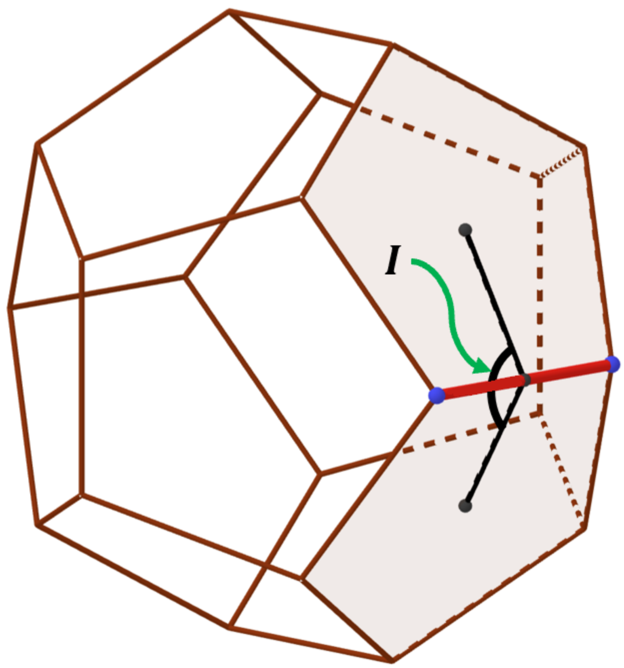
Tuesday Afternoon, 1st Hour

Angle Between Adjacent Faces in a Regular Convex Polyhedron (*i.e.* Platonic solids)

The goal of this lecture is to find a formula for the angle between adjacent faces in a regular convex polyhedron. That is, the angle I in the figure below.



Theorem

Let P be a regular convex polyhedral where each face has m sides and each vertex is formed by the intersection of n planes.

(In the above figure we can see that each face is a pentagon and hence $m = 5$. We can also see that $n = 3$ faces meet at each vertex.)

Let I be the interior measure of the angle between any two adjacent faces of the polyhedron. Then

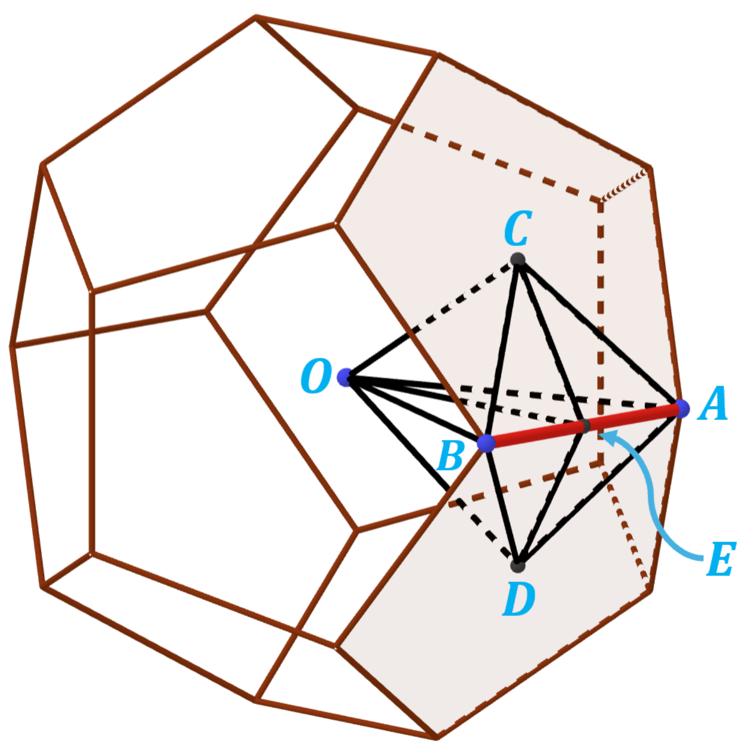
$$\sin\left(\frac{I}{2}\right) = \frac{\cos\left(\frac{\pi}{n}\right)}{\sin\left(\frac{\pi}{m}\right)}.$$

The standard approach for deriving this result involves spherical trigonometry.

Proof

We need to put in some additional edges and vertices.

(i)	Let BA be an edge between two adjacent faces in the polyhedron.
(ii)	Let C and D be the center points of the two faces that share edge BA.
(iii)	Let E be the midpoint of edge BA.
(iv)	Draw edges CE and DE.
(v)	Draw edges BD, AD, BC and CA.
(vi)	Label the angle $\angle CED$ as I.

