Multiplayer Games, Strategic Randomization & Mixed Strategy Nash Equilibrium, Collective Action & Coordination Problems

Microeconomics (SPI 615b)

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FOUR PLAYER GAME

	3	A			B	
4	1 \ 2	L	R	1 \ 2	L	R
X	U	(1,2,-1, 1)	(3,3,1,-1)	U	(0,0,0,0)	(2,1,2,1)
	D	(0,1,-3,0)	(4,2,1,3)	D	(2,1,0,3)	(-1,2,0,2)
	1 \ 2	L	R	1 \ 2	L	R
Y	U	(3,4,0,0)	(1,2,1,2)	U	(1,-1,-2,2)	(3,2,2,-1)
	D	(1,2,-1, -1)	(2,-1,3,1)	D	(2,1,1,1)	(0,3,1,0)

FOUR PLAYER GAME

Player 1 chooses Up or Down

Note that the order of payoffs corresponds with the player number

	3		A			B
4	1\2	L	R	1\2	L	R
X	U	(1,2,-1, 1)	(3,3,1,-1)	U	(0,0,0,0)	(2,1,2,1)
	D	(0,1,-3,0)	(4,2,1,3)	D	(2,1,0,3)	(-1,2,0,2)
	1\2	L	R	1\2	L	R
Y	U	(3,4,0,0)	(1,2,1,2)	U	(1,-1,-2,2)	(3,2,2,-1)
	D	(1,2,-1, -1)	(2,-1,3,1)	D	(2,1,1,1)	(0,3,1,0)

FOUR PLAYER GAME

Player 1 chooses Up or Down & Player 2 chooses Left or Right

Note that L is a dominant strategy in Game ***AY*** but R is a dominant strategy in the other three Games

	3	A			B	
4	1\2	L	R	1\2	L	R
X	U	(1,2,-1, 1)	(3,3,1,-1)	U	(0,0,0,0)	(2,1,2,1)
	D	(0,1,-3,0)	(4,2,1,3)	D	(2,1,0,3)	(-1,2,0,2)
	1\2	L	R	1\2	L	R
Y	U	(3,4,0,0)	(1,2,1,2)	U	(1,-1,-2,2)	(3,2,2,-1)
	D	(1,2,-1, -1)	(2,-1,3,1)	D	(2,1,1,1)	(0,3,1,0)

FOUR PLAYER GAME

Player 1 chooses Up or Down & Player 2 chooses Left or Right

Player 3 chooses Game Column A or B & Player 4 chooses Game Row X or Y

	3	A		B		
4	1 \ 2	L	R	1 \ 2	L	R
X	U	(1,2,-1, 1)	(3,3,1,-1)	U	(0,0,0,0)	(2,1,2,1)
	D	(0,1,-3,0)	(4,2,1,3)	D	(2,1,0,3)	(-1,2,0,2)
	1 \ 2	L	R	1 \ 2	L	R
Y	U	(3,4,0,0)	(1,2,1,2)	U	(1,-1,-2,2)	(3,2,2,-1)
	D	(1,2,-1, -1)	(2,-1,3,1)	D	(2,1,1,1)	(0,3,1,0)

FOUR PLAYER GAME

Nash Equilibria: (U,R,B,X) and (D,R,A,X)

	3	A			B	
4	1\2	L	R	1\2	L	R
X	U	(1,2,-1, 1)	(3,3,1,-1)	U	(0,0,0,0)	(2,1,2,1)
	D	(0,1,-3,0)	(4,2,1,3)	D	(2,1,0,3)	(-1,2,0,2)
	1\2	L	R	1\2	L	R
Y	U	(3,4,0,0)	(1,2,1,2)	U	(1,-1,-2,2)	(3,2,2,-1)
	D	(1,2,-1, -1)	(2,-1,3,1)	D	(2,1,1,1)	(0,3,1,0)

FINDING ALL NASH EQUILIBRIA, INCLUDING MIXED STRATEGIES

		She	
		Bar	Museum
He	Bar	2, 1	0, 0
	Museum	0, 0	1, 2

SYMMETRIC 2X2 GAME WITH ASYMMETRIC PAYOFFS

Column ("She")

