Microeconomics (Andrew Gates)
Practice Problems – Multivariate Optimization / Lagrange Method

Consumer Choice Framework: Utility-maximizing Consumption Choices

Matt enjoys sushi (S) and cocktails (C) with utility function $\mathbf{U_M(S,C) = S*C}$ precisely describing his consumption preferences. Matt has a $60 gift card to a restaurant where the prices of these two goods are P_S = $5 and P_C = $10.

a) Draw a budget constraint and indifference curves depicting Matt's preferences.

b) Is the budget constraint binding in this problem? Briefly explain.

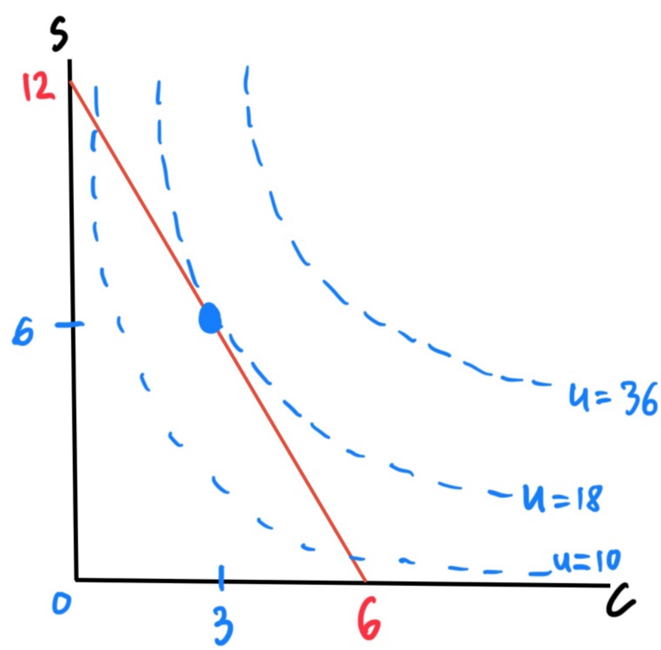
c) Solve Matt's consumer choice optimization problem using a Lagrangian approach.

d) Solve Matt's consumer choice problem using the MRS = MRT approach.

e) Create a table showing each of Matt's possible consumption bundles (integer value allocations only) and the resulting utility from each.

SOLUTION: For constructing the budget constraint, we must find every affordable combination of sushi and cocktails which use all of Matt's purchasing power:

Sushi	Cocktails	Matt's Utility
0	6	**0**
2	5	**10**
4	4	**16**
6	3	**18**
8	2	**16**
10	1	**10**
12	0	**0**



SOLUTION

The budget constraint is binding in this problem, as usual, because Matt can only use the gift card for purchasing these two goods: his utility is monotonically increasing over both, so he will always use up the whole gift card. If it was cash instead of a gift card, the answer would be the same if he placed no value on money but could be different if he also derived utility from the cash, which he might not fully spend in that case.

Using a Lagrangian approach:

$$\mathbf{L}(c, s, \lambda) = c \cdot s + \lambda(60 - 10c - 5s)$$

The first order conditions are:

$$\frac{\partial \mathbf{L}}{\partial c} = s - 10\lambda = 0$$

$$\frac{\partial \mathbf{L}}{\partial s} = c - 5\lambda = 0$$

$$\frac{\partial \mathbf{L}}{\partial \lambda} = 60 - 10c - 5s = 0$$

So from the first two FOCs we have $\frac{c}{5} = \lambda = \frac{s}{10}$ which gives us $s = 2c$ as our optimal consumption ratio. Using the third FOC (budget constraint) we can substitute to obtain the following:

$$60 - 10c - 5(2c) = 0$$
$$c^* = 3, \;\; s^* = 6$$

Using the MRS = MRT approach, we can construct the budget constraint directly: $[\, P_s \cdot s + P_c \cdot c = 100 \,]$ and equating the marginal rate of substitution from the utility function with the marginal rate of transformation from the budget constraint, we have the following:

$$\text{MRS} = -\frac{\frac{\partial U}{\partial c}}{\frac{\partial U}{\partial s}} = \frac{-10}{5} = \text{MRT}$$

$$\frac{-s}{c} = -2$$

$$s = 2c$$

Substituting back into the budget constraint with the prices given in the question, we can solve $5 \cdot (2c) + 10 \cdot c = 100$ which also gets $c^* = 3$ and $s^* = 6$. Intuitively, the logic of this approach is that we are finding the one unique place where the slope of the budget constraint (MRT) is equal to the slope of the indifference curves (MRS). Since there are many indifference curves (which all have the same shape) substituting into the budget constraint allows us to identify exactly which indifference curve we are on.

