Repeated Games & Applications: Sustaining Collusion, Discounting, Comparison of Duopoly Outcomes

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Introduction

- Repeated Games
    - Discount factor and concept.
    - Review of summation formulas.
    - Analysis of sustaining collusion in repeated games.

- Importance of Timing
    - Tech firm duopoly: simultaneous, sequential, and collusive cases.
    - Discussion of concept and applications to policy.

Discount Factor and Arithmetic Summation Formula

- $\delta \in (0, 1)$ is a "discount factor" between zero and one representing the depreciation in utility per period.
  - This assumes that an equivalent amount of utility in the future is worth less than that amount of utility in the present.
  - The fundamental explanation for this assumption is the cost of potential uncertainty.

- Formula for arithmetic summation of a sequence:
  - $\delta + \delta^2 + \delta^3 + ... + \delta^n + \cdots = \sum_{n=1}^{\infty} \delta^n = \frac{\delta}{1-\delta}$

  - Also note that a related formula with a different starting point for the index is the following:

    $1 + \delta + \delta^2 + \delta^3 + ... + \delta^n + \cdots = \sum_{n=0}^{\infty} \delta^n = \frac{1}{1-\delta}$

Re-arranging Summation Formulas

- $\delta + \delta^2 + \delta^3 + ... + \delta^n + \cdots = \sum_{n=1}^{\infty} \delta^n = \frac{\delta}{1-\delta}$
- $1 + \delta + \delta^2 + \delta^3 + ... + \delta^n + \cdots = \sum_{n=0}^{\infty} \delta^n = \frac{1}{1-\delta}$
- To see how these two are related, observe the following: $\delta + \delta^2 + \delta^3 + ... = \delta(1 + \delta + \delta^2 + ...) = \delta\left(\frac{1}{1-\delta}\right)$
- To understand the formula notice that:

$$\begin{aligned} V = \delta^0 + \delta + \delta^2 + \delta^3 + ... = \ & 1 + \delta(1 + \delta + \delta^2 + ...) = 1 + \delta V \\ & \implies V(1-\delta) = 1 \\ & \implies \ V = \left(\frac{1}{1-\delta}\right) \end{aligned}$$

Sustaining Collusion - Duopoly Price War

- Consider the following repeated simultaneous-move game where two competing firms must choose between two possible discrete prices each period:

Firm Y

		Expensive	Cheap
Firm X	High	8 7	9 3
	Low	4 10	5 5

  - What famous game does this look like?
  - What is the Nash Equilibrium in a single-shot version of this game?
  - What is the minimum discount factor required to sustain collusion in an infinitely repeated version of this game?

Sustaining Collusion - Duopoly Price War: Example 1

		Y	
		Exp	Cheap
X	High	8 7	9 3
	Low	4 10	5 5

- This is a Prisoners' Dilemma game and the NE is {Low,Cheap} if it is only played one time.
- When repeated infinitely, we can assess the level of patience required by each firm to sustain (illegal) collusion on the {High,Expensive} pricing outcome which is better for both firms than the NE.
- Suppose firms choose {High,Expensive} as long as no one has ever deviated from the strategy profile, and {Low,Cheap}, otherwise.

Sustaining Collusion - Discount factor required for Firm Y

Firm Y

Firm X	Expensive	Cheap
High	8 7	9 3
Low	4 10	5 5

For firm Y to stay with Expensive pricing, we need her discounted stream of payoffs from "cheating" (undercutting and choosing Cheap pricing) to be less than her discounted stream of payoffs from continuing to collude indefinitely:

$$9 + 5\delta_y + 5\delta_y^{\mathbf{2}} + 5\delta_y^{\mathbf{3}} + \ldots \leq 8 + 8\delta_y + 8\delta_y^{\mathbf{2}} + 8\delta_y^{\mathbf{3}} + \ldots$$

$$1 \leq 3\delta_y + 3\delta_y^{\mathbf{2}} + 3\delta_y^{\mathbf{3}} + \ldots = 3\sum_{n=\mathbf{1}}^{\infty} \delta_y^n$$