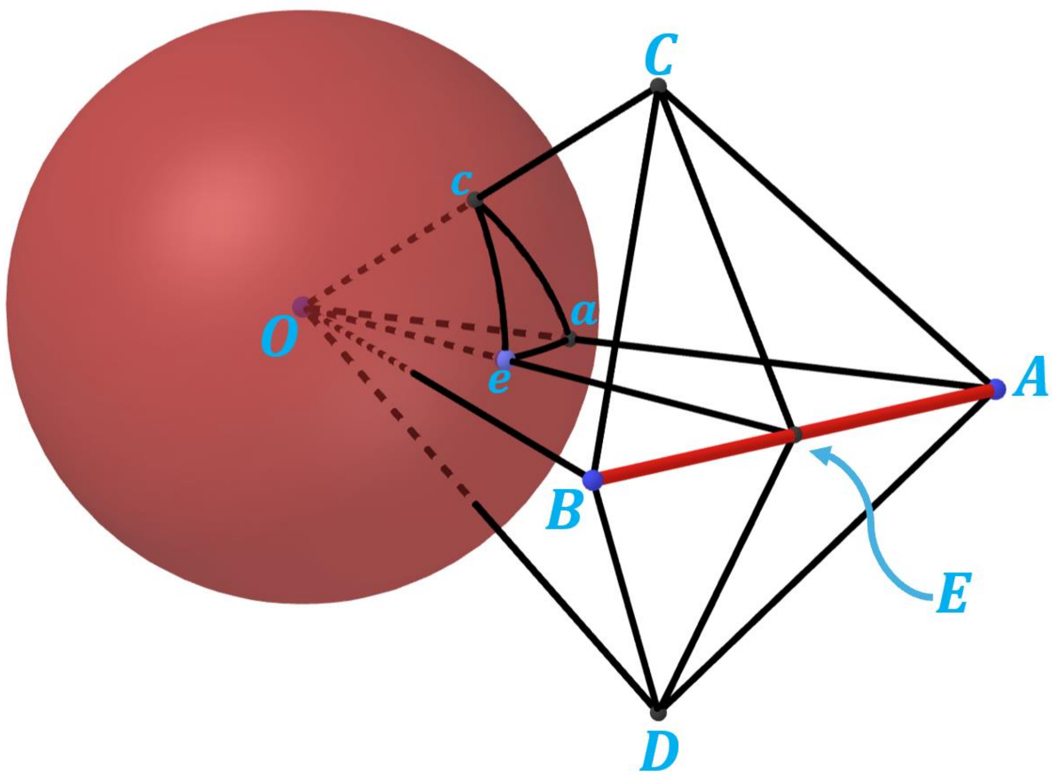


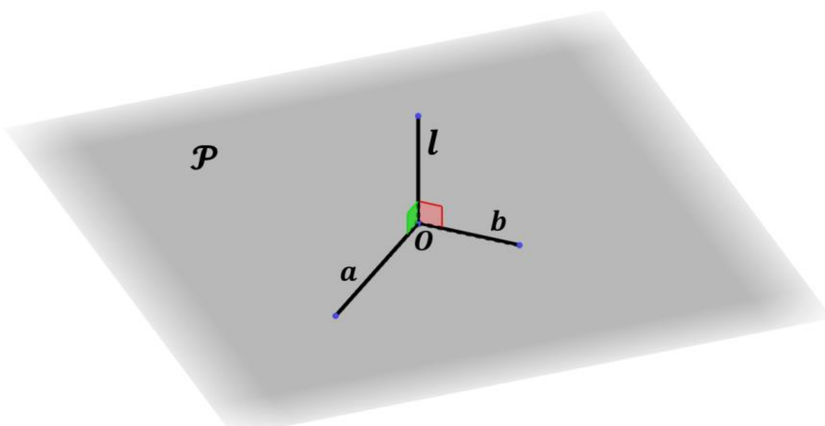
(vi)	Let $\mathcal{P}$ be the plane going through the three points C, D and E. Draw the line in plane $\mathcal{P}$ that is perpendicular to CE. Draw the line in plane $\mathcal{P}$ that is perpendicular to DE. Let O be the intersection point of these two new perpendicular lines in $\mathcal{P}$.

In the above diagram I've just draw in the segments CO and DO on these two new perpendicular lines. (It just gets too messy if I put the entire lines in.)

Also, I did not include the label I for $\angle CED$ for the same reason.

(viii)	Now create a sphere with center point O whose radius is less than OC. (The length of the radius does not matter except that it needs to be less than OC.)
(ix)	Let c, a, e be the points where CO, AO and EO respectively intersect this sphere.
(x)	Draw the spherical triangle Δcae.

