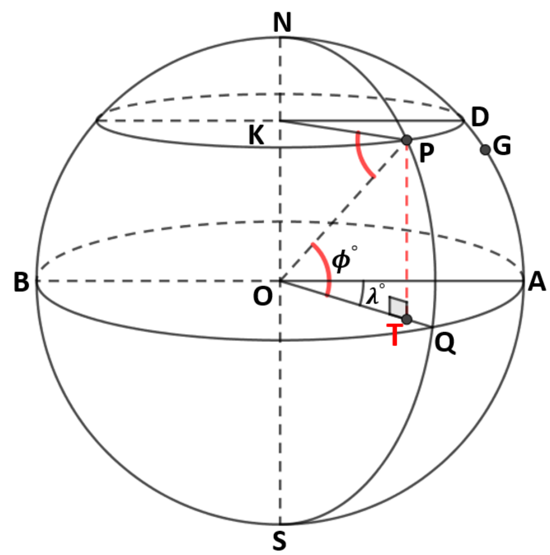
Wednesday Morning

On Monday morning we showed that KP, the radius of the circle of latitude at latitude ϕ, is given

$$\text{KP} = r\cos(\phi)$$

where r is the radius of the sphere.



One of the lead-up questions was about the velocity of Iowa City (latitude $41.6578°$ N) in space due to the rotation of the earth on its axis.

Question 1. How much faster are students at University of Iowa (Iowa City, Iowa) rotating than students at Augsburg University (latitude $\boldsymbol{\phi = 44.9659°}$)?

At latitude ϕ the circumference of their latitude circle is $2\pi r_\phi = 2\pi r\cos(\phi)$ where $r = 3958.8$ miles is the radius of earth. A person at latitude ϕ covers that full circumference every 24 hours. Hence their rotation speed is

$$\frac{2\pi r\cos(\phi)}{24} \text{ miles per hour.}$$

Hence, the rotation speed at Iowa City is

$$\frac{2\pi(3958.8)\cos(41.6578)}{24} = 774.3319277 \text{ miles per hour}$$

and here at Augsburg the rotation speed is

$$\frac{2\pi(3958.8)\cos(44.9659)}{24} = 733.28957 \text{ miles per hour.}$$

Iowans are moving a full 41 miles per hour faster and us Minnesotans.

So, this raises the question:

So, if the earth rotates faster in Iowa, are days in Iowa shorter?

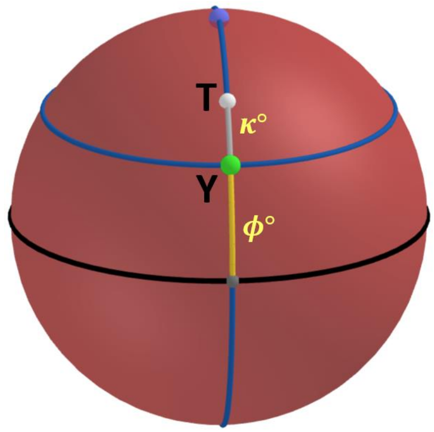
The ***Coriolis*** effect is the term used to describe the impact on objects due to the rotation of earth.

As kids did you every play the game of throwing a kickball around while spinning on a merry-go-round?

What happens? That's the Coriolis effect.

Coriolis Effect in Ballistics

Suppose you are at point Y at latitude $\phi°$ and you shoot a missile at target T that is $\kappa°$ due north of your position (see diagram below). To aim properly you have to account for the fact that the eastward rotation speed of T is slower than the eastward rotation speed of Y.



As we mentioned a point at latitude ϕ travels (rotates) east at

$$\frac{\pi r\cos(\phi)}{12} \text{ miles per hour.}$$

However, the target is at latitude $\phi + \kappa$ and so will rotate east at

$$\frac{\pi r\cos(\phi+\kappa)}{12} \text{ miles per hour.}$$

Suppose it takes t seconds for the missile to reach latitude $\phi + \kappa$. During those t seconds the missile will have moved

$$\left(\frac{\pi r\cos(\phi)}{12} - \frac{\pi r\cos(\phi+\kappa)}{12}\right)\left(\frac{t}{60^2}\right) \text{ miles}$$

to the right of the target.

Eötvös Effect

There is a related effect called the **Eötvös effect** that long-range snipers have to account for (in addition to gravity, wind, humidity, …) that causes bullets to be too high or too low (depending on whether the target is to the east or west respectively of the sniper). If a target is due east of the sniper then in the time it takes the bullet to reach the target, the target will drop down from the sniper's position as earth and the target rotate east. If a target is due west of the sniper then the target will rise up relative to the sniper's position as earth rotates east.

