Firm Profit Maximization: Production of Two Goods

Suppose a factory has six assembly lines which can be used to produce good x or good y. The profit function (which is realistically based on the assumption of nonlinear demand) describing the firm's payoff from production, is $\Pi(x, y) = 2000 \cdot ln(x) + 1000 \cdot ln(y)$. Determine how the firm should use its resources to maximize profits.

SOLUTION: The budget constraint in this case is $x + y \leq 6$ and it is binding because producing more of either good is always monotonically increasing profit because the natural log function is concave increasing. Using this constraint, we can set up our Lagrangian as follows:

$$\begin{aligned}\mathbf{L}(x, y, \lambda) &= \Pi(x, y) + \lambda \cdot g(x, y)\\ &= 2000 \cdot ln(x) + 1000 \cdot ln(y) + \lambda \cdot (6 - x - y)\end{aligned}$$

The three "first order conditions" describing the optimal allocations are obtained by taking the partial derivative of the Lagrangian with respect to the three variables:

$$\frac{\partial \mathbf{L}}{\partial x} = \frac{2000}{x} - \lambda = 0$$

$$\frac{\partial \mathbf{L}}{\partial y} = \frac{1000}{y} - \lambda = 0$$

$$\frac{\partial \mathbf{L}}{\partial \lambda} = 6 - x - y = 0$$

Equating the first two FOCs, we obtain $\frac{2000}{x} = \lambda = \frac{1000}{y}$.

This simplifies to $2000y = 1000x$, which simplifies to $x^* = 2y^*$ as our optimal ratio of inputs. Substituting back into our third FOC, which is our original budget constraint, we can obtain $6 - 2y - y = 0$.

Solving this for y obtains $y^* = 2$ and $x^* = 4$ as our optimal allocations which maximize production output subject to the given constraint. Plugging these values into our original payoff function, we can obtain the maximum payoff value which results from this optimal allocation: $\Pi(4, 2) = 2000 \cdot ln(4) + 1000 \cdot ln(2) = \$3465.74$.

Profit Maximization: Production of Two Goods With Input Costs

Now suppose we have $R(x, y) = 2000 \cdot ln(x) + 1000 \cdot ln(y)$ as our revenue function but there is an additional variable cost involved: each unit of x costs \$150 in materials and each unit of y costs \$750 in materials. We still have the same space constraint of having only 6 assembly lines which can each be used to produce either x or y.

SOLUTION: To solve this, we must account for the costs. Using the given information, we can write the cost function as $C(x, y) = 150x + 750y$, and this is linear for both variables. In order to optimize our production, we now must include these costs in the objective function. We can accomplish this by using the fact that profits equal total revenues minus total costs: $\Pi_{cost}(x, y) = R(x, y) - C(x, y) = 2000 \cdot ln(x) + 1000 \cdot ln(y) - 150x - 750y$.

The Lagrangian equation is:

$$\begin{aligned}\mathbf{L}(x, y, \lambda) &= \Pi(x, y) + \lambda \cdot g(x, y) \\ &= 2000 \cdot ln(x) - 150x + 1000 \cdot ln(y) - 750y + \lambda \cdot (6 - x - y)\end{aligned}$$

With the same constraint as before, the first order conditions are now:

$$\frac{\partial \mathbf{L}}{\partial x} = \frac{2000}{x} - 150 - \lambda = 0$$

$$\frac{\partial \mathbf{L}}{\partial y} = \frac{1000}{y} - 750 - \lambda = 0$$

$$\frac{\partial \mathbf{L}}{\partial \lambda} = 6 - x - y = 0$$

Using the same process to solve, we equate the first two FOCs:

$$\frac{2000}{x} - 150 = \lambda = \frac{1000}{y} - 750$$

$$2000y - 150xy = 1000x - 750xy$$

$$y(2000 + 600x) = 1000x$$

$$y^* = \frac{1000x}{2000 + 600x}$$

Substituting this back into our third FOC, which is once again our original budget constraint, we have $6 - x - (\frac{1000x}{2000+600x}) = 0$. Solving this, we obtain $x^* = -4$ and $x^* = 5$. Since we obviously cannot produce negative amounts of product x, we disregard the negative solution for x. The optimal allocations are $x^* = 5, y^* = 1$. Notice that the cost of y was substantially higher than the cost of x: intuitively this is why we have shifted production to make more of x. The revenue function is concave increasing with respect to each variable, reflecting diminishing returns to productive efficiency: this is why we still optimally allocate one assembly line to producing y. If revenue was linear over quantity with the linear costs here, then we would always allocate all resources to whichever product was more profitable in terms of linear revenue minus linear cost.

Optimization of Production Inputs: Classic Cobb-Douglass

A factory can produce one type of product using capital (K) and labor (L) as its inputs. Its production function is: $F(K, L) = A \cdot K^{\alpha} \cdot L^{\beta}$ and suppose this has a 1:1 relationship with profits. Capital costs \$20 per unit of input and labor costs \$10 per unit of input, with a total budget of \$200. Suppose the technology parameter A equals 100 and the exponent multipliers ("weights") on alpha and beta are equal and sum to one.

SOLUTION: The production function in this case is $F(K, L) = A \cdot K^{\alpha} \cdot L^{\beta} = 100 \cdot K^{0.5} \cdot L^{0.5}$ and the budget constraint is $20K + 10L \leq 200$. Production is monotonically increasing over both types of inputs assuming all non-negative values. The Lagrangian is therefore

$$\mathbf{L}(K, L, \lambda) = 100 \cdot K^{0.5} \cdot L^{0.5} + \lambda(200 - 20K - 10L).$$

$$\frac{\partial \mathbf{L}}{\partial K} = 100(\alpha)K^{\alpha-1}L^{\beta} - 20\lambda = 100(0.5)K^{0.5-1}L^{0.5} - 20\lambda = 0$$

$$\frac{\partial \mathbf{L}}{\partial L} = 100(\beta)K^{\alpha}L^{\beta-1} - 10\lambda = 100(0.5)K^{0.5}L^{0.5-1} - 10\lambda = 0$$

$$\frac{\partial \mathbf{L}}{\partial \lambda} = 200 - 20K - 10L = 0$$

Re-writing the first two first order conditions, we have:

$$\frac{50\sqrt{L}}{\sqrt{K}} - 20\lambda = 0 \;\rightarrow\; \lambda^* = \frac{5\sqrt{L}}{2\sqrt{K}}$$

$$\frac{50\sqrt{K}}{\sqrt{L}} - 10\lambda = 0 \;\rightarrow\; \lambda^* = \frac{5\sqrt{K}}{\sqrt{L}}$$

Equating these with each other, since both are equal to the Lagrangian multiplier λ, we can simplify to obtain $L^* = 2K^*$. Finally, substituting this back into the third FOC, which is also our original budget constraint, we obtain $200 - 20K - 10(2K) = 0$, and solving this yields our solution:

$$K^* = 5, L^* = 10$$

Intuitively, with a symmetric production function (or any objective function or payoff function) we can see that when labor is half the cost of capital we will use twice as much labor as capital. If the costs were equal then optimally we would use equal amounts so long as the objective function was symmetric. With equal costs but a larger exponent multiplier on capital, for example, then we would use more capital. Notice also that indifference curves from this objective function are convex: both inputs are needed to produce anything, but the optimal usage ratio depends on the relative costs of each input compared to their usefulness in making outputs.