Income/Leisure Framework – Sleep Requirements & Asymmetric Preferences

Emily can choose exactly how much she wants to work, but she must sleep exactly 12 hours per day and these hours do not count towards either leisure or income. Emily's utility over income I and leisure L is given by the function $\mathbf{U_E (I,L) = I * L^2}$.

a) Draw Emily's budget constraint and express this mathematically.

b) Find the optimal allocation of her time using whichever approach you want.

c) Draw a graph of her utility over hours spent working (allocated to income).

d) Briefly explain the meaning of the curvature in this graph with words.

e) Find her marginal utility from working 1 more hour if she is working 4 hours.

f) Find her marginal utility from working 1 more hour if she is working 15 hours.

g) Draw out Emily's full income-leisure utility diagram: include a budget constraint and three indifference curves. Label several specific points on each of these indifference curves to show utility and the corresponding amounts of income and leisure on the two axes.

SOLUTION

The budget constraint mathematically is $I + L = 12$ and this also means that the utility equation can be re-written in terms of only one variable, as either $\mathbf{U_E(I) = I \cdot (12 - I)^2}$ in terms of income hours or equivalently as $\mathbf{U_E(L) = (12 - L) \cdot L^2}$ in terms of leisure hours. Graphing either one of these obtains an inverse parabolic concave function reflecting diminishing returns and then eventually decreasing utility beyond the peak, which corresponds with the optimal allocations.

To solve using a Lagrangian:

$$\begin{aligned}\mathbf{L}(I, L, \lambda) &= U_E(I, L) + \lambda \cdot g(I, L) \\ &= I \cdot L^2 + \lambda \cdot (12 - I - L)\end{aligned}$$

The three "first order conditions" describing the optimal allocations are obtained by taking the partial derivative of the Lagrangian with respect to the three variables:

$$\frac{\partial \mathbf{L}}{\partial I} = L^2 - \lambda = 0$$

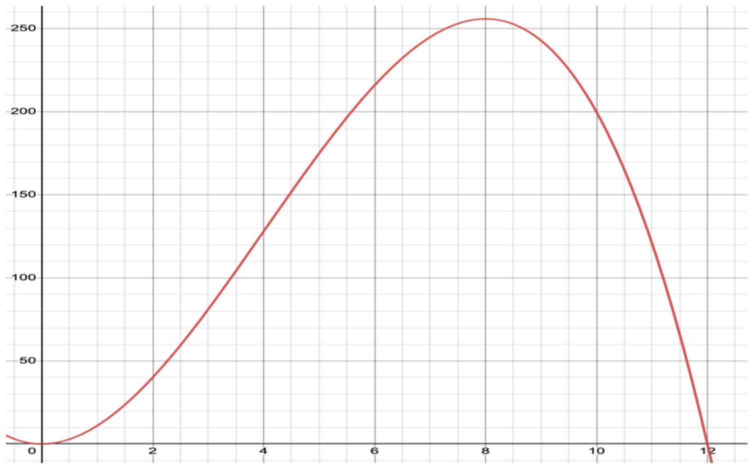
$$\frac{\partial \mathbf{L}}{\partial L} = I \cdot 2L - \lambda = 0$$

$$\frac{\partial \mathbf{L}}{\partial \lambda} = 12 - I - L = 0$$

Equating the first two FOCs we have $I \cdot 2L = \lambda = L^2$, which we can solve to obtain $2I = L$ to find our optimal ratio of time usage. Substituting this back into the budget constraint which is returned by the third FOC, we get $12 - I - (2I) = 0$ which we can solve to obtain $I^* = 4$ and $L^* = 8$.