A Sniper's Problem

A long-range sniper $\phi = 40°$N aims at a target 1500 yards due north of his/her position. If the muzzle velocity of the sniper rifle is 3000 feet per second, how many inches to the right of the target will the bullet hit because of the Coriolis effect?

How would all this change if the sniper was shooting due south? Why?

[Note: 1 mile per hour equals 0.488889 yards per second.]

<u>Solution</u>

Changing from angular distance to physical distance, the sniper is firing from a position

$$\phi\left(\frac{\pi}{180}\right)\cdot r\text{ miles} = 40\cdot\left(\frac{\pi}{180}\right)\cdot(3958.8\text{ miles})\left(\frac{1760\text{ yards}}{\text{mile}}\right) = 4{,}864{,}224.248\text{ yards}$$

north of the equator. Thus, the target is

$$4{,}864{,}224.248 + 1500 = 4865724.248\text{ yards}$$

north of the equator or latitude

$$4865724.248\left(\frac{180}{\pi}\right)\left(\frac{1}{3958.8}\right)\left(\frac{1}{1760}\right) = 40.01233496°$$

(and hence the target is rotating slower than the sniper).

Moving at 3000 feet per second or 1000 yards per second it takes the bullet 1.5 seconds to reach the target 1500 yards away.

But in those 1.5 seconds, the sniper rotates 582.2207503 yards while the target only rotates 582.1155609 yards.

This is a difference of

$$582.2207503 - 582.1155609 = 0.1051894 \text{ yards} = 3.7868184 \text{ inches.}$$

The sniper misses the target approximately 4 inches off to the right.

At latitude $\phi = 40$ the sniper is rotating at

$$\frac{\pi r \cos(\phi)}{12} \text{ miles per hour} = \frac{\pi \cdot 3958.8 \cos(40°)}{12} \cdot 0.488889 \ \frac{\text{yards}}{\text{second}}$$

So, in 1.5 seconds rotates

$$\frac{\pi \cdot 3958.8 \cos(40°)}{12} \cdot 0.488889 \cdot 1.5 = 582.2207503 \text{ yards}$$

At latitude $\phi = 40.01233496$ the target is rotating at

$$\frac{\pi \cdot 3958.8 \cos(40.01233496)}{12} \cdot 0.488889 \ \frac{\text{yards}}{\text{second}}$$

So, in 1.5 seconds the target rotates

$$\frac{\pi \cdot 3958.8 \cos(40.01233496°)}{12} \cdot 0.488889 \cdot 1.5 = 582.1155609 \text{ yards}$$

Pursuit Problem

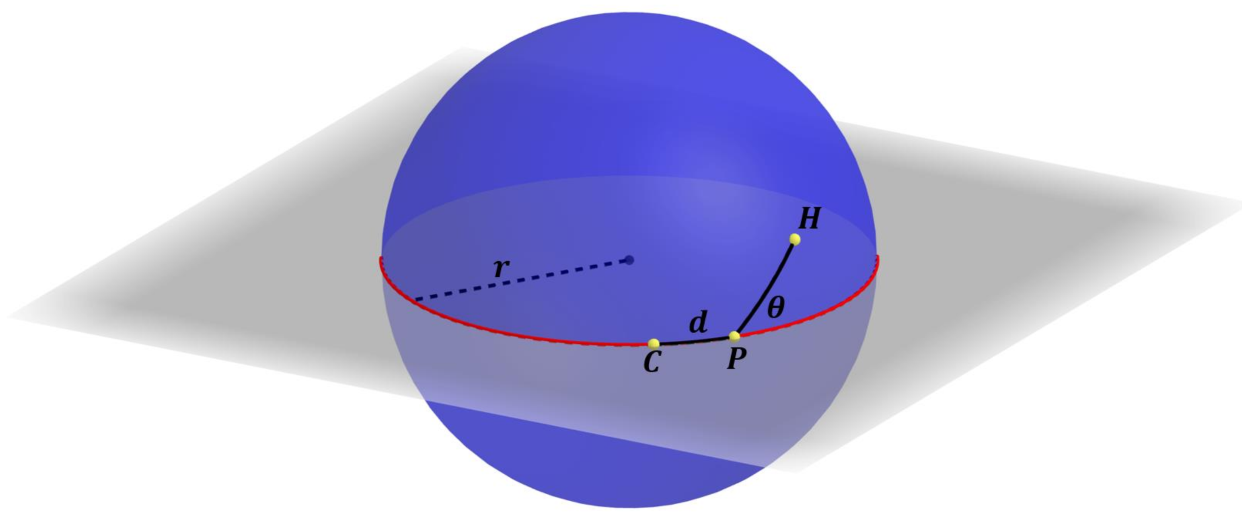
A pirate ship P is initially separated from a chase ship C by d miles measured along a great circle of a sphere of radius r. (By rotational symmetry, we can without loss of generality, assume that the pirate ship is initially located due east of the chase ship on the equator of the sphere.)

