# Solving Utility Functions & Finding Optimal Choice

Andrew's utility function for pizza consumption, where **z** is one slice of pizza: $U^A(z) = -z^2 + 6z$

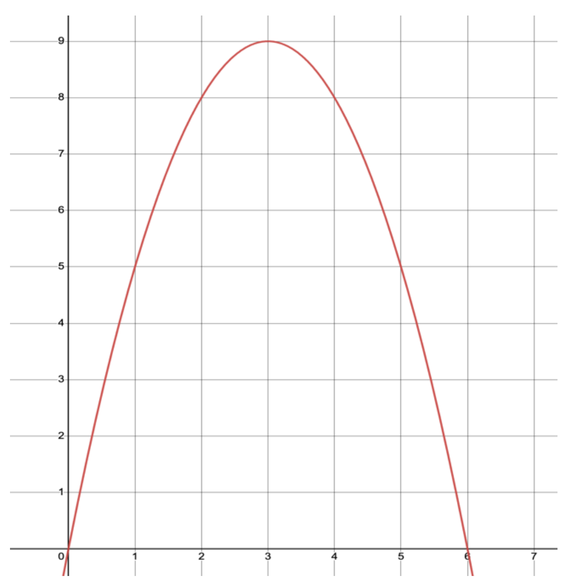
This is sometimes called the "objective function" as it describes the outcome where we want to obtain the best result. In a one-variable optimization problem the independent variable or "choice variable", pizza, is always plotted on the horizontal x-axis and the dependent or outcome variable (utility) is always plotted on the vertical y-axis. This type of function, with a negative squared term and larger positive linear term, is a realistic way to model optimal consumption preferences and its graph has an inverse parabolic shape. Knowing this, the fastest approach is to take the derivative of the utility function with respect to the choice variable and set that derivative equal to zero. The slope of the objective function is zero at the top of the curve, which is our optimal value. In this case, the derivative is $U' = -2z + 6$. This derivative is the function describing the slope (or marginal utility) of the objective function, which means it tells us how utility is changing as the quantity of pizza changes for each specific quantity value z. When the derivative is positive, more pizza increases utility. When the derivative is negative, now more pizza is decreasing utility. Setting the derivative $U' = -2z + 6 = 0$, we can solve for the optimal value: $z^* = 3$, so Andrew will maximize utility by consuming three slices.

## Steps using "plug-in" / graphing approach (without calculus):

1. Enter values for **z** : you cannot have negative pizza so start at zero, and assuming you cannot buy a fraction of a slice, only calculate integer values.

2. Calculate the value of $U^A$ corresponding with each value of **z** and find the largest one
   - Look for a pattern: increasing or decreasing / faster or slower each time? **
   - Notice the optimal consumption choice is $z^* = 3$ because it maximizes $U^A(z)$.

| z | $U^A(z) = -z^2 + 6z$ | *Marginal utility:* |
|---|---|---|
| 0 | 0 | |
| | | +5 |
| 1 | 5 | |
| | | 3 |
| 2 | 8 | |
| | | 1 |
| 3 | 9 | |
| | | -1 |
| 4 | 8 | |
| | | -3 |
| 5 | 5 | |
| | | -5 |
| 6 | 0 | |
| | | -7 |
| 7 | -7 | |



** If marginal utility is increasing at a decreasing rate (concave increasing area of the utility function) then you might be approaching a maximum or converging towards a limit, although this is not always the case, such as the example of a square root function. We will not use complicated polynomial functions in this class, but you need to test each possible value up to a reasonable level: once utility "changes direction" it is safe to assume that you have found a maximum or minimum.

# Solving Utility Functions with a Cost included:

Benefit function: $B(x) = 10x - x^2$
Marginal benefit: $MB(x) = 10 - 2x$

If x is free, then the *optimal consumption choice* is $\mathbf{x_f^* = 5}$ to maximize B(x) at B*=25.

Suppose x instead has a cost of $3 per unit, or equivalently the "cost function" is $C(x) = 3x$.

*Marginal cost*, or the change in cost per unit, is therefore constant here:
MC = *$3 per unit of x* (Note that MC is the derivative of the cost function)

Assume the decision-maker here has lots of money and a linear "utility over money" function U(m) = m , which establishes a fixed exchange rate of 1:1 between utility units and dollars, so every $1 is always worth 1 utility unit at all levels of wealth.

