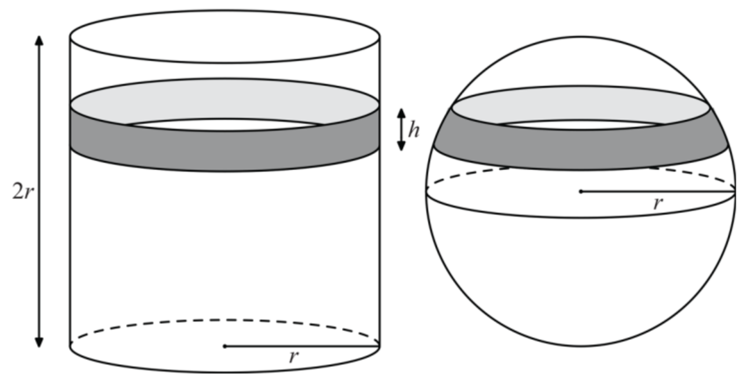
Tuesday Morning, 2nd Hour, June 25

Seeing how nice the formulas for a spherical triangle and more generally a spherical n-gon worked out, perhaps there are other regions on a sphere with equally nice formulas.

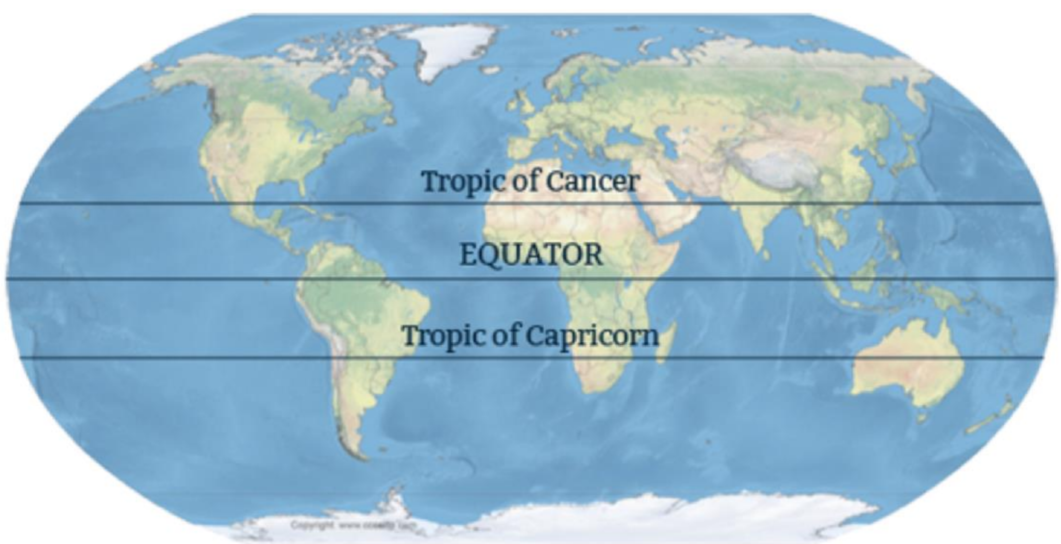
There are! And their discovery dates back to the wizard of Syracuse.

No not Frank Baum of the Wizard of Oz fame who spent much of his life around the corner from Syracuse University, but the other wizard of Syracuse, Archimedes, the ancient mathematician, engineer, astronomer, physicist and inventor spent his entire life in Syracuse (Sicily).



