Proof by Induction

Certain propositions are hard to explain either by a direct proof or by contradiction. So we may sometimes need to rely on a different technique: ***proof by induction***.

These are the steps we follow to create a proof by induction.

Step 1: Frame the proposition. Call it $P(n)$. Our goal is to prove that $P(n)$ is true for all discrete values of n (n is usually defined as a positive integer).

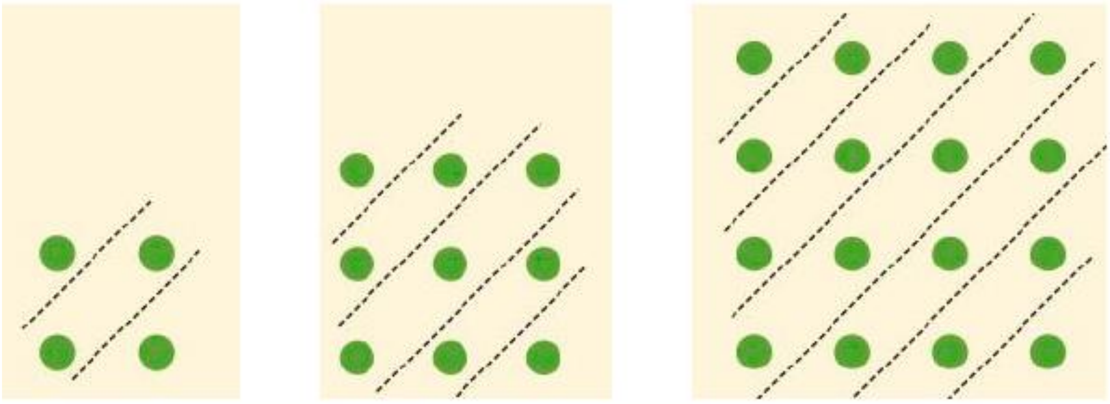
Step 2: Basic Step. Check if the LHS of the proposition equals the RHS when $n = 1$. This proves $P(1)$ is true.

Step 3: Make the assumption. Assume $P(n)$ is true for some value of $n = k$. Thus, assume $P(k)$ is true. Hence, we assume that the LHS of $P(k)$ equals the RHS of $P(k)$.

Step 4: Inductive step. Using the assumption that $P(k)$ is true, can you show that the LHS of $P(k+1)$ equals the RHS of $P(k+1)$. If you can, you would be able to say that if $P(k)$ is true, then it implies that $P(k+1)$ is true.

Step 5: Final statement. $P(k+1)$ must be true if $P(k)$ is true (from Step 4). But also $P(1)$ is true (from Step 2). This implies that $P(2)$ must be true. But then that must imply that $P(3)$ is true. But that means $P(4)$ is also true… and so on and so forth. Hence, $P(n)$ must be true for all positive integer values of n. Hence proved!

Example: The diagrams below suggest a pattern that can lead to a conjecture. It is possible to write a direct proof of this conjecture / proposition. Try it! We will proving the statement by the method of induction.



$1+2+1=4=2^2$ $\quad$ $1+2+3+2+1=9=3^2$ $\quad$ $1+2+3+4+3+2+1=16=4^2$

Proposition
$P(n)$: $1+2+3+\cdots+(n-1)+n+(n-1)+\cdots+3+2+1=n^2$

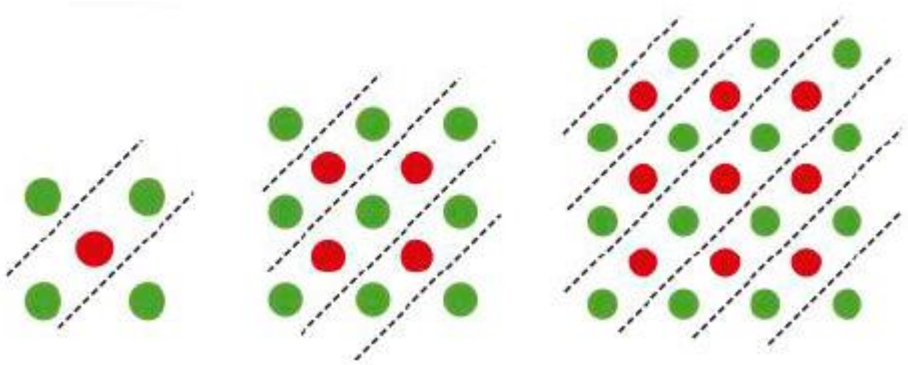
Basic Step
$P(1)$: $LHS = 1 = RHS$. So $P(1)$ is true.

Assumption
Assume $P(n)$ is true for some $n = k$. Assuming $P(k)$ is true,
$1+2+3+\cdots+(k-1)+k+(k-1)+\cdots+3+2+1=k^2$

Inductive Step
Consider $n = k + 1$
$P(k+1)$: $1 + 2 + 3 + \cdots + (k) + (k+1) + (k) + \cdots + 3 + 2 + 1 = (k+1)^2$
$\boldsymbol{LHS} = 1 + 2 + 3 + \cdots + (k) + (k+1) + (k) + \cdots + 3 + 2 + 1$
$= 1 + 2 + 3 + \cdots + (k-1) + (k) + (k+1) + (k) + (k-1) \ldots + 3 + 2 + 1$
$= 1 + 2 + 3 + \cdots + (k-1) + (k) + (k-1) \ldots + 3 + 2 + 1 + (k+1) + (k)$
$= k^2 + (k+1) + (k)$ (Since we assumed $P(k)$ is true, $1 + 2 + \cdots + (k-1) + k + (k-1) + \cdots + 2 + 1 = k^2$)
$= k^2 + 2k + 1$
$= (k+1)^2$
$= \boldsymbol{RHS}$
Hence, if $P(k)$ is true, then $P(k+1)$ must also be true.

Final Statement
We have shown that $P(1)$ is true. Furthermore, we have shown that for some $n = k$, if $P(k)$ is true, then $P(k+1)$ must also be true. Hence, by the principle of mathematical induction we can say that $P(n)$ is true for all values of n, where $n, k \in \mathbb{Z}^+$.

Question 1



Each of the three diagrams above represents a sequence.

a) Write down a sequence based on the line divisions.
b) Write down a sequence based on colour.
c) If the diagrams were to continue what would the next two terms be?
d) Write a conjecture based on your findings.
e) Prove your conjecture using a direct proof.
f) Prove your conjecture using the principle of mathematical induction.

Question 2
Use mathematical induction to prove the following statement:
$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{3}n(n+1)\left(n + \frac{1}{2}\right)$$

Question 3
Use mathematical induction to prove that $11^n - 6$ is a multiple of 5, when $n \in \mathbb{Z}^+$.

Question 4
Use mathematical induction to prove the following statement:
$$1 - 4 + 9 - 16 + \cdots + (-1)^{n+1}n^2 = (-1)^{n+1}\frac{n(n+1)}{2}$$