MSHSML Meet 5, Event B
Study Guide

5B Congruence and Similarity

Ratio and proportion

Segments intercepted by parallel lines

Identification of similar/congruent figures

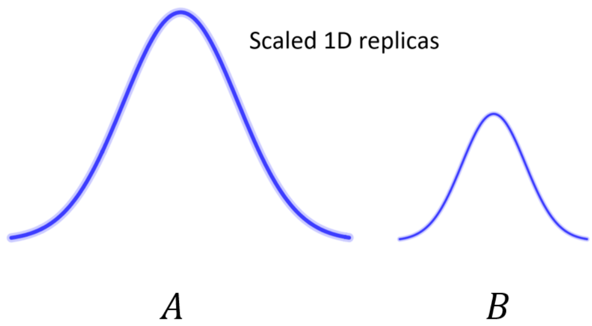
Ratios of areas and volumes

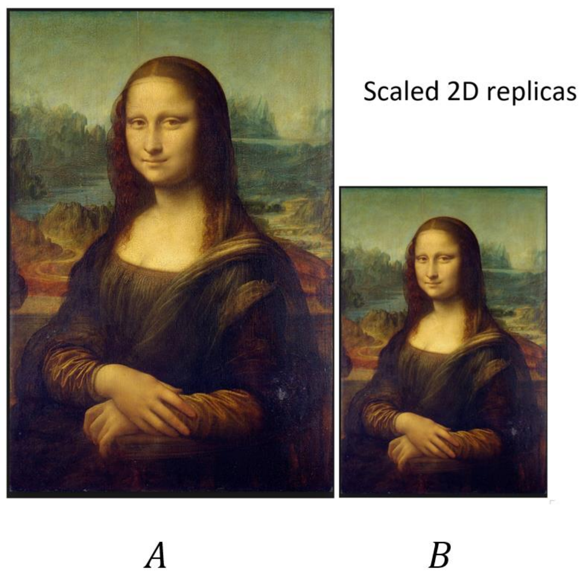
Elementary trigonometric ratios

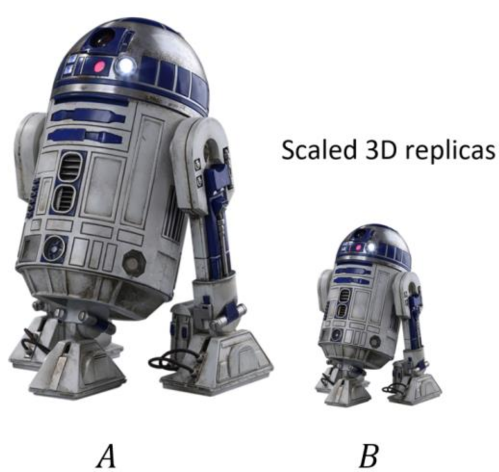
Contents

1. Similarity

Two objects A and B (in number of dimensions) are **similar** if they are scaled replicas of each other.







So all three of the above A, B pairs of replicas are examples of “similar objects” as we use the term in mathematics.

Key Theorem of Similar Objects

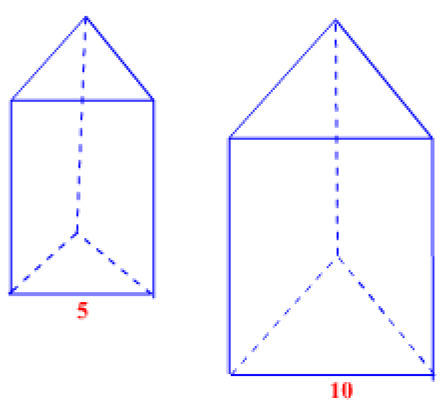
Let $a/b = k$ be the ratio of *any* conveniently measured corresponding one dimensional (*i.e.* length) in two similar objects A and B. Then

- the ratio of *every* corresponding one-dimensional measurement in A and B will have the ratio k
- the ratio of *every* corresponding two-dimensional measurement (*i.e.* area) in A and B will have the ratio k^2
- the ratio of *every* corresponding three-dimensional measurement (*i.e.* volume) in A and B will have the ratio k^3.

k is called the **scale factor** for similar objects A and B.