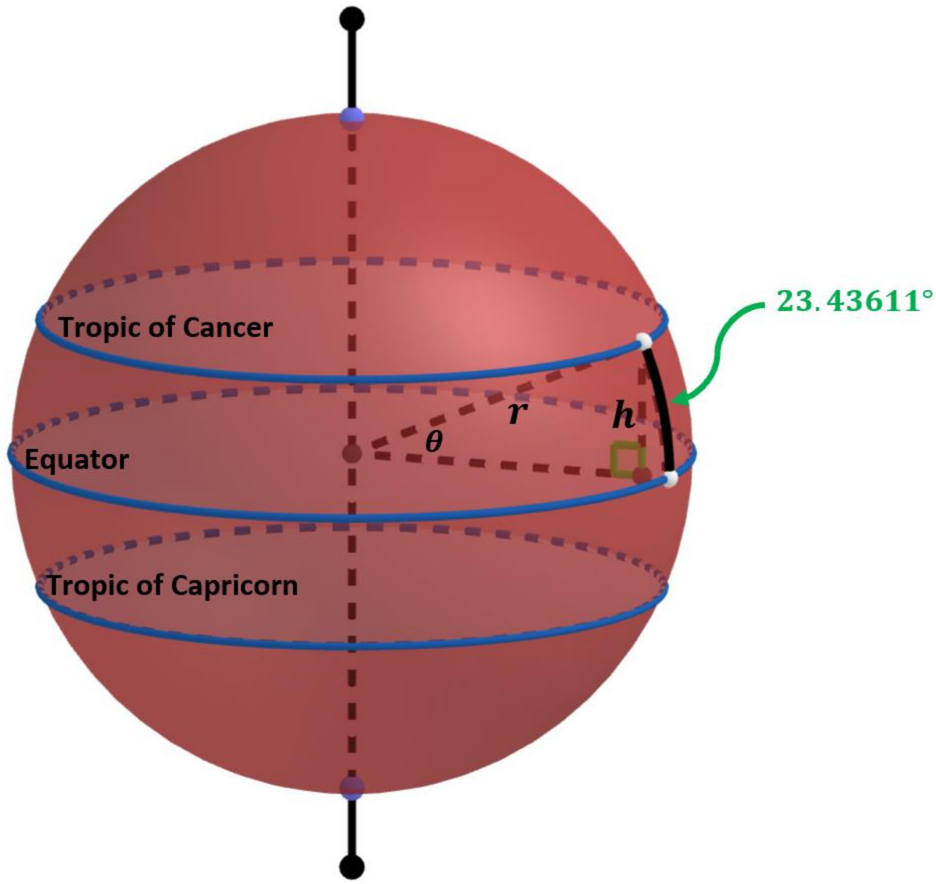
Amazingly, Archimedes was able to prove that the above two bands have equal areas using physical principles of equal weight. His insight allows us to answer the following problems that are usually place under the aegis of integral calculus.

(1) What percentage of the earth's surface lies between the Tropic of Cancer (latitude $23.43611°N$) and the Tropic of Capricorn (latitude $23.4311°S$)?

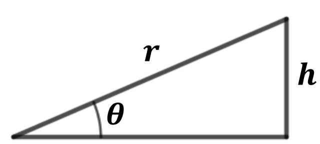


Solution

Because the angular distance between the Tropic of Cancer and the Equator is $23.43611°$ we know that the associated central angle θ in the diagram below has angular distance $23.43611°$.



It follows that the height h of associated right triangle equals $r\sin(\theta) = r\sin(23.43611)$.



$$h = r\sin(\theta)$$

Hence the zone between the Tropic of Cancer and Tropic of Capricorn must have height

$$2h = 2r\sin(23.43611).$$

It follows that this tropical zone makes up

$$\frac{\text{Area of spherical zone}}{\text{Area of entire sphere}} \cdot 100\% = \frac{2\pi r(2h)}{2\pi r(2r)} \cdot 100\%$$