Assume the pirate ship sets sail with an initial angle of θ degrees off the equator with the plan of reaching an island hideout at H as fast as possible.

At that same moment the chase ship begins its pursuit at full speed with the purpose of intercepting the pirate ship as quickly as possible.

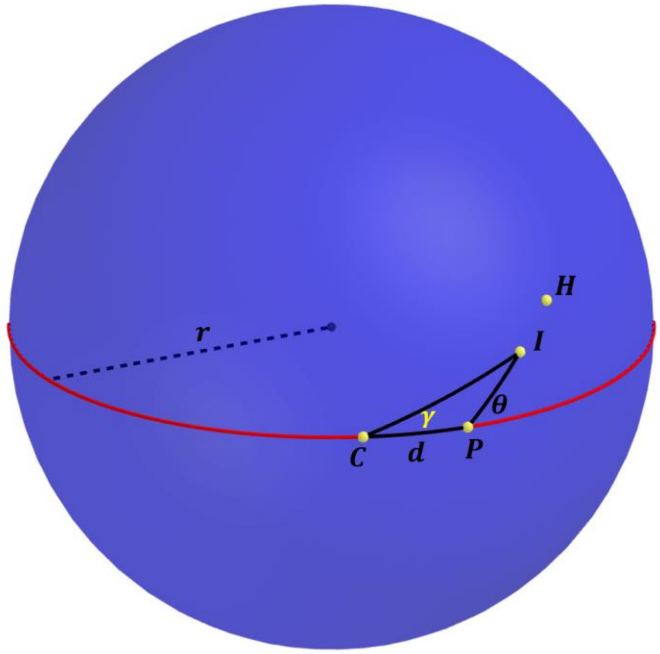
If the chase ship C can sail k times faster than the pirate ship P, what is the optimal pursuit track for the chase ship?

Assume that the distances are great enough that it is necessary to take the curvature of the sphere into account in all calculations. Also assume that both ships can maintain their full speed regardless of external conditions such as winds and obstacles in their path.



Solution

Both ships maintain their maximum speed at all times. Therefore, the goal of minimizing time is equivalent to minimizing distance. Therefore, both ships will travel along a great circle.



So, the problem comes down to finding the initial angle γ for the chase ship so that the two ships will collide at some point I on $\widehat{PH}$.

If we t represent the length of the arc $\widehat{PI}$ then the length of arc $\widehat{CI}$ is kt. From the spherical law of cosines for sides applied to the spherical triangle ΔCPI we have that

$$\cos\left(\frac{kt}{r}\right) = \cos\left(\frac{t}{r}\right)\cos\left(\frac{d}{r}\right) + \sin\left(\frac{t}{r}\right)\sin\left(\frac{d}{r}\right)\cos(180° - \theta)$$
$$= \cos\left(\frac{t}{r}\right)\cos\left(\frac{d}{r}\right) - \sin\left(\frac{t}{r}\right)\sin\left(\frac{d}{r}\right)\cos(\theta).$$

Solving for $\cos(\theta)$ we have

$$\cos(\theta) = \frac{\cos\left(\frac{t}{r}\right)\cos\left(\frac{d}{r}\right) - \cos\left(\frac{kt}{r}\right)}{\sin\left(\frac{t}{r}\right)\sin\left(\frac{d}{r}\right)}.$$

The only unknow in this equation is t and from this equation we can numerically solve for t.

By a second application of the spherical law of cosines for sides we have

$$\cos\left(\frac{t}{r}\right) = \cos\left(\frac{kt}{r}\right)\cos\left(\frac{d}{r}\right) + \sin\left(\frac{kt}{r}\right)\sin\left(\frac{d}{r}\right)\cos(\gamma).$$

Solving for $\cos(\gamma)$ we have

$$\cos(\gamma) = \frac{\cos\left(\frac{t}{r}\right) - \cos\left(\frac{kt}{r}\right)\cos\left(\frac{d}{r}\right)}{\sin\left(\frac{kt}{r}\right)\sin\left(\frac{d}{r}\right)}.$$

Because we solved for t in the previous step, the only unknown in this equation is the angle γ which can be explicated solved by