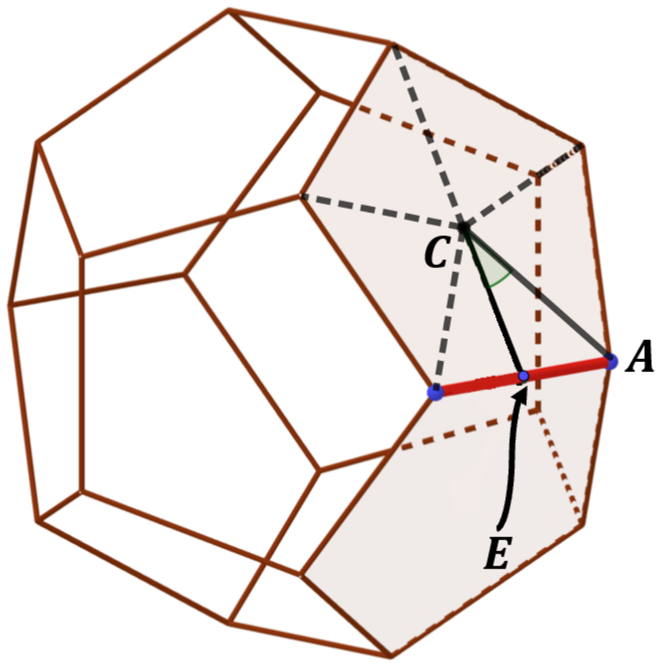


(xi)	$\overline{CE}$ and $\overline{DE}$ are perpendicular to $\overline{AB}$ (It is a standard result in high school geometry that a line drawn from a vertex (C) of an isosceles triangle (ΔCAB) that bisects the opposite side $(\overline{BA})$ is perpendicular to the bisected side.) That is to say, $\overline{CE} \perp \overline{AB}$ and $\overline{DE} \perp \overline{AB}$. (The symbol ⊥ is called the “perp” symbol in geometry and is used between two lines to indicate that those two lines are perpendicular.)
(xii)	$\overline{AB}$ is ***perpendicular to the plane*** containing the points C, E and D. To say that a line l not in plane $\mathcal{P}$ is *perpendicular to plane* $\mathcal{P}$ means, by definition, that line l is perpendicular to ***<u>every</u>*** line in plane $\mathcal{P}$. A sufficient condition to establish that line l is perpendicular to a plane is that l is perpendicular to both of two intersecting lines in that plane at the point of intersection. For example, in the figure below, line l not in $\mathcal{P}$ is perpendicular to $\mathcal{P}$ because l is perpendicular to both of two intersecting lines a and b at their point of intersection O. 
(xiii)	By construction O is a point in the plane containing the three points C, D and E. Hence $\overline{EO}$ is a line in that plane. Hence $\overline{AB} \perp \overline{EO}$ by the result in (xii). That is, $\angle AEO = 90°$.

(xiv)	$\overline{OC}$ is perpendicular to the face (plane) of the polyhedra that has center point C. We know by construction that $\overline{OC} \perp \overline{CE}$ and we could start the whole process over with a different edge (other than $\overline{BA}$) of that same face to establish that $\overline{OC}$ is perpendicular to a second line going from C to the midpoint of this different edge. Hence by the same reasoning as in the previous step, $\overline{OC}$ is perpendicular to *every* line in the face of the polyhedral that has center point C. In particular, it will relevant later that $\angle OCA = 90°$.
(xiv)	Let Q be the plane containing the points A, E and O. We can see that $\angle AEO = 90°$ is the angle between the planes $\mathcal{P}$ (containing the points D, E, C and O) and Q. [Remember the term for an angle between two planes is a *dihedral* angle.] And we can also see that great arc $\widehat{ce}$ is in $\mathcal{P}$ and that great arc $\widehat{ea}$ is in Q. But the dihedral angle between two planes each containing one of two intersecting great arcs is another way of defining the spherical angle between those great arcs. That is, the spherical angle between $\widehat{ce}$ and $\widehat{ea}$ is the same as the dihedral angle between $\mathcal{P}$ and Q which equals $\angle AEO = 90°$. That is spherical $\sphericalangle aec = 90°$ and thus Δaec is a *right* spherical triangle.
(xv)	Now let $\mathcal{W}$ be the plane containing the points A, C and O. By the same reasoning as in (xiv) the spherical angle $\sphericalangle ace$ equals $\angle ACE$, the dihedral angle between planes $\mathcal{W}$ and $\mathcal{P}$. But from symmetry we can establish that $\angle ACE = \pi/m$ where m is the number of sides in each face. (See the argument below.) This shows $$\sphericalangle ace = \angle ACE = \pi/m.$$
(xvi)	What can we say about $\sphericalangle cae$? We will make another visual argument similar to the one that follows next for finding $\sphericalangle ace$ that will show that $$\sphericalangle cae = \angle CAE = \pi/m.$$

Visual Argument for Find $\angle ACE$ (and hence $\sphericalangle ace$)

Notice that in the dodecahedron below that $\angle ACE$ will be one of 10 equal angles going round the point C making up a complete circle.



That is, in the case of the dodecahedron, $10 \cdot \angle ACE = 2\pi$. The number 10 comes from the fact that there are 5 sides in that face and there were two angles for each side.

