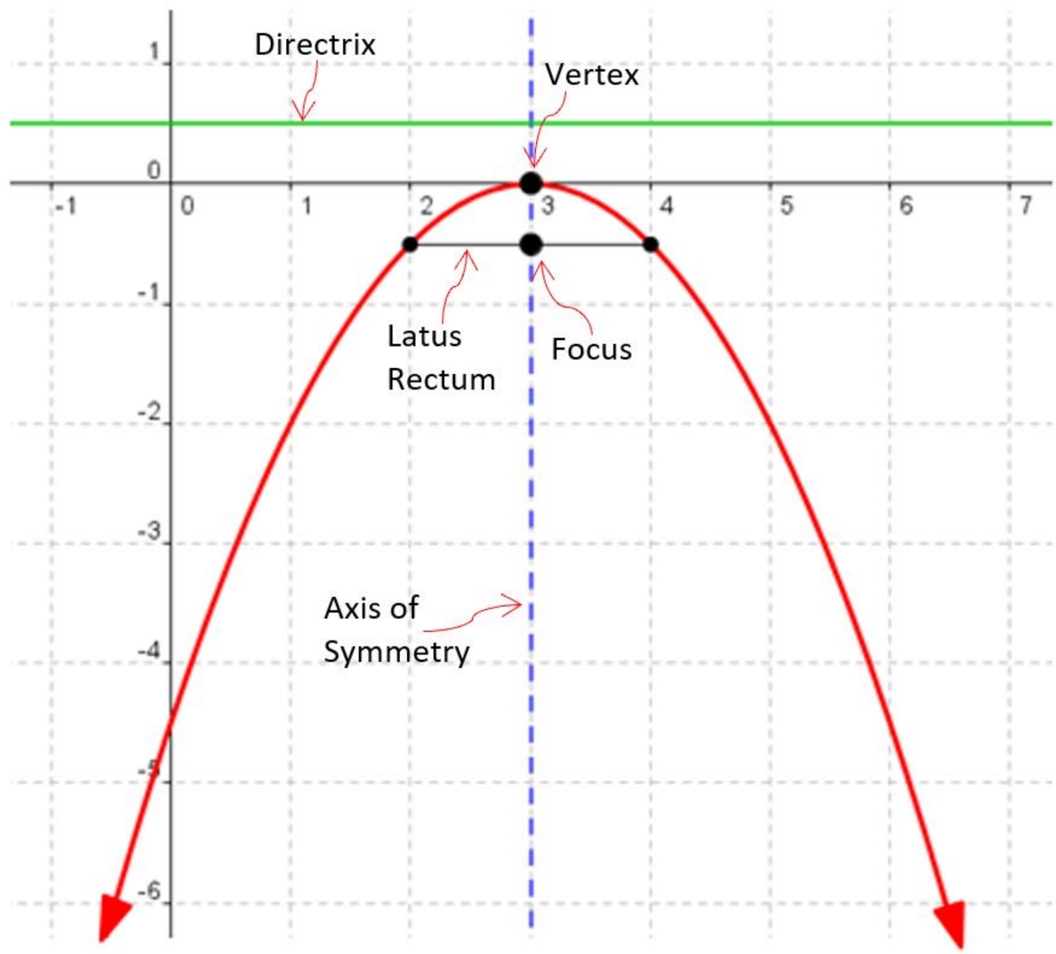
with

$$h = 3, k = 0, \text{ and } a = -1/2.$$

Sketch the graph. Label the appropriate information.

Looking at our chart for a parabola, we can see that the parabola points downwards (*i.e.* ∩) with its vertex at $(3,0)$. The axis of symmetry is the line $x = 3$ and the directrix is the line $y = 1/2$. The focus is located at $(3, -1/2)$. The latus rectum is the line segment connecting the points $(2, -1/2)$ and $(4, -1/2)$. Therefore the length of the latus rectum is 2.

This gives us enough information to make a reasonable sketch. You can plot a few more points on the parabola to make your sketch more accurate.



1.2 Ellipses

Ellipse Centered at $(\boldsymbol{h}, \boldsymbol{k})$		
General Shape	Oval wider than it is tall.	Oval taller than it is wide.
Standard Form of the Equation	$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ with $a > b > 0$	$\frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1$ with $a > b > 0$
Direction of Major (Longer) Axis	Horizontal	Vertical
Foci	$(h+c, k)$ and $(h-c, k)$ where $c^2 = a^2 - b^2$	$(h, k+c)$ and $(h, k-c)$ where $c^2 = a^2 - b^2$
Center	(h, k)	(h, k)
Vertices (Endpoints) of Major Axis	$(h-a, k), (h+a, k)$	$(h, k-a), (h, k+a)$
Vertices (Endpoints) of Minor Axis	$(h, k-b), (h, k+b)$	$(h-b, k), (h+b, k)$
Length of Major (Longer) Axis	$2a$ units	$2a$ units
Length of Minor (Shorter) Axis	$2b$ units	$2b$ units
Sum of the Distances from any Point on the Ellipse to each Focus	$2a$ units	$2a$ units
Distance from the Center to either Foci	$c = \sqrt{a^2 - b^2}$ units	$c = \sqrt{a^2 - b^2}$ units

Exercise 2.

Identify the type of conic, put it in standard form, and sketch the graph. Label the appropriate information for this conic:

$$\mathbf{4x^2 + y^2 - 16x + 15 = 0}$$

<u>Solution</u>

Identify the type of conic.
The coefficients of x^2 and y^2 in $Ax^2 + Cy^2 + Dx + Ey + F = 0$ are the key to identifying the type of conic. In this case $A = 4$ and $C = 1$. So $A \cdot C = 4 > 0$ and $A \neq C$. Therefore this is an ellipse. (See the above chart.)

Put it in standard form.
The key to putting any of the conics in standard form is the "*<u>completing the square</u>*" formula.

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right).$$

Applying this formula to

$$4x^2 - 16x = 4x^2 - 16x + 0$$

we get

$$4x^2 - 16x = 4\left(x + \frac{(-16)}{(2 \cdot 4)}\right)^2 + \left(0 - \frac{(-16)^2}{(4 \cdot 4)}\right) = 4(x-2)^2 - 16.$$

So,

$$\begin{aligned} &4x^2 + y^2 - 16x + 15 = 0 \\ &\quad \Longrightarrow 4(x-2)^2 - 16 + y^2 + 15 = 0 \\ &\quad \Longrightarrow 4(x-2)^2 + y^2 = 1 \\ &\quad \Longrightarrow \frac{(x-2)^2}{(1/2)^2} + \frac{(y-0)^2}{1^2} = 1. \end{aligned}$$

We recognize this from the chart above as an ellipse of the type

$$\frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1, \text{ where } a > b > 0 \quad (i.e. \text{ taller than it is wide})$$