Example 1

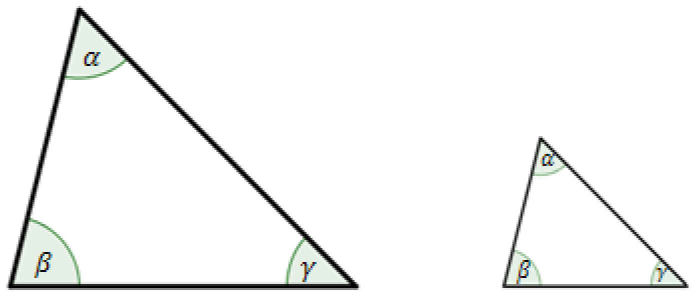
If the two prisms below are similar solids and because we know that the front sides of the bases have lengths in the ratio $1:2$ or $1/2$, then by the key theorem for similarity we can immediately state that



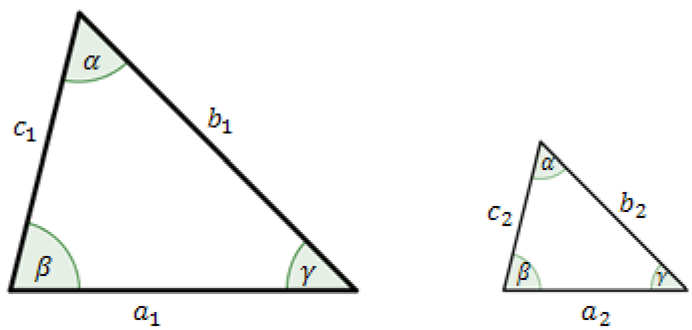
- the ratio of their base perimeters is $1:2$ or $1/2$
- the ratio of their heights is $1:2$ or $1/2$
- the ratio of lateral areas (*i.e.* surface areas) is $1^2:2^2$ or $1/4$
- the ratio of their volumes is $1^3:2^3$ or $1/8$.

1.1 Similar Triangles

☞ **Similar Triangles**: two triangles with matching angles. That is, the triangles have the same shape but are not necessarily the same size.



The sides of similar triangles opposite matching angles are called **corresponding sides**. That is, for the two triangles below,



we would say that the two sides opposite the angle α, namely sides a_1 and a_2 are corresponding sides (of the two similar triangles). In the same way, we would say that the two sides opposite the angle β, namely sides b_1 and b_2 are corresponding sides. Finally, sides c_1 and c_2 are corresponding sides.

☞ **The symbol $\sim$ means "*is similar to*".**

For example, you will often see

$$\Delta ABC \sim \Delta DEF$$

as shorthand for saying that the triangle with angles A, B and C is similar to the triangle with angles D, E and F.

☞ **The order the angles are listed tells us which angles and which sides are corresponding**.

When you write

$$\Delta ABC \sim \Delta DEF$$

it is understood that

$$\Delta \boldsymbol{A}BC \sim \Delta \boldsymbol{D}EF \implies \angle A = \angle D$$

$$\Delta A\boldsymbol{B}C \sim \Delta D\boldsymbol{E}F \implies \angle B = \angle E$$

$$\Delta AB\boldsymbol{C} \sim \Delta DE\boldsymbol{F} \implies \angle C = \angle F$$

and

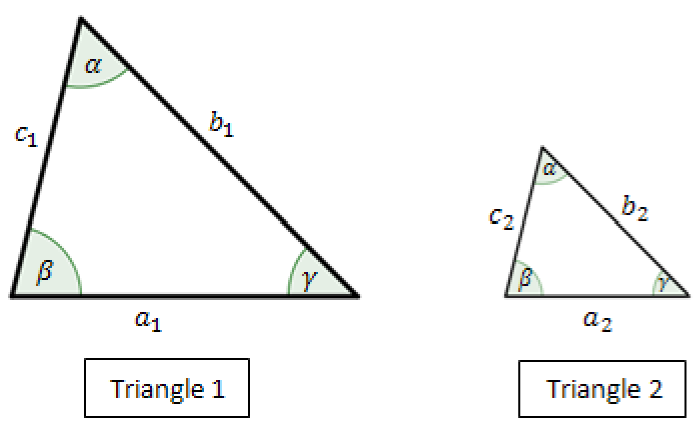
$$\Delta \boldsymbol{AB}C \sim \Delta \boldsymbol{DE}F \implies \text{side } \overline{AB} \text{ corresponds to side } \overline{DE}$$