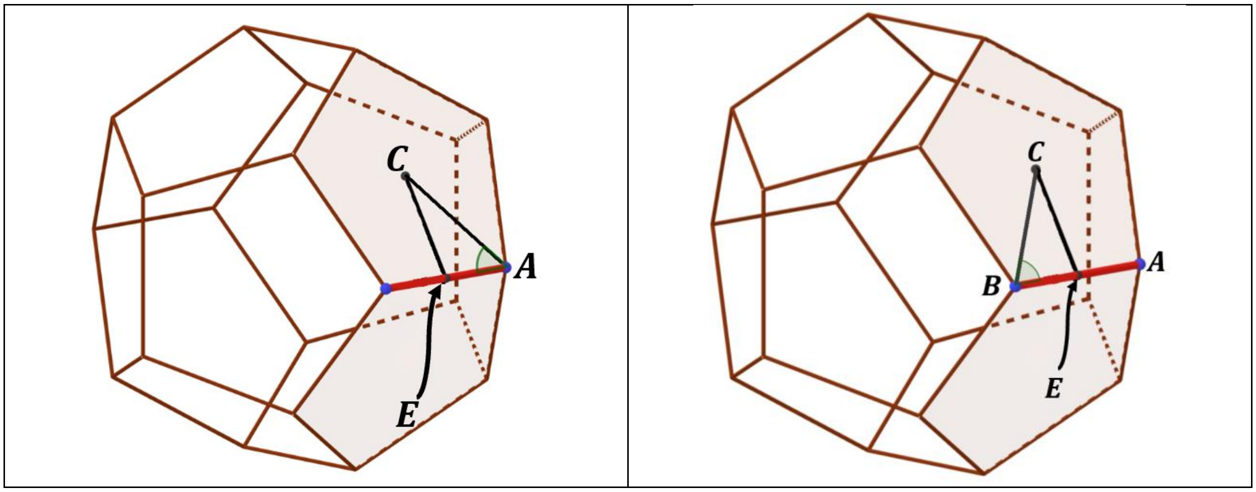
For general, when there are m sides to each face we would have

$$2m \cdot \angle ACE = 2\pi$$

and thus

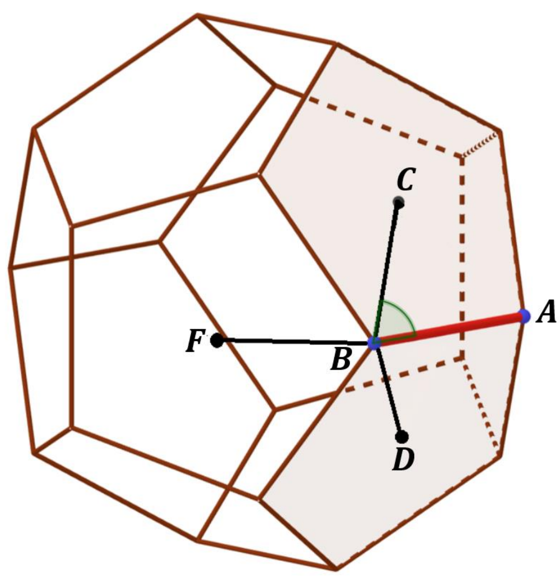
$$\angle ACE = \frac{\pi}{m}.$$

Visual Argument to Find $\angle CAE$ (and hence $\sphericalangle cae$)



The goal is to find $\angle CAE$ but for better visualization I will find $\angle CBE$, which is equal because these are base angles of the isosceles triangle ΔCBA.

Again, for better visualization I got rid of the unnecessary center line $\overline{CE}$ and accordingly switched notation so $\angle CBE$ became $\angle CBA$.



Next, we can see by symmetry that

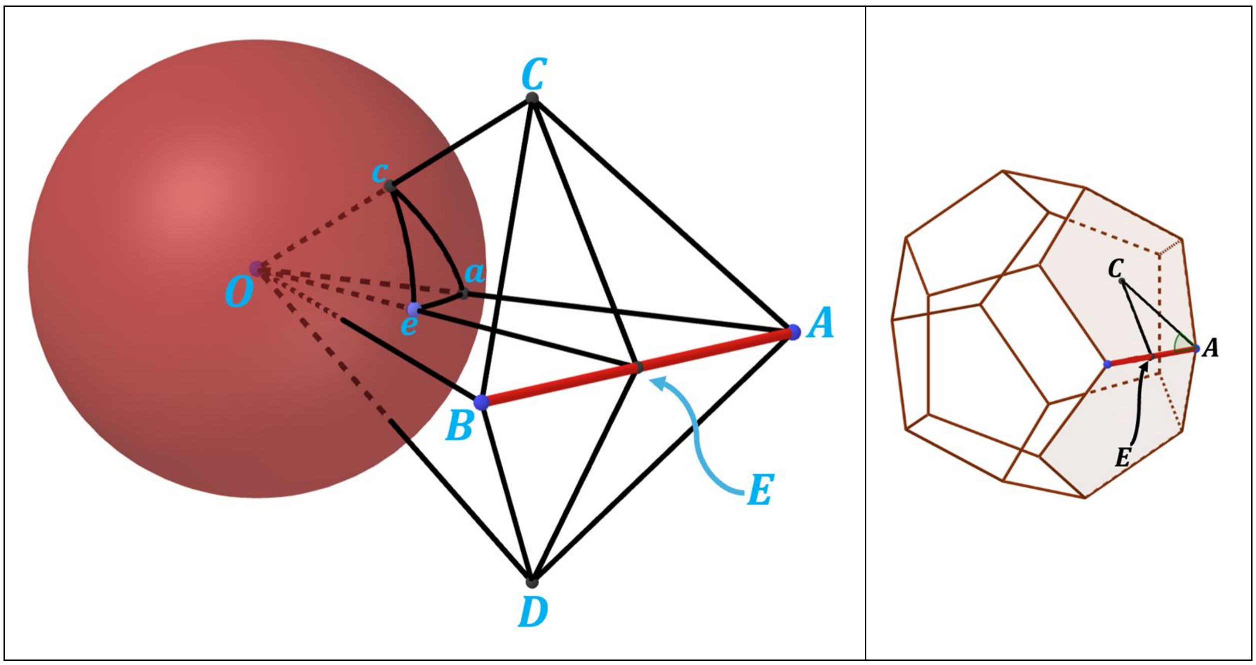
$$\angle CBA = \frac{1}{2}\angle CBD.$$

Note that the diagram above distorts the perspective some and make it look like $\angle CBD$, $\angle CBF$ and $\angle FBD$ are angles of a plane, but of course the vertex B is elevated (in a sense) from vertices C, F, D. So, we ***cannot*** just immediate conclude that