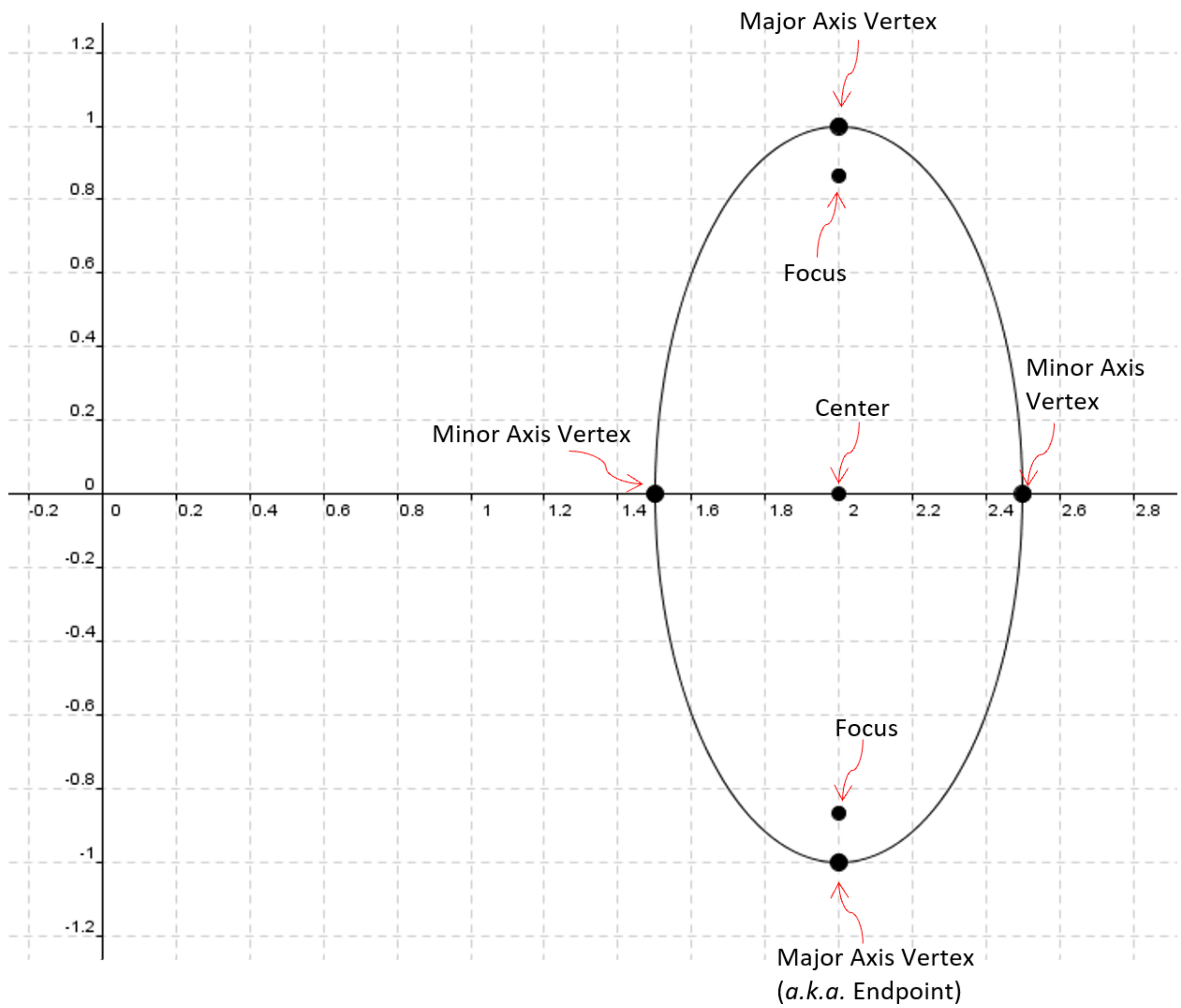
with

$$h = 2, k = 0, a = 1, b = 1/2, c^2 = a^2 - b^2 = 3/4, c = \sqrt{3}/2.$$

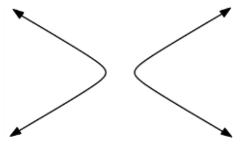
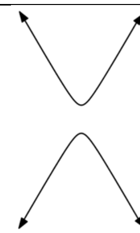
***Sketch the graph. Label the appropriate information*.**

Looking at our chart for an ellipse, we can see that the major axis is vertical (*i.e.* the ellipse is taller than it is wide). The vertices (endpoints) of the major axis are $(2,-1)$ and $(2,1)$. The vertices of the minor axis are $(3/2,0)$ and $(5/2,0)$. The center is at $(2,0)$. The foci are located at $(2,\sqrt{3}/2)$ and $\left(2,-\sqrt{3}/2\right)$. The length of the major axis is 2 and the length of the minor axis is 1.

This gives us enough information to make a reasonable sketch. You can plot a few more points on the ellipse to make your sketch more accurate.



1.3 Hyperbolas

Hyperbola Centered at $(\boldsymbol{h}, \boldsymbol{k})$		
General Shape		
Standard Form of the Equation	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ with $a > 0, b > 0$	$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$ with $a > 0, b > 0$
Center	(h, k)	(h, k)
Vertices (The point on each branch of the hyperbola nearest the center.)	$(h-a, k), (h+a, k)$	$(h, k-a), (h, k+a)$
Direction of Transverse Axis (Line segment connecting the vertices.)	Horizontal	Vertical
Endpoints of Transverse Axis (*i.e.* the vertices of the hyperbola)	$(h-a, k), (h+a, k)$	$(h, k-a), (h, k+a)$
Endpoints of Conjugate Axis (Line segment going through the center and perpendicular to the transverse axis.)	$(h, k-b), (h, k+b)$	$(h-b, k), (h+b, k)$

Four corners of the "fundamental rectangle"	$(-a+h, b+k)$ $(a+h, b+k)$ $(-a+h, -b+k)$ $(a+h, -b+k)$	$(-b+h, a+k)$ $(b+h, a+k)$ $(-b+h, -a+k)$ $(b+h, -a+k)$
Equation of the Asymptotes	$y-k=\pm\frac{b}{a}(x-h)$	$y-k=\pm\frac{a}{b}(x-h)$
Foci	$(h+c, k)$ and $(h-c, k)$ where $c^2=a^2+b^2$	$(h, k+c)$ and $(h, k-c)$ where $c^2=a^2+b^2$
Length of Transverse Axis	$2a$ units	$2a$ units
Length of Conjugate Axis	$2b$ units	$2b$ units

Note: Each oblique asymptote of the hyperbola goes through opposite corners of the "fundamental rectangle". So once you draw this rectangle, it is easy to see where the asymptotes go. Just draw in the two lines going through opposite corners of this rectangle.

Perhaps the easiest way to make a reasonable sketch of a hyperbola is to draw the fundamental rectangle, the asymptotes and the vertices all before drawing in the branches of the hyperbola.

Then draw in the branches with the given vertices that "close in" on the asymptotes already drawn in.

Exercise 3.

Identify the type of conic, put it in standard form, and sketch the graph. Label the appropriate information for this conic:

$$\boldsymbol{x^2-25y^2+25=0}$$

Solution

Identify the type of conic.

