MSHSML Meet 5, Event A
Study Guide

5A Puzzle Problems (20 minutes)

Word problems, one or more variables

Max-min problems not requiring calculus

Problems found in "brain-teaser" type books

Logic puzzles, including the use of Venn Diagrams

Contents

1. Logic puzzles, including the use of Venn Diagrams

Problem 1.

At Jefferson Summer Camp, 60% of the children play soccer, 30% of the children swim, and 40% of the soccer players swim. To the nearest whole percent, what percent of the non-swimmers play soccer? (Source: AMC 10A, 2009, Problem 18)

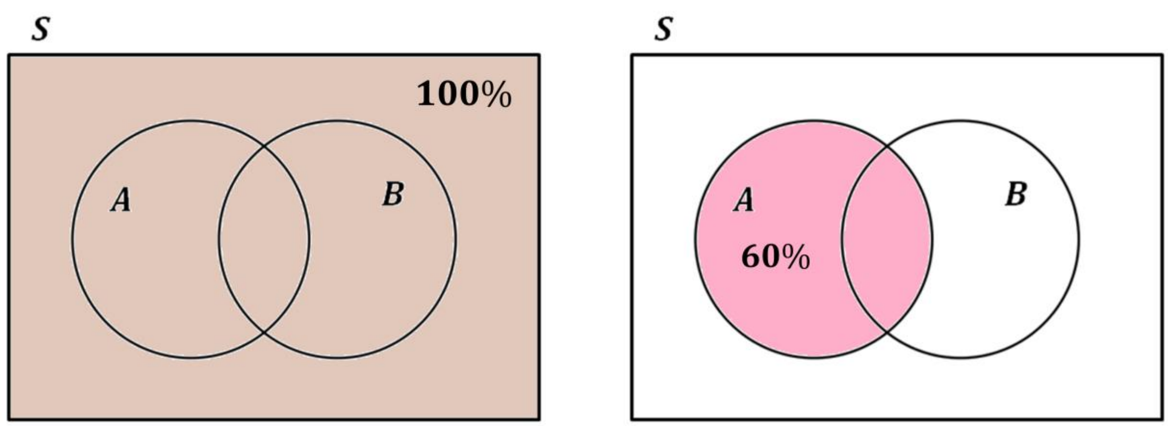
Solution

Let's introduce the symbols

S: all children A: children who play soccer B: children who swim

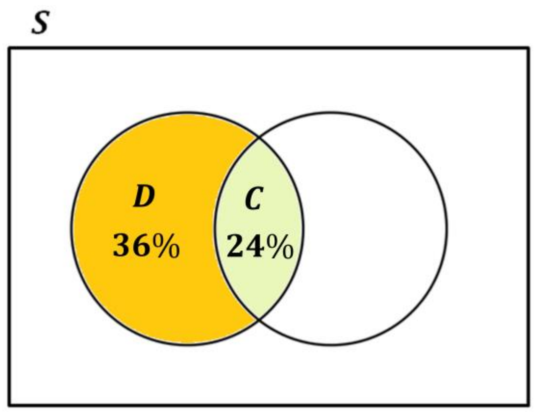
C: children who play soccer and swim D: children who play soccer but do not swim.

The key to this problem is pay close attention to the phrase following the word "of". Consider the difference in 60% of *all children* play soccer versus 40% of *all soccer players* swim. In the Venn diagrams below this translates to 60% of the brown region versus 40% of the light red region.

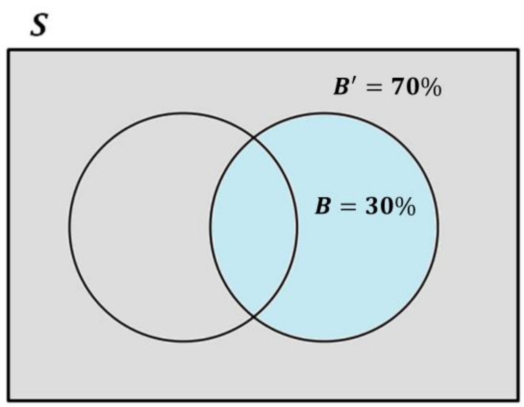


We can express C: children who play soccer and swim in the notation $C = A \cap B$.

What percentage of *all children* are $C = A \cap B$? We are given that it is 40% of the light red. That is C is 60% of 40% or 24%. This immediately tells us that the percentage of *all* children that play soccer but do not swim is $36\% = 60\% - 24\%$.



Remember that we are also told the 30% of all children are swimmers and hence B': children who do not swim makes up 70% of all children.



The original question was, to the nearest whole percent, what percent of the non-swimmers play soccer? This is asking for what percent of B': non-swimmers (the above grey region) is D: non-swimmers who play soccer (the above gold region). That is, find x such that $x\%$ of $70\% = 36\%$.

$$x = \frac{36}{70} \approx 0.5142.$$

So, to the nearest whole percent, 51% of all non-swimmers play soccer.

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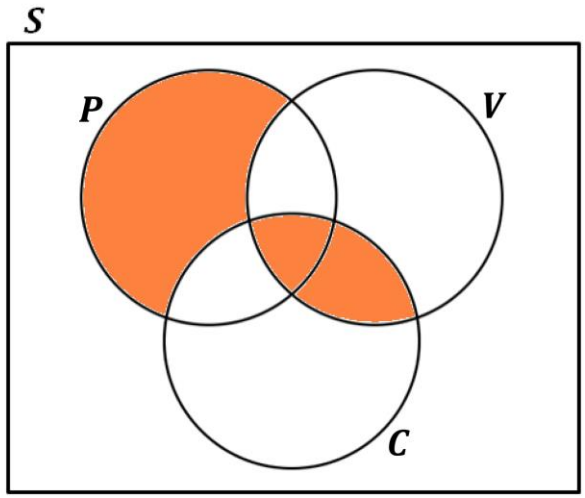
In the Venn diagram of Problems 2-4, if a region is shaded, then it is empty. If a region is not shaded then it contains at least one element. (Source: http://www.math.wsu.edu/mathlessons/html/venndiagrams/Logic/Examples/welcome.html)

Problem 2.

Consider the information contained in the following Venn diagram about P: People, V: Vampires and C: Cheese-eaters all contained in a universal set S.



Based on this information determine if the following conclusions are True, False or Impossible to tell.

(a) All people are vampires or eat cheese.
(b) All people are also vampires.
(c) Some cheese-eaters are not people.
(d) Some people are cheese-eating vampires.
(e) No vampire eats cheese.

<u>Solution</u>

(a) True (b) False (c) True (d) False (e) True

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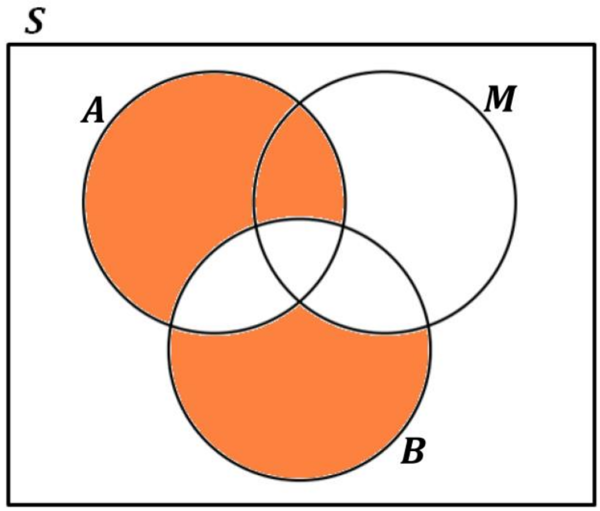
Problem 3.

Consider the information contained in the following Venn diagram about A: Absent-minded people, M: Mathematicians and B: Bungee jumpers all contained in a universal set S.



Based on this information determine if the following conclusions are True, False or Impossible to tell.

(a) All absent-minded mathematicians are bungee jumpers.
(b) Some bungee jumpers are not absent-minded.
(c) Some absent-minded people do not bungee jump.
(d) No bungee jumpers are mathematicians.
(e) All absent-minded bungee jumpers are mathematicians.

Solution

(a) True (b) True (c) False (d) False (e) False

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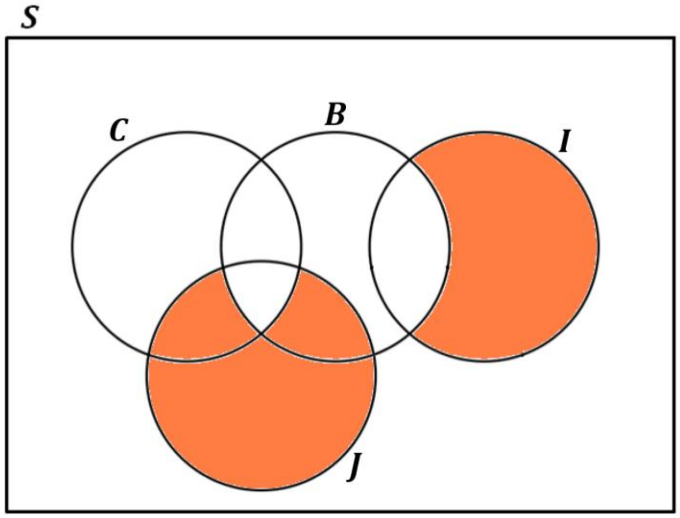
Problem 4.