$$= \frac{h}{r} \cdot 100\% = \sin(23.43611) \cdot 100\%$$

$$= 39.77262159\%$$

of the Earth's surface

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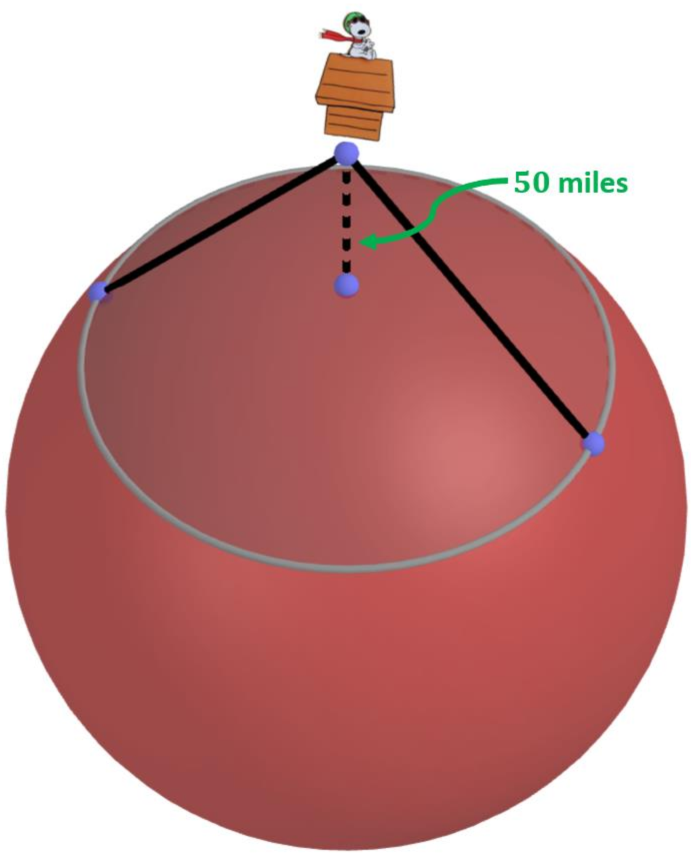
(2) How many square miles of the Earth's surface can Snoopy see from his flying doghouse if he is in orbit 50 miles above the surface? Take the radius of the Earth as 3,958.8 miles.

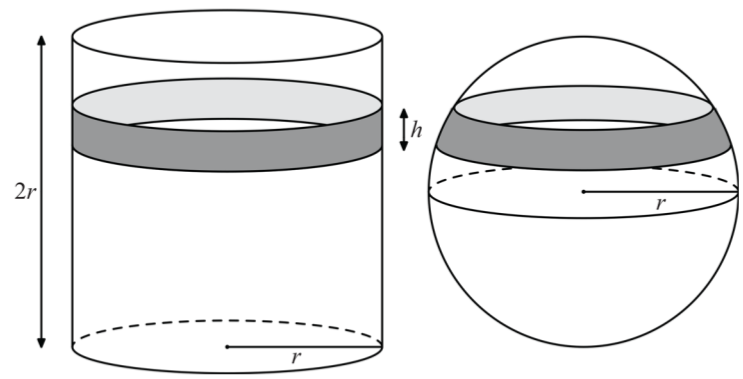


The shaded cap is Snoopy's *horizon circle*. He can see all of the Earth inside the horizon circle and none of the Earth outside the horizon circle. A line from Snoopy to a point of the horizon circle is therefore *tangent* to the sphere.

Solution

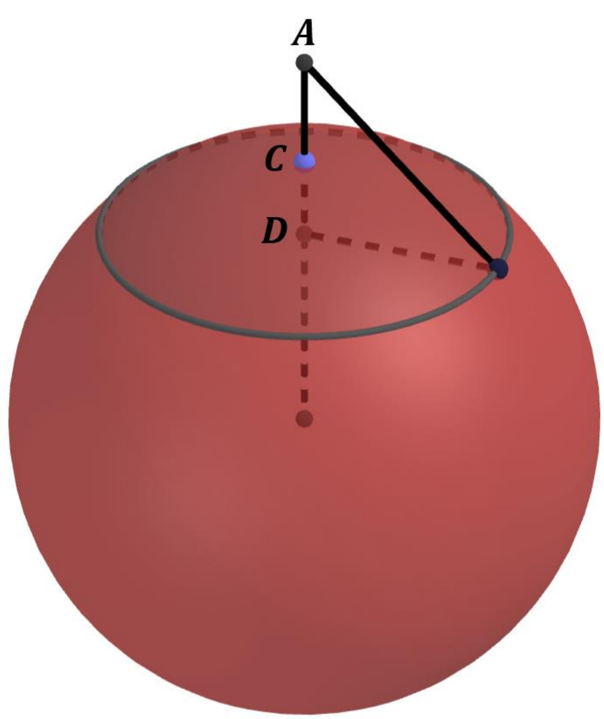
The value we need to find first is the length of CD, the height of the spherical zone in this problem.