Row "He"	Left	Right
Up	1 8	0 0
Down	7 7	8 1

USA VS. RUSSIA: MSE & INTUITION

		Attack	Peace
USA	Attack	-3000 -3000	0 200
	Peace	200 0	0 0

- In this symmetric game, both sides can gain from attacking when the other side chooses Peace
- This incentive to choose Attack must be weighed against the extremely bad outcome for each side if both attack
- The two pure strategy NE outcomes are **{Attack, Peace}** and **{Peace, Attack}** … but this outcome does not make sense and fails to adequately model the world if the game is repeated over and over

USA VS. RUSSIA: MSE & INTUITION

		Attack	Peace
USA	Attack	-3000 -3000	0 200
	Peace	200 0	0 0

- If we set ***a*** as the probability of USA choosing attack and ***r*** is the probability of Russia choosing attack, then we can equate the expected payoffs for each to find the mixed strategy Nash equilibrium

- $EV_{USA}[Attack] = -3000(r) + 200(1-r) = 200 – 3200r$
- $EV_{USA}[Peace] = 0(r) + 0(1-r) = 0$
- Equating these two expected values and solving obtains **r*= 1/16**
- At this exact point of Russia's attack probability, USA is exactly indifferent between its two options.

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USA VS. RUSSIA: MSE & INTUITION

USA		Attack	Peace
	Attack	-3000 -3000	0 200
	Peace	200 0	0 0

- The unique MSE in this game is **{1/16, 1/16}** , meaning both sides will randomize and commit to attacking with exactly a 1 in 16 chance, which is set before the game begins and then beyond their control as it is literally left up to chance
- This implies that there is a 1/256 probability of both sides choosing Attack every time this game is played

FOOTBALL: PLAY-CALLING GAME

		Defense	
		Cover	Line
Offense	Rush	7, -7	-1, 1
	Pass	0, 0	12, -12

FOOTBALL MSE OUTCOME

Offense \ Defense	Cover	Line
Rush	7, -7	-1, 1
Pass	0, 0	12, -12

There is no pure strategy NE in this game. If c = Pr(cover) and r = Pr(rush) , then the MSE here is **r* = 3/5 , c* = 13/20** : this means the offense will optimally rush 60% of the time and the defense will optimally cover against passing 65% of the time, which is close to what we often see in the NFL.

N-PLAYER INNOVATION GAME (COORDINATION PROBLEM)

Suppose the pharmaceutical industry in Copyland is comprised of n identical firms (with equal market share) which are all equally capable of inventing a new vaccine technology, which is certain to succeed but requires a costly investment in research and testing. All n firms must simultaneously and individually choose whether or not to spend C billion dollars to develop the technology. Firms operate independently and they cannot communicate, collaborate, or share research costs. Copyland has no intellectual property ("IP") or patent protection laws, so if at least one firm chooses to incur cost C to develop the technology, then all firms will obtain benefit $\frac{V}{n}$ billion dollars from being able to sell this new vaccine. Payoffs for each firm are defined as benefit minus individual cost incurred and all firms are completely self-interested. If zero firms develop the vaccine, then payoffs are zero for all firms.

(i) What is the likelihood that this vaccine will be developed? (find a specific mathematical equation $p(n)$ for this probability)

N-PLAYER GAME: INNOVATION

Solution(i):

The players are each of the n firms, the possible actions are Develop or No Investment, and the payoffs are either $\frac{V}{n}$, $\left(\frac{V}{n} - C\right)$, or zero. Let x_i denote the probability that firm i chooses to develop the vaccine. To find a Nash Equilibrium with identical firms (symmetric behavior) in mixed strategies, each of them must randomize between the two possible actions to be indifferent between them, just like in the matrix examples. The probability that some other firm j does not invest in development is $(1 - x_j)$, and the probability that firm j and firm k both do not invest in development is $(1 - x_j) \cdot (1 - x_k)$. Since it is given that all firms are identical, we know that $x_i = x_j = x_k = \ldots = x$ for all n firms. If we have three rival firms j, k, w, then the probability that none of them invest in development is $(1 - x_j) \cdot (1 - x_k) \cdot (1 - x_w) = (1 - x)^3$. So for n firms, the probability that none will invest in developing the vaccine is $(1 - x) \cdot (1 - x) \cdot (1 - x)\ldots$ n times, which we can express as $(1 - x)^n$.