Consider the information contained in the following Venn diagram about C: Contestants on "Survivors", B: Bug-eaters, I: Immunity winners and J: joe all contained in a universal set S.



Based on this information determine if the following conclusions are True, False or Impossible to tell.

(a) Some immunity winners do not eat bugs.
(b) All bug-eating "Survivor" contestants win immunity.
(c) Some bug-eaters are not "Survivor" contestants.
(d) Joe is a survivor contestant.
(e) Joe eats bugs.

<u>Solution</u>

(a) False (b) False (c) True (d) False (e) True

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Problem 5.

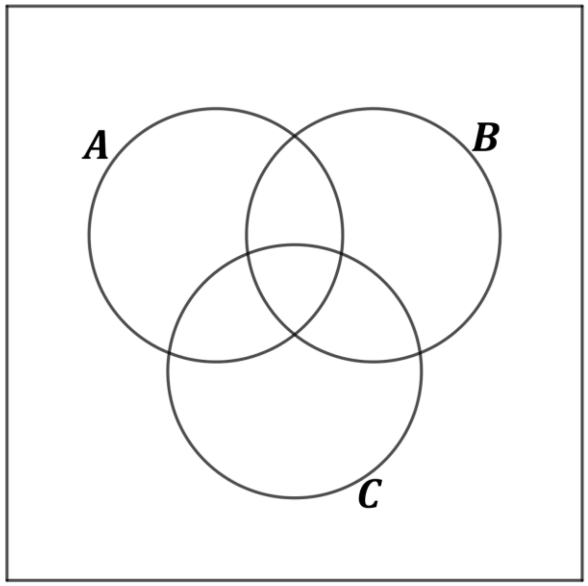
Fill in the Venn Diagram that would represent this data.

100 people seated at different tables in a Mexican restaurant were asked if their party had ordered any of the following items: margaritas, chili con queso, or quesadillas.

23 people had ordered none of these items
11 people had ordered all three of these items
29 people had ordered chili con queso or quesadillas but did not order margaritas
41 people had ordered quesadillas
46 people had ordered at least two of these items
13 people had ordered margaritas and quesadillas but had not ordered chili con queso
26 people had ordered margaritas and chili con queso

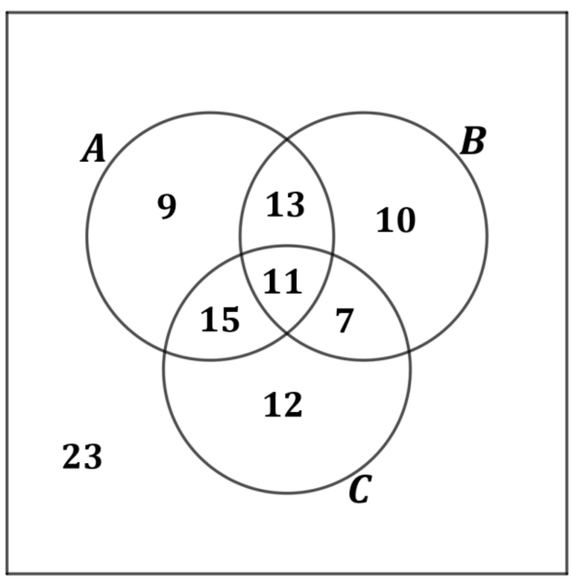
Define events A, B and C as

A: ordered margaritas
B: ordered quesadillas
C: ordered chili con queso



Source: http://www.math.tamu.edu/~kahlig/venn/toons/toons.html

Solution



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2. Problems found in "brain-teaser" type books

Problem 6.

Solve the following cryptarithm. The rules for cryptarithms are:

- Each letter or symbol represents only one digit throughout the problem;
- When letters are replaced by their digits, the resultant arithmetical operation must be correct;
- The numerical base, unless specifically stated, is 10;
- Numbers must not begin with a zero;
- There must be only one solution to the problem.

		D	O	U	B	L	E
		D	O	U	B	L	E
+				T	O	I	L
	T	R	O	U	B	L	E

Solution

D=7, O=9, U=8, B=0, L=6, E=4, T=1, I=3, R=5

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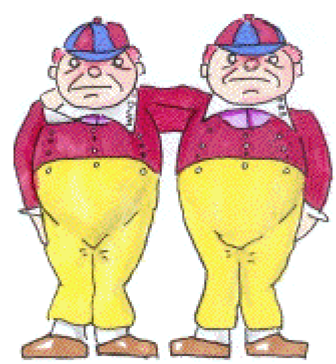
Problems 7-8.

In Problems 7-8, Tweedledum and Tweedledee look alike, but Tweedledum lies on Monday, Tuesday, and Wednesday, whereas Tweedledee lies on Thursday, Friday, and Saturday. They both tell the truth on Sunday. You come upon the two of them, and they make the following statements. In each case, determine who is who, and what day it is. Source: http://www2.stetson.edu/~efriedma/puzzle/logic/



(7) A: I am Tweedledum.
B: I am Tweedledee.

A is ____________________ B is ____________________ It is ____________________

Solution

A and B claim to be different people, so they are either both lying or both telling the truth. Therefore it is Sunday, A is Tweedledum, and B is Tweedledee.

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(8) A: I will lie tomorrow.
B: I lied yesterday, and I will lie tomorrow.

A is ____________________ B is ____________________ It is ____________________

Solution

If it were Sunday, A would not make the statement he did. Therefore it is not Sunday, and A is telling the truth. One of them is lying since it isn't Sunday, so B is lying, and it is Monday. That means A is Tweedledee and B is Tweedledum.

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Problems 9.

There are three boxes. One is made of gold, one of silver and one of lead. There is a prize in one of the boxes. Each box has an inscription but only one of the inscriptions is true.

Gold Box Inscription: The prize is in this box.
Silver Box Inscription: The prize is not in this box.
Lead Box Inscription: The prize is not in the gold box.

Which box contains the prize? Source: *What is the Name of this Book*? By Raymond Smullyan

Solution

The inscriptions on the gold and lead boxes contract each other so one of them is true. But only one of the inscriptions is true, therefore the inscription on the silver box *must* be false. So the prize must be in the silver box.

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3. Word problems, one or more variables

Problem 10.

Find A, B, C, D and E if each letter takes on a different value in $\{1,2,3,4,5\}$ and if

(i) $D - 2 = C$
(ii) $E + C = D$
(iii) $A + E = C$.

Solution

$D - 2 = C \Longrightarrow D \geq 3$ (otherwise $C < 1$). So cross out $D = 1$ and $D = 2$ in the grid below because these are impossible values for D. Similarly, we can rule out $C = 4$ and $C = 5$ (otherwise $D > 5$).

	A	B	C	D	E
1				×	
2				×	
3					
4			×		
5			×		

Then $A + E = C \Longrightarrow C \geq 3$ because A and E are distinct integers each ≥ 1. So $C \neq 1$ and $C \neq 2$. Therefore, $C = 3$ because that is the only possible value for C left.

	A	B	C	D	E
1			✗	✗	
2			✗	✗	
3			✓		
4			✗		
5			✗		

Because only one letter can equal 3 we can rule out $A = 3, B = 3, D = 3$ and $E = 3$.

	A	B	C	D	E
1			✗	✗	
2			✗	✗	
3	✗	✗	✓	✗	✗
4			✗		
5			✗		

But $D - 2 = C \Longrightarrow D - 2 = 3 \Longrightarrow D = 5$. But then $E + C = D \Longrightarrow E + 3 = 5 \Longrightarrow E = 2$. So we can cross out $D = 4, E = 1, E = 4$, and $E = 5$ because each letter takes on a single value. But we can also cross out $A = 2, B = 2, A = 5$ and $B = 5$ because no two letters can take on the same value.

	A	B	C	D	E
1			✗	✗	✗
2	✗	✗	✗	✗	✓
3	✗	✗	✓	✗	✗
4			✗	✗	✗
5	✗	✗	✗	✓	✗

Now we can solve for A because we can see that

$$A + E = C \Longrightarrow A + 2 = 3 \Longrightarrow A = 1.$$

But that forces $B = 4$ because it is the only possible value left for B. So our final answer is

$$A = 1, B = 4, C = 3, D = 5, E = 2.$$

Note: While this problem is small enough that it really is not necessary to resort to filling in the grid as a aid in keeping track of what values we can rule out at a given point in the process. But I've done other problems just like this one that without filling in the grid as I go along it was very easy to get lost.

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Problem 11.

Five women including a geologist and one man named Peter, were recently invited as experts to an international conference held at the United Nations on the state of the environment. Based on the following information can you determine who is engaged in each profession?

1. Karen debated Lori and the meteorologist at the beginning of the conference.
2. Peter is not the physicist.
3. Mary is not the urban planner.
4. Joan is neither the meteorologist nor the biologist.
5. At the end of the conference, the six experts had a general discussion around a table. The debaters were: the physicist, Karen, Joan, the zoologist, the female urban planner, and Paula.

Solution

I think you will find it goes faster with less chance of making a mistake if you fill out a grid to keep track of what is and is not possible after reading each piece of information.

The first sentence, "five women including a geologist", and the information that Peter is the only man, allows us to rule out Peter as the geologist.

