Income/Leisure Framework – Sleep Requirements & Asymmetric Preferences

Emily can choose exactly how much she wants to work, but she must sleep exactly 12 hours per day and these hours do not count towards either leisure or income. Emily's utility over income I and leisure L is given by the function $\mathbf{U_E (I,L) = I * L^2}$.

a) Draw Emily's budget constraint and express this mathematically.

b) Find the optimal allocation of her time using whichever approach you want.

c) Draw a graph of her utility over hours spent working (allocated to income).

d) Briefly explain the meaning of the curvature in this graph with words.

e) Find her marginal utility from working 1 more hour if she is working 4 hours.

f) Find her marginal utility from working 1 more hour if she is working 15 hours.

g) Draw out Emily's full income-leisure utility diagram: include a budget constraint and three indifference curves. Label several specific points on each of these indifference curves to show utility and the corresponding amounts of income and leisure on the two axes.

SOLUTION

The budget constraint mathematically is $I + L = 12$ and this also means that the utility equation can be re-written in terms of only one variable, as either $\mathbf{U_E(I) = I \cdot (12 - I)^2}$ in terms of income hours or equivalently as $\mathbf{U_E(L) = (12 - L) \cdot L^2}$ in terms of leisure hours. Graphing either one of these obtains an inverse parabolic concave function reflecting diminishing returns and then eventually decreasing utility beyond the peak, which corresponds with the optimal allocations.

To solve using a Lagrangian:

$$\begin{aligned} \mathbf{L}(I, L, \lambda) &= U_E(I, L) + \lambda \cdot g(I, L) \\ &= I \cdot L^2 + \lambda \cdot (12 - I - L) \end{aligned}$$

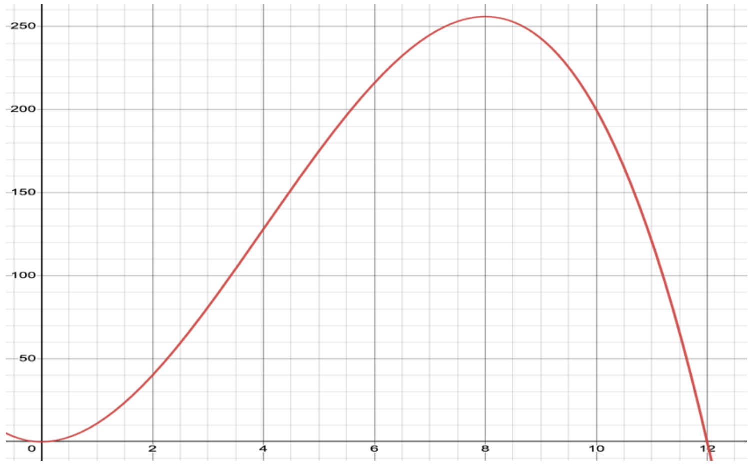
The three "first order conditions" describing the optimal allocations are obtained by taking the partial derivative of the Lagrangian with respect to the three variables:

$$\begin{aligned} \frac{\partial \mathbf{L}}{\partial I} &= L^2 - \lambda = 0 \\ \frac{\partial \mathbf{L}}{\partial L} &= I \cdot 2L - \lambda = 0 \\ \frac{\partial \mathbf{L}}{\partial \lambda} &= 12 - I - L = 0 \end{aligned}$$

Equating the first two FOCs we have $I \cdot 2L = \lambda = L^2$, which we can solve to obtain $2I = L$ to find our optimal ratio of time usage. Substituting this back into the budget constraint which is returned by the third FOC, we get $12 - I - (2I) = 0$ which we can solve to obtain $I^* = 4$ and $L^* = 8$.

Emily cannot work 15 hours because she only has 12 hours to allocate. Her marginal utility from working one more hour if she is working 4 hours is the difference in utility that results: $U_E(4, 8) = 256$ and $U_E(5, 7) = 5 \cdot 7^2 = 245$ so her marginal utility from this fifth hour spent working is -11.

Emily's Utility over hours spent on leisure (U plotted over L):



Emily's Utility over hours spent working (U plotted over I):

Emily's Income/Leisure Diagram: