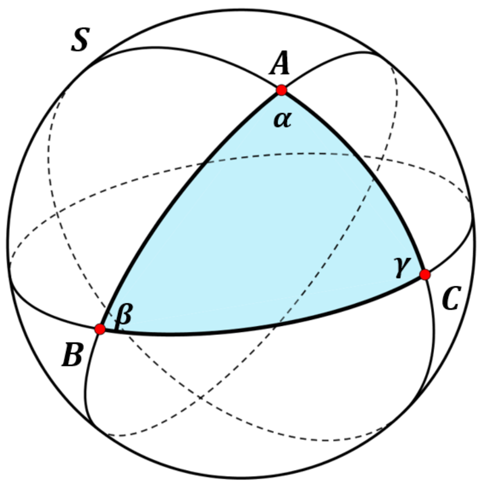
Tuesday Morning, 1st Hour, June 25

Area of a Spherical Triangle and Spherical Excess

Area of a Spherical Triangle Knowing Its Three Angles (Girard's Theorem)

The area of a spherical triangle with angles α, β and γ on a sphere with radius r equals

$$\text{Area} = (\alpha + \beta + \gamma - \pi) \cdot r^2 = \text{E} \cdot r^2.$$

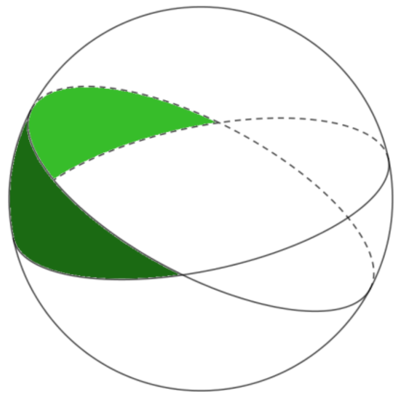


Note 1: The factor $\text{E} = \alpha + \beta + \gamma - \pi$ is called the **spherical excess** of a spherical triangle. It is the amount by which the sum of the angles of a spherical triangle exceeds π (i.e. $180°$).

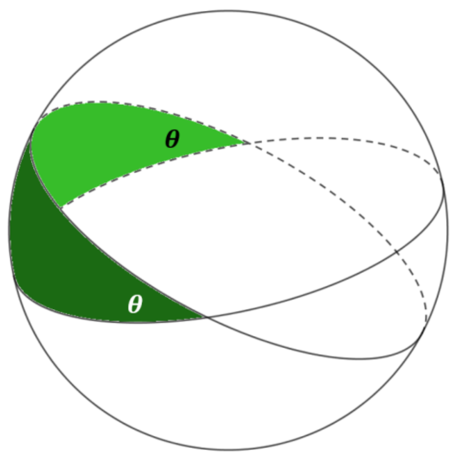
Note 2: Girard's Theorem highlights a distinction between planar and spherical triangles. On a sphere (with a fixed radius) a set of three angles of a triangle fixes area. Obviously, that is not the case for planar triangles where a set of three angles fixes shape (all planar triangles with the same angles are similar) but not the area (similar triangles can be arbitrarily big or small).

Proof

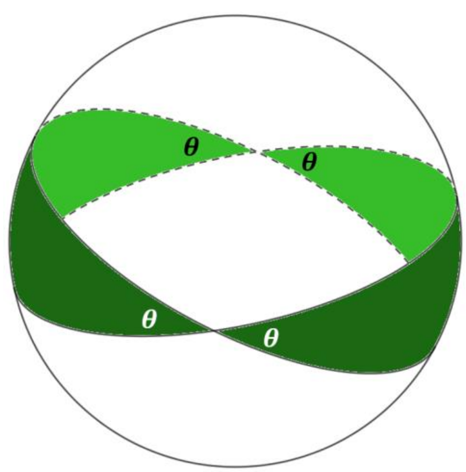
The region bounded by two great circles as shown below is called a spherical **lune** (also called a biangle or digon).



The angle θ between two great circles at an intersection point equals the angle between the tangents to the two great circles at that point. We note that by the symmetry of a sphere the angle at both vertices are equal.



We also note that a lune and its antipodal version (the lune on the opposite side of the sphere) are congruent.



Our first step in finding the area of a spherical triangle is to find the area of a (spherical) lune.

Let A_θ be the area of a lune with angle θ on a sphere with radius r. A lune with $\theta = 2\pi$ is the entire sphere and hence has area $A_{2\pi} = 4\pi r^2$.

By a proportionality argument*

$$\frac{\theta}{2\pi} = \frac{A_\theta}{A_{2\pi}}$$

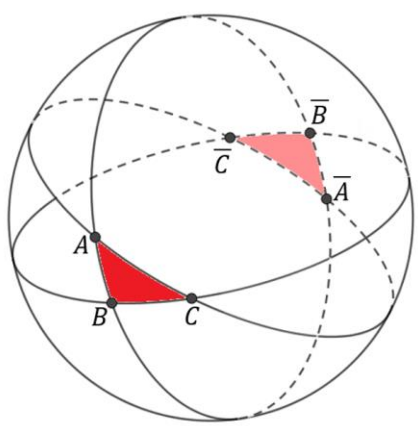
or

$$A_\theta = \frac{\theta}{2\pi} \cdot 4\pi r^2 = 2\theta r^2.$$

(*Reasoning "by proportionality" is generally accepted as valid on face value in spherical trigonometry textbooks but actually it requires justification. The typical justification uses surface integration which is part of a multivariate calculus course. But following tradition we will accept a proportionality argument here without justification.)

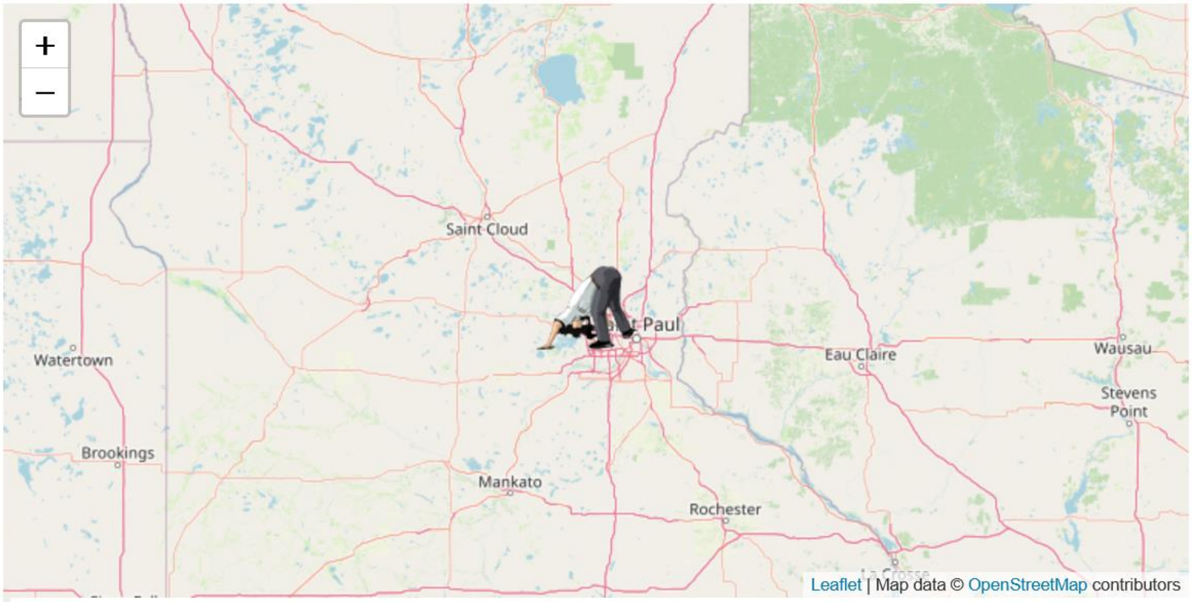
And because a lune and its antipodal version are congruent, the antipodal lune also has area $2\theta r^2$.

Now consider a general spherical triangle $\Delta_s ABC$ and its congruent antipodal spherical triangle $\Delta_s \overline{A}\,\overline{B}\,\overline{C}$.



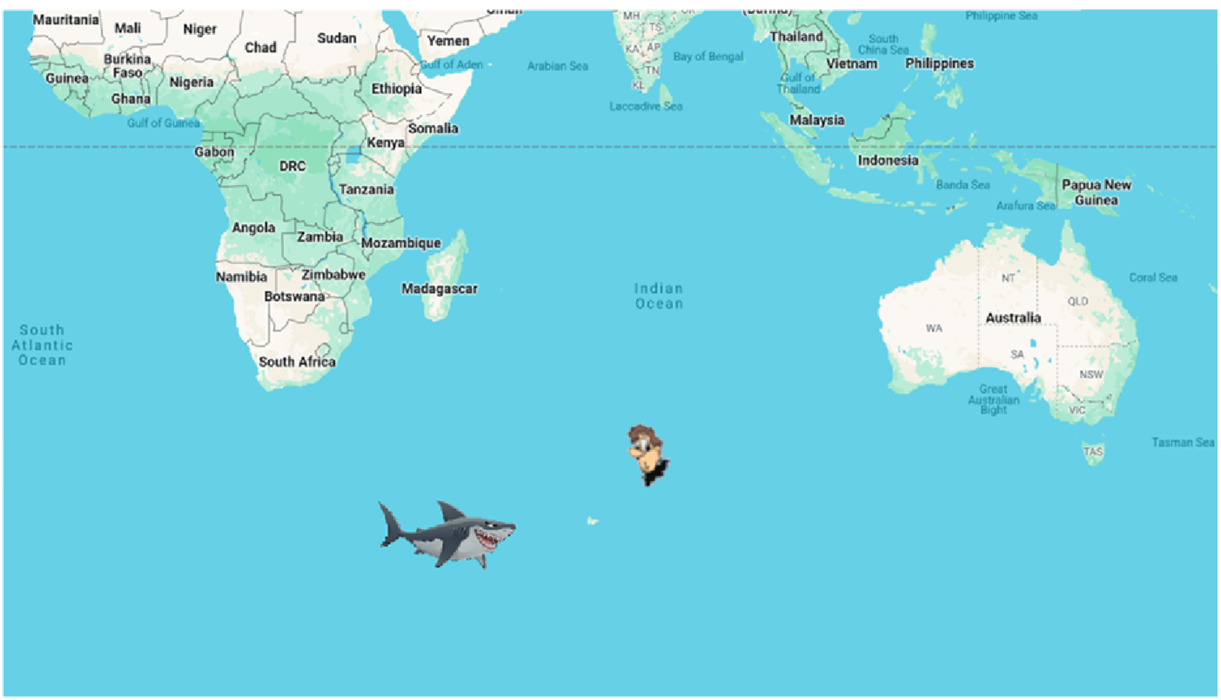
Caution: Don't confuse <u>antipodal</u> spherical triangles with <u>polar</u> spherical triangles (see Lecture 1). They are two ***different concepts*** and need to be kept separate.

Ever Wondered Where Your Antipodal Self Is?