	Geologist	Meteorologist	Physicist	Urban Planner	Biologist	Zoologist
Karen						
Lori						
Mary						
Joan						
Paula						
Peter	×					

Karen debated Lori and the meteorologist, therefore Karen and Lori are not the meteorologist. Statements 2,3 and 4 tell us that Peter is not the physicist, Mary is not the urban planner, Joan is neither the meteorologist nor the biologist. Updating our grid we have

	Geologist	Meteorologist	Physicist	Urban Planner	Biologist	Zoologist
Karen		X				
Lori		X				
Mary				X		
Joan		X			X	
Paula						
Peter	X		X			

Statement 5 tells us that Karen, Joan and Paula are not the physicist, zoologist or urban planner. Statement 5 also tells us that the urban planner is female and hence Peter is not the urban planner. Updating our grid again we have

	Geologist	Meteorologist	Physicist	Urban Planner	Biologist	Zoologist
Karen		X	X	X		X
Lori		X				
Mary				X		
Joan		X	X	X	X	X
Paula			X	X		X
Peter	X		X	X		

At this point we can see that Lori must be the urban planner (because we have ruled out everybody else for that job and we know *somebody* has that job) and Joan must be the geologist. Therefore, we can rule out Lori as the geologist, physicist and biologist because Lori can only have one job and we can rule out Karen, Lori, Mary and Paula as the geologist because there is only one geologist.

	Geologist	Meteorologist	Physicist	Urban Planner	Biologist	Zoologist
Karen	✗	✗	✗	✗		✗
Lori	✗	✗	✗	✓	✗	✗
Mary	✗			✗		
Joan	✓	✗	✗	✗	✗	✗
Paula	✗		✗	✗		✗
Peter	✗		✗	✗		

At this point we can see that Mary must be the physicist and Karen must be the biologist.

	Geologist	Meteorologist	Physicist	Urban Planner	Biologist	Zoologist
Karen	✗	✗	✗	✗	✓	✗
Lori	✗	✗	✗	✓	✗	✗
Mary	✗	✗	✓	✗	✗	✗
Joan	✓	✗	✗	✗	✗	✗
Paula	✗		✗	✗	✗	✗
Peter	✗		✗	✗	✗	

And finally this means that Peter is the zoologist and Paula is the meteorologist.

	Geologist	Meteorologist	Physicist	Urban Planner	Biologist	Zoologist
Karen	✗	✗	✗	✗	✓	✗
Lori	✗	✗	✗	✓	✗	✗
Mary	✗	✗	✓	✗	✗	✗
Joan	✓	✗	✗	✗	✗	✗
Paula	✗	✓	✗	✗	✗	✗
Peter	✗	✗	✗	✗	✗	✓

Karen (geologist), Lori (urban planner), Mary (physicist), Joan (geologist), Paula (meteorologist) and Peter (zoologist).

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4. Max-min problems not requiring calculus

(Author's Note: As far as I can tell there has never been a "max-min problem without calculus" question on any past MSHSML test. Nonetheless, I have included this section because the topic is specifically listed in the official MSHSML Meet 5 Test A syllabus so it is possible this type of question might show up in the future. Additionally, "max-min problems without calculus" *have* shown up on AMC 10/12 exams and so this material will hopefully be useful to those students planning on taking the AMC 10/12 exam.)

4.1 Discriminant Approach

Problem 12.

Find the maximum value of $3x - y$ if (x, y) are real number pairs on the ellipse $\frac{x^2}{4} + y^2 = 1$. For what (x, y) pair(s) does $3x - y$ achieve its maximum value?

Solution

Let $z = 3x - y$ so that $y = 3x - z$. The pair (x, z) is real for all real (x, y).

Make the substitution $y = 3x - z$ in the equation for the ellipse to get $\frac{x^2}{4} + (3x - z)^2 = 1$.

We can rewrite this equation as

$$x^2 + 4(3x - z)^2 = 4$$

$$\Leftrightarrow x^2 + 4(9x^2 - 6xz + z^2) = 4$$

$$\Leftrightarrow x^2 + 36x^2 - 24xz + 4z^2 - 4 = 0$$

$$\Leftrightarrow (37)x^2 + (-24z)x + (4z^2 - 4) = 0.$$

In this form we can see that x is a root of a quadratic equation for each fixed real value of z.

But the roots of a quadratic equation are real numbers if and only the discriminant of that quadratic is greater than or equal to 0.

That is, $\Delta = (-24z)^2 - 4(37)(4z^2 - 4) = 592 - 16z^2 \geq 0$. Simplifying we have

$$592 - 16z^2 \geq 0$$

$$\Leftrightarrow 37 - z^2 \geq 0$$

$$\Leftrightarrow z^2 \leq 37$$

$$\Leftrightarrow -\sqrt{37} \leq z \leq \sqrt{37}.$$

So, the maximum value of $z = 3x - y = \sqrt{37} \approx 6.083$.

Solving the quadratic $(37)x^2 + (-24z)x + (4z^2 - 4) = 0$ and substituting this value for z we find that

$$x = \frac{-(-24z) \pm \sqrt{(-24z)^2 - 4(37)(4z^2 - 4)}}{2(37)}$$

$$= \frac{-(-24z) \pm \sqrt{592 - 16z^2}}{2(37)}$$

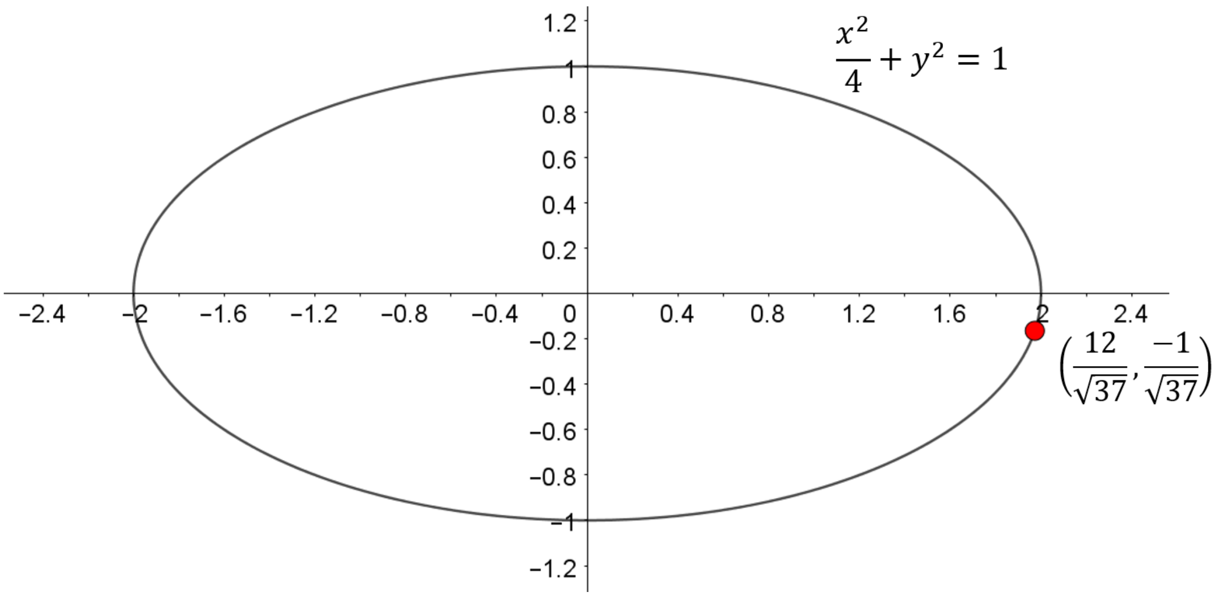
$$= \frac{24\left(\sqrt{37}\right) \pm \sqrt{0}}{2(37)}$$

$$= \frac{12}{\sqrt{37}} \approx 1.973$$

and then

$$y = 3x - z = 3\left(\frac{12}{\sqrt{37}}\right) - \left(\sqrt{37}\right) = \frac{36 - 37}{\sqrt{37}} = \frac{-1}{\sqrt{37}} \approx -0.164.$$

So, of all points (x, y) on the ellipse $\frac{x^2}{4} + y^2 = 1$, the point $\left(\frac{12}{\sqrt{37}}, \frac{-1}{\sqrt{37}}\right)$ will maximize the value of $z = 3x - y$ and the maximum value of z equals $\sqrt{37}$.



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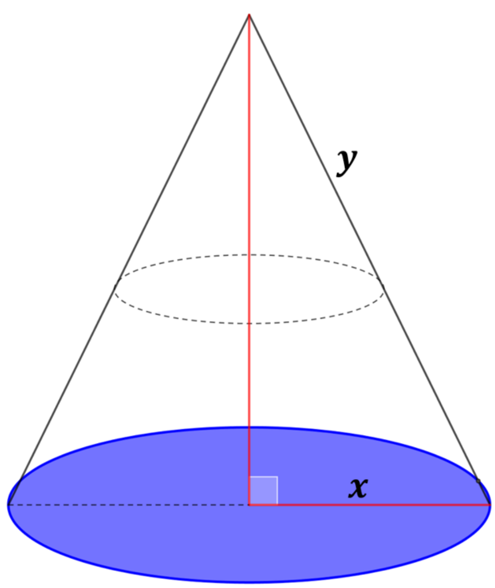
Problem 13.

Find the relationship between the radius x and slant height y which maximize the volume of a right circular cone with a fixed total surface area s.