[Utility over money is a straight line in this case, which is consistent with risk-neutral preferences]***

Total utility over x is now Benefit minus Cost:
$U(x) = B(x) - C(x) = [10x - x^2] - [3x] = 7x - x^2$

| x | B(x) | MB(x) | | MC(x) | C(x) | U(x) = B(x)-C(x) | |
|---|---|---|---|---|---|---|---|
| 0 | 0 | | | | 0 | 0 | |
| | | +9 | | 3 | | | |
| 1 | 9 | | | | 3 | 6 | |
| | | +7 | | 3 | | | |
| 2 | 16 | | | | 6 | 10 | |
| | | +5 | | 3 | | | |
| 3 | 21 | | | | 9 | 12 | |
| | | +3 | | 3 | | | **U(3.5)= 12.25** |
| 4 | 24 | | | | 12 | 12 | |
| | | +1 | | 3 | | | |
| 5 | 25 | | | | 15 | 10 | **B(5)= 25** |
| | | -1 | | 3 | | | |
| 6 | 24 | | | | 18 | 6 | |

The solution to maximize U(x) by taking both cost and benefit into account is $\mathbf{x_c^* = 3.5}$

This is also the point where the marginal benefit equals the marginal cost: increasing x any further beyond this point would increase cost more than it would increase benefit.

If x is not divisible (like cars or plane tickets) then the decision-maker is *indifferent* between x* = 3 and x* = 4 with U*(3) = U*(4) = 12. Being indifferent between these two options means the decision maker is equally happy in either case.

*** In quantitative problems you should not assume "risk averse preferences" or decreasing utility over money unless this is explicitly stated. In subjective / written problems you should generally assume that people are rational, selfish, and that most people are risk-averse.

# Common Functions in Microeconomics & Partial Derivatives

a) $U(x,y) = x^3 + y$

Derivatives: $U'(x) = 3x^2$ $U'(y) = 1$

This is convex increasing over x and linear increasing over y, with no direct interaction between x and y. If this represented preferences, then this person would derive more utility from each of these two things, but exponentially more utility from x and the same amount of positive utility per unit (1 util per unit) from y.

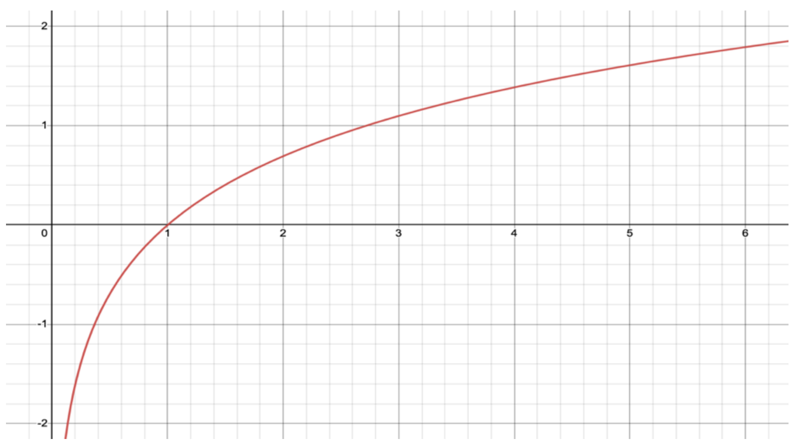
b) $U(z) = 47zk$ where z is a choice variable and $k > 0$ is some fixed parameter

Derivative: $U'(z) = 47k$ This is linear increasing in z and linearly increased by parameter k.

c) $U(y) = \ln(y)$

Derivative: $U'(y) = 1/y$

This is a common monotonic concave increasing function, with the following graph:




d) $F(K,L) = A K^{\alpha} L^{\beta}$

Derivatives: $U'(K) = A\alpha K^{(\alpha - 1)} L^{\beta}$ $U'(L) = A\beta K^{\alpha} L^{(\beta - 1)}$

This is a Cobb-Douglass Production function, which is often used to characterize concave increasing production output over capital (K) and labor (L) with a technology parameter (A) and the relative efficiency parameters alpha and beta (numbers between zero and one).

e) $Y(x) = \sqrt{(4x)}$

Derivative: $Y'(x) = 2 x^{(-1/2)}$ This is also a monotonic, concave, increasing function.