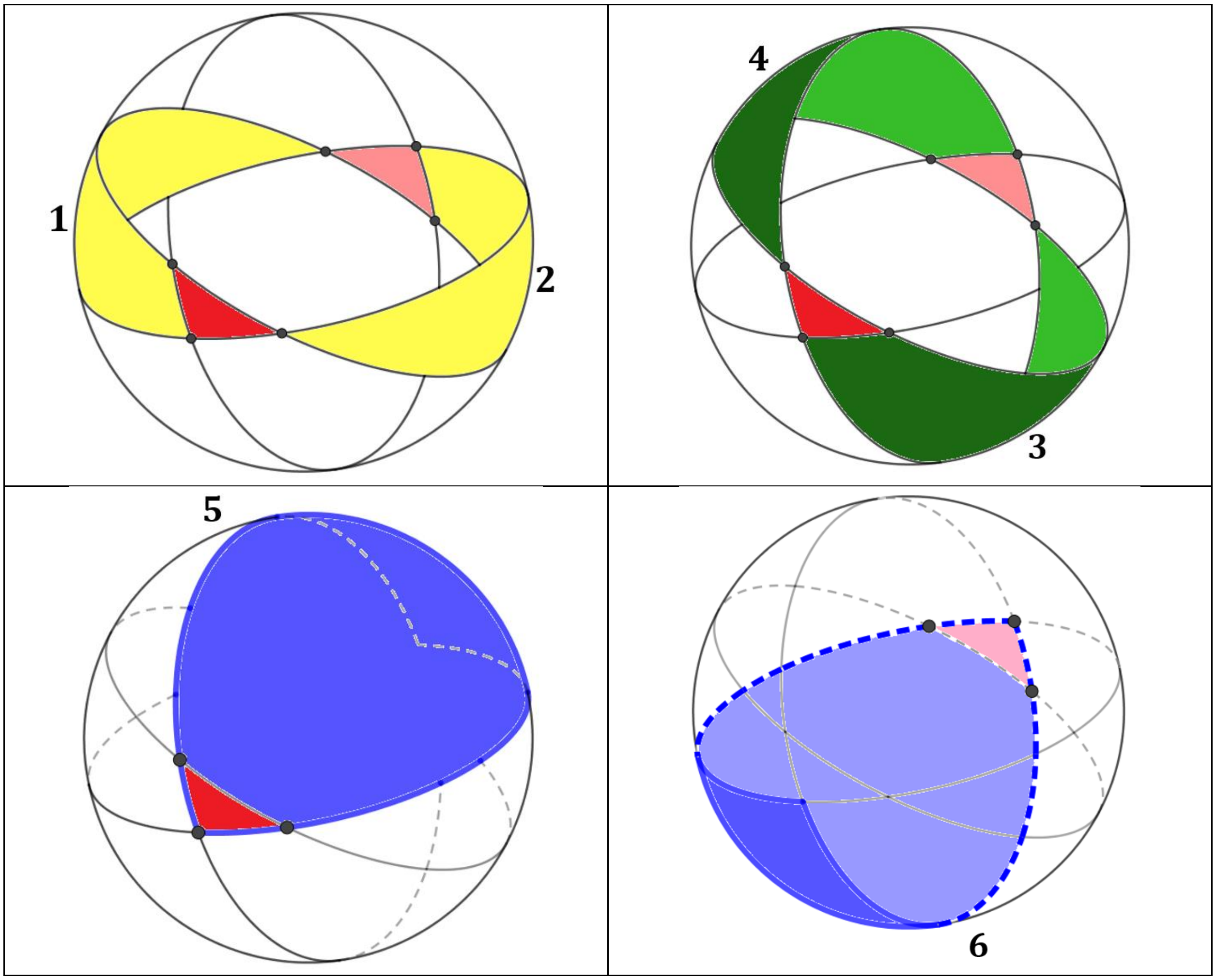
$(44° \, 58' \, 38.3" \, N, 93° \, 15' \, 55.7" \, W)$ https://www.antipodesmap.com/

Oh, no! Try to keep your head above water and watch out for sharks. If we went to the other side of the globe from here, we would find ourselves in a real hotspot for Great Whites.

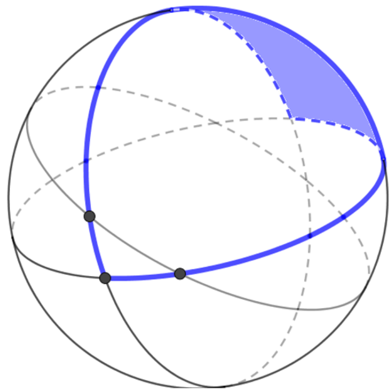


$(44° \, 58' \, 38.3" \, S, 86° \, 44' \, 4.3" \, E)$

Let $\angle BAC = \angle \overline{B}\,\overline{A}\,\overline{C} = \alpha$, $\angle ABC = \angle \overline{A}\,\overline{B}\,\overline{C} = \beta$ and $\angle BCA = \angle \overline{B}\,\overline{C}\,\overline{A} = \gamma$ and consider the six spherical lunes created by the three great circles forming the sides of these two congruent antiopodal spherical triangles.



For clarity, note that lune #5 stretches around the backside of the sphere and includes the region



While it might be hard to picture from these 2-D sketches, these six lunes completely cover the sphere.

However, they are not disjoint (they have overlap).

In fact, the union of these six lunes includes the red spherical triangle $\Delta_s ABC$ three times and includes the congruent antipodal pink spherical triangle $\Delta_s \overline{A}\,\overline{B}\,\overline{C}$ three times.

It follows that

$$\begin{aligned}\text{Area(Sphere)} &= \sum_{i=1}^{6} \text{Area(Lune } i) - 2 \cdot \text{Area}(\Delta_s ABC) - 2 \cdot \text{Area}\left(\Delta_s \overline{A}\,\overline{B}\,\overline{C}\right) \\ &= \sum_{i=1}^{6} \text{Area(Lune } i) - 4 \cdot \text{Area}(\Delta_s ABC)\end{aligned}$$

because $\Delta_s ABC$ and its antipode $\Delta_s \overline{A}\,\overline{B}\,\overline{C}$ are congruent and hence have the same area.

Filling in the values we've already found for the areas of the sphere and each lune we have

$$4\pi r^2 = 2\gamma r^2 + 2\gamma r^2 + 2\alpha r^2 + 2\alpha r^2 + 2\beta r^2 + 2\beta r^2 - 4 \cdot \text{Area}(\Delta_s ABC).$$

or

$$4 \cdot \text{Area}(\Delta_s ABC) = 2\gamma r^2 + 2\gamma r^2 + 2\alpha r^2 + 2\alpha r^2 + 2\beta r^2 + 2\beta r^2 - 4\pi r^2.$$

Simplifying we see that

$$\text{Area}(\Delta_s ABC) = (\alpha + \beta + \gamma - \pi)r^2 = \mathrm{E} \cdot r^2.$$

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Girard's Theorem allows one to find the area of a spherical triangle knowing only its three angles. There is a companion result which allows us to find the area of a spherical triangle knowing only its three sides.

Area of a Spherical Triangle Knowing Only Its Three Sides

$$\text{Area}(\Delta_s ABC) = 4r^2 \tan^{-1}\left(\sqrt{\tan\left(\frac{s}{2r}\right)\tan\left(\frac{s-a}{2r}\right)\tan\left(\frac{s-b}{2r}\right)\tan\left(\frac{s-c}{2r}\right)}\right)$$

where $2s = a + b + c$.

Proof

Let $\mathrm{E} = \alpha + \beta + \gamma - \pi$ represent the spherical excess of a spherical triangle. Then **L'Huilier's Theorem** (proof omitted) for spherical triangles states that

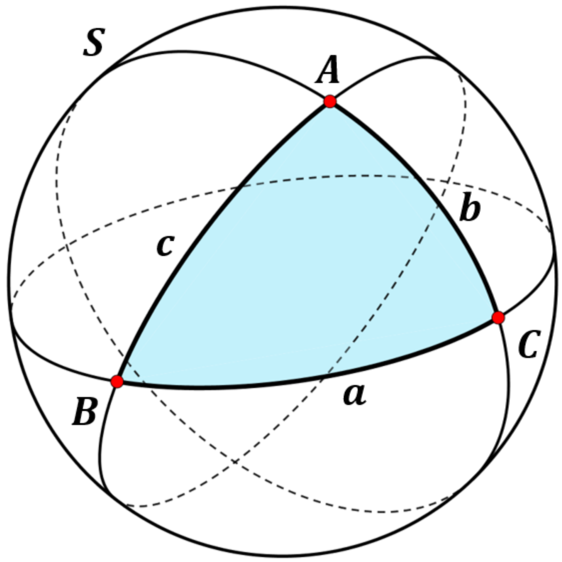
$$\tan^2\left(\frac{\mathrm{E}}{4}\right) = \tan\left(\frac{s}{2r}\right)\tan\left(\frac{s-a}{2r}\right)\tan\left(\frac{s-b}{2r}\right)\tan\left(\frac{s-c}{2r}\right)$$

where $2s = a + b + c$.

Combining this with Girard's Theorem we see that.

$$\text{Area}(\Delta_s ABC) = \mathrm{E} \cdot r^2 = 4r^2 \tan^{-1}\left(\sqrt{\tan\left(\frac{s}{2r}\right)\tan\left(\frac{s-a}{2r}\right)\tan\left(\frac{s-b}{2r}\right)\tan\left(\frac{s-c}{2r}\right)}\right)$$

where $2s = a + b + c$.



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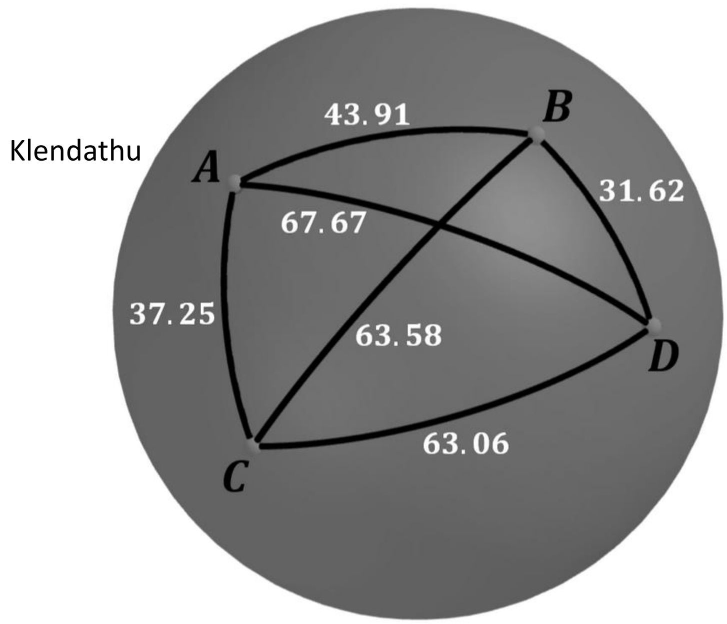
It is constructive at this point to contrast this result with **Heron's formula** for the area of a planar triangle based on the three sides of a planar triangle.

$$\text{Area}(\Delta ABC) = \sqrt{s(s-a)(s-b)(s-c)}$$

where $2s = a + b + c$.

Watch Out Johnny Rico

Starship officer Johnny Rico used the enhanced jump jet on his powered armor suit to bounce around the spherical planet Klendathu, the home world of the dreaded underground Bugs. He was able to determine the following distances between four suspected Bug hives (A, B, C and D) as part of his scouting mission.



