UNIQUE FACTORIZATION

Q) Is it True that Unique FACTORIZATION Holds True for the Numbers 1, 2, 3, 7, 11, 19, 43, 67 and 163 ?

Yes—but not in the ordinary integers, where unique factorization holds for **every** positive integer. What you're referring to is a famous result in algebraic number theory involving the nine **Heegner numbers**. Heegner number

These special numbers are:

$$1,\ 2,\ 3,\ 7,\ 11,\ 19,\ 43,\ 67,\ 163$$

$1, 2, 3, 7, 11, 19, 43, 67, 163$

What Makes Them Special?

For each such number d, the ring

$$\mathbb{Z}[\sqrt{-d}]$$

(or, more precisely, the full ring of integers in the field $\mathbb{Q}(\sqrt{-d})$) has **unique factorization**.

That means every nonzero element can be factored into irreducible elements in essentially only one way, just like ordinary integers.

$\mathcal{O}_{\mathbb{Q}(\sqrt{-d})}$ is a UFD

Why Only These Nine?

This remarkable theorem was proved through the combined work of mathematicians such as Kurt Heegner, Alan Baker, and Harold Stark.

No other positive integers d have this property.

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The Famous Near-Integer

The largest one, 163, is especially legendary because

$$e^{\pi\sqrt{163}}$$

is astonishingly close to an integer.

$e^{\pi\sqrt{163}} \approx 262537412640768743.99999999999925$

Mathematics occasionally likes to show off.

Precise Statement

> The imaginary quadratic fields $\mathbb{Q}(\sqrt{-d})$ have class number one exactly for
>
> $$d = 1, 2, 3, 7, 11, 19, 43, 67, 163.$$

And class number one is equivalent to unique factorization in their rings of integers.

HEEGNER NUMBER	PRIME FACTORIZATION	POSITIVE FACTORS	NUMBER OF FACTORS
1	1	1	1
2	2	1, 2	2
3	3	1, 3	2
7	7	1, 7	2
11	11	1, 11	2
19	19	1, 19	2
43	43	1, 43	2

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67	67	1, 67	2
163	163	1, 163	2

HEEGNER NUMBER	SQUARE (n^2)	CUBE (n^3)
1	1	1
2	4	8
3	9	27
7	49	343
11	121	1331
19	361	6859
43	1849	79507
67	4489	300763
163	26569	4330747