Note: The result of Archimedes, that

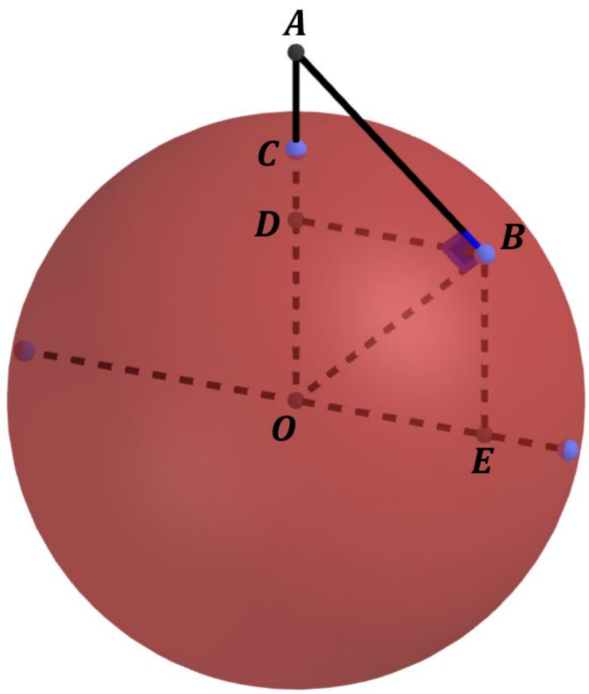


these two shaded bands have the same area is true even if we take them to the top of the cylinder and sphere. In the case when the band (zone) is at the top of the sphere it becomes a ***cap*** (that is the term used in solid geometry).

So we need to find $\overline{CD}$, the height of the cap on the sphere.



Because $\overline{AB}$ is tangent to the sphere (by definition of a horizon) we know that $\overline{OB}$ is perpendicular to $\overline{AB}$ (radii are necessarily perpendicular to tangents). So ΔAOB is a right (planar) triangle.



We see that $\overline{OC} = 3{,}958.8$ because it is a radius of the Earth. And we are given that $\overline{AC} = 50$. Therefore, $\overline{OA} = 4008.8$ miles. And we know $\overline{OB} = 3{,}958.8$ because it is also a radius of the Earth. Using (plane) trigonometry we can determine that

$$\cos(\angle AOB) = \frac{\overline{OB}}{\overline{OA}} = \frac{3958.8}{4008.8}$$

and

$$\angle AOB = \cos^{-1}\left(\frac{3958.8}{4008.8}\right) = 9.058741573°.$$

But we can see that $\angle BOE = 90° - \angle AOB$. Therefore

$$\cos(\angle BOE) = \cos(90° - 9.058741573°) = \frac{\overline{OE}}{\overline{OB}}$$

and

$$\overline{OE} = \overline{OB} \cdot \cos(90° \; - 9.058741573°) = 3958.8 \cdot (0.1574469941) = 623.3011602.$$

Therefore $\overline{DB} = \overline{OE} = 623.3011602$. Finally, we can see that

$$\angle ABD = 90° - \angle DAB = \angle 90° - (90° - \angle AOB) = \angle AOB = 9.058741573°.$$

Therefore,

$$\begin{aligned}\overline{AD} &= \tan(\angle ABD) \cdot \overline{BD} \\ &= \tan(9.058741573°) \cdot 623.3011602 \\ &= 99.37637194 \text{ miles.}\end{aligned}$$

So,

$$\overline{CD} = \overline{AD} - \overline{CA} = 99.37637194 - 50 = 49.37637194 \text{ miles.}$$

Plugging this into the result from Archimedes, we have that the area that Snoopy can see is

$$\begin{aligned}2\pi r \cdot h &= 2\pi(3958.8)(49.37637194) \\ &= 1{,}228{,}181.654 \text{ square miles.}\end{aligned}$$

I wonder if the lyric "**I can see for miles and miles and miles and miles and miles**" is playing in Snoopy's head (like it is mine)?

Does anybody knno *Who* sang that?

To give this result some perspective, what *percentage* of the Earth's surface would Snoopy be able to see?

$$\frac{1228181.654}{4\pi(3958.8)^2} \cdot 100 \approx 0.6\ \%.$$

Don't let the diagram fool you. It is not drawn to scale. If it was drawn to scale then Snoopy's 50 mile altitude would be negligible off the surface of the Earth.

I guess Snoopy will have to go much higher if we wants to see more of the world.

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This begs the question, how much higher? So, here's the next question.

Suppose you want to place a satellite into orbit so that it can see $100 \cdot p\,\%$ of Earth. How many miles high would the satellite have to be?