The coefficients of x^2 and y^2 in $Ax^2 + Cy^2 + Dx + Ey + F = 0$ are the key to identifying the type of conic. In this case $A = 1$ and $C = -25$. We note that $A \cdot C = -25 < 0$, so this is an hyperbola. (See the above chart.)

Put it in standard form.

$$x^2 - 25y^2 = -25$$

$$\Rightarrow \frac{x^2}{-25} - \frac{25y^2}{-25} = 1$$

$$\Rightarrow \frac{y^2}{1^2} - \frac{x^2}{5^2} = 1$$

$$\Rightarrow \frac{(y-0)^2}{1^2} - \frac{(x-0)^2}{5^2} = 1$$

We recognize this from the chart above as a hyperbola of the type

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1, \text{ where } a > 0, b > 0$$

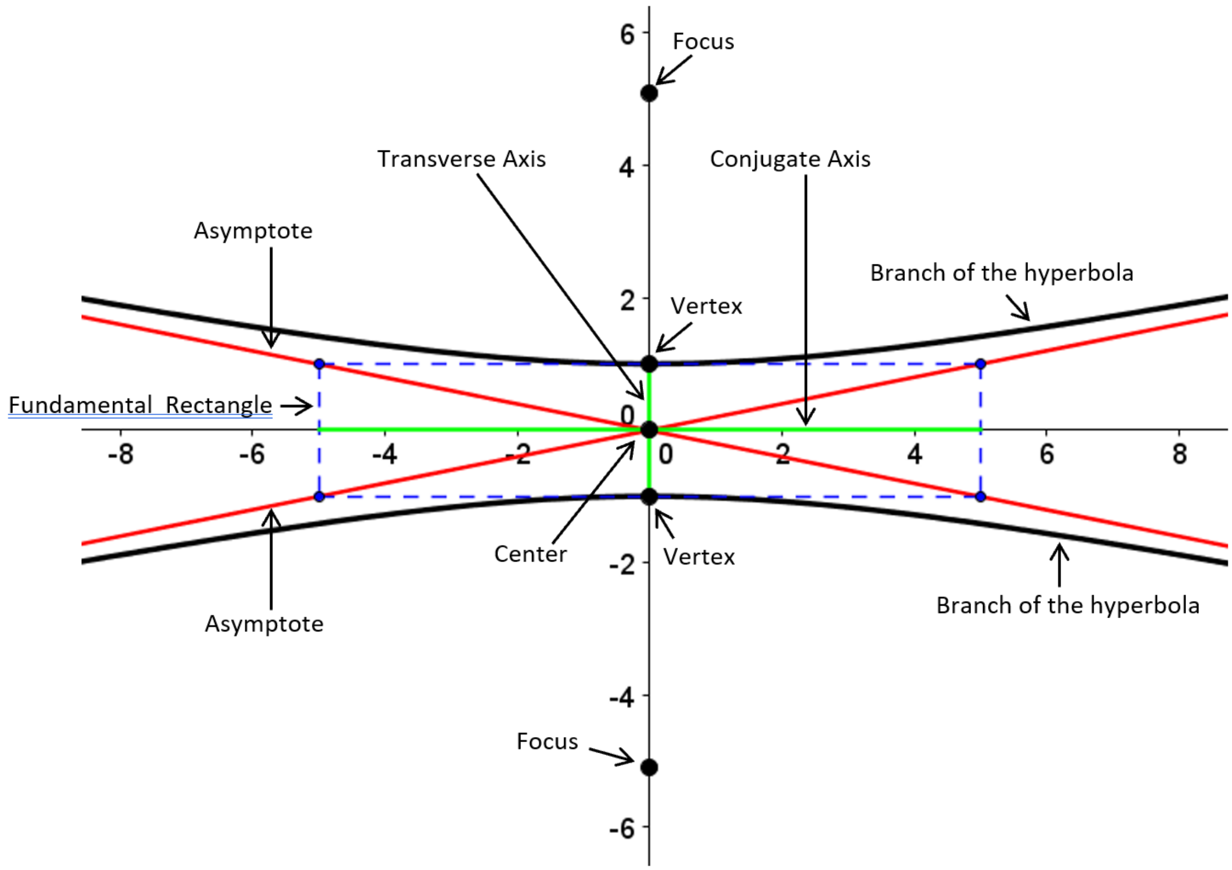
with

$$h = 0, k = 0, a = 1, b = 5, \text{ and } c = \sqrt{a^2 + b^2} = \sqrt{26}.$$

***Sketch the graph. Label the appropriate information*.**

Looking at our chart for a hyperbola, we can see that it is centered at $(0,0)$, *i.e.* the origin. The vertices are $(0,-1)$ and $(0,1)$. The transverse axis is vertical and connects the vertices. The conjugate axis is horizontal, goes through the center at $(0,0)$ and has endpoints $(-5,0)$ and $(5,0)$. The four corners of the fundamental rectangle are $(-5,1), (5,1), (-5,-1), (5,-1)$. The asymptotes are the two lines going through the opposite corners of the fundamental rectangle and have equations $y = x/5$ and $y = -x/5$. The foci are located at $(0,\sqrt{26})$ and $(0,-\sqrt{26})$. Finally, we note that the transverse axis has length 2 and the conjugate axis has length 10.

This gives us enough information to make a reasonable sketch. You can plot a few more points on the parabola to make your sketch more accurate.



Remember that these three examples all belong to the special case when $B = 0$ in the general equation for conics

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

We will come back to situation when $B \neq 0$ but now we move on to the third way of describing conic sections – as the locus of points in a given plane meeting certain conditions. The phrase "*locus* of points" may be new to you. You can substitute "the *collection* of all points". But in the context of conic sections the term locus is more commonly used than "collection" or "set".

1.4 Locus Definitions of Conic Sections

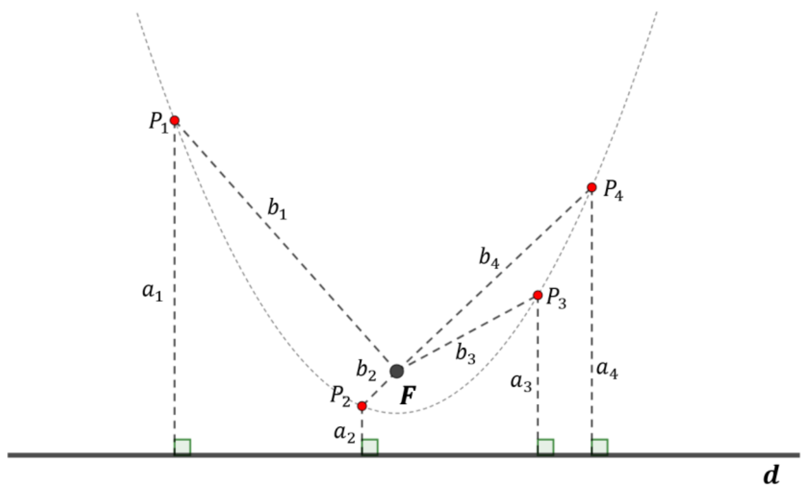
1.4.1 Parabola – Locus Definition

Parabola: the locus (set) of all points in a plane that are equidistant from a fixed line and a fixed point not on that line.