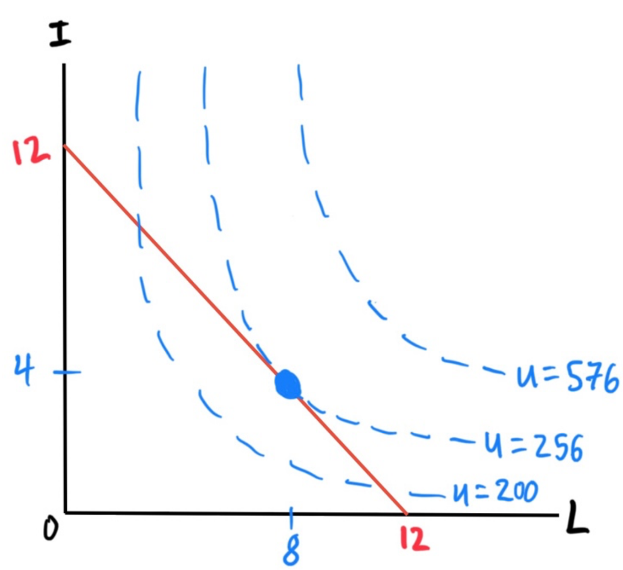
Emily cannot work 15 hours because she only has 12 hours to allocate. Her marginal utility from working one more hour if she is working 4 hours is the difference in utility that results: $U_E(4, 8) = 256$ and $U_E(5, 7) = 5 \cdot 7^2 = 245$ so her marginal utility from this fifth hour spent working is -11.

Emily's Utility over hours spent on leisure (U plotted over L):



Emily's Utility over hours spent working (U plotted over I):

Emily's Income/Leisure Diagram:



Supreme Side Hustle: Business Optimization with Space Constraints

Frank has a lucrative small business re-selling limited edition shoes and jackets from Nike and Supreme. He has 200 cubic feet to store these goods in his apartment: each box of Nike shoes (**N**) takes up one cubic foot and each Supreme jacket (**J**) takes up two cubic feet. Frank can buy one pair of Nike shoes for $200 and he can buy a Supreme jacket for $400. The revenue function describing how much money Frank can get from re-selling these items is **R(N,J) = 9000 ln(N) + 600 J** where ln denotes the natural logarithm, as usual. Assume Frank has no financial constraints.

SOLUTION

Frank generates revenue by re-selling, which are monotonic concave increasing for Nikes and monotonic linear increasing for jackets. He also has a cost to acquire each type of product, which is linear for both. This means that buying and re-selling more jackets will always generate a $200 profit per unit, but there is a decreasing benefit to selling Nikes since the concave revenue function indicates diminishing marginal revenue: the returns are decreasing for Nikes but profit is constant for jackets. The objective function for profits, including revenues and costs, subject to the space constraint of his storage, is:

$$\Pi(N,J) = R(N,J) - C(N,J) = 9000 \cdot ln(N) + 600J - 200N - 400J$$

The budget constraint, which is binding because profit is monotonically increasing over at least one good, is $1N + 2J \leq 200$. Conceptually, this means that Frank will always profit from selling more and therefore he will always use up every last foot of space in his apartment to store the goods. The Lagrangian is:

$$\mathbf{L}(N,J,\lambda) = R(N,J) - C(N,J) + \lambda g(N,J)$$

$$= 9000 \cdot ln(N) + 600J - 200N - 400J + \lambda(200 - N - 2J)$$

The partial derivatives with respect to the three choice variables here are:

$$\frac{\partial L}{\partial N} = \frac{9000}{N} - 200 - \lambda = 0$$

$$\frac{\partial L}{\partial J} = 600 - 400 - 2\lambda = 0$$

$$\frac{\partial \mathbf{L}}{\partial \lambda} = 200 - 2J - N = 0$$

Solving the second FOC, we obtain $\lambda^* = 100$, and substituting this into the first FOC to solve for N, we obtain $N^* = 30$. Using the third FOC with $N^* = 30$, we obtain $J^* = 85$.

If the cost of Nikes doubles, this will reduce the profit per unit from selling Nikes, so he will need to re-optimize to figure out how much he should shift from Nikes to jackets. We now have the following Lagrangian: $\mathbf{L}(N,J,\lambda) = 9000 \cdot ln(N) + 600J - 400N - 400J + \lambda(200 - N - 2J)$.