Solution

The total surface area s of the cone above equals the lateral surface area plus the base area and is given by

$$s = \pi xy + \pi x^2.$$

Solving for y we have

$$y = \frac{s}{\pi x} - x.$$

The altitude of this cone equals

$$\text{Altitude} = \sqrt{y^2 - x^2} = \sqrt{\left(\frac{s}{\pi x} - x\right)^2 - x^2}.$$

The volume V of this cone is given by

$$V = \frac{1}{3}(\pi x^2)\left(\sqrt{\frac{s^2}{\pi^2 x^2} - \frac{2s}{\pi}}\right).$$

We can take advantage of the fact that V and V^2 are necessarily maximized at the same value(s) of x. Working with V^2 is computationally easier because it will not have the square root operation in the formula.

$$V^2 = \frac{1}{9}\left(\frac{\pi^2 x^4 s^2}{\pi^2 x^2} - \frac{2\pi^2 x^4 s}{\pi}\right) = \frac{2\pi s}{9}\left(\frac{s}{2\pi}x^2 - x^4\right).$$

Let $r = \frac{s}{2\pi}x^2 - x^4$. Then the value of x where $V^2 = \left(\frac{2\pi s}{9}\right) \cdot r$ is maximized is the same value of x where $r = \frac{s}{2\pi}x^2 - x^4$ is maximized.

Let r_{max} equal the maximum value of r. So, our goal now is to find the real value of x such that $\frac{s}{2\pi}x^2 - x^4 = r_{\text{max}}$.

(To be clear, we do not know the value of r_{max} at this point. We have just given the label r_{max} to whatever the maximum value of $\frac{s}{2\pi}x^2 - x^4$ turns out to equal.)

We note that

$$\frac{s}{2\pi}x^2 - x^4 = r_{\max} \Leftrightarrow x^4 - \frac{s}{2\pi}x^2 - r_{\max} = 0$$

which we recognize as a quadratic equation in the variable x^2. The real value(s) of x where

$$x^4 - \frac{s}{2\pi}x^2 - r_{\max} = 0$$

are the real roots of this quadratic equation. Because we are looking for real roots, the discriminant Δ of this quadratic equation (in the variable x^2) must be nonnegative.

That is,

$$\Delta = \left(\frac{-s}{2\pi}\right)^2 - 4(1)(r_{\max}) \geq 0.$$

Simplifying and solving for $r_{\max}$ we have

$$r_{\max} \leq \frac{s^2}{16\pi}.$$

So $r_{\max}$, the largest possible value of $r = \frac{s}{2\pi}x^2 - x^4$, equals $\frac{s^2}{16\pi}$.

Now we can work backwards to find the value(s) of x such that

$$x^4 - \frac{s}{2\pi}x^2 - r_{\max} = x^4 - \frac{s}{2\pi}x^2 - \left(\frac{s^2}{16\pi}\right) = 0.$$

By the quadratic formula we find that

$$x^2 = \frac{-\left(\frac{-s}{2\pi}\right) \pm \sqrt{\left(\frac{-s}{2\pi}\right)^2 - 4(1)\left(\frac{s^2}{16\pi}\right)}}{2(1)} = \frac{s}{4\pi}$$

and

$$x = \sqrt{\frac{s}{4\pi}} = \frac{\sqrt{s}}{2\sqrt{\pi}}.$$

From here we can solve for y to get

$$y = \frac{s}{\pi x} - x = \frac{s}{\pi\left(\sqrt{\frac{s}{4\pi}}\right)} - \sqrt{\frac{s}{4\pi}} = \frac{3\sqrt{s}}{2\sqrt{\pi}}.$$

Thus, the maximum volume is achieved when

$$\frac{y}{x} = \frac{\frac{3\sqrt{s}}{2\sqrt{\pi}}}{\frac{\sqrt{s}}{2\sqrt{\pi}}} = \frac{3}{1}.$$

That is, for any fixed total surface area s, the slant height will be three times longer than the radius of the cone with maximum volume.

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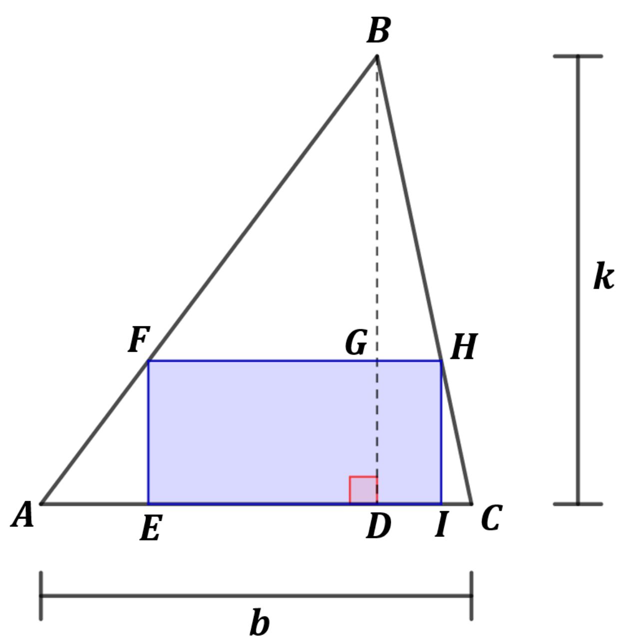
Problem 14.

Find the dimensions of the rectangle $EFHI$ with maximum area that can be inscribed in the triangle ABC with fixed base length $AC = b$ and fixed altitude length $BD = k$. (Triangle ABC is **not** necessarily equilateral or isosceles.)



Source: *A Treatise on Problems of Maxima and Minima Solved by Algebra* by Ramchundra, page 13.

Solution

$\overline{FH} \parallel \overline{AC}$ (recall the symbol ∥ means "parallel to"). Therefore ΔFBH and ΔABC are similar triangles.

Hence,

$$\frac{FH}{AC} = \frac{BG}{BD}.$$

Let $DG = x$. Then we can express the above relation as

$$\frac{FH}{b} = \frac{k - x}{k}$$

or

$$FH = \frac{b(k - x)}{k}.$$

Hence the area of the blue rectangle equals

$$DG \cdot FH = x\left(\frac{b(k - x)}{k}\right).$$

Let y represent the maximum area of rectangle $EFHI$. Then our goal comes down to finding the value of x such that

$$x\left(\frac{b(k - x)}{k}\right) = y.$$

We can rewrite this as the quadratic equation

$$(-b)x^2 + (bk)x - ky = 0.$$

Let h equal the value of x that satisfies this quadratic equation. Then h is a real root of this quadratic equation. But the discriminant Δ must be nonnegative in order for this quadratic equation to have a real root.

That is,

$$\Delta = (bk)^2 - 4(-b)(-ky) \geq 0.$$

Simplifying we find

$$y \leq \frac{b^2k^2}{4bk} = \frac{bk}{4}.$$

This proves that $bk/4$ is an upper bound on the area of the inscribed rectangle but we still need to demonstrate that there exists an $x = h$ where the area equals $bk/4$.

Going back to the quadratic equation and substituting $x = h$ and $y = bk/4$ we have

$$(-b)h^2 + (bk)h - k\left(\frac{bk}{4}\right) = 0.$$

Now we can solve for h to find

$$h = \frac{-(bk) \pm \sqrt{\Delta}}{2(-b)}.$$

But we have already seen that $y = bk/4$ makes the discriminant $\Delta = 0$. So, when $y = bk/4$ we find

$$h = \frac{-bk}{-2b} = \frac{k}{2} = \frac{\text{altitude of } \Delta\text{ABC}}{2}.$$

That is, to maximize the area of the inscribed rectangle we set the height of the rectangle DG to equal one-half the height of the triangle BD.

Of course, this also requires that

$$BG = BD - DG = BD - \frac{1}{2}BD = \frac{1}{2}BD.$$

Now recall from our similar triangles that

$$\frac{FH}{AC} = \frac{BG}{BD} = \frac{\frac{1}{2}BD}{BD} = \frac{1}{2}.$$

Thus, the width of the rectangle FH must be one-half the base width of the triangle ABC in order to maximize the area of the inscribed rectangle.

It follows that the

maximum area of the rectangle

$$= \left(\frac{1}{2}k\right)\left(\frac{1}{2}b\right) = \frac{1}{2}\left(\frac{1}{2}bk\right)$$

$$= \frac{1}{2}(\text{area of } \Delta ABC).$$

In summary, the inscribed rectangle with the maximum area will have a

- height equal to one half the height (altitude) of the triangle
- width equal to one half the base of the triangle
- area equal to one half the area of the triangle.

■

4.2 AM GM Approach

This approach uses the AM GM inequality which states that the arithmetic mean (AM) is greater than or equal to the geometric mean (GM).

AM GM Inequality

For all nonnegative real numbers $x_1, x_2, \dots, x_n$

$$\text{AM} = \frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 \cdot x_2 \cdots x_n} = \text{GM}$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$.

Problem 15.

Find the maximum value of ab^2c^3 among all positive numbers a, b, c such that $a + b + c = 30$. For what value(s) of (a, b, c) is the maximum of ab^2c^3 attained?

Solution

If we apply the AM GM inequality to $(a + b + c)/3$ we get

$$10 = \frac{a + b + c}{3} \geq \sqrt[3]{abc}\,.$$

But we need the right-hand side to involve ab^2c^3 instead of abc.

