Formal Market Competition Models - Advanced: Monopoly, Cournot Duopoly, Stackelberg Duopoly

Andrew Gates

Advanced Microeconomics / Industrial Organization

2026

Monopolist Profit Maximization - Abstract Formulation

- Market demand is given by: $Q_D = a - b \cdot p$ where a is the "baseline demand parameter", $b > 0$ is demand elasticity, $p \in (0, a)$ is the price.
- With the demand curve re-arranged as $p = \frac{a-Q}{b}$, graphically we have a/b as the intercept and $-1/b$ as the slope
- Firm cost function and production capabilities: constant marginal cost: $c \in (0, a)$ and fixed costs assumed to be zero for simplicity
  - The standard assumption is that demand firms observe the market demand curve but cannot influence it (so no marketing or manipulating consumer preferences) and either cannot observe "who is who" on the demand curve, or is not legally allowed to price discriminate (so it sets one market price for all consumers)
- Profit margin per unit is revenue (price received) minus cost, and total profit is this times the total quantity sold, with market quantity equal to individual firm quantity ($Q = q$) in this case of a monopolist:

$$\Pi = p \cdot q - c \cdot q = (p - c) \cdot Q$$

Substituting the demand equation into the profit function obtains:

$$\Pi = (p - c)Q = (p - c)(a - bp) = ap - ac - bp^2 + bpc$$

This function is inversely parabolic: as $p \to c$ or as $p \to \frac{a}{b}$, profit approaches zero in either case... intuitively there is no margin per unit as price gets too low and there are no units sold as price gets too high. Monopolistic optimization is finding the perfect trade-off between these to maximize profit, which is obtained mathematically by differentiating with respect to the relevant choice variable, which in this case is price:

$$\frac{\partial \Pi}{\partial p} = a - 2bp + bc = 0$$

so optimal monopoly price is

$$p_M = \frac{\frac{a}{b} + c}{2} = \frac{a + bc}{2b}$$

Note that a monopolist can control price or quantity, which are inter-dependent in this case: specifying either inherently specifies the other one as this choice is simply moving along the demand curve to choose a point

$$Q_M = a - p_m = a - \left(\frac{\frac{a}{b} + c}{2}\right) = \left(\frac{a - bc}{2}\right)$$ is monopolist output quantity.

Monopoly profit is therefore:

$$\Pi_M = pq - cq = \left(\frac{a + bc}{2b}\right)\left(\frac{a - bc}{2}\right) - c\left(\frac{a - bc}{2}\right) = \frac{(a - bc)^2}{4b}$$

Cournot Competition

- In this case, competing firms (indexed as $i, j, ...$) non-cooperatively and simultaneously each choose their individual quantity for a homogeneous product. A good example of this is oil production.
- The total output in the market is $\sum_i q_i = Q$ and in the case of two firms, for example, the total market quantity would be $Q = q_i + q_j$
- Price is still given by the market's inverse demand function ($p = \frac{a-Q}{b}$) which is again exogenously determined by consumer preferences with price and quantity of course inversely related.
- Firm i therefore chooses its individual quantity q_i to maximize its own individual profit $\Pi_i = (p - c)q_i$ and will do so by accounting for the equivalent optimization processes of other firms, as the market price determining every firm's revenue depends on the market quantity Q, the sum of all of the firms' quantity supply choices

Firm i seeks to maximize profit by choosing q_i, and we can re-write this as:

$$\Pi_i = (p - c)q_i = \left(\frac{a - [\sum_i q_i]}{b} - c \right) q_i = \left(\frac{a - \left[q_i + \sum_{j \neq i} q_j \right]}{b} - c \right) q_i$$

$$\Pi_i = \frac{aq_i - q_i^2 - q_i q_j}{b} - cq_i$$

Differentiating profit with respect to quantity, which the the choice variable here, we obtain:

$$\frac{\partial \Pi_i}{\partial q_i} = \frac{a - 2q_i - q_j}{b} - c = 0$$

Re-arranging this FOC:

$$a - 2q_i - q_j - bc = 0$$

- With identical firms, we can observe that $q_i = q_j = \hat{q}$ in a symmetric pure strategy Nash Equilibrium, so we can re-write this as:

$$a - bc = 2q_i + q_j = 3\hat{q}$$

The NE quantity for each firm and Cournot market quantity are:

$$\hat{q} = \frac{a - bc}{3} \qquad Q = 2(\hat{q}) = \frac{2(a - bc)}{3}$$

- Substituting this in for market price:

$$p = \frac{a - Q}{b} = \frac{a - 2\left(\frac{a-c}{3}\right)}{b} = \frac{a + 2bc}{3b}$$

which allows the calculation of profits:

$$\Pi_i = \Pi_j = (p_C - c)\hat{q} = \left(\frac{a + 2bc}{3b} - c\right)\left(\frac{a - bc}{3}\right) = \frac{(a - bc)^2}{9b}$$