The fixed point is called the **focus** and the fixed line the **directrix**.

Note: whenever we say "the distance between" what is meant is "the ***shortest*** distance between".

Suppose we take F as our fixed point (focus) and d as our fixed line (directrix). For this chosen focus and directrix each of the red points P_1, P_2, P_3 and P_4 would belong to our described locus (points in the plane containing F and d which are equidistant from F and d).



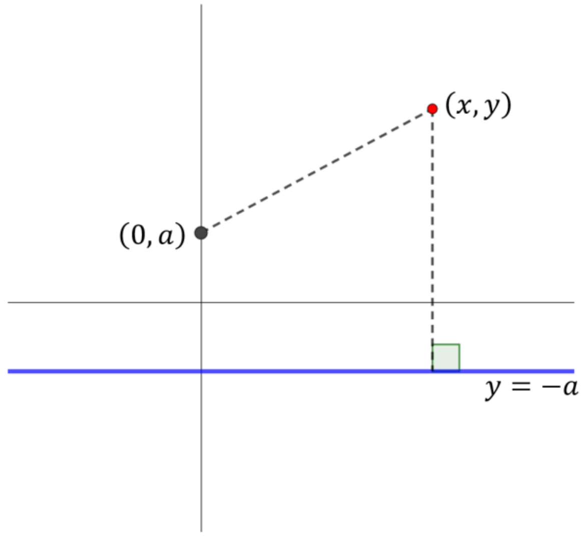
In particular, notice that $a_1 = b_1$ where a_1 is the shortest (*i.e.* perpendicular) distance between point P_1 and our chosen directrix line d and b_1 is the shortest (*i.e.* straight line) distance between P_1 and our chosen focus point F.

Similarly, $a_2 = b_2$, $a_3 = b_3$ and $a_4 = b_4$. You can imagine that if we continued to fill in our "red points" we would generate the lightly colored shape shown in the figure above.

But is this really a parabola? Let's work out the details for the following special case.

Example 4.

Suppose we let F (focus) have coordinates $(0, a)$ and we let d (directrix) be the line $y = -a$ for some $a > 0$.



For the point (x, y) to belong to the locus of all points equidistant from the focus $F(0, a)$ and the directrix $y = -a$ it would be necessary that

$$\sqrt{(x-0)^2 + (y-a)^2} = |y - (-a)|.$$

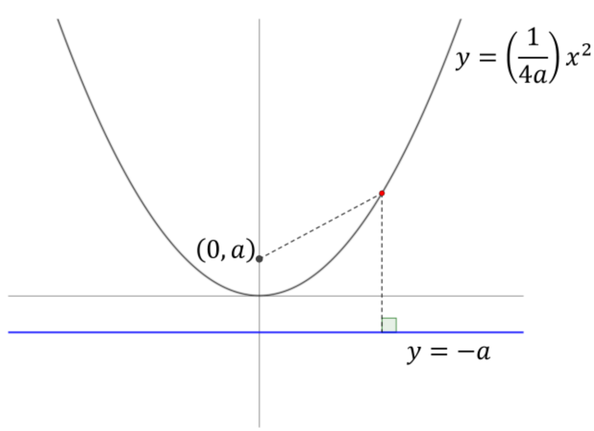
Squaring both sides we get

$$x^2 + (y-a)^2 = (y+a)^2$$
$$x^2 + y^2 - 2ay + a^2 = y^2 + 2ay + a^2$$
$$4ay = x^2$$

or

$$y = \frac{1}{4a}x^2.$$

We can use our charts above to identify this as a parabola with focus $(0, a)$ and directrix $y = -a$ as we required.



What about for a general focus point F and a general directrix line d? Remember that the general formula for the (perpendicular) distance between the line $ax + by + c = 0$ and the point (x_1, y_1) is

$$\text{distance}\big(ax + by + c = 0, (x_1, y_1)\big) = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

and the general formula for the (straight line) distance between the points (x_0, y_0) and (x_1, y_1) is

$$\text{distance}\big((x_0, y_0), (x_1, y_1)\big) = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}.$$

Therefore, if we take our general **focus** point F as $(\boldsymbol{x_0}, \boldsymbol{y_0})$ and our general **directrix** as the line $\boldsymbol{ax + by + c = 0}$ then all points (x, y) such that

$$\boxed{\frac{|ax + by + c|}{\sqrt{a^2 + b^2}} = \sqrt{(x - x_0)^2 + (y - y_0)^2}}$$

is our locus of all points (x, y) such that these two distances are equal.

1.4.2 Ellipse – Locus Definition

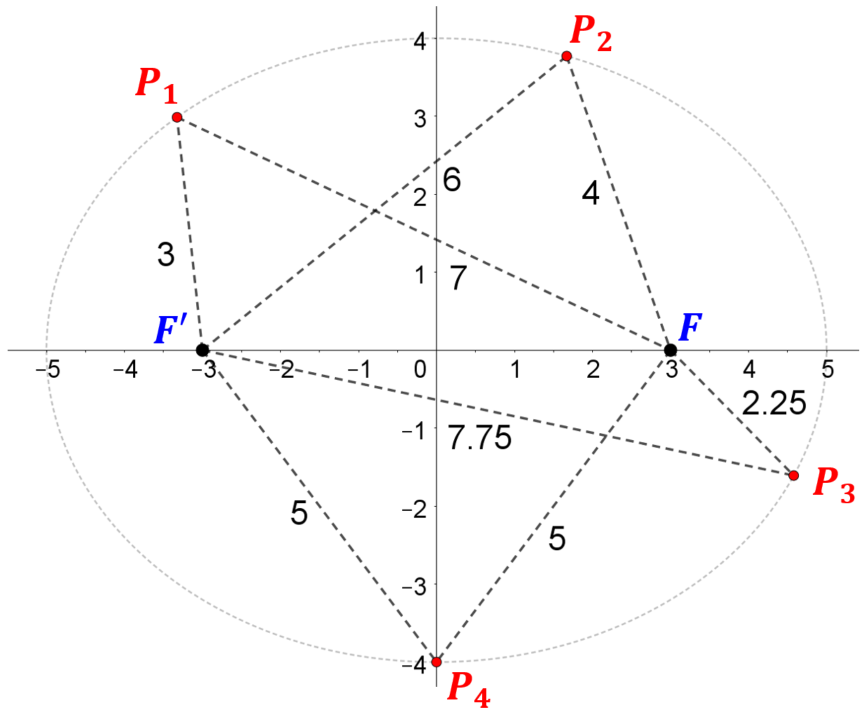
Ellipse: the locus (set) of all points P in a plane such that the <u>sum</u> of the distances of P from two fixed points F and F' equals some fixed constant.