$$\angle CBD + \angle CBF + \angle FBD = 2\pi.$$

This turns out to be true, but we have to show why.

To start, we have already made the argument why $\angle CAE = \sphericalangle cae.$



If we had the patience for it, we could draw in the spherical angles $\sphericalangle cbe, \sphericalangle cbd, \sphericalangle cbf$ and $\sphericalangle fbd$ and repeat the same arguments to establish that

$$\angle CBE = \sphericalangle cbe,\ \angle CBD = \sphericalangle cbd,\ \angle CBF = \sphericalangle cbf \text{ and } \angle FBD = \sphericalangle fbd.$$

Then we could note that

$$\angle CBD + \angle CBF + \angle FBD = \sphericalangle cbd + \sphericalangle cbf + \sphericalangle fbd$$

and when drawn as spherical angles it can immediately conclude that $\sphericalangle cbd$, $\sphericalangle cbf$ and $\sphericalangle fbd$ do form a complete circle around the spherical vertex b and there sum is immediately seen to equal 2π.

Furthermore, because of the symmetry in a regular convex polyhedral each of these three angles must be equal.

So, we can finally say that

$$\angle CBD = \sphericalangle cbd = 2\pi/3.$$

What about for the general convex polyhedron where each vertex is formed by the intersection of n planes? The argument we have made does not depended on the fact that $n = 3$ planes intersect at each vertex. The argument is valid for general n.

Retracing our steps from the original question we see that in general

$$\angle CAE = \angle CBE = \angle CBA = \frac{1}{2}\angle CBD = \frac{1}{2}\left(\frac{2\pi}{n}\right) = \frac{\pi}{n}$$

Hence, we also have $\sphericalangle cae = \pi/n$.

To this point we have established that the right spherical triangle Δcea has $\sphericalangle aec = \pi/2$, $\sphericalangle ace = \pi/m$ and $\sphericalangle cae = \pi/n$.

Now consider the special case (R8) of the spherical Pythagorean Theorem which in the notation of this problem states that

$$\cos(\sphericalangle cae) = \sin(\sphericalangle ace) \cdot \cos\left(\frac{\widehat{ce}}{r}\right).$$

We have taken $r = 1$ for this problem so this will become

$$\cos\left(\frac{\pi}{n}\right) = \sin\left(\frac{\pi}{m}\right) \cdot \cos(\widehat{ce})$$

$\angle EOC$ is the the central angle that corresponds to $\widehat{ce}$. Thus, $\widehat{ce} = \angle EOC \cdot r = \angle EOC$.

We already have established that ΔOCE is a right angle. Therefore

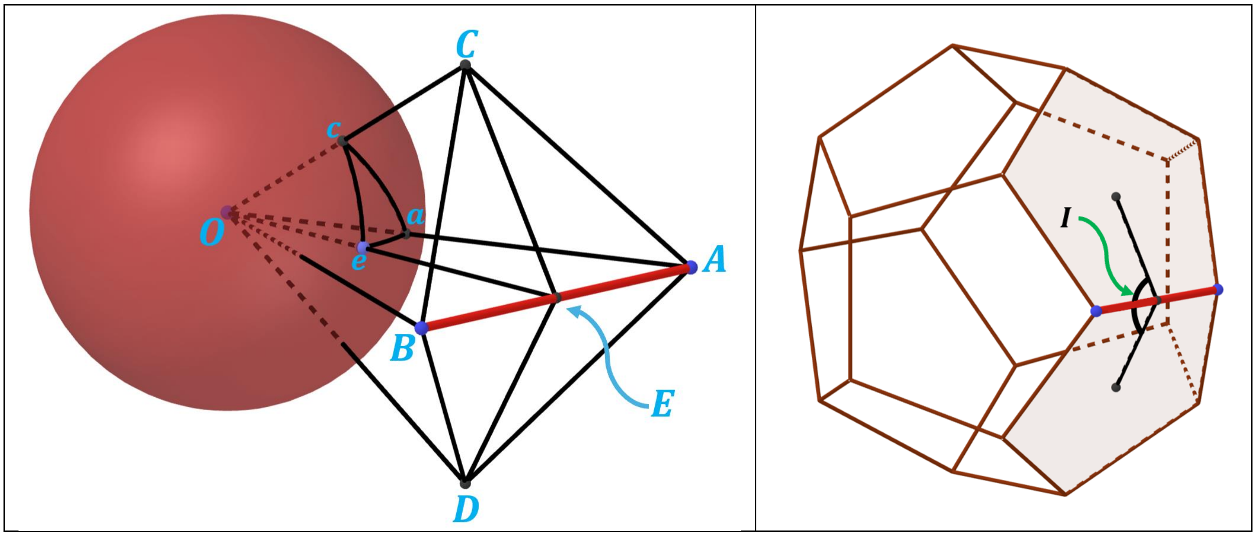
$$\angle OEC = \frac{\pi}{2} - \angle CEO.$$