However, suppose we rewrite $a+b+c$ in the form

$$a+b+c=a+\frac{b}{2}+\frac{b}{2}+\frac{c}{3}+\frac{c}{3}+\frac{c}{3}.$$

Then when we apply the AM GM inequality we get

$$\frac{30}{6}=\frac{a+\frac{b}{2}+\frac{b}{2}+\frac{c}{3}+\frac{c}{3}+\frac{c}{3}}{6}\geq\left(a\cdot\frac{b}{2}\cdot\frac{b}{2}\cdot\frac{c}{3}\cdot\frac{c}{3}\cdot\frac{c}{3}\right)^{1/6}=\left(\frac{ab^2c^3}{36}\right)^{1/6}.$$

So now we know that

$$\left(\frac{ab^2c^3}{36}\right)^{1/6}\leq 5$$

or

$$ab^2c^3\leq 36\cdot 5^6$$

with equality if and only if

$$a=\frac{b}{2}=\frac{b}{2}=\frac{c}{3}=\frac{c}{3}=\frac{c}{3}.$$

That is, the maximum value of ab^2c^3 is $36\cdot 5^6$ and this is achieved when

$$b=2a \text{ and } c=3a.$$

Thus,

$$30=a+b+c=a+2a+3a=6a\Longrightarrow a=5,b=10,c=15.$$

That is, the maximum value of ab^2c^3 is attained for $(a,b,c)=(5,10,15)$. As a check we see that

$$ab^2c^3=(5)(10)^2(15)^3=(5)(2^2 5^2)(3^3 5^3)=36\cdot 5^6$$

as it should.

■

Problem 16.

Find the maximum value of $x^3(4a-x)^5$ if x and a are positive numbers such that $x\leq 4a$.

(*Higher Algebra*, Hall and Knight, Chapter XIX Inequalities, Problem 10, page 219.)

Solution

To end up with an $x^3(4a-x)^5$ factor in the GM side of and AM GM inequality we would need

$$\Big(x+x+x\Big)+\Big((4a-x)+(4a-x)+(4a-x)+(4a-x)+(4a-x)\Big)$$

on the AM side. But if we apply the AM GM inequality with this expression on the AM side we end up with

$$\frac{\Big(x+x+x\Big)+\Big((4a-x)+(4a-x)+(4a-x)+(4a-x)+(4a-x)\Big)}{8}\geq\sqrt[8]{x^3(4a-x)^5}.$$

We would like the left hand-side to be a ***constant*** but unfortunately

$$\text{AM side}=\frac{3x+5(4a-x)}{8}=\frac{10a-x}{4}$$

contains the variable x. But notice that if we could manage to remove both the 3 and 5 in the numerator of the AM side we would get

$$\frac{x+(4a-x)}{8}=\frac{a}{2}$$

which is a constant. But how can we accomplish this? Suppose we make the AM side

$$\frac{\left(\frac{x}{3}+\frac{x}{3}+\frac{x}{3}\right)+\left(\frac{4a-x}{5}+\frac{4a-x}{5}+\frac{4a-x}{5}+\frac{4a-x}{5}+\frac{4a-x}{5}\right)}{8}$$

Now when we apply the AM GM inequality we have

$$\frac{\left(\frac{x}{3}+\frac{x}{3}+\frac{x}{3}\right)+\left(\frac{4a-x}{5}+\frac{4a-x}{5}+\frac{4a-x}{5}+\frac{4a-x}{5}+\frac{4a-x}{5}\right)}{8}\geq\sqrt[8]{\frac{x^3(4a-x)^5}{3^3 5^5}}.$$

The GM side now includes the constant factor $\frac{1}{3^3 5^5}$ which we don't want but the AM side

$$\frac{3\left(\frac{x}{3}\right)+5\left(\frac{4a-x}{5}\right)}{8}=\frac{4a}{8}=\frac{a}{2}$$

is a constant like we need it to be. Can we manipulate this to isolate the $x^3(4a-x)^5$? At this point we have

$$\sqrt[8]{\frac{x^3(4a-x)^5}{3^3 5^5}} \leq \frac{a}{2}$$

$$\Leftrightarrow \frac{x^3(4a-x)^5}{3^3 5^5} \leq \left(\frac{a}{2}\right)^8$$

$$\Leftrightarrow x^3(4a-x)^5 \leq (3^3 5^5)\left(\frac{a}{2}\right)^8 = \left(\frac{3^3 5^5}{2^8}\right)a^8.$$

So $\left(\frac{3^3 5^5}{2^8}\right)a^8$ is an upper bound for $x^3(4a-x)^5$. But before we can declare this is the maximum value of $x^3(4a-x)^5$ we have to find a value of x where $x^3(4a-x)^5$ actually achieves this upper bound.

But recall that the AM GM inequality gives us the additional information that this upper bound is achieved if and only if

$$\frac{x}{3} = \frac{x}{3} = \frac{x}{3} = \frac{4a-x}{5} = \frac{4a-x}{5} = \frac{4a-x}{5} = \frac{4a-x}{5} = \frac{4a-x}{5}$$

or more simply

$$\frac{x}{3} = \frac{4a-x}{5}.$$

Solving this equation for x we have

$$5x = 12a - 3x$$

$$8x = 12a$$

$$x = \left(\frac{3}{2}\right)a.$$

If we want to check we could note that

$$\left(\frac{3}{2}a\right)^3\left(4a - \frac{3a}{2}\right)^5 = \frac{3^3 a^3}{2^3} \cdot \frac{5^5 a^5}{2^5} = \left(\frac{3^3 5^5}{2^8}\right)a^8$$

as required.

In summary, for all positive a and x with $x \leq 4a$, we have that the maximum value of $x^3(4a-x)^5$ equals $\left(\frac{3^3 5^5}{2^8}\right) a^8$ and $x^3(4a-x)^5$ actually achieves this maximum value when $x = \left(\frac{3}{2}\right) a$.

Note: The restriction $x \leq 4a$ was necessary to ensure that all eight terms on the AM side of this application were positive (a requirement of the AM GM inequality).

■

4.3 Using Both the Discriminant and the AM GM Approach

Problem 17.

I. Use the discriminant approach to find the minimum value of $\frac{1}{a} + \frac{1}{b}$ over all positive real a and b numbers such that $a + 2b = 3$.

II. Use the AM GM approach to find the minimum value of $\frac{1}{a} + \frac{1}{b}$ over all positive real a and b numbers such that $a + 2b = 3$.

Solutions

I. The Discriminant Approach

Let v equal the minimum value of $\frac{1}{a} + \frac{1}{b}$ over all positive real a and b such that $a + 2b = 3$.

Suppose (a, b) is a pair of positive real numbers such that $a + 2b = 3$ and $\frac{1}{a} + \frac{1}{b} = v$. It follows that

$$v = \frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab}$$

$$\Leftrightarrow vab = a + b$$

$$\Leftrightarrow vab - a - b = 0.$$

Now we can substitute $a = 3 - 2b$ to get

$$0 = v(3-2b)b - (3-2b) - b = (-2v)b^2 + (3v+1)b + (-3).$$

That is, $x = b$ is a positive real root of the quadratic equation

$$(-2v)x^2 + (3v+1)x + (-3) = 0.$$

This means that the discriminant of this quadratic is nonnegative. Thus,

$$\Delta = (3v+1)^2 - 4(-2v)(-3) \geq 0$$

or

$$9v^2 - 18v + 1 \geq 0.$$

By the quadratic formula we can determine that $\Delta = 0$ at

$$v = \frac{-(-18) \pm \sqrt{(-18)^2 - 4(9)(1)}}{2(9)} = \frac{18 \pm \sqrt{288}}{18} = \frac{3 \pm 2\sqrt{2}}{3}$$

and $\Delta \geq 0$ if and only if $v \in \left(0, \frac{3-2\sqrt{2}}{3}\right) \cup \left(\frac{3+2\sqrt{2}}{3}, \infty\right)$.

But a and b are positive real numbers that satisfy $a + 2b = 3$. Hence $0 \leq a \leq 3$ and $0 \leq b \leq 3/2$. It follows that

$$v = \frac{1}{a} + \frac{1}{b} \geq \frac{1}{3} + \frac{1}{(3/2)} = \frac{1}{3} + \frac{2}{3} = 1.$$

So

$$v \notin \left(0, \frac{3-2\sqrt{2}}{3}\right) \approx (0, 0.057)$$

Hence the minimum value of $\frac{1}{a} + \frac{1}{b}$ over all positive real a and b numbers such that $a + 2b = 3$ equals

$$\boxed{v = \frac{3+2\sqrt{2}}{3} \approx 1.943.}$$

Now we can work backwards to find the value(s) of a and b such that lead to this minimum value.