Each of the fixed points is called a **focus**. When together we call them the two **foci** (the plural of focus).

Note: In order not to have an empty set for our locus, our fixed constant must be greater than the distance between foci F and F'.

Example 5.

Suppose we take F' as the point $(-3,0)$ and F as the point $(3,0)$. Furthermore suppose we want the <u>sum</u> of the distances from points in our locus to each foci to equal 10.



We can see in this figure that P_1 belongs to our locus because $\text{dist}(P_1, F') + \text{dist}(P_1, F) = 3 + 7 = 10$. And the point P_2 belongs because $\text{dist}(P_2, F') + \text{dist}(P_2, F) = 6 + 4 = 10$. Similarly for P_3 and P_4

$$\text{dist}(P_3, F') + \text{dist}(P_3, F) = 7.75 + 2.25 = 10$$

$$\text{dist}(P_4, F') + \text{dist}(P_4, F) = 5 + 5 = 10.$$

Can we derive the expected equation for an ellipse from this? We set this up so that a point $P(x, y)$ will belong to our locus provided $\text{dist}(P, F') + \text{dist}(P, F) = 10$.

That is

$$\sqrt{\left(x - (-3)\right)^2 + (y - 0)^2} + \sqrt{(x - 3)^2 + (y - 0)^2} = 10$$

$$\Longrightarrow \sqrt{(x + 3)^2 + y^2} = 10 - \sqrt{(x - 3)^2 + y^2}$$

$$\Longrightarrow \left(\sqrt{(x + 3)^2 + y^2}\right)^2 = \left(10 - \sqrt{(x - 3)^2 + y^2}\right)^2$$

$$\Longrightarrow \quad (x+3)^2 + y^2 = 100 + (x-3)^2 + y^2 - 20\sqrt{(x-3)^2 + y^2}$$

$$\Longrightarrow \quad 3x - 25 = -5\sqrt{(x-3)^2 + y^2} \quad \text{(after cancellations)}$$

$$\Longrightarrow \quad (3x-25)^2 = \left(-5\sqrt{(x-3)^2 + y^2}\right)^2$$

$$\Longrightarrow \quad 16x^2 + 25y^2 = 400 \quad \text{(after cancellations)}$$

$$\Longrightarrow \quad \frac{x^2}{5^2} + \frac{y^2}{4^2} = 1.$$

Looking back at our chart for ellipses we can see that this is the equation for an ellipse with foci at $F' = (-3,0)$ and $F = (3,0)$ and where dist$(P, F') +$ dist$(P, F) = 10$.

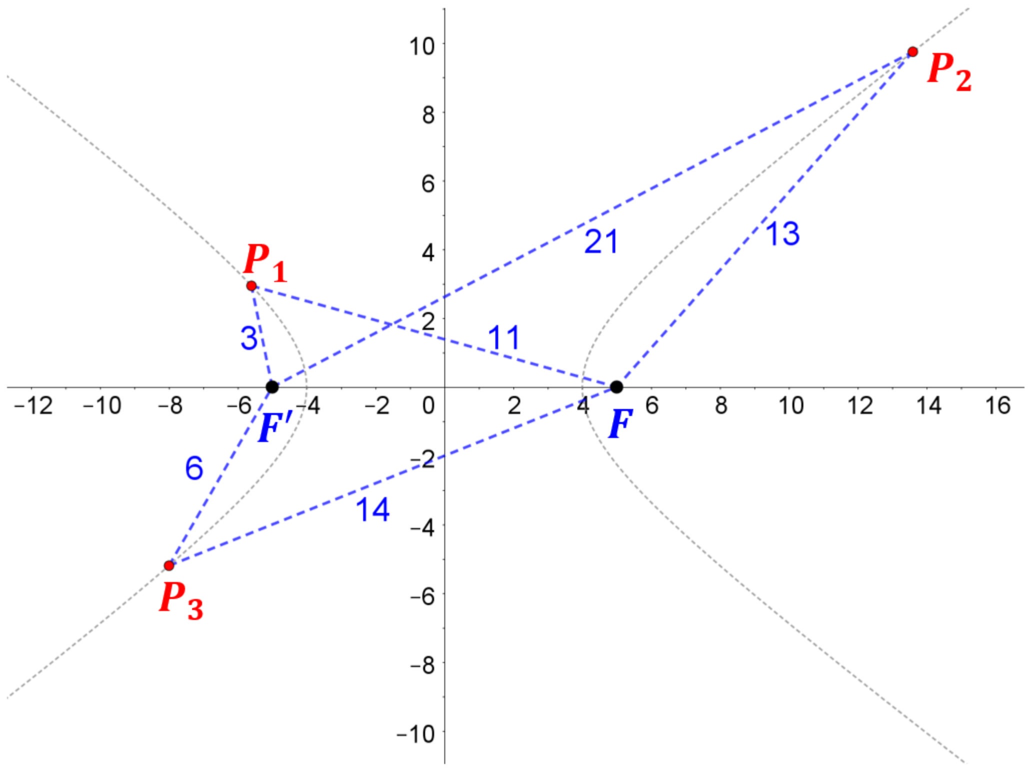
1.4.3 Hyperbola – Locus Definition

Hyperbola: the locus (set) of all points P in a plane such that the difference of the distances of P from two fixed points F and F' (foci) equals some fixed constant.

Note: In order not to have an empty set for our locus, our fixed constant must be less than the distance between foci F and F'.

Example 6.

Suppose we take F' as the point $(-5,0)$ and F as the point $(5,0)$. Furthermore suppose we want the difference of the distances from points in our locus to each foci to equal 8.



We can see in this figure that P_1 belongs to our locus because $|\text{dist}(P_1, F') - \text{dist}(P_1, F)| = |3 - 11| = 8$. And the point P_2 belongs because $|\text{dist}(P_2, F') - \text{dist}(P_2, F)| = |21 - 13| = 8$. In the same way, $|\text{dist}(P_3, F') - \text{dist}(P_3, F)| = |6 - 14| = 8$.

Can we derive the equation for our hyperbola from this? We set this up so that a point $P(x, y)$ will belong to our locus provided $|\text{dist}(P, F') - \text{dist}(P, F)| = 8$.

That is

$$\sqrt{\left(x - (-5)\right)^2 + (y - 0)^2} - \sqrt{(x - 5)^2 + (y - 0)^2} = \pm 8$$

$$\Rightarrow \sqrt{(x + 5)^2 + y^2} = \pm 8 + \sqrt{(x - 5)^2 + y^2}$$

$$\Rightarrow \left(\sqrt{(x + 5)^2 + y^2}\right)^2 = \left(\pm 8 + \sqrt{(x - 5)^2 + y^2}\right)^2$$

$$\Rightarrow (x + 5)^2 + y^2 = 64 + (x - 5)^2 + y^2 \pm 16\sqrt{(x - 5)^2 + y^2}$$

$$\Longrightarrow \quad 5x - 16 = \pm 4\sqrt{(x-5)^2 + y^2}$$

$$\Longrightarrow \quad 9x^2 - 16y^2 = 3^2 \cdot 4^2$$

$$\Longrightarrow \quad \frac{x^2}{4^2} - \frac{y^2}{3^2} = 1$$

Looking back at our chart for hyperbolas we can see that this is the equation for a hyperbola with foci at $F' = (-5,0)$ and $F = (5,0)$ and where $|\text{dist}(P, F') - \text{dist}(P, F)| = 8$.