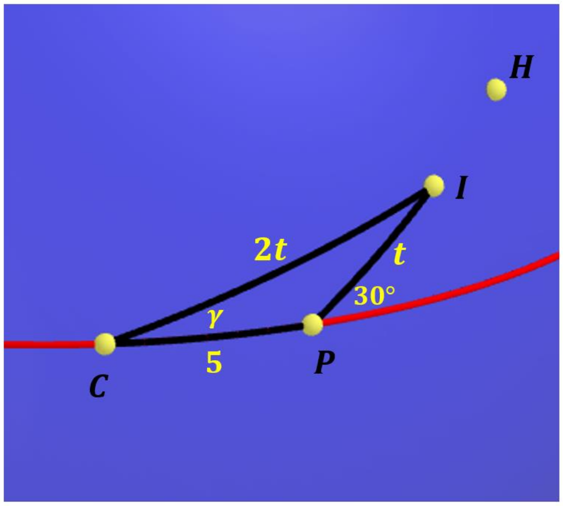
$$\gamma = \cos^{-1}\left(\frac{\cos\left(\frac{t}{r}\right) - \cos\left(\frac{kt}{r}\right)\cos\left(\frac{d}{r}\right)}{\sin\left(\frac{kt}{r}\right)\sin\left(\frac{d}{r}\right)}\right).$$

■

Example 1.

Following the set up as detailed above, assume the sphere for the two ships is our earth with radius $r = 3959$ miles, that the two ships start out 5 miles apart, that the chase ship's maximum speed is twice that of the pirate ship and that the pirate ship sets out with an initial angle of $30°$ off the equator. Find the initial angle γ that the chase ship should take so as to overtake the pirate ship as quickly as possible. How far will the pirate ship go and how far will the chase ship go before colliding?

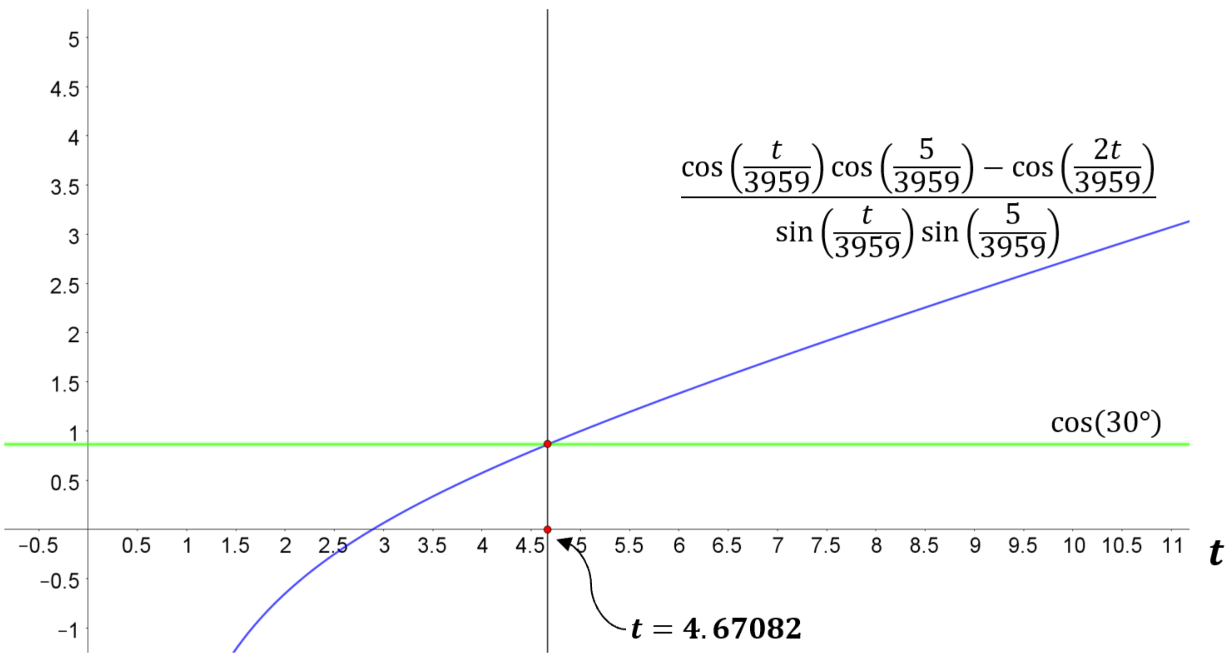
<u>Solution</u>



The first step in the solution is to numerically solve for t in the equation

$$\cos(30°) = \frac{\cos\left(\frac{t}{3959}\right)\cos\left(\frac{5}{3959}\right) - \cos\left(\frac{2t}{3959}\right)}{\sin\left(\frac{t}{3959}\right)\sin\left(\frac{5}{3959}\right)}.$$

You can you use any numerical software package to do this. I used the software Geogebra to graph the left-hand side expression and then determined where this graph equaled $\cos(30°)$. In this way I found $t = 4.67082$ miles. (I have ignored all potential issues of numerical accuracy in this calculation.)