We have already established that $x = b$ is a positive root of

$$(-2v)x^2 + (3v + 1)x + (-3) = 0.$$

By the quadratic formula

$$b = \frac{-(3v + 1) \pm \sqrt{\Delta}}{2(-2v)}$$

But we have already established that $\Delta = 0$ at $v = \frac{3+2\sqrt{2}}{3}$. Therefore,

$$\begin{aligned} b &= \frac{3v + 1}{4v} = \frac{3}{4} + \frac{1}{4v} = \frac{3}{4} + \frac{3}{4(3 + 2\sqrt{2})} \\ &= \frac{3}{4} + \frac{3(3 - 2\sqrt{2})}{4(3 + 2\sqrt{2})(3 - 2\sqrt{2})} \\ &= \frac{6 - 3\sqrt{2}}{2} \approx 0.879. \end{aligned}$$

Solving for a we have

$$a = 3 - 2b = 3 - 2\left(\frac{6 - 3\sqrt{2}}{2}\right) = -3 + 3\sqrt{2} \approx 1.243.$$

■

II. The AM GM Approach

If we apply the AM GM inequality to $\frac{1}{a} + \frac{1}{b}$ directly we find

$$\frac{1}{a} + \frac{1}{b} \geq 2\sqrt{\frac{1}{a} \cdot \frac{1}{b}}$$

with no clear path towards further simplification.

As is often the case, having success with the AM GM approach requires finding a clever way to rewrite the functions involved.

It *sometimes* helps to take the product of the constraint and the objective function and then simplify.

$$\frac{1}{a}+\frac{1}{b}=1\cdot\left(\frac{1}{a}+\frac{1}{b}\right)=\left(\frac{a+2b}{3}\right)\left(\frac{1}{a}+\frac{1}{b}\right)$$

$$=\left(\frac{a+2b}{3a}\right)+\left(\frac{a+2b}{3b}\right)$$

$$=\frac{1}{3}+\left(\frac{2b}{3a}\right)+\left(\frac{a}{3b}\right)+\frac{2}{3}=1+\left(\frac{2b}{3a}\right)+\left(\frac{a}{3b}\right).$$

Now look at what happens when we apply the AM GM inequality to the last two terms in this new expression.

$$\left(\frac{2b}{3a}\right)+\left(\frac{a}{3b}\right)\geq\sqrt{\left(\frac{2b}{3a}\right)\left(\frac{a}{3b}\right)}=\frac{\sqrt{2}}{3}$$

with equality if and only if

$$\frac{2b}{3a}=\frac{a}{3b}.$$

The right-hand side of the AM GM inequality is now a constant! So, we have established that

$$\frac{1}{a}+\frac{1}{b}\geq 1+\frac{\sqrt{2}}{3}=\frac{3+2\sqrt{2}}{3}$$

which agrees with our answer from the using the discriminant method.

But we still need to find a pair (a,b) where $\frac{1}{a}+\frac{1}{b}$ actually equals this lower bound. Fortunately, the AM GM inequality also tells us that

$$\frac{1}{a}+\frac{1}{b}=1+\frac{\sqrt{2}}{3}=\frac{3+2\sqrt{2}}{3}$$

if and only if

$$\frac{2b}{3a}=\frac{a}{3b}\ \text{ or }\ 2b^2=a^2.$$

Combining this with the constraint $a + 2b = 3$ we have a system of two equations in two unknowns: $2b^2 = a^2$ and $a = 3 - 2b$.

Substituting the second equation into the first we have

$$2b^2 = (3 - 2b)^2 = 9 - 12b + 4b^2$$

or

$$2b^2 - 12b + 9 = 0.$$

Solving this quadratic equation we have

$$b = \frac{-(-12) \pm \sqrt{(-12)^2 - 4(2)(9)}}{2(2)} = \frac{12 \pm \sqrt{144 - 72}}{4}$$

$$= \frac{12 \pm 6\sqrt{2}}{4} = \frac{6 \pm 3\sqrt{2}}{2}.$$

But

$$b = \frac{6 + 3\sqrt{2}}{2} \approx 5.121$$

is not possible if $a + 2b = 3$ and $a \geq 0$. So

$$b = \frac{6 - 3\sqrt{2}}{2} \approx 0.879.$$

We have already seen that $a^2 = 2b^2$. Therefore

$$a = \sqrt{2}b = \frac{6\sqrt{2} - 3(2)}{2} = 3\sqrt{2} - 3 \approx 1.243.$$

As a check, we note that for this (a, b) pair

$$a + b = \left(3\sqrt{2} - 3\right) + \left(\frac{6 - 3\sqrt{2}}{2}\right) = 3$$

and

$$\frac{1}{a}+\frac{1}{b}=\frac{1}{3\sqrt{2}-3}+\frac{2}{6-3\sqrt{2}}$$

$$=\frac{-3\sqrt{2}-3}{(3\sqrt{2}-3)(-3\sqrt{2}-3)}+\frac{2(6+3\sqrt{2})}{(6-3\sqrt{2})(6+3\sqrt{2})}$$

$$=\frac{3+3\sqrt{2}}{9}+\frac{6+3\sqrt{2}}{9}=\frac{9+6\sqrt{2}}{9}=\frac{3+2\sqrt{2}}{3}$$

as required.

■

5. Extra Problems

2016-17 Meet 5, Individual Event A

3. Two trains are traveling at constant speeds on parallel tracks. One train is twice as long as the other. If they are traveling in the same direction, it takes 60 seconds for the entire shorter train to pass by the entire length of the longer one. If they are traveling in opposite directions, it takes 15 seconds for the entire shorter train to pass by the entire length of the longer one. If the shorter train is length a, determine how far it travels in one second in terms of a.

Let r_1 be the speed of the short train and r_2 be the speed of the longer train. Traveling in the same direction, it takes 60 seconds for the shorter train to travel a distance of $3a$ farther than the longer train. This means $3a = 60(r_1 - r_2)$. In opposite directions, the trains are passing each other at a rate of $r_1 + r_2$, so our equation is $3a = 15(r_1 + r_2)$. Multiplying the second equation by 4 and adding it to the first equation gives $15a = 120\, r_1 \Rightarrow r_1 = \frac{a}{8}$.

4. Two people play a game, taking turns drawing a ball from an urn randomly, without replacing them. The urn starts out with 3 black balls and 4 white balls. If a player draws a black ball, their turn is completed. If a player draws a white ball, they must also (nonrandomly) take two black balls from the urn before they complete their turn. A player loses when they cannot complete their turn. If you draw first, determine exactly the probability you will win the game.

Given the rules of the game, there are only two ways you can win the game. The first way you can win is if you draw a white ball and your opponent draws a white. You win because they can not complete their move by removing 2 black balls. The probability of this event happening is $\frac{4}{7}\cdot\frac{3}{4}=\frac{3}{7}$. *The second way you can win is if you draw a black ball, your opponent draws a black ball, and you draw a black ball. You win because you will leave 4 white balls in the urn leaving your opponent unable to complete their turn. This is done with a probability of* $\frac{3}{7}\cdot\frac{1}{3}\cdot\frac{1}{5}=\frac{1}{35}$. *Your probability of winning is* $\frac{3}{7}+\frac{1}{35}=\frac{16}{35}$.

2016-17 Meet 5, Team Event

3. Who knew Batman hides inside the Incredible Hulk! Each letter of $\frac{\begin{array}{r}BANNER\\ -BRUCE\end{array}}{WAYNE}$ stands for one of the digits 0-9. Fill in the letters of *BANNER* with the correct digits.

3. *First, note the BA – B = W implies that B = 1 since the result of the subtraction is a single digit. This also implies that A can only be 0 since 1 is taken. It is also important to note that N – R = 0 implies that N = R + 1 since N and R are different digits. Therefore, W = 9. Looking at the units digit, R – E = E means that R and E could be 4 and 2, 6 and 3, 8 and 4, 2 and 6, 4 and 7, or 6 and 8. Since N = R + 1, we can remove 8 and 4. From the other five pairs, R must be 4 and E must be 2. The other four pairs force C to be a digit already taken. Therefore, BANNER =* $\boxed{105524}$.

2015-16 Meet 5, Individual Event A

2. The Young Triplets have an annoying habit. Whenever a question is asked of the three of them, two tell the truth and the third lies. When I asked them which of them was born first, they replied as follows:

Al: Bob was born first.
Bob: I am not the oldest.
Carl: Al is the oldest.
Which of the Young Triplets was born first?

Al and Bob contradict one another. This means either Al or Bob is the single liar and Carl is telling the truth. Therefore, Al was born first..

Al: Bob was born first
Bob: I am not the oldest
Carl: Al is the oldest

Al	Truth Teller	Bob oldest	

Bob	Truth Teller	Bob not oldest	Contradiction. Bob cannot be the oldest and not the oldest at the same time.
Carl	Liar	Al is not oldest	

So (Al, Bob, Carl) = (Truth Teller, Truth Teller, Liar) is not the true state of affairs.

Al: Bob was born first
Bob: I am not the oldest
Carl: Al is the oldest

Al	Truth Teller	Bob oldest	Contradiction. Bob and Al cannot both be the oldest.
Bob	Liar	Bob oldest	
Carl	Truth Teller	Al is oldest	

So (Al, Bob, Carl) = (Truth Teller, Liar, Truth Teller) is not the true state of affairs.

Therefore (Al, Bob, Carl) = (Liar, Truth Teller, Truth Teller) must be the true state of affairs. So Carl is telling the truth when he says Al is the oldest.