$$\Delta A\boldsymbol{BC} \sim \Delta D\boldsymbol{EF} \implies \text{side } \overline{BC} \text{ corresponds to side } \overline{EF}$$

$$\Delta \boldsymbol{A}B\boldsymbol{C} \sim \Delta \boldsymbol{D}E\boldsymbol{F} \implies \text{side } \overline{AC} \text{ corresponds to side } \overline{DF}.$$

☞ **Ratios of Corresponding Sides of Similar Triangles are all the same.**

If Triangle 1 and Triangle 2 are similar triangles



as illustrated, then

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}.$$

For example, *if* side a_1 is twice as long as side a_2, then side b_1 must be twice as long as side b_2 and side c_1 must be twice as long as side c_2.

Note that if the corresponding sides of one triangle are all twice as long as their corresponding side on the other triangle, then $a_1 = 2 \cdot a_2$, $b_1 = 2 \cdot b_2$ and $c_1 = 2 \cdot c_2$. So in this example we would have that

$$\frac{a_1}{a_2} = 2, \qquad \frac{b_1}{b_2} = 2, \qquad \frac{c_1}{c_2} = 2.$$

The general case of this result tells us that if the corresponding sides of one triangle are all k times as long as their corresponding side on the other triangle, then $a_1 = k \cdot a_2$, $b_1 = k \cdot b_2$ and $c_1 = k \cdot c_2$. This would mean that

$$\frac{a_1}{a_2} = k, \qquad \frac{b_1}{b_2} = k, \qquad \frac{c_1}{c_2} = k$$

from which we have that

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}.$$

Remember that if two objects A and B are similar then *every* corresponding one-dimensional measurement in A and B will be in the same ratio.

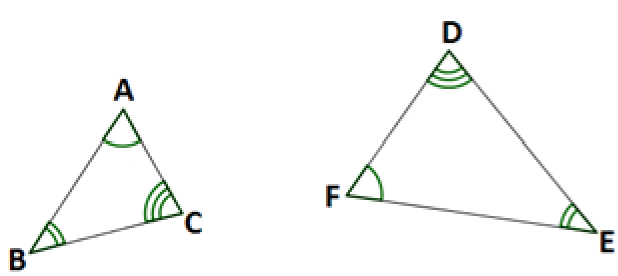
In particular, if two triangles are similar, then

- their corresponding medians have the same ratio as their corresponding sides
- their corresponding angle bisectors have the same ratio as the corresponding sides

1.1.1 Proving Two Triangles are Similar

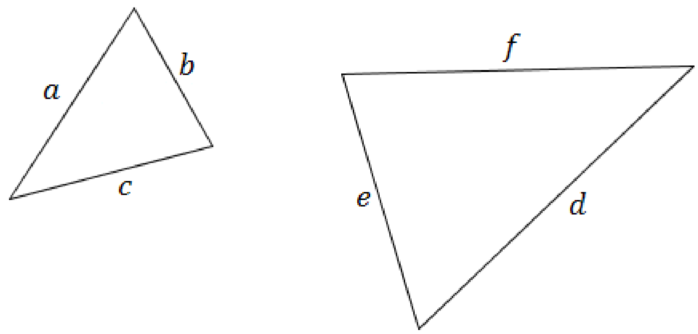
To prove that two triangles are similar, you only need to show that one of these three rules is true:

Rule 1 All three pairs of angles are equal. (AAA Rule)