$$\cos\left(\frac{\pi}{n}\right) = \sin\left(\frac{\pi}{m}\right) \cdot \cos(\widehat{ce})$$

$$= \sin\left(\frac{\pi}{m}\right) \cdot \cos(\angle EOC)$$

$$= \sin\left(\frac{\pi}{m}\right) \cdot \cos\left(\frac{\pi}{2} - \angle OEC\right)$$

Now look back to the beginning of this lecture to see how we defined I, the angle between two adjacent faces in our convex regular polyhedral.



Looking at these two figures sided by side we can see that $\angle OEC = \frac{I}{2}$.

So, we can conclude that

$$\cos\left(\frac{\pi}{n}\right) = \sin\left(\frac{\pi}{m}\right) \cdot \cos\left(\frac{\pi}{2} - \angle OEC\right)$$

$$= \sin\left(\frac{\pi}{m}\right) \cdot \sin(\angle OEC)$$

$$= \sin\left(\frac{\pi}{m}\right) \cdot \sin\left(\frac{I}{2}\right)$$

or

$$\sin\left(\frac{I}{2}\right) = \frac{\cos\left(\frac{\pi}{n}\right)}{\sin\left(\frac{\pi}{m}\right)}.$$

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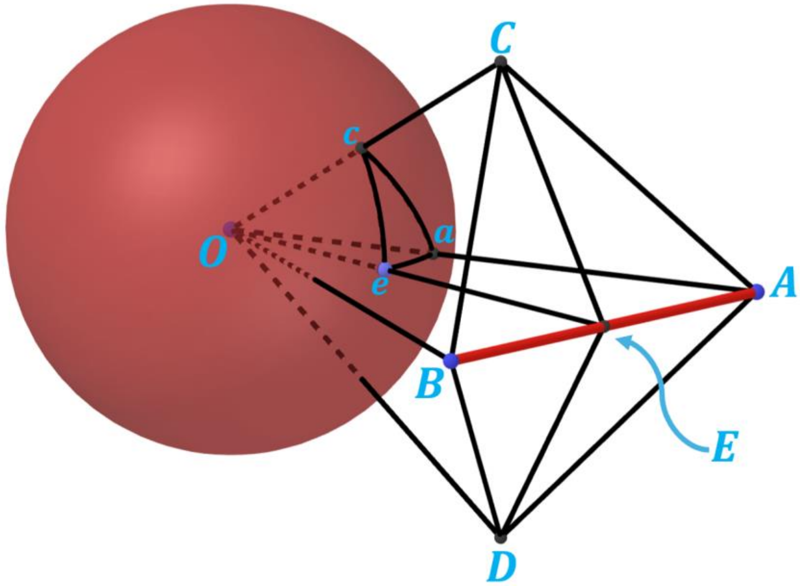
Insphere and Circumsphere

We can use this last result to find the radius r of the insphere (the largest sphere that fits inside) of a regular polyhedron and the radius R of the circumsphere (the smallest sphere that surrounds) of a regular polyhedron.

Using the notation in our previous diagram,



the radius r of the insphere is OC and the radius R of the circumsphere is OA.

Finding r.

Consider the right triangle ΔCEA (with right angle at E). It follows from this triangle that

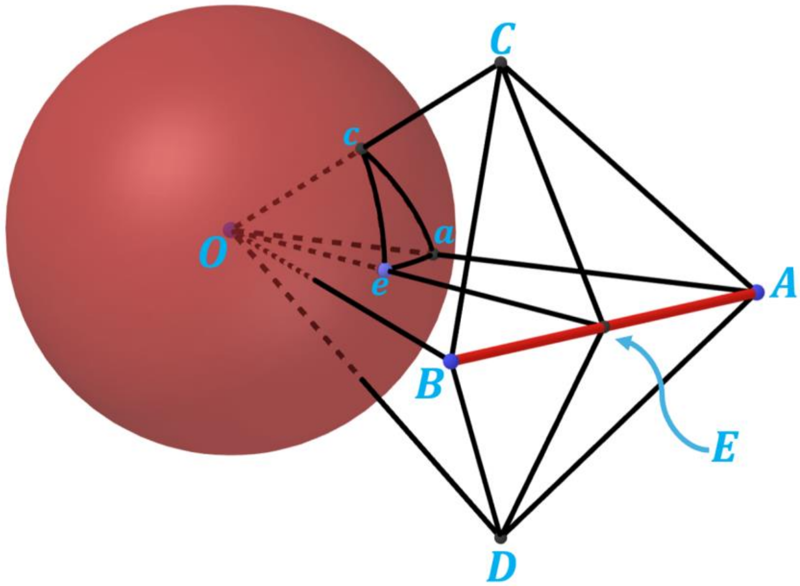
$$CE = AE \cdot \cot(\angle ACE) = AE \cdot \cot\left(\frac{180°}{m}\right)$$

Now consider the right triangle ΔOCE (with right angle at C). It follows from this triangle that

$$r = OC = CE \cdot \tan(\angle CEO) = CE \cdot \tan\left(\frac{I}{2}\right) = AE \cdot \cot\left(\frac{180°}{m}\right) \cdot \tan\left(\frac{I}{2}\right).$$

Finding R.