The second step is to find the initial angle γ of the chase ship from the formula

$$\gamma = \cos^{-1}\left(\frac{\cos\left(\frac{4.67082}{3959}\right) - \cos\left(\frac{2 \cdot 4.67082}{3959}\right)\cos\left(\frac{5}{3959}\right)}{\sin\left(\frac{2 \cdot 4.67082}{3959}\right)\sin\left(\frac{5}{3959}\right)}\right)$$

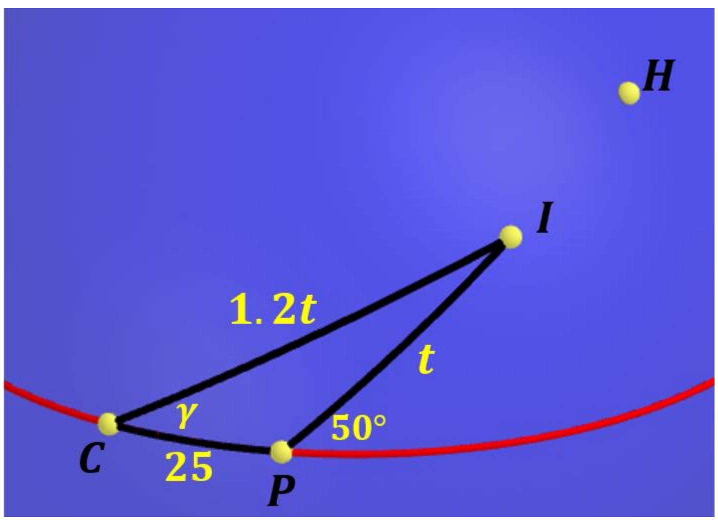
$$= 14.47843°.$$

So, we have determined that in order for the chase ship to intercept the pirate ship as fast as possible, the chase ship should steer an initial angle of $14.47843°$ off the equator, the pirate ship will go $t = 4.67082$ miles and the chase ship will go $2t = 9.34164$ miles before the two ships collide.

Example 2.

Once again, we will follow the set up as detailed above, assume the sphere for the two ships is our earth with radius $r = 3959$ miles.   But in this example assume that the two ships start out 25 miles apart, that the chase ship's maximum speed is only 1.2 times that of the pirate ship and that the pirate ship sets out with an initial angle of $50°$ off the equator. Again, the question is to find the initial angle γ that the chase ship should take so as to overtake the pirate ship as quickly as possible. How far will the pirate ship go and how far will the chase ship go before colliding?

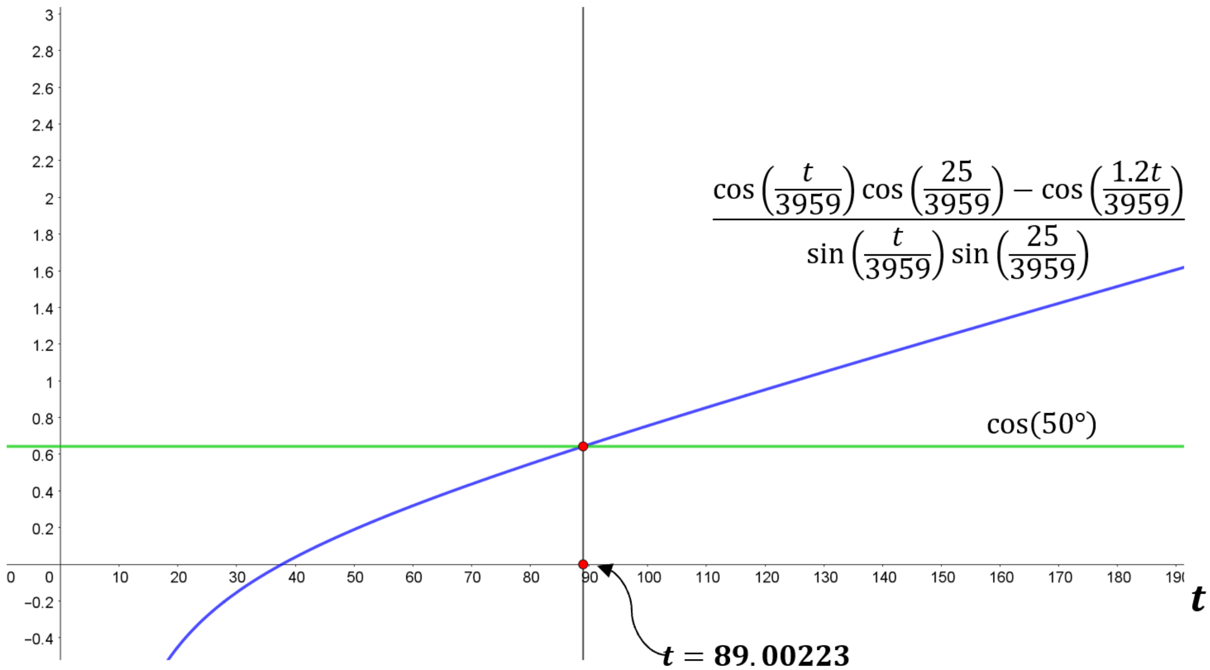
<u>Solution</u>



As in the previous example the first step in the solution is to numerically solve for t in the equation