1.5 Rotated Conics: The Case of $B \neq 0$.

Up to this point our discussion has been limited to conics where the graph of the conic is symmetric to a line parallel or perpendicular to the x-axis. That is, conics with no xy term. More specially, where $B = 0$ in the general equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

Now we will consider the case of $B \neq 0$. That is, *rotated conics*.

First, remember how the value of the discriminant $B^2 - 4AC$ determined whether the conic is a hyperbola, ellipse or parabola.

Ellipse	$B^2 - 4AC < 0$
Parabola	$B^2 - 4AC = 0$
Hyperbola	$B^2 - 4AC > 0$

These rules did not require $B = 0$. They are valid for all possible values of B.

Let's begin our discussion with an example.

Example 7.

Classify $xy = 3$ as an ellipse, parabola or hyperbola and then provide a graph.

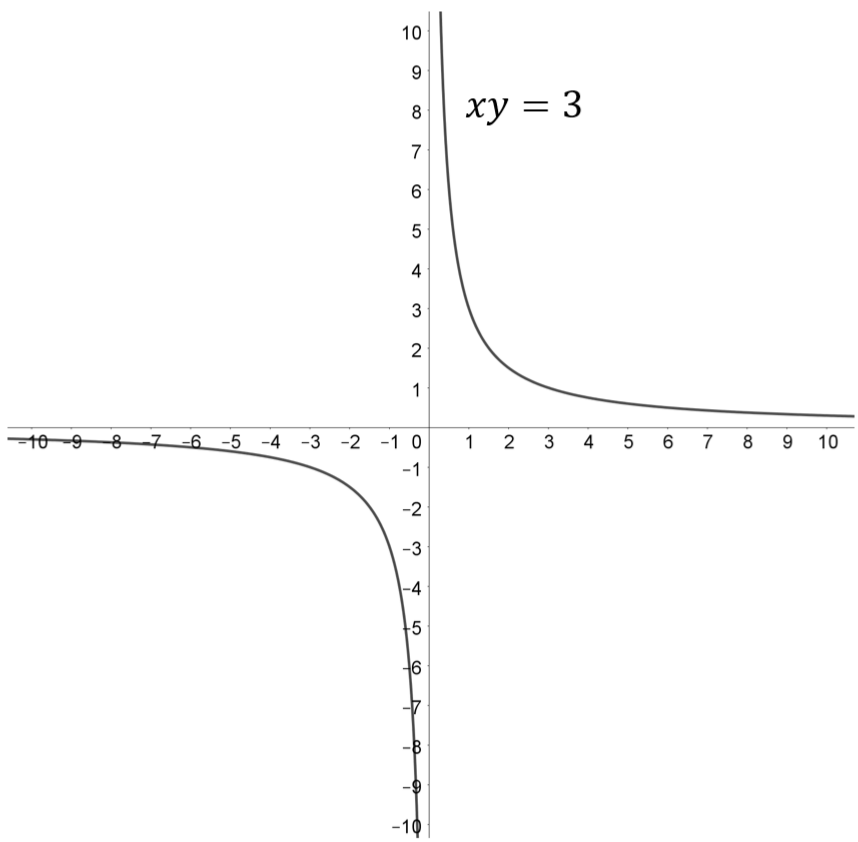
Solution

We can immediately see that $A = C = D = E = 0$, $B = 1$ and $F = -3$. Therefore, the discriminant $B^2 - 4AC = 1 > 0$ and $xy = 3$ must be the equation of a hyperbola.

We need to find all the components (center, vertices, foci, asymptotes, etc.) of this hyperbola.

However, the chart we used in Exercise 3 to determine all the components of the hyperbola $x^2 - 25y^2 + 25 = 0$ is no longer valid. That chart applies exclusively to the case of hyperbolas with $B = 0$.

If we plug the equation $xy = 3$ (or $y = 3/x$) into a graphing calculator we get the familiar picture.



But how do we determine all the components? The necessary approach (no matter which of the three types of conics you have) is basically this.

Step 1. Use the discriminant $B - 4AC$ to determine what type of conic you have.

Step 2. (Somehow?) rotate the graph through an acute angle until it is symmetric to a line parallel and/or perpendicular to the x-axis.

Step 3. Use our above charts for $B = 0$ (for whichever type of conic you have) to find the components in this new form.

Step 4. (Somehow?) use these answers to determine the components in the original form.

Just eyeballing the above graph of $xy = 3$ it appears that we would need to rotate the graph (clockwise) by $\theta = 45°$ in order to make the graph symmetric with the y-axis. But is this exactly right?

Determining θ, the exact angle of rotation needed.

The necessary angle of rotation θ is the acute angle θ that satisfies

$$\cot(2\theta) = \frac{A - C}{B}.$$

- If $\frac{A-C}{B} > 0$, then 2θ is in the first quadrant, and θ is between $(0°, 45°)$.
- If $\frac{A-C}{B} < 0$, then 2θ is in the second quadrant, and θ is between $(45°, 90°)$.
- If $\frac{A-C}{B} = 0$ (*i.e.* $A = C$), then $\theta = 45°$.

Determining the equation for the rotated conic.

The equation for the conic after rotating by the acute angle θ is determined by

- Replace all x's (in the original equation) with $x\cos(\theta) - y\sin(\theta)$.
- Replace all y's (in the original equation) with $x\sin(\theta) + y\cos(\theta)$.

This new conic will no longer have an xy term (*i.e.* $B = 0$) and will be symmetric to a line parallel and/or perpendicular to the x-axis.

Example 8.

Find the acute angle of rotation needed so that the hyperbola $xy = 3$ will be symmetric to a line parallel and/or perpendicular to the x-axis. Determine the equation of this rotated hyperbola, find the foci of this hyperbola and provide a graph.

Solution

In our example $A = C = 0$, so $\theta = 45°$. Also, $\cos(45°) = \sqrt{2}/2$ and $\sin(45°) = \sqrt{2}/2$. So we need to replace all x's in $xy = 3$ with $\left(\sqrt{2}/2\right)(x - y)$ and we need to replace all y's in $xy = 3$ with $\left(\sqrt{2}/2\right)(x + y)$.