Use this information to find the radius of the planet Klendathu. (FYI. This story line comes from Robert Heinlein’s 1959 science fiction novel *Starship Troopers*.)

Solution

Calculate the area of the quadrilateral in two ways and equate:

$$\text{Area}(\Delta_S ACD) + \text{Area}(\Delta_S ABD) = \text{Area}(\Delta_S ABC) + \text{Area}(\Delta_S DBC)$$

We will use the second formula for area because we have information about the sides but not the angles.

	$\frac{s}{2}$	$\frac{s-a}{2}$	$\frac{s-b}{2}$	$\frac{s-c}{2}$
$\Delta_s ACD$	83.99	23.37	10.465	8.16
$\Delta_S ABD$	71.6	13.845	19.99	1.965
$\Delta_S ABC$	72.37	14.23	4.395	17.56

$\Delta_S DBC$	79.13	23.755	7.775	8.035

$$\text{Area}(\Delta_s ACD) = 4r^2 \tan^{-1}\left(\sqrt{\tan\left(\frac{83.99}{r}\right)\tan\left(\frac{23.37}{r}\right)\tan\left(\frac{10.465}{r}\right)\tan\left(\frac{8.185}{r}\right)}\right)$$

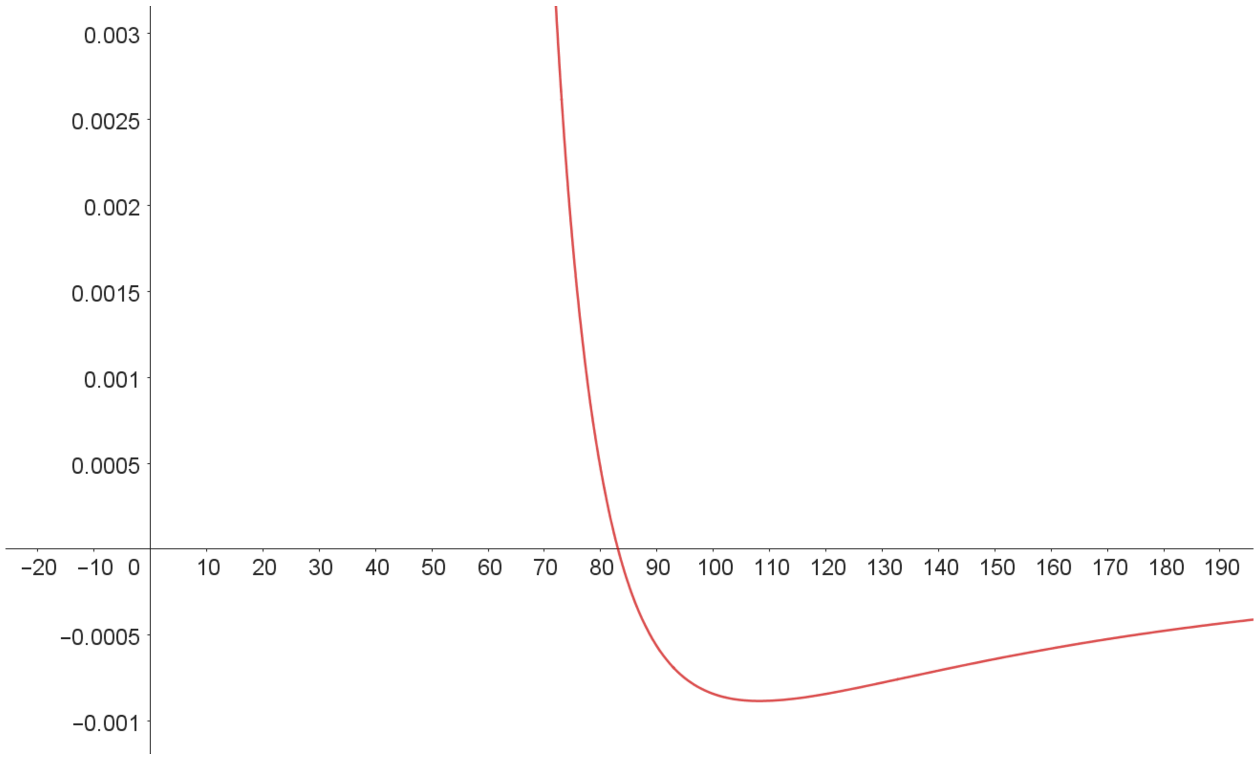
$$\text{Area}(\Delta_s ABD) = 4r^2 \tan^{-1}\left(\sqrt{\tan\left(\frac{71.6}{r}\right)\tan\left(\frac{13.845}{r}\right)\tan\left(\frac{19.99}{r}\right)\tan\left(\frac{1.965}{r}\right)}\right)$$

$$\text{Area}(\Delta_s ABC) = 4r^2 \tan^{-1}\left(\sqrt{\tan\left(\frac{72.37}{r}\right)\tan\left(\frac{14.23}{r}\right)\tan\left(\frac{4.395}{r}\right)\tan\left(\frac{17.56}{r}\right)}\right)$$