$$\cos(50°) = \frac{\cos\left(\frac{t}{3959}\right)\cos\left(\frac{25}{3959}\right) - \cos\left(\frac{1.2t}{3959}\right)}{\sin\left(\frac{t}{3959}\right)\sin\left(\frac{25}{3959}\right)}.$$

You can use any numerical software package to do this. I used the software Geogebra to graph the left-hand side expression and then determined where this graph equaled $\cos(30°)$. In this way I found $t = 4.67082$ miles. (Once again, I have ignored all potential issues of numerical accuracy in this calculation.)



The second step is to find the initial angle γ of the chase ship from the formula

$$\gamma = \cos^{-1}\left(\frac{\cos\left(\frac{89.00223}{3959}\right) - \cos\left(\frac{1.2\cdot 89.00223}{3959}\right)\cos\left(\frac{25}{3959}\right)}{\sin\left(\frac{1.2\cdot 89.00223}{3959}\right)\sin\left(\frac{25}{3959}\right)}\right)$$

$$= 39.67217°.$$

So, we have determined that in order for the chase ship to intercept the pirate ship as fast as possible, the chase ship should steer an initial angle of $39.67217°$ off the equator, the pirate ship will go $t = 89.00223$ miles and the chase ship will go $1.2t = 106.802676$ miles before the two ships collide.

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Satellite Wants to Communicate With Both New York City and Los Angeles

How high of an orbit would a communications satellite need to have in order to be able to communicate with both New York City and Los Angeles?

<u>Solution</u>

On Tuesday we found that at height t miles above the surface the horizon radius of the satellite is

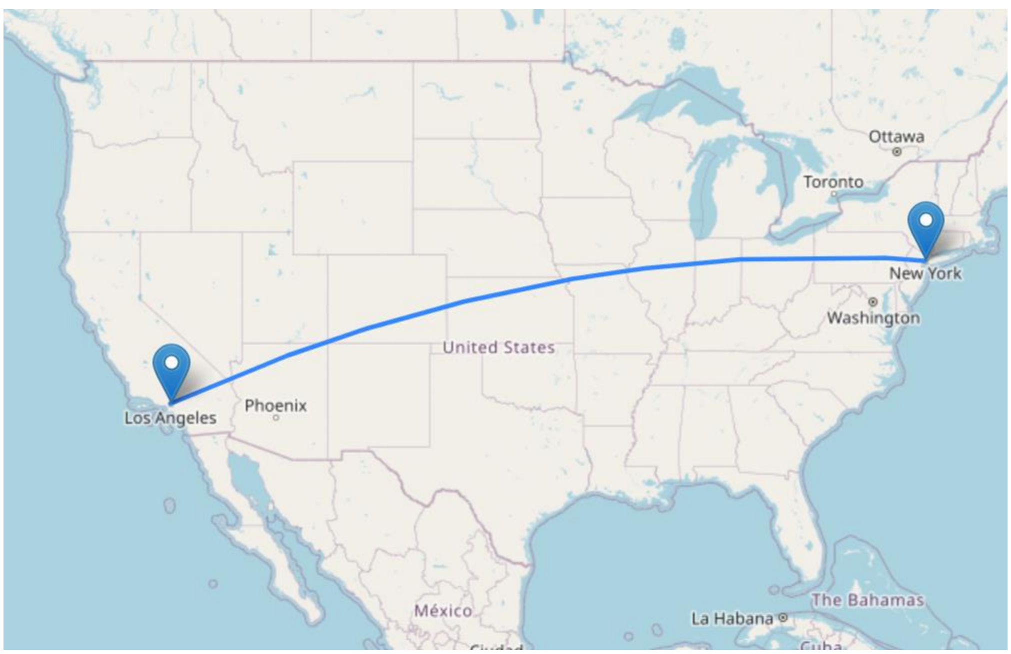
$$\overline{DB} = \frac{r}{r+t}\sqrt{(r+t)^2 - r^2}\text{ miles}$$