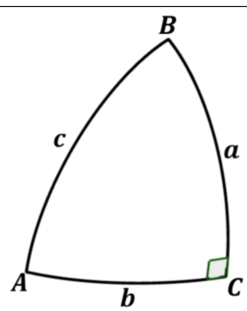
Looking again at our previous figure we can see that the central angle associated with great arc $\widehat{ac}$ is $\angle COA$. Hence, $\widehat{ac} =$ (radius of the sphere) $\cdot \angle COA$. And as we are assuming our sphere has radius 1 we have $\widehat{ac} = \angle COA$.



Thus,

$$\cos(\widehat{ac}) = \cos(\angle COA) = \frac{OC}{OA} = \frac{r}{R}.$$

Now look back at the Pythagorean relationship (R10) for right spherical triangles.

	(R10) $\cos\left(\frac{c}{r}\right) = \cot(A)\cot(B)$ where again we have $r = 1$ in this problem.

Applied to the right spherical triangle Δaec with spherical right angle at e, we find

$$\cos(\widehat{ca}) = \cot(\sphericalangle cae) \cdot \cot(\sphericalangle ace).$$

But we have already established that

$$\sphericalangle cae = 180°/n \text{ and } \sphericalangle ace = 180°/m.$$

Therefore,

$$\cos(\widehat{ca}) = \cot\left(\frac{180°}{n}\right) \cdot \cot\left(\frac{180°}{m}\right)$$

and

$$\frac{r}{R} = \cot\left(\frac{180°}{n}\right) \cdot \cot\left(\frac{180°}{m}\right).$$

Solving for R, we have

$$\begin{aligned} R &= \tan\left(\frac{180°}{n}\right) \cdot \tan\left(\frac{180°}{m}\right) \cdot r \\ &= \tan\left(\frac{180°}{n}\right) \cdot \tan\left(\frac{180°}{m}\right) \cdot AE \cdot \cot\left(\frac{180°}{m}\right) \cdot \tan\left(\frac{I}{2}\right) \\ &= AE \cdot \tan\left(\frac{180°}{n}\right) \cdot \tan\left(\frac{I}{2}\right). \end{aligned}$$

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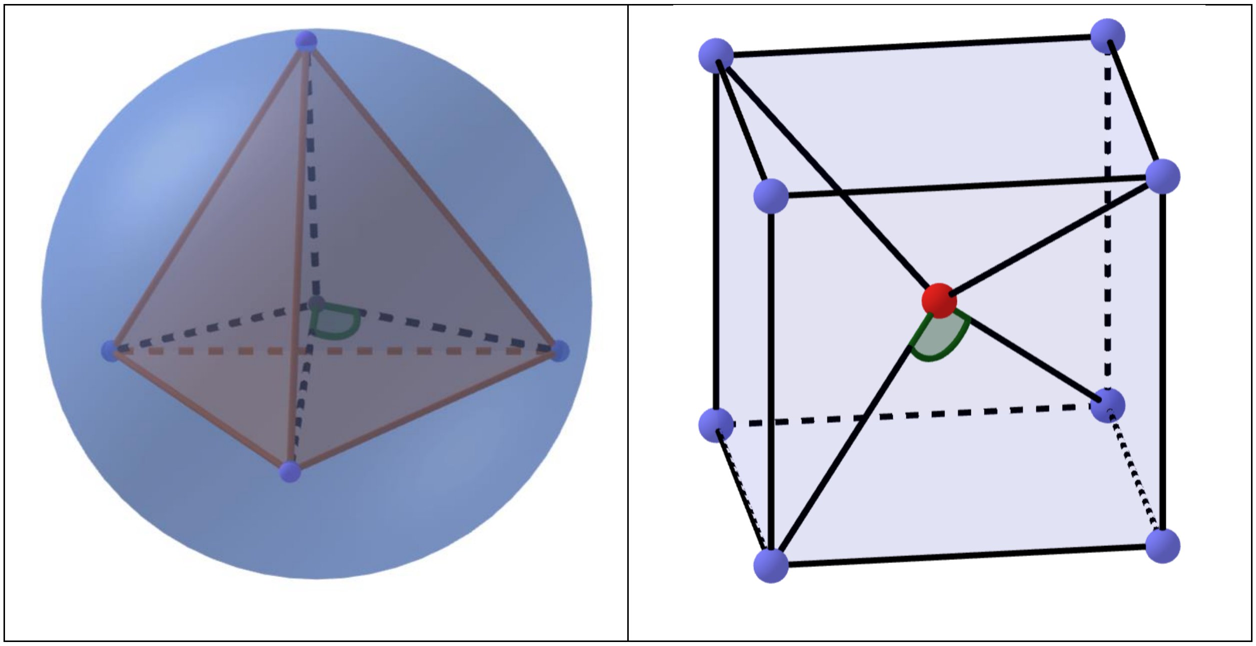
Central Angle in a Tetrahedron



You might have seen this molecular structure on the right side in your chemistry class, its methane, CH_4. **What is the bond angle in methane?**

We can take advantage of the work we put into angles and radii of the general regular polyhedral to solve this chemistry problem.