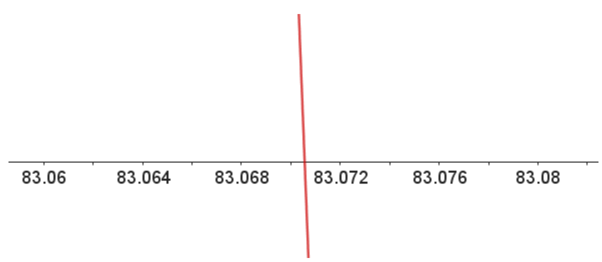
$$\text{Area}(\Delta_s DBC) = 4r^2 \tan^{-1}\left(\sqrt{\tan\left(\frac{79.13}{r}\right)\tan\left(\frac{23.755}{r}\right)\tan\left(\frac{7.775}{r}\right)\tan\left(\frac{8.035}{r}\right)}\right)$$

$$\begin{aligned}
0 &= \text{Area}(\Delta_S ACD) + \text{Area}(\Delta_S ABD) - \text{Area}(\Delta_S ABC) - \text{Area}(\Delta_S DBC) \\
&= 4r^2 \tan^{-1}\left(\sqrt{\tan\left(\frac{83.99}{r}\right)\tan\left(\frac{23.37}{r}\right)\tan\left(\frac{10.465}{r}\right)\tan\left(\frac{8.185}{r}\right)}\right) \\
&\quad +4r^2 \tan^{-1}\left(\sqrt{\tan\left(\frac{71.6}{r}\right)\tan\left(\frac{13.845}{r}\right)\tan\left(\frac{19.99}{r}\right)\tan\left(\frac{1.965}{r}\right)}\right) \\
&\quad -4r^2 \tan^{-1}\left(\sqrt{\tan\left(\frac{72.37}{r}\right)\tan\left(\frac{14.23}{r}\right)\tan\left(\frac{4.395}{r}\right)\tan\left(\frac{17.56}{r}\right)}\right) \\
&\quad -4r^2 \tan^{-1}\left(\sqrt{\tan\left(\frac{79.13}{r}\right)\tan\left(\frac{23.755}{r}\right)\tan\left(\frac{7.775}{r}\right)\tan\left(\frac{8.035}{r}\right)}\right)
\end{aligned}$$

Now we can divide out the common factor of $4r^2$ and what we have left is a equation with one unknown, namely r, the radius of Klendathu. This is a messy equation involving transcendental functions so we have no hope of solving this to get an exact answer. But graphing calculators will us to find r to any number of decimal places we want.



This plot using Geogebra shows $r \approx 83$. Zooming in further reveals



that the radius r is close to 83.07 miles.

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Exercise

Suppose the area of an equilateral spherical triangle is one fourth the area of the sphere it is drawn on. What are the angles and sides of this spherical triangle?

<u>Solution</u>

Let's start with a more fundamental question. Equilateral means "equal sides". We know that for planar triangles if the sides are equal in length then the angles will also be equal in measure.

That is, for planar triangles

equilateral ⟺ equiangular.

Is the same true for spherical triangles?

The answer is "Yes" and this follows immediately from the Spherical Law of Sines.

$$\frac{\sin\left(\frac{a}{r}\right)}{\sin(A)} = \frac{\sin\left(\frac{b}{r}\right)}{\sin(B)} = \frac{\sin\left(\frac{c}{r}\right)}{\sin(C)}.$$

Equilateral means $a = b = c$. But in this case it follows directly from the Spherical Law of Sines that $A = B = C$ (and vice versa).

Now suppose the sphere has radius r. Then the surface area of the sphere equals $4\pi r^2$. Let θ be the angle and let x be the side of the equilateral spherical triangle.

By Girard's Theorem (the formula for the area of a spherical triangle just knowing the angles) we have that

$$\text{Area of Triangle} = \frac{\pi r^2 E}{180} = \frac{\pi r^2(3\theta - 180)}{180}.$$

And if the area of the triangle is one-fourth the area of the sphere, then

$$\frac{\pi r^2(3\theta - 180)}{180} = \frac{1}{4}(4\pi r^2)$$

which allows us to solve for θ. In this case, solving for θ yields that $\theta = 120°$. (Note that the value of θ depends on what percentage the area of the triangle is of the sphere.)

We can use the Spherical Law of Cosines for Angles to find x, the side length of the equilateral spherical triangle.

Plugging in $120°$ for each angle in the Spherical Law of Cosines for Angles we have

$$\cos(120°) = -\cos(120°)\cos(120°) + \sin(120°)\sin(120°)\cos\left(\frac{x}{r}\right)$$

or

$$\cos\left(\frac{x}{r}\right) = \frac{\cos(120°) + \cos^2(120°)}{\sin^2(120°)} = \frac{\left(-\frac{1}{2}\right) + \left(-\frac{1}{2}\right)^2}{\left(\frac{\sqrt{3}}{2}\right)^2} = \frac{-\frac{1}{4}}{\frac{3}{4}} = -\frac{1}{3}.$$