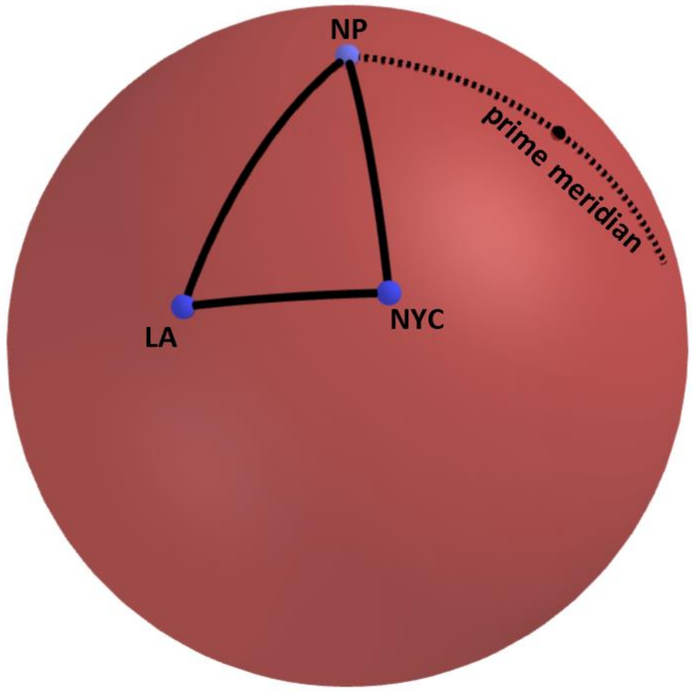
and a satellite can see

$$\frac{t}{2(r+t)} \cdot 100\%$$

of the Earth's surface.

We need to find the great circle distance between NYC (JFK 40.6446° N, 73.7797°W) and Los Angeles (LAX 33.9422° N, 118.4036°W) and then find the orbit that have both cities on or in the horizon of the satellite.





We need to find the great circle distance between NYC (JFK 40.6446° N, 73.7797°W) and Los Angeles (LAX 33.9422° N, 118.4036°W) and then find the orbit that have both cities on or in the horizon of the satellite.

$$\cos(b) = \cos(90 - 33.9422)\cos(90 - 40.6446) + \sin(90 - 33.9422)\sin(90 - 40.6446)\cos(118.4036 - 73.7797) = 0.811709817$$

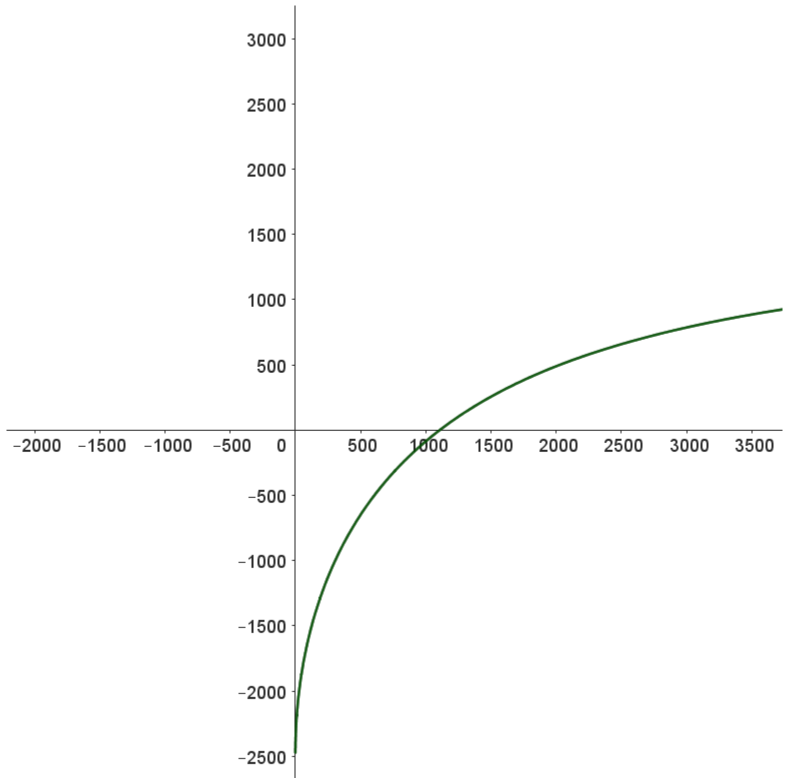
$$b = \cos^{-1}(0.811709817) = 35.73667687°$$