3. **Given positive integers a and b, the units digit of b is 8, but Ralph thought it was 6 and got 4740 for the product of a and b. Natalie thought the units digit of b was 3 and got 4695 for the product. Determine the integer values a and b.**

Let a and b be the positive integers. Since Ralph thought the units digit of b was 6, he used b –2 in his multiplication. Therefore, $a(b-2) = 4740 \Rightarrow ab = 4740 + 2a$. *Natalie thought the units digit of b was 3, so she actually used b – 5 in her multiplication. Therefore,* $a(b-5) = 4695 \Rightarrow ab = 4695 + 5a$. *Equating and solving, we obtain* $4695 + 5a = 4740 + 2a \Rightarrow 3a = 45 \Rightarrow a = 15$. *Then* $b = 318$.

2014-15 Meet 5, Individual Event A

3. A small stream is flowing into a pond at a constant rate. A pack of 12 elephants can empty the pond in 4 minutes, while a pack of 9 elephants would do so in 6 minutes. How long would it take 6 elephants to drink the pond dry?

Let S = rate of stream, E = rate of 1 elephant. Then $S+12E=\frac{1}{4}$ *pond/minute, while* $S+9E=\frac{1}{6}$ *pond/minute. Subtracting the second equation from the first yields* $3E=\frac{1}{4}-\frac{1}{6}=\frac{1}{12}$, *so* $E=\frac{1}{36}$. *By substitution,* $S+12\left(\frac{1}{36}\right)=\frac{1}{4} \Rightarrow S=\frac{1}{4}-\frac{1}{3}=-\frac{1}{12}$, *and so* $S+6E=-\frac{1}{12}+6\left(\frac{1}{36}\right)=\frac{1}{12}$ *pond/minute.*

2013-14 Meet 5, Individual Event A

Note: For alphametics problems #2 and #3 below, each letter represents a different digit, and leading digits may not be zero.

2. A master sushi chef, frustrated while judging a competition of apprentice chefs, angrily scrawls "LESS EEL" onto one apprentice's *unagi* judging form. The master then smiles as he realizes that by adding just two symbols, the statement becomes an alphametic puzzle; i.e., $\underline{L}\,\underline{E} + \underline{S}\,\underline{S} = \underline{E}\,\underline{E}\,\underline{L}$. Solve this puzzle, writing as your answer the 4-digit integer $\underline{L}\,\underline{E}\,\underline{S}\,\underline{S}$.

$$\begin{array}{r} LE \\ +\ SS \\ \hline EEL \end{array}$$

Looking at the three-digit result (EEL), the first digit, E, must represent 1. Then the "ones" column requires that either 1 + S = L, or else 1 + 9 = 10 (with S = 9 and L = 0). In the latter case, the tens column has 1 (carried) + 0 + 9 = 11, which is false. So 1 + S = L. Tens column: L + S = 11 --> (1 + S) + S = 11 --> S = 5, so L = 6 and $\underline{L}\,\underline{E}\,\underline{S}\,\underline{S} = 6155$.

3. Last summer, after tripping over some exposed tree roots and skinning my knee, I got an infection from some dirt that was stuck in the wound. Fortunately, a doctor was able to clean things out and purge the infection. Solve the alphametic $\sqrt{\underline{D}\,\underline{R}} + \sqrt[3]{\underline{R}\,\underline{I}\,\underline{D}} = \underline{I}\,\underline{T}$, writing as your answer the 4-digit integer $\underline{D}\,\underline{I}\,\underline{R}\,\underline{T}$.

Begin by looking at the three-digit perfect cubes: 125, 216, 343, 512, and 729. Throw out 343 because it contains a repeated digit. We now seek a paired units digit and hundreds digit that juxtapose to form the two-digit perfect square $\underline{D}\,\underline{R}$. *125 --> 51 (no), 216 --> 62 (no), 512 --> 25 (yes), 729 --> 97 (no). So* $\sqrt{\underline{D}\,\underline{R}} + \sqrt[3]{\underline{R}\,\underline{I}\,\underline{D}} = \sqrt{25} + \sqrt[3]{512} = 5 + 8 = 13 = \underline{I}\,\underline{T}$, *and* $\underline{D}\,\underline{I}\,\underline{R}\,\underline{T} = 2153$.

4. A blind woman is trying to find the bus station, located at some integer on the number line. Starting at 0, she asks a stranger which direction to walk, and the stranger says to walk in the positive direction. At each integer there is another stranger whom the blind woman asks for direction. Strangers at even integers always tell the truth, while strangers at odd integers lie the first time they are asked, and tell the truth thereafter. However, the woman assumes everyone is helpful, and follows all directions she is given. If the woman finally arrives at the bus station after traveling 101 units, at what integer is the station located?

Since 0 is even, the first stranger is telling the truth, and the station is located at a positive integer. The woman first walks to 1, where she is lied to, and told to walk back to 0. There, the first stranger again tells her to walk to 1, where this time she is told the truth, and walks to 2. The stranger at 2 tells her to walk to 3, where she is lied to again. This pattern can be grouped into blocks of four movements, each of the form truth/lie/truth/truth. The woman's path is 1/0/1/2 --> 3/2/3/4 --> 5/4/5/6 . . . etc. After every four units of movement, the woman has arrived at the next even integer. It takes 100 units to arrive at 50, so the station is located at 51.

2012-13 Meet 5, Individual Event A

2. Among one hundred applicants for a certain technical position, it was discovered that ten had never taken a course in chemistry or in physics. Seventy-five had taken at least one course in chemistry; eighty-three had taken at least one course in physics. How many of the applicants had taken courses in <u>both</u> chemistry and physics?

<u>101 Puzzles in Thought & Logic</u>, C.R. Wylie Jr., #28

90 applicants have taken either chemistry, or physics, or both. Adding 75 to 83 gives us quite a bit more than 90, but this excess is because we are counting twice the applicants who have taken both types of courses. The excess is 75 + 83 – 90 = 68 applicants.

4. The chicken nuggets at DonMickey's restaurants come in packages of 6, 9, or 20. What is the <u>largest</u> total number of nuggets that <u>cannot</u> be purchased using some combination of these packages? ***Adapted from <u>Mathematics Teacher</u>, March 17, 2006***

Any multiple of 3 (greater than 3) can be created by packages of 6 and 9 nuggets. Since 36 = 9 + 9 + 9 + 9, 38 = 20 + 9 + 9, and 40 = 20 + 20, any even number ≥ 36 can be derived by adding groups of 6 to each of these quantities. Also, adding 9 to those three quantities gives us the basis for all odds ≥ 45. Consider the next greatest odd number, 43: it is not a multiple of 3; 43 – 20 = 23 is not a multiple of 3, and 43 – 20 – 20 = 3 is not purchasable. 43 can't be done!

2011-12 Meet 5, Individual Event A

1. Let N be a positive integer. Exactly three of the following statements are true. Which statement is false?

a) N is divisible by 2.
b) N is divisible by 4.
c) N is divisible by 12.
d) N is divisible by 24.

*If **a** were false, then **b**, **c**, and **d** would necessarily be false. Similarly for **b** (it would make **c** and **d** false), and for **c** (it would make **d** false). The only statement that can be false by itself is **d**.*

4. On a board, Mike has written several positive integers so that the difference between any two consecutively-written integers is the same. Then Mike substituted letters for digits, so that each letter corresponds to the same digit each time it is written. The result looks as follows:

$$\underline{T}, \underline{E}\,\underline{L}, \underline{E}\,\underline{K}, \underline{L}\,\underline{A}, \underline{C}\,\underline{C}$$

What positive integer does $\underline{L}\,\underline{A}$ represent?

Let d be the difference between any pair of consecutively-written integers. Since $\underline{E}\,\underline{L}$ *and* $\underline{E}\,\underline{K}$ *start with the same digit, we can conclude that* $d < 10$. *In that case, we must have* $E = 1$, $L = 2$, $C = 3$. *Thus the second integer is* 12 *and the fifth integer is* 33. *The difference between those two is* 21, *and should be equal to 3d. Thus, d = 7 and the numbers are:* 5, 12, 19, $\boxed{26}$, 33.

2009-10 Event 5A

1. Compute, in simplified form, the value of the 8th expression in this pattern:

1	2 + 3	4 + 5 + 6	7 + 8 + 9 + 10	...

The next expression in the pattern sums 11 through 15, the 6th expression sums 16 through 21, the 7th expression sums 22 through 28, and so the 8th expression = 29 + ... + 36 = 260.

2. In the pattern ..., *a*, 2, *b*, *a*, ..., each number is equal to twice the previous number plus the number two places back. Compute the value of *a*.

By definition, b = 2(2) + a = 4 + a, and a = 2(b) + 2. Substituting for the value of b, $a = 2(4+a)+2 = 10+2a$, *and* ***a*** **=** *–10.*

2008-09 Event 5A

1. The six-digit number **2 1 7 X 8 5** , when divided by 9, leaves a remainder of 2. What is the value of the obscured digit, **X** ?

If **2 1 7 X 8 5** *leaves a remainder of 2 when divided by 9, then* **2 1 7 X 8 3** *must be divisible by 9, and the sum of its digits must also be divisible by 9.* $2+1+7+X+8+3=21+X$, *which is only divisible by 9 if X = 6.*

2. A single digit is placed in each empty square in the grid *(Figure 2)* so that each row and each column contain exactly one of each of the digits 1, 2, 3, 4, and 5. What digit must be placed in the square at the bottom right corner?

	5	4		
1	3			
		5	3	
2		3	1	
				?