Our new equation becomes

$$\left(\frac{\sqrt{2}}{2}\right)(x - y) \cdot \left(\frac{\sqrt{2}}{2}\right)(x + y) = 3$$

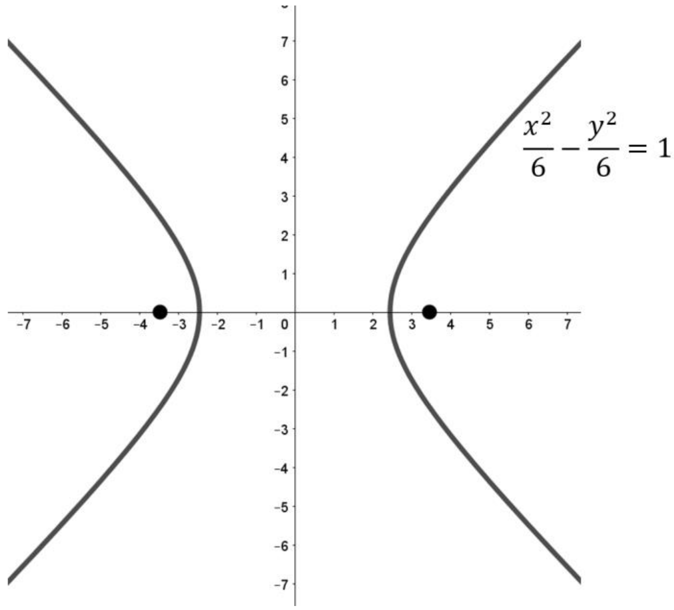
$$\Rightarrow \quad \left(\frac{2}{4}\right)(x^2 - xy + xy - y^2) = 3$$

$$\Rightarrow \quad \frac{x^2}{6} - \frac{y^2}{6} = 1.$$

Now that we have an equation that no longer has an xy term (*i.e.* $B = 0$) we can use our chart for hyperbolas to determine the location of the foci. Comparing our equation against the standard form for hyperbolas we can see that $h = k = 0$ and $a^2 = b^2 = 6$.

Therefore, $c^2 = a^2 + b^2 = 12$ and $c = \sqrt{12} = 2\sqrt{3}$.

From our chart, the foci are located at $(h - c, k) = \left(-2\sqrt{3}, 0\right)$ and $(h + c, k) = \left(2\sqrt{3}, 0\right)$.



Determining the coordinates of a point (*e.g.* foci or vertex) in the original conic.

The point (x, y) in the conic after rotation by an acute angle of θ corresponds to the point

$$(x\cos(\theta) - y\sin(\theta)\,, x\sin(\theta) + y\cos(\theta)\,)$$

in the conic before rotation.

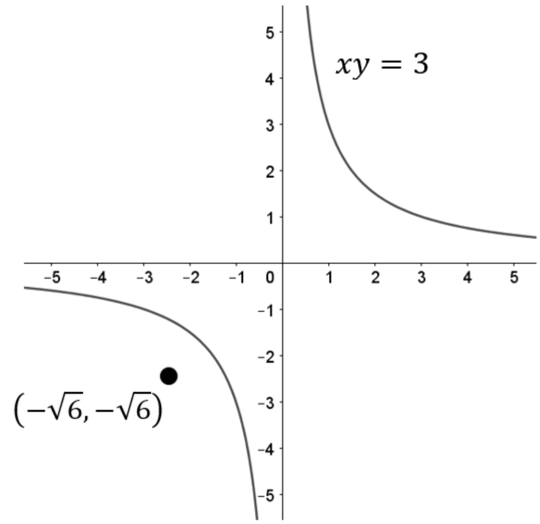
Example 9.

Determine exactly the focal point of $xy = 3$ that lies in the third quadrant. (2016-17, Meet 4, Team Event, Problem 6).

<u>Solution</u>

We can see from our work in Example 8 that what the problem is asking for the coordinates of the focus of the hyperbola $xy = 3$ which corresponds to the focus with coordinates $\left(-2\sqrt{3}, 0\right)$ of the hyperbola $x^2 - y^2 = 6$ which resulted from a rotation of $\theta = 45°$ of the the original hyperbola $xy = 3$.

$$\left(-2\sqrt{3}, 0\right) \longrightarrow \left(\left(-2\sqrt{3}\right)\cos(45°) - 0\sin(45°)\,, \left(-2\sqrt{3}\right)\sin(45°) + 0\cos(45°)\,\right)$$
$$= \left(\left(-2\sqrt{3}\right)\left(\frac{\sqrt{2}}{2}\right), \left(-2\sqrt{3}\right)\left(\frac{\sqrt{2}}{2}\right)\right) = \left(-\sqrt{6}, -\sqrt{6}\right).$$



Example 10.

Find the length of the line segment from the endpoint in the second quadrant of the minor axis to the foci in the first quadrant of the ellipse

$$41x^2 - 24xy + 34y^2 - 25 = 0.$$

Solution

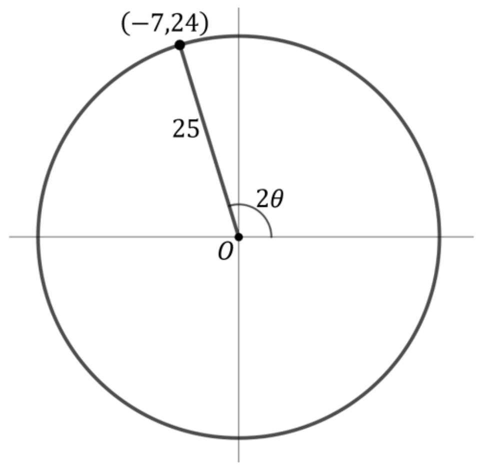
First, let's answer this question for the ellipse rotated to remove the xy term. A quick check of the discriminant verifies this really is an ellipse.

$$B^2 - 4AC = (-24)^2 - 4(41)(34) = -5000 < 0 \Longrightarrow \text{Ellipse}$$

The acute angle of rotation is that angle θ such that

$$\cot(2\theta) = \frac{A - C}{B} = \frac{41 - 34}{-24} = -\frac{7}{24}.$$

Because $0 < 2\theta < 180°$ we know x is negative and y is positive which gives us the following circle diagram.



It follows that $\cos(2\theta) = -7/25$. But for all acute angles θ

$$\cos(\theta) = \sqrt{\frac{1+\cos(2\theta)}{2}} \quad \text{and} \quad \sin(\theta) = \sqrt{\frac{1-\cos(2\theta)}{2}}.$$

Therefore,

$$\cos(\theta) = \sqrt{\frac{1+\cos(\theta)}{2}} = \sqrt{\frac{1+(-7/25)}{2}} = \sqrt{\frac{9}{25}} = \frac{3}{5}$$

and

$$\sin(\theta) = \sqrt{\frac{1-\cos(\theta)}{2}} = \sqrt{\frac{1-(-7/25)}{2}} = \sqrt{\frac{16}{25}} = \frac{4}{5}.$$