Therefore,

$$x = r\cos^{-1}(-1/3).$$

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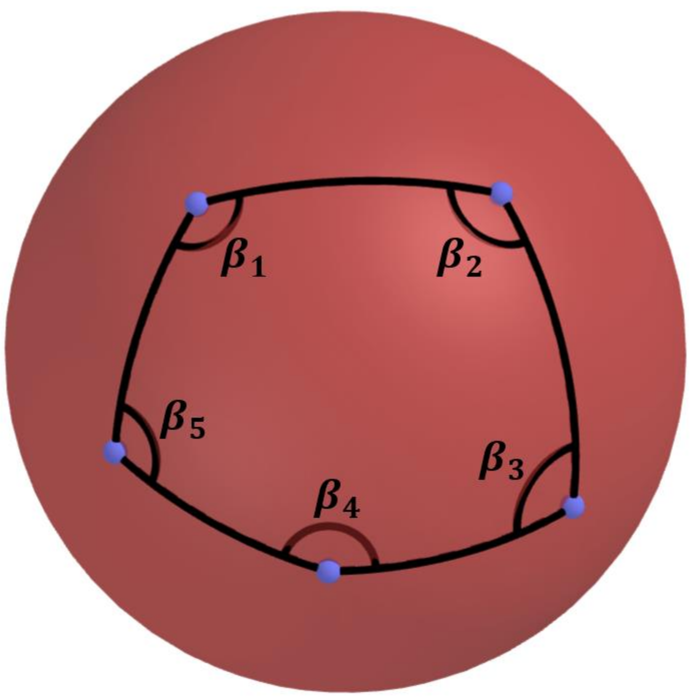
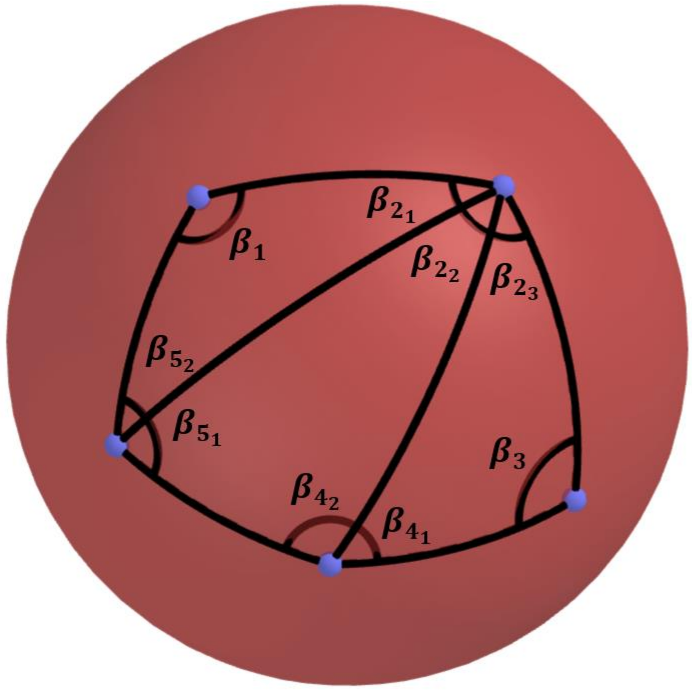
Area of a Spherical Polygon Knowing Its Angles

Girard's Theorem for the area of a spherical triangle can be extended to find the area of a spherical polygon with known angles. We will illustrate the idea on the pentagon shown below with general angles $\beta_1, \ldots, \beta_5$.



Now "triangulate" this pentagon by adding 3 arcs going from any one vertex to the 3 vertices it is not already connected to. (In general, if you start with an n side polygon you will need to add $n - 2$ great arcs to "triangulate" the polygon.)



This will split some of the angles into smaller angles. The area of the spherical polygon equals the sum of the 3 spherical triangles formed. By Girard's Theorem, we have

$$\text{Area of Spherical Pentagon} = \text{Area of Triangle } 1 + \text{Area of Triangle } 2 + \text{Area of Triangle } 3$$

$$= \left(\beta_{5_2} + \beta_1 + \beta_{2_1} - \pi\right) \cdot r^2 + \left(\beta_{2_2} + \beta_{5_1} + \beta_{4_2} - \pi\right) \cdot r^2 + \left(\beta_{2_3} + \beta_3 + \beta_{4_1} - \pi\right) \cdot r^2$$

$$= \beta_1 r^2 + \left(\beta_{2_1} + \beta_{2_2} + \beta_{2_3}\right) r^2 + \beta_3 r^2 + \left(\beta_{4_1} + \beta_{4_2}\right) r^2 + \left(\beta_{5_1} + \beta_{5_2}\right) r^2 - 3\pi r^2$$

$$= (\beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5 - 3\pi) r^2.$$

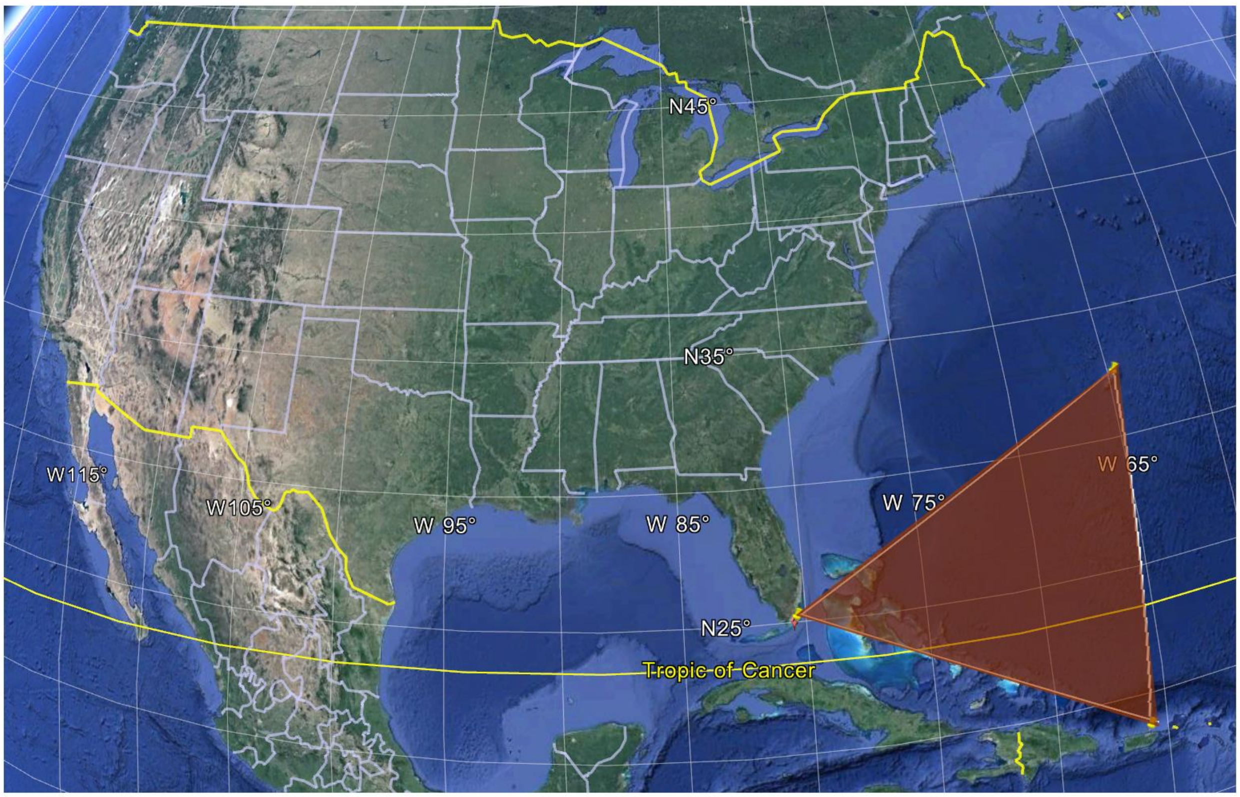
And more generally if we started with an n sided spherical polygon, its formula for area becomes n - gon