The first FOC will change but the other two will not. Solving this, we get $\frac{\partial L}{\partial N} = \frac{9000}{N} - 400 - \lambda = 0$ and therefore $N^* = 18$ now. Substituting this into the other two FOCs now obtains $J^* = 91$.

Frank's original profit was $\Pi(30,85) = 9000ln(30) + 600(85) - 200(30) - 400(85) = \$41,610$.
With Nike shoes doubling in price, Frank's profit is now $\Pi_D(18,91) = 9000ln(18) + 600(91) - 400(18) - 400(91) = \$37,013$. We can see that he has shifted towards re-selling fewer Nikes and more jackets in response to this price change, which has also reduced his profits. If he did not re-allocate, then his new profits would be $\Pi(30,85) = 9000ln(30) + 600(85) - 400(30) - 400(85) = \$35,611$.

Firm Profit Maximization: Production of Two Goods

Suppose a factory has six assembly lines which can be used to produce good x or good y. The profit function (which is realistically based on the assumption of nonlinear demand) describing the firm's payoff from production, is $\Pi(x, y) = 2000 \cdot ln(x) + 1000 \cdot ln(y)$. Determine how the firm should use its resources to maximize profits.

SOLUTION: The budget constraint in this case is $x + y \leq 6$ and it is binding because producing more of either good is always monotonically increasing profit because the natural log function is concave increasing. Using this constraint, we can set up our Lagrangian as follows:

$$\begin{aligned} \mathbf{L}(x, y, \lambda) &= \Pi(x, y) + \lambda \cdot g(x, y) \\ &= 2000 \cdot ln(x) + 1000 \cdot ln(y) + \lambda \cdot (6 - x - y) \end{aligned}$$

The three "first order conditions" describing the optimal allocations are obtained by taking the partial derivative of the Lagrangian with respect to the three variables:

$$\frac{\partial \mathbf{L}}{\partial x} = \frac{2000}{x} - \lambda = 0$$

$$\frac{\partial \mathbf{L}}{\partial y} = \frac{1000}{y} - \lambda = 0$$

$$\frac{\partial \mathbf{L}}{\partial \lambda} = 6 - x - y = 0$$

Equating the first two FOCs, we obtain $\frac{2000}{x} = \lambda = \frac{1000}{y}$.

This simplifies to $2000y = 1000x$, which simplifies to $x^* = 2y^*$ as our optimal ratio of inputs. Substituting back into our third FOC, which is our original budget constraint, we can obtain $6 - 2y - y = 0$.

Solving this for y obtains $y^* = 2$ and $x^* = 4$ as our optimal allocations which maximize production output subject to the given constraint. Plugging these values into our original payoff function, we can obtain the maximum payoff value which results from this optimal allocation: $\Pi(4, 2) = 2000 \cdot ln(4) + 1000 \cdot ln(2) = \$3465.74$.

Profit Maximization: Production of Two Goods With Input Costs

Now suppose we have $R(x, y) = 2000 \cdot ln(x) + 1000 \cdot ln(y)$ as our revenue function but there is an additional variable cost involved: each unit of x costs \$150 in materials and each unit of y costs \$750 in materials. We still have the same space constraint of having only 6 assembly lines which can each be used to produce either x or y.

SOLUTION: To solve this, we must account for the costs. Using the given information, we can write the cost function as $C(x, y) = 150x + 750y$, and this is linear for both variables. In order to optimize our production, we now must include these costs in the objective function. We can accomplish this by using the fact that profits equal total revenues minus total costs: $\Pi_{cost}(x, y) = R(x, y) - C(x, y) = 2000 \cdot ln(x) + 1000 \cdot ln(y) - 150x - 750y$.