To remove the xy term in

$$41x^2 - 24xy + 34y^2 - 25 = 0$$

we need to make the substitutions

$$x \longrightarrow x\cos(\theta) - y\sin(\theta) = x\left(\frac{3}{5}\right) - y\left(\frac{4}{5}\right)$$

and

$$y \longrightarrow x\left(\frac{4}{5}\right) + y\left(\frac{3}{5}\right)$$

in

$$41x^2 - 24xy + 34y^2 - 25 = 0.$$

This gives us

$$41\left(x\left(\frac{3}{5}\right)-y\left(\frac{4}{5}\right)\right)^2-24\left(x\left(\frac{3}{5}\right)-y\left(\frac{4}{5}\right)\right)\left(x\left(\frac{4}{5}\right)+y\left(\frac{3}{5}\right)\right)$$

$$+34\left(x\left(\frac{4}{5}\right)+y\left(\frac{3}{5}\right)\right)^2-25=0.$$

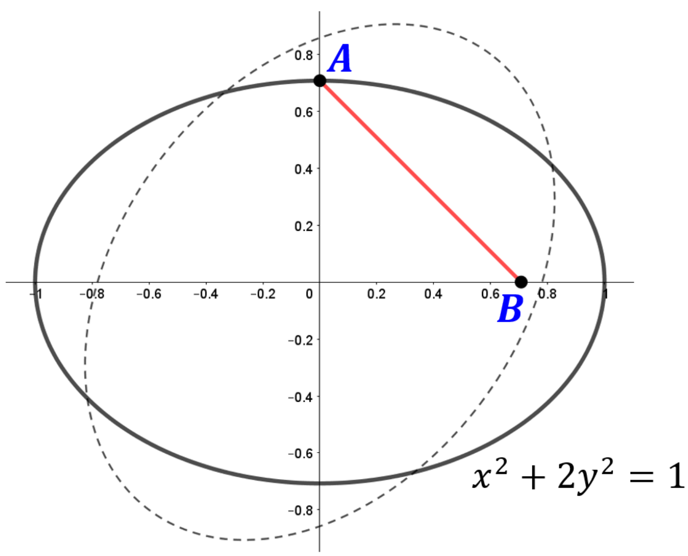
We can ignore all xy terms in this because we know that in the end they will drop out. There are no x terms or y terms so we just need to collect our x^2 and y^2 terms. Simplifying, we have

$$\left(\frac{41(9)-24(12)+34(16)}{25}\right)x^2+\left(\frac{41(16)+24(12)+34(9)}{25}\right)y^2-25=0$$

$$25x^2+50y^2-25=0$$

$$x^2+2y^2=1 \text{ or } \frac{x^2}{1}+\frac{y^2}{0.5}=1.$$

As $1>0.5$ we know that the minor axis is vertical. To answer the given question, but applied to this new ellipse $x^2+2y^2=1$ we need to find the length of the red line segment $\overline{AB}$ in the diagram below.



From our chart for ellipses we know that for an ellipse in standard form

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

with $a>b>0$ the point A has coordinates $(h,k+b)$ and point B has coordinates $(h+c,k)$ where $c^2=a^2-b^2$. There $A=\left(0,0+\sqrt{1/2}\right)=\left(0,\sqrt{1/2}\right)$ and $B=\left(0+\sqrt{1-(1/2)},0\right)=\left(\sqrt{1/2},0\right)$. The distance between these two points equals

$$\text{dist}(A,B) = \sqrt{\left(\sqrt{1/2} - 0\right)^2 + \left(0 - \sqrt{1/2}\right)^2} = 1.$$

Now let's rotate points A and B to find their coordinates back in the original ellipse.

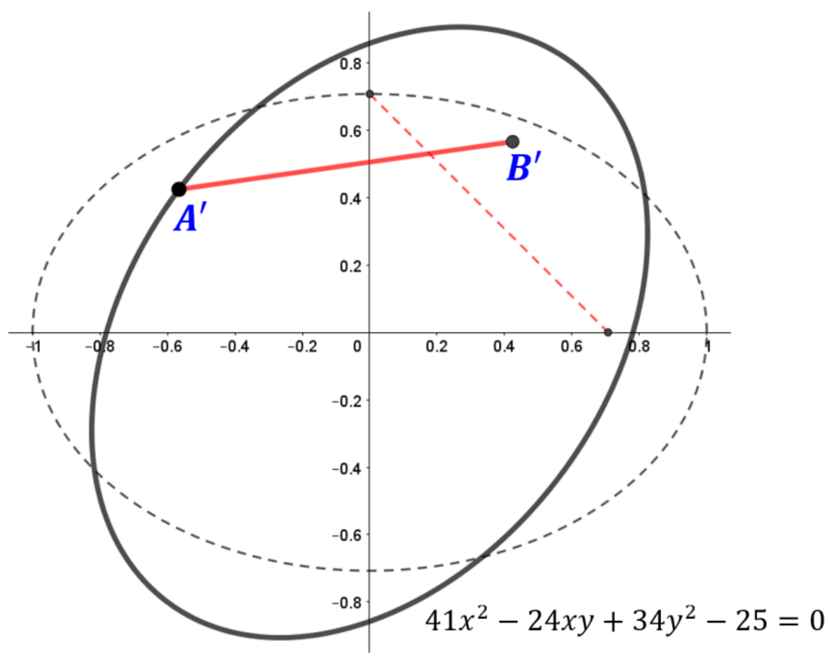
$$A' = \left(0 \cdot \cos(\theta) - \sqrt{1/2} \cdot \sin(\theta), 0 \cdot \sin(\theta) + \sqrt{1/2} \cdot \cos(\theta)\right)$$

$$= \left(-\sqrt{1/2} \cdot \left(\frac{4}{5}\right), \sqrt{1/2} \cdot \left(\frac{3}{5}\right)\right)$$

and

$$B' = \left(\sqrt{1/2} \cdot \cos(\theta) - 0 \cdot \sin(\theta), \sqrt{1/2} \cdot \sin(\theta) + 0 \cdot \cos(\theta)\right)$$

$$= \left(\sqrt{1/2} \cdot \left(\frac{3}{5}\right), \sqrt{1/2} \cdot \left(\frac{4}{5}\right)\right).$$



Now we are ready to answer the original question, namely find the length of $\overline{A'B'}$.

$$\text{dist}(A',B') = \sqrt{\left(\left(\sqrt{1/2} \cdot \left(\frac{3}{5}\right)\right) - \left(-\sqrt{1/2} \cdot \left(\frac{4}{5}\right)\right)\right)^2 + \left(\left(\sqrt{1/2} \cdot \left(\frac{4}{5}\right)\right) - \left(\sqrt{1/2} \cdot \left(\frac{3}{5}\right)\right)\right)^2}$$

$$= \sqrt{\left(\sqrt{1/2} \cdot \left(\frac{7}{5}\right)\right)^2 + \left(\sqrt{1/2} \cdot \left(\frac{1}{5}\right)\right)^2} = \sqrt{\frac{49}{50} + \frac{1}{50}} = 1.$$

Interesting!

$$\text{dist}(A, B) = \text{dist}(A', B').$$

Is this just some weird coincidence? No! This is an example of what is called an ***invariant***. The distance between any two points does not change when you rotate both points about the origin.

The value of knowing this is that we could have stopped once we found dist(A, B) which because of invariance must equal dist(A', B'). Taking advantage of invariance in a situation such as this can save you a lot of time!