<u>Solution</u>

Archimede's formula for the area of a spherical cap is $2\pi r h_t$ where r is the radius of Earth and h_t is the vertical height of the cap when the satellite is t miles above the surface.

[We are adding the subscript to h making it h_t to make it clear that h is a function of t.]

We will continue with the notation from the last problem only leaving t general (instead of 50 miles as in the last problem).

$$\cos(\angle AOB) = \frac{\overline{OB}}{\overline{OA}} = \frac{r}{r+t} \tag{1}$$

$$\angle AOB = \cos^{-1}\left(\frac{r}{r+t}\right) \tag{2}$$

$$\begin{aligned}\frac{\overline{OE}}{\overline{OB}} &= \cos(\angle BOE) = \cos(90° - \angle AOB) = \sin(\angle AOB) \\ &= \sin\left(\cos^{-1}\left(\frac{r}{r+t}\right)\right) = \sqrt{1-\left(\frac{r}{r+t}\right)^2}.\end{aligned} \tag{3}$$

[Remember that in general, $\sin(cos^{-1}(\theta)) = \sqrt{1-\theta^2}$.]

$$\begin{aligned}\overline{OE} = \overline{OB} \cdot \cos(\angle BOE) &= r\sqrt{1-\left(\frac{r}{r+t}\right)^2} \\ &= \frac{r}{r+t}\sqrt{(r+t)^2 - r^2}\end{aligned} \tag{4}$$

$$\overline{DB} = \overline{OE} = \frac{r}{r+t}\sqrt{(r+t)^2 - r^2} \tag{5}$$

$$\angle ABD = \angle AOB = \cos^{-1}\left(\frac{r}{r+t}\right) \tag{6}$$

$$\overline{AD} = \tan\left(\cos^{-1}\left(\frac{r}{r+t}\right)\right)\cdot\left(\frac{r}{r+t}\right)\cdot\sqrt{(r+t)^2-r^2} \tag{7}$$

$$= \frac{\sqrt{(r+t)^2-r^2}}{r}\cdot\left(\frac{r}{r+t}\right)\sqrt{(r+t)^2-r^2}$$

$$= \frac{(r+t)^2-r^2}{r+t}$$

$$h_t = \overline{CD} = \overline{AD} - \overline{CA} = \frac{(r+t)^2-r^2}{r+t} - t \tag{8}$$

$$= \frac{(r+t)^2-r^2-t(r+t)}{r+t}$$

$$= \frac{rt}{r+t}.$$

So, at height t the spherical cap for the satellite will have area

$$2\pi r h_t = 2\pi r\left(\frac{rt}{r+t}\right).$$

and at height t the satellite can see

$$\frac{2\pi r h_t}{4\pi r^2}\cdot 100\% = \frac{2\pi r\left(\frac{rt}{r+t}\right)}{4\pi r^2}\cdot 100\% = \frac{t}{2(r+t)}\cdot 100\%$$

of the Earth.

Just as a check we note that all the results from the previous particular case when $t = 50$ comport with these general results.

$$h_{50} = \frac{3958.8\cdot 50}{3958.8+50} = 49.37637198$$

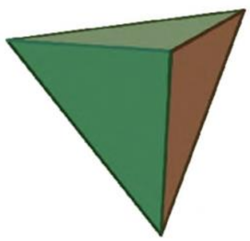
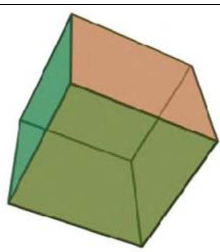
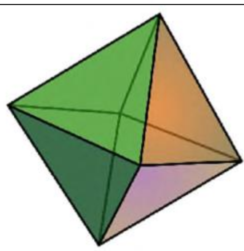
$$2\pi r h_{50} = 2\pi(3958.8)(49.37637198) = 1228181.655$$

$$100\cdot\frac{2\pi r h_{50}}{4\pi r^2} = 100\cdot\frac{h_{50}}{2r} = 100\cdot\frac{49.37637198}{2(3958.8)} = 0.6236280183.$$

Euler's Polyhedral Formula

Let P be a convex polyhedral. Let v be the number of vertices, e be the number of edges and f be the number of faces of P.