We just determined that the formula for the radius R of the circumsphere is

$$R = AE \cdot \tan\left(\frac{180°}{n}\right) \cdot \tan\left(\frac{I}{2}\right)$$

with

$$\sin\left(\frac{I}{2}\right) = \frac{\cos\left(\frac{180°}{n}\right)}{\sin\left(\frac{180°}{m}\right)}$$

and where AE is half of the length of an edge, n is the number of planes that meet at a vertex and m is the number sides a face has.

In the case of a tetrahedron we can see by inspection that $n = 3$ and $m = 3$. Therefore,

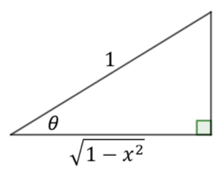
$$\sin\left(\frac{I}{2}\right) = \frac{\cos\left(\frac{180°}{n}\right)}{\sin\left(\frac{180°}{m}\right)} = \frac{\cos(60°)}{\sin(60°)} = \frac{1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}}.$$

Hence

$$\frac{I}{2} = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right).$$

Therefore,

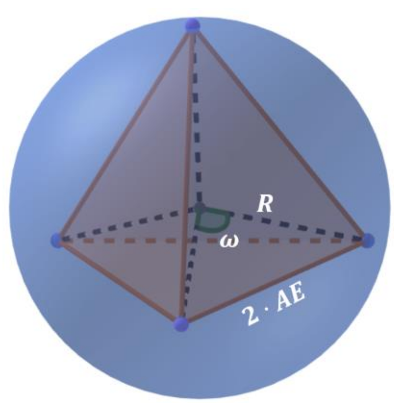
$$\tan\left(\frac{I}{2}\right) = \tan\left(\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)\right) = \frac{\frac{1}{\sqrt{3}}}{\sqrt{1-\left(\frac{1}{\sqrt{3}}\right)^2}} = \frac{\frac{1}{\sqrt{3}}}{\frac{\sqrt{2}}{\sqrt{3}}} = \frac{1}{\sqrt{2}}.$$

[Recall that in general, $\tan\left(\cos^{-1}(\theta)\right) = \frac{\theta}{\sqrt{1-\theta^2}}$, ]

So, in the case of a tetrahedron the radius R of the circumsphere is

$$R = AE \cdot \tan(60°) \cdot \frac{1}{\sqrt{2}} = AE \cdot \frac{\sqrt{3}}{\sqrt{2}} = AE \cdot \frac{\sqrt{6}}{2}.$$

Our updated diagram looks as follows.



We can use the (planar) Law of Cosines to solve for this missing angle.

$$(2 \cdot AE)^2 = \left(AE \cdot \frac{\sqrt{6}}{2}\right)^2 + \left(AE \cdot \frac{\sqrt{6}}{2}\right)^2 - 2\left(AE \cdot \frac{\sqrt{6}}{2}\right)\left(AE \cdot \frac{\sqrt{6}}{2}\right)\cos(\omega)$$

$$4 = \frac{6}{4} + \frac{6}{4} - 2\left(\frac{6}{4}\right)\cos(\omega)$$

$$1 = -3\cos(\omega)$$

$$\cos(\omega) = \frac{1}{-3}$$

$$\omega = \cos^{-1}(-1/3) \approx 109.4712206°.$$

Homework

105. Surface of Regular Polyhedron.

The area of the surface of a regular polyhedron can be expressed in terms of r and the angle POI, the value of which angle is obtained from the equation—

$$\cos \mathrm{POI} = \cot \frac{\pi}{m} \cot \frac{\pi}{n}.$$

Call this angle θ.

$$\text{Then } \mathrm{IP} = r \tan \theta,$$

$$\text{and the area of the triangle PIQ} = \tfrac{1}{2}(r \tan \theta)^2 \sin \frac{2\pi}{n};$$

$$\therefore \text{the area of the polygon} = \frac{n}{2} r^2 \tan^2 \theta \sin \frac{2\pi}{n};$$

$$\therefore \text{the surface of the polyhedron} = \frac{n\mathrm{P}}{2} r^2 \tan^2 \theta \sin \frac{2\pi}{n}$$

$$= \mathbf{Lr^2 \tan^2 \theta \sin \frac{2\pi}{n}};$$

where L is the least common multiple of 2, m, and n, and is equal to the number of edges in the polyhedron.

105. Surface of Regular Polyhedron.

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Call this angle θ.

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