$$\text{Area of Spherical } n\text{ - gon} = \left(\beta_1 + \beta_2 + \cdots + \beta_n - (n-2)\pi\right) r^2.$$

Homework

(1) Colorado extends from $37°$ to $40°$ N and from $102°$ to $107°$ W. What is the area of Colorado in statute square miles?

(2) The Bermuda Triangle is a region deemed as having an unusually high number of disappearances of ships and planes. It extends from Puerto Rico $(18.5°N, 66°W)$, Bermuda $(32.3°N, 64.9°W)$ to the southern tip of Florida $(25°N, 80.5°W)$. Just how large is the (*spherical*) Bermuda Triangle in square miles? Assume the Earth's radius is 3,958.8 miles.



(3) Find the area of a spherical triangle A with angles of $50°, 110°$ and $80°$ on a S sphere with radius $r = 5$. Find the area of A', the polar dual of spherical triangle A on sphere S.

(4) Find the angles of a equilateral spherical triangle whose area is equal to the area of a great circle.

(5) The area of a spherical triangle is one tenth the area of the surface of the sphere on which it lies. Two angles of the triangle are $96°$ and $87°$. Find the third angle.

(6) On a sphere with an area of 72 square inches, find the area of a spherical quadrilateral whose angles are $100°, 120°, 140°$ and $160°$.

Some Solutions

(4) Find the angles of an equilateral spherical triangle whose area is equal to the area of a great circle.

Solution

Suppose this equilateral spherical triangle is on a sphere of radius r.

Recall that an equilateral spherical triangle is also equiangular. Let that angle be $\beta°$. Then the area of that equilateral spherical triangle

$$\text{Area} = \left(\beta°\left(\frac{\pi}{180}\right) + \beta°\left(\frac{\pi}{180}\right) + \beta°\left(\frac{\pi}{180}\right) - \pi\right) \cdot r^2$$
$$= \left(\frac{3\pi\beta°}{180} - \pi\right) \cdot r^2$$

The area of a great circle on a sphere of radius r is πr^2.

Equating the area of the spherical triangle with the area of the great circle we have

$$\left(\frac{3\pi\beta°}{180} - \pi\right) \cdot r^2 = \pi r^2$$

$$\frac{3\pi\beta°}{180} - \pi = \pi$$

Dividing through by the common factor π we have

$$\frac{3\beta°}{180} - 1 = 1$$

or

$$3\beta° = 2(180)$$

$$\beta^\circ = \frac{2}{3}(180) = 120^\circ.$$

(5) The area of a spherical triangle is one tenth the area of the surface of the sphere on which it lies. Two angles of the triangle are 96° and 87°. Find the third angle.

<u>Solution</u>

Suppose the spherical triangle is on a sphere with radius r.

Let the third angle of this spherical triangle be γ°. Then the area of this spherical triangle equals

$$\text{Area } \Delta = \left(96^\circ\left(\frac{\pi}{180}\right) + 87^\circ\left(\frac{\pi}{180}\right) + \gamma^\circ\left(\frac{\pi}{180}\right) - \pi\right) \cdot r^2$$

The surface area of sphere of radius r is $4\pi r^2$.

We are given that

$$\text{Area } \Delta = \frac{1}{10} \cdot \text{Area Sphere}$$

$$\left(96^\circ\left(\frac{\pi}{180}\right) + 87^\circ\left(\frac{\pi}{180}\right) + \gamma^\circ\left(\frac{\pi}{180}\right) - \pi\right) \cdot r^2 = \frac{1}{10}(4\pi r^2)$$

Multiply through by 180 and divide through by πr^2.

$$96^\circ + 87^\circ + \gamma^\circ - 180 = 4 \cdot 18$$

$$\gamma^\circ = 4(18) + 180 - 96 - 87 = 69^\circ$$

(6) On a sphere with an area of 72 square inches, find the area of a spherical quadrilateral whose angles are $100°, 120°, 140°$ and $160°$.

<u>Solution</u>

Surface area of a sphere with radius r is $4\pi r^2$ which is given to be 72 square inches.

$$r^2 = \frac{72}{4\pi} = \frac{18}{\pi}$$

Area of spherical quadrilateral with angles $100°, 120°, 140°$ and $160°$ equals

$$\big(100 + 120 + 140 + 160 - (4-2)\pi\big)r^2$$

$$= (520 - 2\pi)r^2$$

$$= (520 - 2\pi)\left(\frac{18}{\pi}\right)$$

$$= 9246.902664 \text{ square inches}$$