So, for example,

	Name	Number of Vertices, v	Number of Edges, e	Number of Faces, f
	Tetrahedron	4	6	4
	Hexahedron (cube)	8	12	6
	Octohedron	6	12	8

Suppose the edges and faces of a convex polyhedron were stretchable and that you somehow could blow air into the polyhedron evenly in all directions so it would stretch but not lose its general shape. What would happen?

You would get a sphere that was “tiled” with spherical polygons. But notice that while the shape has changed from a polyhedron to a sphere, the number of edges, the number of faces and the number of vertices is the same in both shapes.

For convenience let’s assume the sphere generated has a radius $r = 1$.

Let v, e and f denote the number of vertices, edges and faces of P respectively. Let $R_1, R_2, \ldots, R_f$ be the spherical polygons covering the sphere.

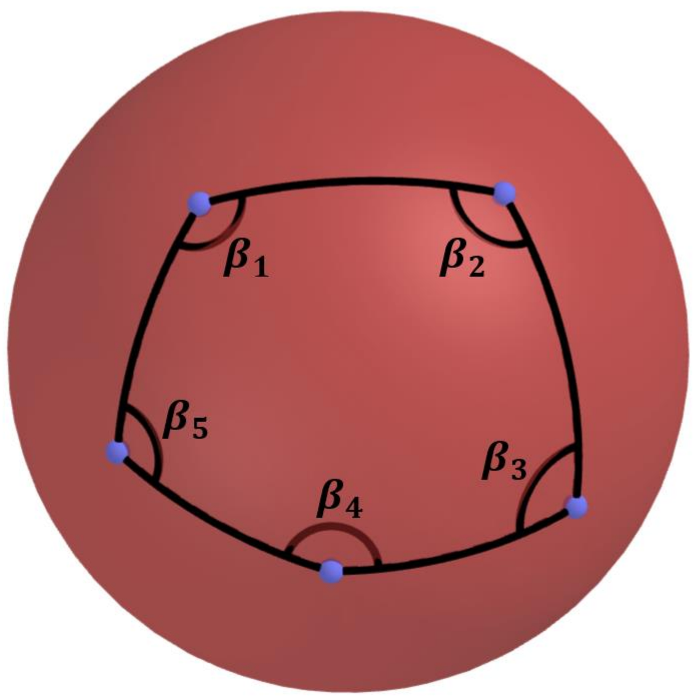
These spherical polygons together make up the entire sphere with no overlapping regions. Therefore,

$$\text{Area}(R_1) + \text{Area}(R_2) + \cdots + \text{Area}\left(R_f\right) = \text{Area(sphere)}$$

Let n_i be the number of edges of R_i and let α_{i_j}, $j = 1,2, \ldots, n_i$, be the number of degrees in the j^{th} interior angle of R_i.

Now think back to our previous example on the area of a spherical penetagon. Recall how as part of that derivation we showed

$$\text{Area of Spherical Pentagon} = (\beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5 - 3\pi)r^2.$$



Using that same approach we now have

$$Area(R_i) = \left(\sum_{j=1}^{n_i} \left(\alpha_{i_j}\right)\right) - (n_i - 2)\pi$$

Therefore,

$$\text{Area(sphere)} = \sum_{i=1}^{f} \text{Area}(R_i)$$

$$= \sum_{i=1}^{f} \left(\left(\sum_{j=1}^{n_i} \left(\alpha_{i_j}\right)\right) - (n_i - 2)\pi \right)$$

$$= \sum_{i=1}^{f} \left(\sum_{j=1}^{n_i} \left(\alpha_{i_j}\right)\right) - \sum_{i=1}^{f} (n_i\pi) + \sum_{i=1}^{f} (2\pi).$$

Let's consider each of these three terms separately:

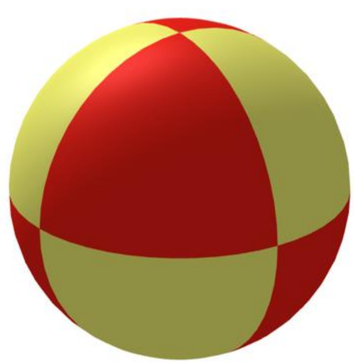
The first term is summing the degrees of all angles of all spherical polygons.