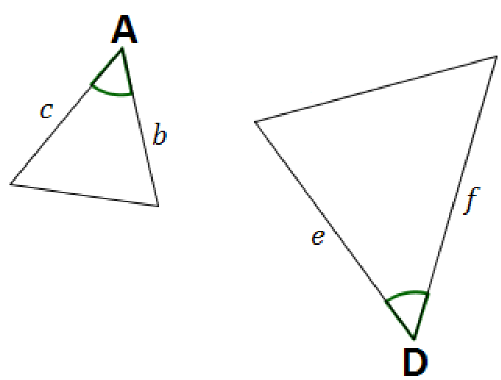
$$\angle A = \angle F$$
$$\angle B = \angle E$$
$$\angle C = \angle D$$

Rule 2 All pairs of corresponding sides are in the same ratio. (SSS Rule)



$$\frac{a}{d} = \frac{b}{e} = \frac{c}{f}$$

Rule 3 Two pairs of corresponding sides are in the same ratio and the angles *between* each pair of sides are also equal. (SAS Rule)



$$\frac{c}{e} = \frac{b}{f} \quad \text{and} \quad \angle A = \angle D$$

1.1.2 Proving Two Right Triangles are Similar

To prove that two right triangles are similar, you only need to show that <u>one</u> of these three rules is true:

Rule 1 their legs are proportional

Rule 2 a leg and a hypotenuse of one triangle are proportional to a leg and a hypotenuse of the other;

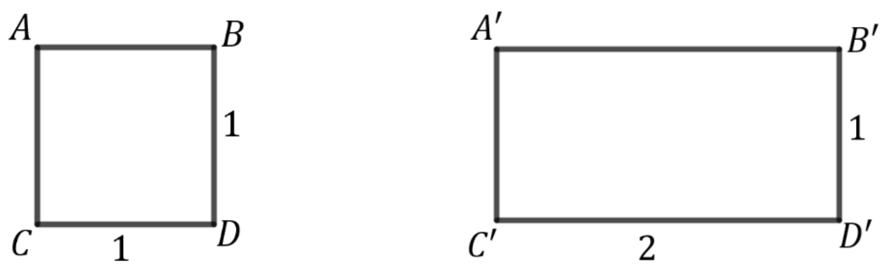
Rule 3 one non-right angle of one triangle is equal to one non-right angles of other

1.2 Similar Polygons

Similar polygons (# sides > 3) are polygons for which corresponding angles are equal **AND** the ratios of pairs of corresponding sides are all equal.

It is important to see how this definition differs from the definition of similar triangles. For two triangles, if corresponding angles are equal then it is always true that the ratios of pairs of corresponding sides will also be equal.

But for polygons with more than 3 sides this is not always true. Consider a 1×1 unit square and a 2×1 rectangle (width $= 2$, height $= 1$).

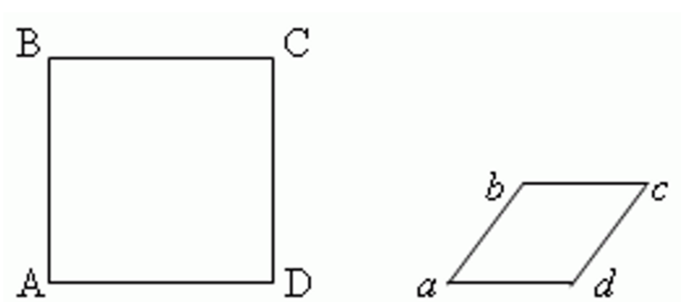


While corresponding angles in $ABCD$ and $A'B'C'D'$ are equal it is not true that corresponding sides are in the same ratio. In particular

$$\frac{1}{2} = \frac{CD}{C'D'} \neq \frac{BD}{B'D'} = \frac{1}{1}.$$

Also notice that while proportionality of corresponding sides is enough to prove two triangles are similar it is not enough to show to polygons with more than 3 sides are similar.

Consider for example a square and a rhombus. Clearly corresponding sides are in the same proportion but a square and a rhombus are not necessarily similar quadrilaterals as seen in the diagram shown below.