SOLUTION CONT. (N-PLAYER INNOVATION GAME)

Either someone develops the vaccine or nobody develops the vaccine - exactly one of these two statements is always true, so the probability that someone develops the vaccine is therefore equal to one minus the probability that nobody develops the vaccine. The probability that at least one firm will develop is therefore $p(n) = 1 - [(1-x)^n]$. Now consider the point of indifference for firm i in choosing between the two possible actions:

$$(\frac{V}{n} - C) \cdot [1] = \frac{V}{n} \cdot [1 - (1-x)^{(n-1)}]$$

If firm i develops, then it receives a certain payoff $(\frac{V}{n} - C)$ no matter what choices the other firms make.

If firm i does not develop, then its expected payoff is $\frac{V}{n}$ times the probability that at least one of the $(n-1)$ other firms does develop the vaccine technology.

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We can then use algebra to solve for x:

$$(V - Cn) \cdot [1] = V \cdot [1 - (1-x)^{(n-1)}]$$

$$1 - (1-x)^{(n-1)} = \frac{V - Cn}{V}$$

$$1 - (1-x)^{(n-1)} = 1 - \frac{Cn}{V}$$

$$(1-x)^{(n-1)} = \frac{Cn}{V}$$

$$(1-x) = \left(\frac{Cn}{V}\right)^{1/(n-1)}$$

$$x = 1 - \left(\frac{Cn}{V}\right)^{1/(n-1)}$$

SOLUTION CONT. (N-PLAYER INNOVATION GAME)

$$x = 1 - \left(\frac{Cn}{V}\right)^{1/(n-1)}$$

The probability of any one arbitrary firm i developing the vaccine is x, so the probability that at least one of the n firms develops is equal to one minus the probability that zero firms develop:

$$p(n) = 1 - [1-(x)]^n = 1 - \left[1 - \left(1 - \left(\frac{Cn}{V}\right)^{1/(n-1)}\right)\right]^n = 1 - \left(\frac{Cn}{V}\right)^{n/(n-1)} = 1 - \left(\frac{C}{(\frac{V}{n})}\right)^{n/(n-1)}$$

Notice this depends on the cost-benefit ratio ($\frac{C}{V}$) of investing and the number of firms which are making this simultaneous and independent decision. If the payoff value increases, then the probability of each firm (and therefore at least one firm) choosing to develop will increase. At the same time, this decreases as there is an increase in the number of identical firms splitting the market value, while the individual cost of choosing develop stays the same. As the cost of development goes up and/or as the number of firms increases, the individual and aggregate probabilities of development both decrease. This type of coordination issue is a problematic but unfortunately very common behavior scenario in markets and policymaking.

N-PLAYER INNOVATION GAME (FOUR FIRMS)

Suppose the pharmaceutical industry in Copyland is comprised of n identical firms (with equal market share) which are all equally capable of inventing a new vaccine technology, which is certain to succeed but requires a costly investment in research and testing. All n firms must simultaneously and individually choose whether or not to spend C billion dollars to develop the technology. Firms operate independently and they cannot communicate, collaborate, or share research costs. Copyland has no intellectual property ("IP") or patent protection laws, so if at least one firm chooses to incur cost C to develop the technology, then all firms will obtain benefit $\frac{V}{n}$ billion dollars from being able to sell this new vaccine. Payoffs for each firm are defined as benefit minus individual cost incurred and all firms are completely self-interested. If zero firms develop the vaccine, then payoffs are zero for all firms.

(ii) Suppose now the industry has four identical firms named Phrizer ("P"), Mofirma ("M"), Jensen ("J"), and Zenika ("Z"). If $V = 10$ and $C = 5$, what is the probability that this vaccine will be invented now?