The second term can be written as

$$\sum_{i=1}^{f} (n_i\pi) = \pi \sum_{i=1}^{f} n_i = \pi \sum_{i=1}^{f} \left(\text{Number of edges in the i}^{\text{th}} \text{ spherical polygon}\right)$$

$$\stackrel{?}{=} \pi \cdot \text{Number of edges in the entire spherical polygon.}$$

But are we double counting here? Yes. Adding the number of edges in each spherical polygon double counts the total number of edges because each edge occurs in exactly two polygons.



Notice how every edge in the above spherical polygon occurs in exactly one red shaded spherical triangle and exactly one yellow shaded spherical triangle.

So,

$$\sum_{i=1}^{f} \left(\text{Number of edges in the i}^{\text{th}}\text{ spherical polygon}\right)$$

$$= 2 \cdot \text{Total Number of Edges}$$

$$= 2e.$$

Hence,

$$\sum_{i=1}^{f} (n_i \pi) = 2\pi e.$$

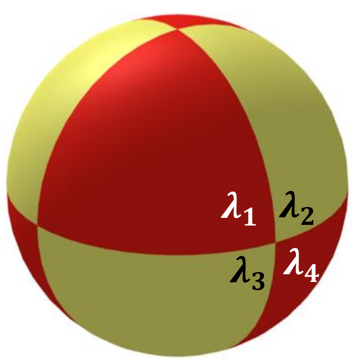
What about the third term?

$$\sum_{i=1}^{f} (2\pi) = 2\pi + 2\pi + \cdots + 2\pi = 2\pi f.$$

That is, the third term is just adding f copies of 2π.

Let's look again at the first term. We noted that the first term is the **sum of the degrees of all angles in the entire spherical polygon**.

So somewhere in that global sum we would be including $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4$.



But what does this sum equal? When you combine these spherical angles you are getting one entire circle. That is, $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 2\pi$.

And if we did this at every single vertex we would always get 2π.

Hence,

$$\sum_{i=1}^{f}\left(\sum_{j=1}^{n_i}\left(\alpha_{i_j}\right)\right) =$$

$$= \text{Sum of the degrees of all spherical angles in the entire spherical polygon}$$

$$= 2\pi + 2\pi + 2\pi + \cdots + 2\pi$$

$$= 2\pi \cdot (\text{number of vertices in the spherical polygon})$$

$$= 2\pi v$$

Bringing this all together, we've just concluded that

$$\text{Area(sphere)} = \sum_{i=1}^{f}\left(\sum_{j=1}^{n_i}\left(\alpha_{i_j}\right)\right) - \sum_{i=1}^{f}(n_i\pi) + \sum_{i=1}^{f}(2\pi)$$

$$= 2\pi v - 2\pi e + 2\pi f$$

where v, e and f denote the number of vertices, edges and faces in the spherical polygon.

But we also know that in general Area(sphere) $= 4\pi r^2$ and because we have taken $r = 1$ in this problem, we have

$$4\pi = \text{Area(sphere)} = 2\pi v - 2\pi e + 2\pi f$$

or after cancelling where we can we have

$$v - e + f = 2$$

for the sphere covered by spherical polygons.

But what about for the original polyhedron we started with before we "pumped air" into it to turn it into a sphere covered with spherical polygons?

Remember we already noticed that the values of v, e and f are the same for the polyhedron as for the sphere covered with spherical polygons.

That is,

$$v - e + f = 2$$

is true for the polyhedron as well as for sphere covered with spherical polygons.

Conclusion:

Euler's Polyhedral Formula. If a convex polyhedron with v vertices, e edges and f faces, it is always true that $$v - e + f = 2.$$

If you are paying heed to every word you might have noticed we glossed over the word "convex" in a convex polyhedron.

What is convex? In this context it (roughly) means that none of the faces (sides) of the polyhedron are pushed inwards towards the center.

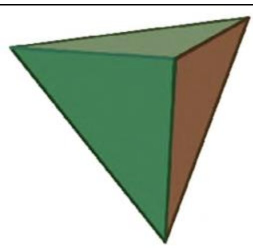
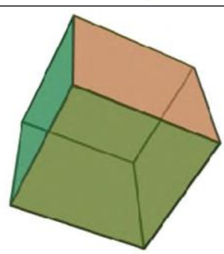
What goes wrong if some side(s) are pushed inwards?