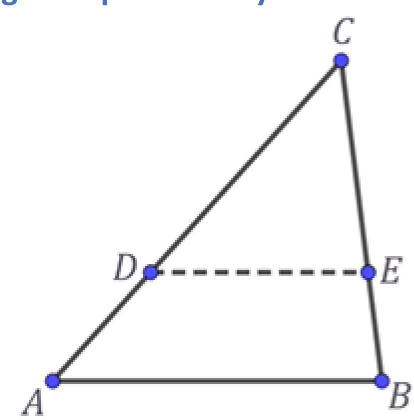
In simple language, similar polygons are polygons with the "same shape", where one polygon is just "an enlarged version of the other".

☞ **If two polygons are similar, their corresponding sides, altitudes, medians, diagonals, and perimeters are all in the same ratio.**

☞ **If two polygons are similar, the ratio of their areas is equal to the square of the ratio of their corresponding sides.**

2. Segments Intercepted by Parallel Lines

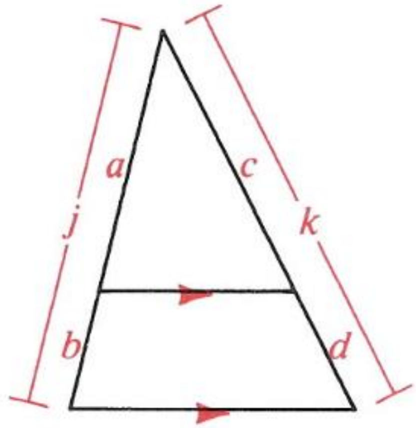
2.1 Triangle Proportionality Theorem (*a.k.a.* the Side Splitter Theorem)



$$\frac{AD}{CD} = \frac{BE}{CE} \text{ if and only if } DE \parallel AB$$

Notation: $DE \parallel AB$ means $\overline{DE}$ and $\overline{AB}$ are parallel.

It follows from the Triangle Proportionality Theorem that …

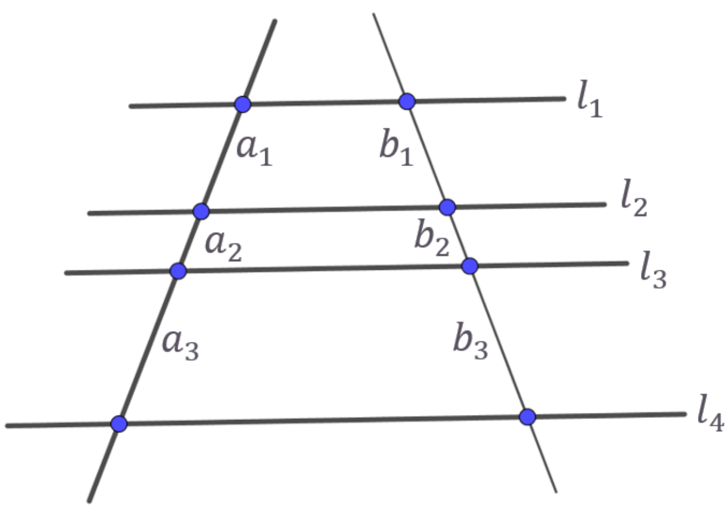


$$\frac{a}{j}=\frac{c}{k} \qquad \frac{a}{c}=\frac{j}{k} \qquad \frac{b}{j}=\frac{d}{k}$$

$$\frac{a}{b}=\frac{c}{d} \qquad \frac{a}{c}=\frac{b}{d} \qquad \frac{b}{d}=\frac{j}{k}$$

2.2 Extended Side Splitter Theorem (multiple parallel lines)

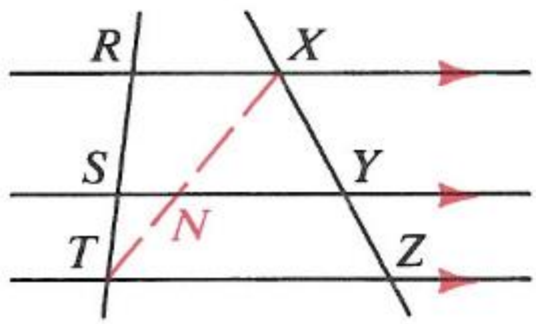
If lines l_1, l_2, l_3 and l_4 are parallel, then $\frac{a_1}{b_1}=\frac{a_2}{b_2}=\frac{a_3}{b_3}$.