(iii) If the government can offer to subsidize the market, increasing the value of V by some amount S to raise the likelihood of development for the new vaccine technology, precisely how much does it need to spend to meet the policy goal of having at least a $\frac{15}{16}$ probability of development?

SOLUTIONS: N-PLAYER INNOVATION GAME (FOUR FIRMS)

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Solution(ii):

The probability is zero if the individual benefit is smaller than the individual cost of development.

Solution(iii): For there to be any chance of vaccine development in Copyland, the individual benefit for the firms must be larger than the cost of development, so $\frac{V+S}{n} > C$. With four firms and a cost of 5, we need $V + S > 5 * 4$ so with $V = 10$ the government must subsidize the market by at least 10 billion to have any chance of someone developing the vaccine.

N-PLAYER INNOVATION GAME (POLICY INTERVENTION)

Suppose the pharmaceutical industry in Copyland is comprised of n identical firms (with equal market share) which are all equally capable of inventing a new vaccine technology, which is certain to succeed but requires a costly investment in research and testing. All n firms must simultaneously and individually choose whether or not to spend C billion dollars to develop the technology. Firms operate independently and they cannot communicate, collaborate, or share research costs. Copyland has no intellectual property ("IP") or patent protection laws, so if at least one firm chooses to incur cost C to develop the technology, then all firms will obtain benefit $\frac{V}{n}$ billion dollars from being able to sell this new vaccine. Payoffs for each firm are defined as benefit minus individual cost incurred and all firms are completely self-interested. If zero firms develop the vaccine, then payoffs are zero for all firms.

(iii) If the government can offer to subsidize the market, increasing the value of V by some amount S to raise the likelihood of development for the new vaccine technology, precisely how much does it need to spend to meet the policy goal of having at least a $\frac{15}{16}$ probability of development?

Solution(iii): For there to be any chance of vaccine development in Copyland, the individual benefit for the firms must be larger than the cost of development, so $\frac{V+S}{n} > C$. With four firms and a cost of 5, we need $V + S > 5 * 4$ so with $V = 10$ the government must subsidize the market by at least 10 billion to have any chance of someone developing the vaccine.

To obtain a $\frac{15}{16}$ probability of the vaccine being invented, we need

$$p(n) = \frac{15}{16} = 1 - \left(\frac{Cn}{V+S}\right)^{4/3}$$

so rearranging this yields:

$$\left(\frac{Cn}{V+S}\right)^{4/3} = \frac{1}{16}$$

and therefore

$$\frac{Cn}{V+S} = \frac{1}{8}$$

To obtain a $\frac{15}{16}$ probability of the vaccine being invented, we need

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so rearranging this yields:

$$\left(\frac{Cn}{V+S}\right)^{4/3} = \frac{1}{16}$$

and therefore

$$\frac{Cn}{V+S} = \frac{1}{8}$$

When $n = 4$ and $C = 5$, we need $V + S = 160$ so if $V = 10$ then the government must subsidize the market by $S = 150$ to achieve this sufficiently high probability of the vaccine technology being developed by at least one of the four firms.

With $n = 4, C = 5, V + S = 160$ we have the probability of development:

$$p(4) = 1 - (1-x)^4 = 1 - \left(\frac{5}{\frac{160}{4}}\right)^{4/(4-1)} = 1 - \left(\frac{1}{8}\right)^{4/3} = \frac{15}{16}$$

With these values, the probability of each firm i developing would be:

$$x = 1 - \left(\frac{20}{160}\right)^{1/(4-1)} = 1 - \left(\frac{1}{8}\right)^{1/3} = 1 - \frac{1}{2} = \frac{1}{2}$$

The probability that none of the four firms develop is $(\frac{1}{2})^4 = \frac{1}{16}$

INNOVATION GAME: DISCUSSION QUESTIONS

- **What are some of the reasons why governments establish intellectual property laws and protections for patent rights?**
  - **Are there drawbacks to this?**
  - **What are the effects on consumers?**
- **Do you see any pros/cons of centralized vs. decentralized industry organization and implications for decision-making?**
- **What would be a more efficient mechanism for the government to use money to intervene in this situation?**