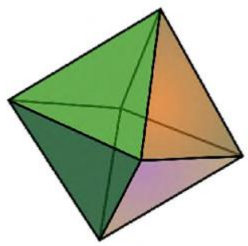
For one thing, the idea of "pumping air" into the polyhedron to turn it into a sphere covered with spherical polygons does not work.

The pushed in part(s) don't get pushed out to form a sphere and the whole argument fails.

By the way, does Euler's Polyhedral Formula give the correct result for the three convex polyhedron we began the proof with. Namely the tetrahedron, the cube and the octahedron.

Let's check.

	Name	Number of Vertices, v	Number of Edges, e	Number of Faces, f	$v - e + f$
	Tetrahedron	4	6	4	$4 - 6 + 4 = \mathbf{2}$
	Hexahedron (cube)	8	12	6	$8 - 12 + 6 = \mathbf{2}$

	Octohedron	6	12	8	$6 - 12 + 8 = \mathbf{2}$

Success!

Exactly 5 Platonic Solids

If V is the number of vertices in a Platonic solid and F is the number of faces, and if each vertex has degree d and each face has degree k, then

$$dV = 2E \text{ and } kF = 2E$$

$V = \left(\frac{2}{d}\right)E$ and $F = \left(\frac{2}{k}\right)E$, and then plug them into Euler's Formula, $V + F - E = 2$, to get

$$\left(\frac{2}{d} + \frac{2}{k} - 1\right)E = 2$$

But E is positive, and so is 2. Thus $(\frac{2}{d} + \frac{2}{k} - 1)$ must be positive. That is,

$$\frac{2}{d} + \frac{2}{k} > 1.$$

Now, you can turn this into

$$(d-2)(k-2) < 4.$$

A polyhedron cannot have faces with fewer sides than a triangle, so k is greater than or equal to 3.

Furthermore, a polyhedron cannot have a vertex of degree 1 or 2. So d is greater than or equal to 3.

If $k = 3$, then

$$(d-2)(3-2) < 4$$

$$d - 2 < 4$$

$$d < 6 \text{ or } d \leq 5.$$

Similarly, $k \leq 5$. So there are only a handful of possible cases for d and k. At this point we have

$$3 \leq d \leq 5 \text{ and } 3 \leq k \leq 5.$$

Investigate all nine of these possibilities. It will turn out that only five of these nine actually generate a solid.

For example, the case $d = 5, k = 5$ is not possible because in this case

$$\frac{2}{d} + \frac{2}{k} \ngtr 1.$$

Similarly, $(d = 5, k = 4), (d = 4, k = 5)$ and $(d = 4, k = 4)$ are not possible.

The remaining five possibilities generate the five Platonic solids.

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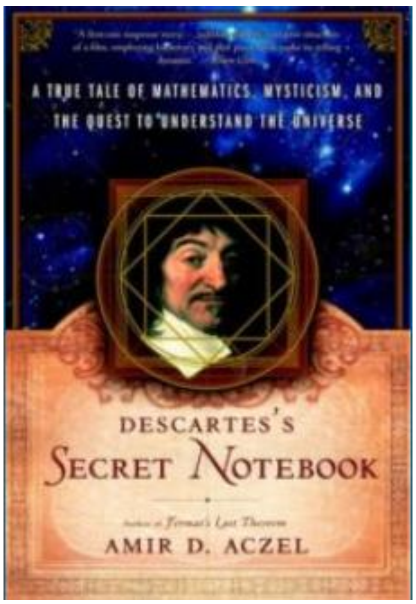
Anybody play Dungeons and Dragons?

P.S.

If you want a fun summer read that has a mathematical element to it, consider Amir Aczel’s

Descartes' Secret Notebook : A True Tale of Mathematics, Mysticism, and the Quest to Understand the Universe

Euler’s Polyhedral formula $v - e + f = 2$ is central to the book. It isn’t often that a result in mathematics generates such adventure and intrigue as this result has.



<u>Homework</u>

(1) Find the area in square miles of the Earth’s surface within the Arctic Circle (circle of latitude $66°32'N$).

(2) How high does a satellite have to be in order to see both Los Angeles and New York City?