Theorem

If three parallel lines intersect two transversals, then they divide the transversals proportionally. That is,

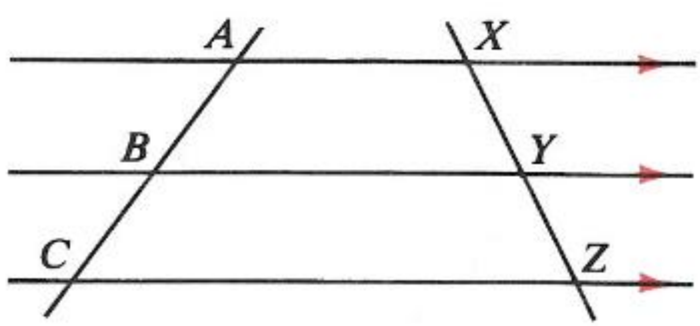
$$RS \parallel SY \parallel TZ \Longrightarrow \frac{RS}{ST} = \frac{XY}{YZ}$$



Theorem

If three parallel lines cut off congruent segments on one transversal, then they cut off congruent segments on every transversal. That is,

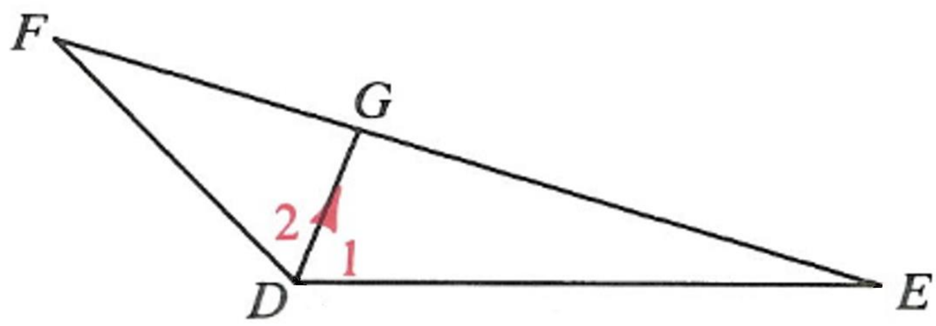
$$AX \parallel BY \parallel CZ \text{ and } AB = BC \Longrightarrow XY = YZ.$$



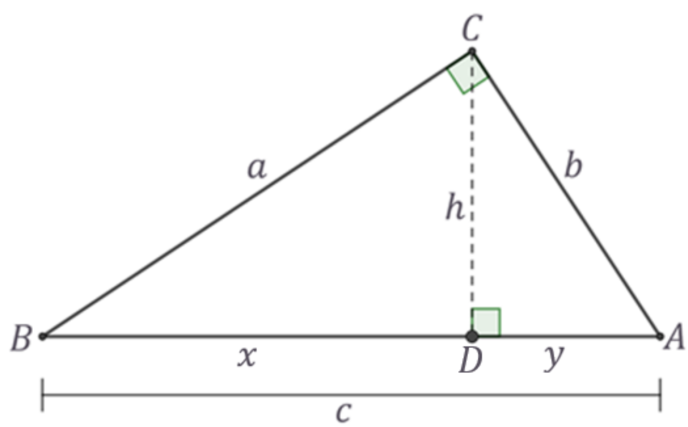
2.3 Triangle Angle-Bisector Theorem

If a ray bisects an angle of a triangle, then it divides the opposite side into segments proportional to the other two sides. That is, in ΔDEF

$$\angle 1 = \angle 2 \Longrightarrow \frac{GF}{GE} = \frac{DF}{DE}.$$



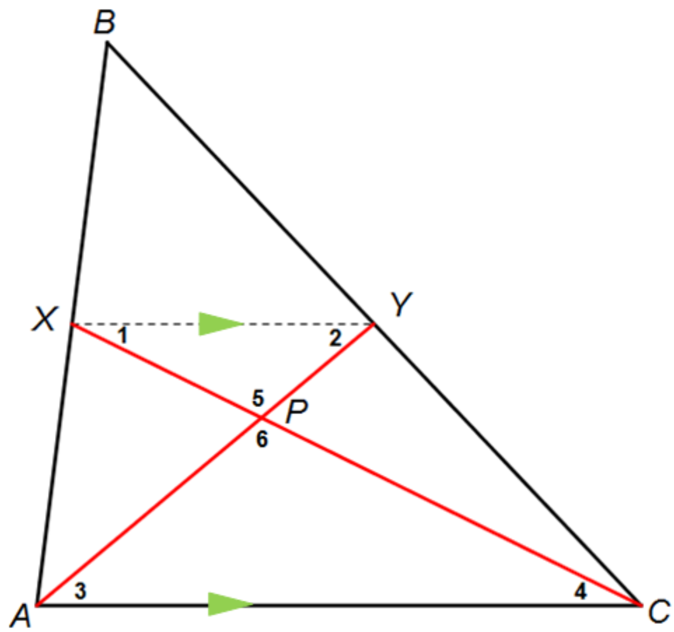
2.4 Altitude to Hypotenuse Theorem (Three Similar Triangles)



$$\Delta ACB \sim \Delta ADC \sim \Delta CDB$$

2.5 Medians Intersect in Two to One Ratio

Medians of a triangle intersect in a point that divides each median in the ratio $2:1$.

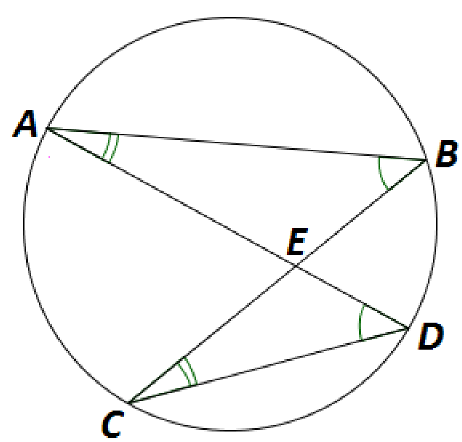


Let X be the midpoint of $\overline{AB}$ and let Y be the midpoint of $\overline{BC}$. Let P be the point of intersection of medians $\overline{AY}$ and $\overline{CX}$.

Then

$$\frac{XP}{PC} = \frac{YP}{PA} = \frac{1}{2}.$$

☞ In the following type of diagram where $\overline{AB}$ and $\overline{CD}$ are any chords of a circle that do not cross,



then it is always the case that $\angle ABC = \angle ADC$ and $\angle BCD = BAD$. This implies that $\Delta ABE \sim \Delta CDE$.

3. Related but Non-Similar Solids

Example 2

The length of a rectangular box is tripled while the other two dimensions are not changed. By what factor is the volume of the box increased?

Solution

$$Volume = length \times width \times height$$

$$V_1 = l_1 \times w \times h$$

$$V_2 = l_2 \times w \times h$$

$$l_2 = 3 \cdot l_1$$

$$\therefore \quad V_2 = l_2 \times w \times h = (3 \cdot l_1) \times w \times h = 3 \cdot (l_1 \times w \times h) = 3 \cdot V_1$$

So the volume of the box is increased by a factor of 3 (*i.e.* the volume is also tripled) when just the length is tripled.

Example 3

The length and width of a rectangular box are tripled while the height is left unchanged. By what factor is the volume of the box increased?

Solution

$$Volume = length \times width \times height$$

$$V_1 = l_1 \times w_1 \times h$$

$$V_2 = l_2 \times w_2 \times h$$

$$l_2 = 3 \cdot l_1$$

$$w_2 = 3 \cdot w_1$$

$$\therefore \quad V_2 = l_2 \times w_2 \times h = (3 \cdot l_1) \times (3 \cdot w_1) \times h = 3^2 \cdot (l_1 \times w_1 \times h) = 3^2 \cdot V_1$$

So the volume of the box is increased by a factor of 3^2 when both the length and width are tripled.