The Lagrangian equation is:

$$\begin{aligned} \mathbf{L}(x, y, \lambda) &= \Pi(x, y) + \lambda \cdot g(x, y) \\ &= 2000 \cdot ln(x) - 150x + 1000 \cdot ln(y) - 750y + \lambda \cdot (6 - x - y) \end{aligned}$$

With the same constraint as before, the first order conditions are now:

$$\frac{\partial \mathbf{L}}{\partial x} = \frac{2000}{x} - 150 - \lambda = 0$$

$$\frac{\partial \mathbf{L}}{\partial y} = \frac{1000}{y} - 750 - \lambda = 0$$

$$\frac{\partial \mathbf{L}}{\partial \lambda} = 6 - x - y = 0$$

Using the same process to solve, we equate the first two FOCs:

$$\frac{2000}{x} - 150 = \lambda = \frac{1000}{y} - 750$$

$$2000y - 150xy = 1000x - 750xy$$

$$y(2000 + 600x) = 1000x$$

$$y^* = \frac{1000x}{2000 + 600x}$$

Substituting this back into our third FOC, which is once again our original budget constraint, we have $6 - x - (\frac{1000x}{2000+600x}) = 0$. Solving this, we obtain $x^* = -4$ and $x^* = 5$. Since we obviously cannot produce negative amounts of product x, we disregard the negative solution for x. The optimal allocations are $x^* = 5, y^* = 1$. Notice that the cost of y was substantially higher than the cost of x: intuitively this is why we have shifted production to make more of x. The revenue function is concave increasing with respect to each variable, reflecting diminishing returns to productive efficiency: this is why we still optimally allocate one assembly line to producing y. If revenue was linear over quantity with the linear costs here, then we would always allocate all resources to whichever product was more profitable in terms of linear revenue minus linear cost.

Optimization of Production Inputs: Classic Cobb-Douglass

A factory can produce one type of product using capital (K) and labor (L) as its inputs. Its production function is: $F(K, L) = A \cdot K^{\alpha} \cdot L^{\beta}$ and suppose this has a 1:1 relationship with profits. Capital costs \$20 per unit of input and labor costs \$10 per unit of input, with a total budget of \$200. Suppose the technology parameter A equals 100 and the exponent multipliers ("weights") on alpha and beta are equal and sum to one.

SOLUTION: The production function in this case is $F(K, L) = A \cdot K^{\alpha} \cdot L^{\beta} = 100 \cdot K^{0.5} \cdot L^{0.5}$ and the budget constraint is $20K + 10L \leq 200$. Production is monotonically increasing over both types of inputs assuming all non-negative values. The Lagrangian is therefore

$$\mathbf{L}(K, L, \lambda) = 100 \cdot K^{0.5} \cdot L^{0.5} + \lambda(200 - 20K - 10L).$$

$$\frac{\partial \mathbf{L}}{\partial K} = 100(\alpha)K^{\alpha-1}L^{\beta} - 20\lambda = 100(0.5)K^{0.5-1}L^{0.5} - 20\lambda = 0$$

$$\frac{\partial \mathbf{L}}{\partial L} = 100(\beta)K^{\alpha}L^{\beta-1} - 10\lambda = 100(0.5)K^{0.5}L^{0.5-1} - 10\lambda = 0$$

$$\frac{\partial \mathbf{L}}{\partial \lambda} = 200 - 20K - 10L = 0$$

Re-writing the first two first order conditions, we have:

$$\frac{50\sqrt{L}}{\sqrt{K}} - 20\lambda = 0 \;\rightarrow\; \lambda^* = \frac{5\sqrt{L}}{2\sqrt{K}}$$

$$\frac{50\sqrt{K}}{\sqrt{L}} - 10\lambda = 0 \;\rightarrow\; \lambda^* = \frac{5\sqrt{K}}{\sqrt{L}}$$

Equating these with each other, since both are equal to the Lagrangian multiplier λ, we can simplify to obtain $L^* = 2K^*$. Finally, substituting this back into the third FOC, which is also our original budget constraint, we obtain $200 - 20K - 10(2K) = 0$, and solving this yields our solution:

$$K^* = 5, L^* = 10$$

Intuitively, with a symmetric production function (or any objective function or payoff function) we can see that when labor is half the cost of capital we will use twice as much labor as capital. If the costs were equal then optimally we would use equal amounts so long as the objective function was symmetric. With equal costs but a larger exponent multiplier on capital, for example, then we would use more capital. Notice also that indifference curves from this objective function are convex: both inputs are needed to produce anything, but the optimal usage ratio depends on the relative costs of each input compared to their usefulness in making outputs.