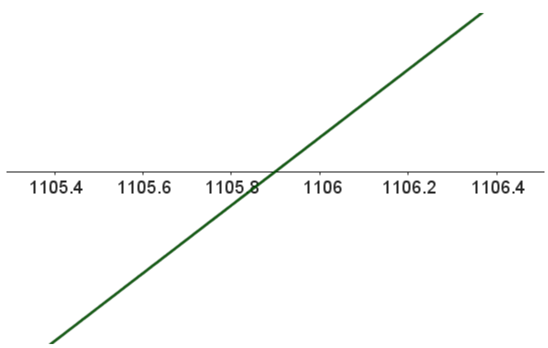
$$35.73667687° = 35.73667687 \cdot \left(\frac{\pi}{180}\right) \cdot 3958.8 = 2469.193326 \text{ miles}$$

So we need to find the height t such that

$$\frac{3958.8}{3958.8 + t}\sqrt{(3958.8 + t)^2 - 3958.8^2} = 2469.193326$$

$t \approx 1105.9$ miles



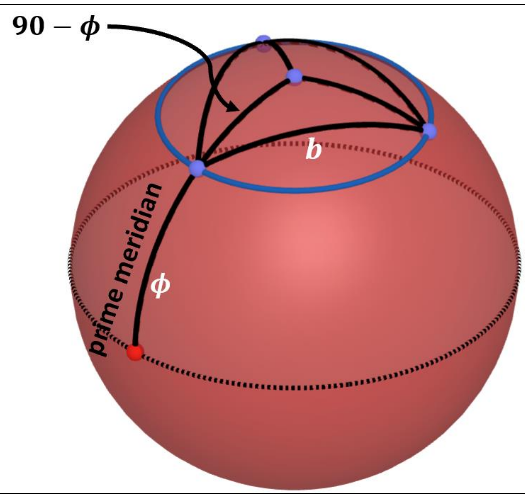


Three Points Equidistant from Each Other and the North Pole

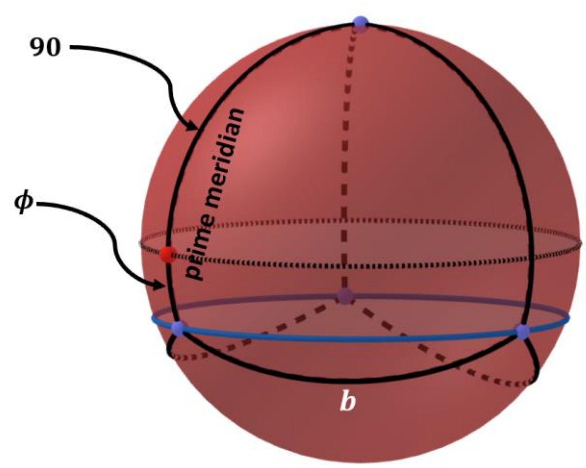
What is the latitude of three points on Earth equally distant from each other and from the North Pole?

<u>Soution</u>

There are two cases to consider – the three points are above the equator versus the three points are below the equator.



Case 1.



Case 2.

Without loss of generality, we can assume that one of the three points is on the prime meridian.

In Case 1, the coordinates of the three points would be $(\phi, 0)$, $(\phi, 120)$ and $(\phi, -120)$. In this case the angular distance of each point from the North Pole is $90 - \phi$ and in this case the angular distance *between* the three points is determined through the spherical law of cosines for sides. In this case we have

$$\begin{aligned}\cos(b) &= \cos(90-\phi)\cos(90-\phi) + \sin(90-\phi)\sin(90-\phi)\cos(120-0)\\ &= \sin^2(\phi) - \frac{\cos^2(\phi)}{2}.\end{aligned}$$

In Case 2, the coordinates of the three points would be $(-\phi, 0)$, $(-\phi, 120)$ and $(-\phi, -120)$. In this case the angular distance of each point from the North Pole is $90 + \phi$.

And in this case we again find

$$\cos(b) = \cos(90+\phi)\cos(90+\phi) + \sin(90+\phi)\sin(90+\phi)\cos(120-0)$$

$$= (-\sin(\phi))(-\sin(\phi)) + (\cos(\phi))(\cos(\phi))\left(-\frac{1}{2}\right)$$
$$= \sin^2(\phi) - \frac{\cos^2(\phi)}{2}.$$

So, in Case 1, we want to find ϕ such that

$$\frac{\pi}{2} - \phi = \cos^{-1}\left(\sin^2(\phi) - \frac{\cos^2(\phi)}{2}\right).$$

Plotting this equation we find the only solution is

$$\phi \approx -19.47°.$$

While in Case 2, we want to find ϕ such that

$$\frac{\pi}{2} + \phi = \sin^2(\phi) - \frac{\cos^2(\phi)}{2}.$$

Plotting this equation we find the only solution is again

$$\phi \approx 19.47°.$$

So the only solution is a latitude of 19.47° south of the equator.