- Notice that in the Cournot duopoly scenario, we have market quantity:

$$Q = q_i + q_j = 2\hat{q} = 2\left(\frac{a - bc}{3}\right) > \frac{a - bc}{2} = q_M$$

  so the market has a larger quantity under this Cournot duopoly scenario than it would have with a monopoly.

- Note that Cournot duopoly profit is lower than the monopolist's profit.

Stackelberg Duopoly

- Now consider a sequential version of this quantity competition: Firm 1 (the "leader") moves first, but otherwise the same as Cournot.

- Firm 2 (the "follower") will set its quantity q_2 as a function of the leader's quantity q_1 to maximize its profit:

$$\max_{q_2}(\Pi_2) = \left(\frac{a - [\sum_i q_i]}{b} - c\right) q_2 = \left(\frac{a - [q_2 + q_1]}{b} - c\right) q_2$$

$$= \frac{aq_2 - q_2^2 - q_1 q_2}{b} - cq_2$$

- Differentiating with respect to the follower firm's choice variable (its quantity), we obtain the FOC which will define its best response:

$$\frac{\partial \Pi_2}{\partial q_2} = \frac{a - 2q_2 - q_1}{b} - c = 0$$

- Firm 2 (the "follower") sets q_2 as a function of q_1 to maximize its profit, and its best response function (obtained from the FOC) is:

$$R_2\left(q_1\right) = \left(\frac{a - bc - q_1}{2}\right)$$

- Firm 1 (the "leader") correctly predicts this in setting its output first to maximize its own profit by choosing its quantity:

$$\max_{q_1}(\Pi_1) = \left(\frac{a - [\sum_i q_i]}{b} - c\right) q_1 = \left(\frac{a - [q_1 + R_2(q_1)]}{b} - c\right) q_1$$

$$= \frac{aq_1 - q_1^2 - \left(\frac{aq_1 - bcq_1 - q_1^2}{2}\right)}{b} - cq_1 = \frac{(a - bc)q_1 - q_1^2}{2b}$$

- Differentiating and setting equal to zero, we have the FOC:

$$\frac{\partial \Pi_1}{\partial q_1} = 0 = \frac{a - bc - 2q_1}{2b}$$

and solving this obtains $q_1^* = \frac{a-bc}{2}$ and $q_2^* = \frac{a-bc-q_1}{2} = \frac{a-bc}{4}$

- With identical firms, the leader produces more and secures more profit, so moving first is advantageous.

$$\begin{bmatrix} q_1^* = \frac{a-bc}{2} \\ \\ q_2^* = \frac{a-bc}{4} \end{bmatrix}$$

Comparison of Models

$$\left[Q^{\text{total}}_{\text{Stack}} = \frac{3}{4}(a - bc)\right] > \left[Q^{\text{total}}_{\text{Cournot}} = \frac{2}{3}(a - bc)\right] > \left[Q^{\text{total}}_{\text{Monop}} = \frac{1}{2}(a - bc)\right]$$

- Quantity supplied to the market is the lowest with a (private) monopolist, higher with a duopoly, and highest with an asymmetric duopoly Stackelberg scenario where one firm is a first-mover.
- Assuming that government / policymakers and consumers want as close to the free market competitive equilibrium quantity as possible, they benefit from having more players and circumstances less conducive to allowing collusive behaviors.

Discussion & Insights

- Monopoly is worst for consumers, with the smallest total market quantity and highest profits per firm.
- With more firms profits decline and total market output increases.
- Assuming no externalities, it is inefficient to have a quantity below the free market equilibrium level because people who value a product at an amount higher than the cost of production are unable to acquire this product. Remember that demand curve represents "willingness to pay" and supply curve represents "willingness to sell".
- Under-producing certain types of products may create strategic issues (transportation, defense, food...) or involve negative externalities in addition to the market failure caused by the inefficiency.

Extra Practice Questions

Consider a Cournot scenario with J $>$ 2 firms which each have constant unit production cost c and face a market with inverse demand function $p(q) = a - bq$ with $a > c \geq 0$ and demand elasticity parameter $b > 0$

Note that a is still a baseline demand parameter and elasticity b can be kept "normalized" (equal to 1 for simplicity)...

a) Calculate the monopoly outcome for $J = 1$.

b) Find the quantity and profit outcomes when J=3.

c) Determine what happens as $J \to \infty$ and how this affects consumers.