Figure 2

> *In the left-most column, the only possible spot for the digit 5 is at the bottom. Now, in the bottom row, there is only one possible spot for the digit 3 ... the bottom right corner.*

Credit: Mathematics Teacher, Nov. 2006 Calendar Question #2.

3. In the addition problem shown in *Figure 3*, each letter stands for a distinct non-zero digit. If the problem is mathematically correct, what is the largest possible value for the three-digit number ***ONE***?

```
  ONE
+ ONE
-----
  TWO
```

Figure 3

> *We concentrate on the hundreds digit (the "**O**") of **ONE**. The largest it can be is the digit 4. If **O = 4**, then E = 2 or 7. We prefer 7, carrying a 1. Now make N as large as possible.* $N = 9 \Rightarrow W = 9$, $N = 8 \Rightarrow W = 7$, *and **N** cannot be 7. Trying* **E** = 2, $N = 9 \Rightarrow T = 9$, *but* ***N = 8*** $\Rightarrow W = 6, T = 9$. *Yes!*

2. Four men were being questioned by the police about a robbery.

   "Jack did it," said Alan
   "George did it,' said Jack.
   "It wasn't me," said Sid.
   "Jack is a liar if he said I did it," said George.

   Only one had spoken the truth. Who committed the robbery?

2. If A did it, the truth was told by S, G.
Similarly, $G \Rightarrow J, S$; $J \Rightarrow A, S, G$; $S \Rightarrow G$
Sid is, therefore, the guilty one.

2005-06 Event 5A

2. In the two additions,

$$A + B = C$$
$$C + D = EA$$

each of the five letters represents a distinct digit, *EA* being a two digit number. What is the value of $B+D$?

2. Surely $E = 1$

Then $(C) + D = 10 + A$

so $(A + B) + D = 10 + A$

$B + D = 10$

3. With one straight line, you can slice a pie into two pieces; a second cut that crosses the first one will produce four pieces; and a third cut can produce as many as seven pieces (Figure 3). What is the largest number of pieces that you can get with seven straight cuts?

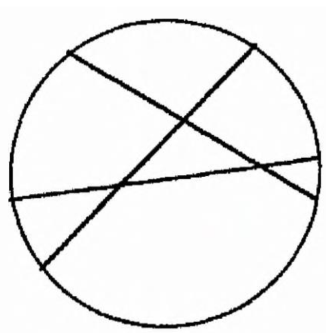


Figure 3

3. The second cut adds two pieces; the third cut adds three. Convince yourself that this is the pattern

Number of cuts	Number of pieces
0	1
1	2
2	4
3	7
4	11
5	16
6	22
7	29

4. (a) Amy, Beth, and Christine toss a coin 15, 16 and 17 times respectively. Which girl is least likely to get more heads than tails?
(b) Amy, Beth, and Christine toss a coin 18, 19 and 20 times respectively. Which girl is least likely to get more heads than tails?

4. (a) For an even number of tosses, the probability of getting the same number of heads and tails reduces the probability of one being tossed more than another. Example with $n = 6$

$$\text{prob}(3 \text{ heads}) = \frac{20}{64}$$

$$\text{prob}(\text{less than } 3) = \text{prob}(\text{more than } 3) = \frac{22}{64}$$

Beth, with an even number of tosses, is least likely to get more heads than tails.

(b) Note $\begin{cases} \text{pr--}\ (4 \text{ heads in } 8 \text{ tosses}) = \frac{70}{256} \\ \text{prob}\ (5 \text{ heads in } 10 \text{ tosses}) = \frac{63}{256} \end{cases}$

The higher the number of even tosses, the lower the probability of same number of heads and tails, so the higher the probability

2001-02 Event 5A

4. Display 4 shows an alphametric in which each of the letters X, Y, and Z in the indicated addition represents a distinct digit. Find X, Y, and Z.

$$\begin{array}{r} XXXX \\ YYYY \\ ZZZZ \\ \hline YXXXZ \end{array}$$

Display 4

4. Since $X+Y+Z \le 24$, the carry into 10^4 column is 1 or 2; $Y=1$ or $Y=2$.
Since $X+Y+Z = Z$ or $10+Z$,
$X+Y=0$ (impossible) or
$X+Y=10$; so $X+Y=10$
Consider two cases:

$Y=1$ or $Y=2$
$X=9$ $\quad$ $X=8$

In either case, $10 \le X+Y+Z < 20$, so the carry from the unit's column is 1.
It follows that $X = Z+1$.
Try $X=8$, $Y=2$, $Z=7$; It doesn't work
Try $X=9$, $Y=1$, $Z=8$; this works!

2000-01 Event 5A

4. A cryptographer devises the following method of encoding positive integers. First the integer is expressed in base 5. Second, a 1-to-1 correspondence is established between the digits that appear in expressions base 5 and the elements of the set {V,W,X,Y,Z}. Using this correspondence, the cryptographer finds that three consecutive integers in increasing order are coded as VYZ, VYX, VVW, respectively. What is the base-10 expression for the integer encoded as XYZ? [AHSME, 1987. 16]

4.

$$\begin{array}{r} VYZ \\ +1 \\ \hline VYX \\ +1 \\ \hline VVW \end{array}$$

Since $X+1$ results in a "carry," $W=0$ and $X=4$

Then $Z+1=4$, so $Z=3$

Now Y must be 1 or 2

but $Y+1=V\neq 3$, so $Y\neq 2$

$Y=1$. Then $V=2$

$$XYZ = 413_5 = 4\cdot 5^2 + 1\cdot 5 + 3$$

$$= 100 + 5 + 3 = 108_{10}$$

1999-2000 Event 5A

1. What is the last digit (the units digit) of 1999^{2000}?

	LAST DIGIT
1999^1	9
$(1999)^2$	1
$(1999)^3$	9
$(1999)^4$	1 etc.

Last digits alternate between 9 and 1; even powers correspond to 1

1998-99 Event 5A

4. Complete the grid shown below by filling in the squares with the digits $1, 2, 3, 4, 5$, so that each digit occurs once in each row, column, and each of the two main diagonals.

1	2	3	4	
5				2

1	2	3	4	5
2	5	4	1	3
4	3	2	5	1
5	4	1	3	2
3	1	5	2	4

Begin by looking at the diagonals. The number 2 must appear in the middle square. The other 2's are then forced. Continue next with the 5's which are forced, then the other numbers.

1996-97 Event 5A

1. The integer 25! ends with a string of 0's. How many?

1. Equivolently, how many times is 10 a factor? There are plenty of 2's ? How many times is 5 a factor?

1.... 5.... 10.... 15.... 20..... 25

(1) (2) (3) (4) (6)

1995-96 Event 5A

1. What digit substituted for * will make the following addition correct?

$$\begin{array}{r} 5\,9\,7\,* \\ \underline{2\,*\,6\,0} \\ *\,*\,3\,* \end{array}$$

1. A 1 will be carried into thousands column, so * = 8

5A 87-88

2. [From Friedland, Puzzles in Logic] In Figure 2, you are given three views of the same cube. How many dots are there on the bottom face (opposite the six) in view 1 ?

Our last two puzzles come from [Hunter and Madachy, **Mathematical Diversions**]

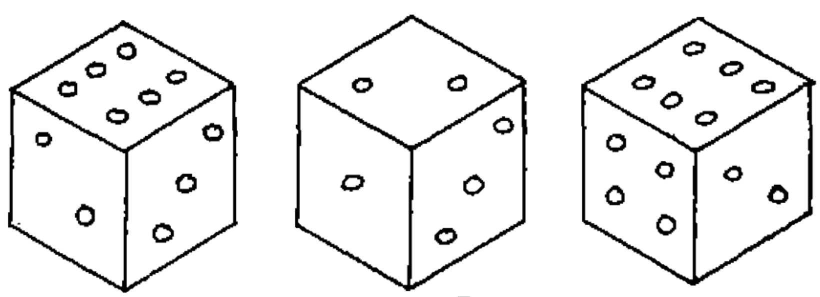


Figure 2

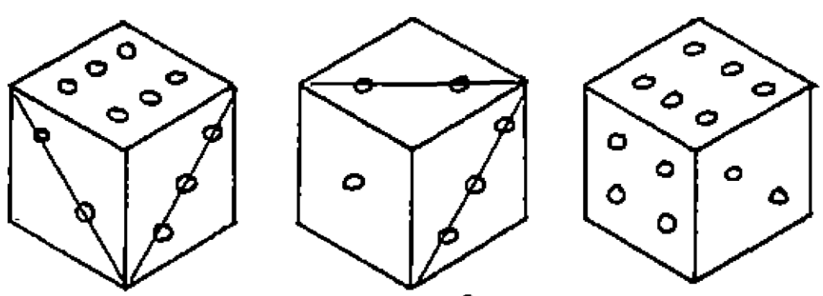
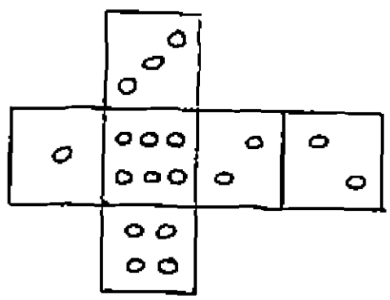


Figure 2

2. Note that this is not a die. Views 1 and 2 in Fig. 2 do not show the same  because the V's they determine (see Answer sheet) open respectively on  and  "Unfolded," we must have:



The face opposite the six shows two dots.