[Hint: compare to Bertrand price competition...]

Discussion - Competition

- Implications for consumer and producer surplus?
- There are many policy actions to address inefficiencies arising from market power.
- Incidence of subsidy and value capture: what is the problem with government subsidizing production to increase quantity to a level closer to the efficient level?
- We can also evaluate potential market failure scenarios (even with many firms) arising from asymmetric information instead of market power...

Cournot Duopoly Allowing for Different Costs

- Now suppose the two firms still choose quantities simultaneously and non-cooperatively, but firm i has constant marginal cost c_i and firm j has constant marginal cost c_j: these could be equal or different
- Market inverse demand (facing both firms) is still:

$$p = \frac{a - Q}{b} \qquad \text{where} \qquad Q = q_i + q_j$$

- Firm i solves $\Pi_i = (p - c_i)q_i$
- Firm j solves: $\Pi_j = (p - c_j)q_j$

$\rightarrow$ With cost asymmetry firms may choose different equilibrium quantities

Profit Functions with Asymmetric Costs

- Substituting inverse demand into firm i's profit function gives:

$$\Pi_i = \left(\frac{a - q_i - q_j}{b} - c_i\right) q_i = \frac{aq_i - q_i^2 - q_i q_j}{b} - c_i q_i$$

- Similarly, firm j's profit function is:

$$\Pi_j = \left(\frac{a - q_i - q_j}{b} - c_j\right) q_j = \frac{aq_j - q_i q_j - q_j^2}{b} - c_j q_j$$

- Each firm takes the other firm's quantity as given and chooses its own quantity to maximize profit.

First-Order Conditions and Best Responses

- FOC: Differentiate firm i's profit with respect to q_i:

$$\frac{\partial \Pi_i}{\partial q_i} = \frac{a - 2q_i - q_j}{b} - c_i = 0$$

- Intuition: Profit is inverse parabolic over quantity, so a firm's output (supply) quantity level (the choice variable within firm i's control) that maximizes its profits is obtained by setting the derivative of its profit with respect to its quantity equal to zero.
- Re-arranging obtains firm i's best response function, which specifies its optimal quantity choice as a function of the parameters and the quantity chosen by other firm(s):

$$R_i(q_j) = q_i = \frac{a - bc_i - q_j}{2}$$

Asymmetric Cost Cournot: Nash Equilibrium Solution

- The two best response equations are:

$$q_i = \frac{a - bc_i - q_j}{2} \qquad \text{and} \qquad q_j = \frac{a - bc_j - q_i}{2}$$

- Multiplying both equations by 2 and re-arranging:

$$2q_i + q_j = a - bc_i \qquad \text{and} \qquad q_i + 2q_j = a - bc_j$$

- Solving this system obtains the optimal Cournot NE quantities:

$$q_i^* = \frac{a - 2bc_i + bc_j}{3}$$

$$q_j^* = \frac{a - 2bc_j + bc_i}{3}$$

Equilibrium Price and Interpretation

- Total market quantity is:

$$Q^* = q_i^* + q_j^* = \frac{2a - bc_i - bc_j}{3}$$

- Substituting into inverse demand obtains the market equilibrium price:

$$p^* = \frac{a - Q^*}{b} = \frac{a - \left(\frac{2a - bc_i - bc_j}{3}\right)}{b} = \frac{a + bc_i + bc_j}{3b}$$

- Implications:
  - Lower-cost firms produce more than the higher-cost firms
  - Market price depends on all firms' costs

Equilibrium Profits and Special Case

- Each firm's equilibrium profit is still:

$$\Pi_i^* = (p^* - c_i)q_i^* \qquad \text{and} \qquad \Pi_j^* = (p^* - c_j)q_j^*$$

- For firm i:

$$p^* - c_i = \frac{a + bc_i + bc_j}{3b} - c_i = \frac{a - 2bc_i + bc_j}{3b}$$

so:

$$\Pi_i^* = \left(\frac{a - 2bc_i + bc_j}{3b}\right)\left(\frac{a - 2bc_i + bc_j}{3}\right) = \frac{(a - 2bc_i + bc_j)^2}{9b}$$

- Similarly:

$$\Pi_j^* = \frac{(a - 2bc_j + bc_i)^2}{9b}$$

- If $c_i = c_j = c$, these expressions become the symmetric cost case