$$1 \leq \frac{3\delta_y}{1 - \delta_y}$$

$$1[1 - \delta_y] \leq 3\delta_y$$

$$\delta_y^* \geq \frac{1}{4}$$

Sustaining Collusion - Discount factor required for Firm X

Firm Y

Firm X	Expensive	Cheap
High	8 7	9 3
Low	4 10	5 5

For X to stay with High, we need his discounted stream of payoffs from cheating to be less than his discounted stream of payoffs from colluding indefinitely:

$$10 + 5\delta_x + 5\delta_x^2 + 5\delta_x^3 + \ldots \leq 7 + 7\delta_x + 7\delta_x^2 + 7\delta_x^3 + \ldots$$

$$3 \leq 2\delta_x + 2\delta_x^2 + 2\delta_x^3 + \ldots$$

$$3 \leq \frac{2\delta_x}{1 - \delta_x}$$

$$\delta_x^* \geq \frac{3}{5}$$

Notice that Firm X has a larger potential relative gain from cheating for one period: X requires a higher discount factor in order to sustain collusion here. Firm Y is "more patient" about the timing of its profit realization in this sense.

Sustaining Collusion - Duopoly Price War: Example 2

- Consider the following repeated simultaneous-move game where two competing firms must choose between two possible discrete prices each period:

		Firm H	
		Expensive	Cheap
Firm G	High	6 7	9 3
	Low	4 9	7 5

  - What is the Nash Equilibrium in a single-shot version of this game?
  - What is the minimum discount factor required to sustain collusion in an infinitely repeated version of this game?

Sustaining Collusion - Duopoly Price War: Ex 2 Solution

		Firm H	
		Expensive	Cheap
Firm G	High	6 7	9 3
	Low	4 9	7 5

- The unique NE is {Low, Cheap} once again.
- Once again the lesser price option is a strictly dominant strategy for both firms, but now the payoff from Cheap is higher than the payoff from Expensive for Firm H: this prevents collusion.
- Why might this be the case?

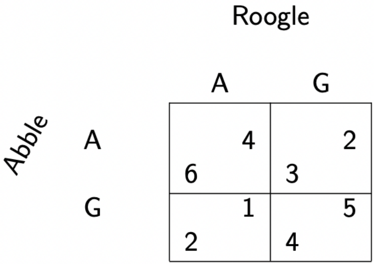
Tech Duopoly and Product Compatibility

- Consider two firms deciding whether to make compatible products by choosing one of two charging technologies, called A and G:

		Roogle	
		A	G
Abble	A	4 6	2 3
	G	1 2	5 4

  - Find all NE for a single-shot version of this simultaneous game
  - What if one firm has a faster R&D team and gets to move first?

Tech Duopoly: One-Shot Simultaneous-Move Outcomes



		Roogle	
		A	G
Abble	A	4 6	2 3
	G	1 2	5 4

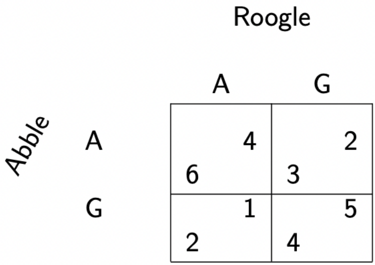
- There are no dominant strategies for either firm.
- The best outcome for each firm is both firms matching on its preferred technology (A for Abble and G for Roogle).
- The worst possible outcome in this game for both firms is each firm choosing the other firm's charging technology.
- The two pure strategy Nash Equilibria are {A,A} and {G,G}
- There is also a mixed strategy NE...

Tech Duopoly: One-Shot Mixed Strategy Equilibrium

		Roogle	
		A	G
Abble	A	4 6	2 3
	G	1 2	5 4

- Let g = Pr {Roogle chooses A}. Let a = Pr{ Abble chooses A}.
- Abble is indifferent when $6g + 3(1 - g) = 2g + 4(1 - g)$ which obtains $g^* = 1/5$.
- Roogle is indifferent when $4a + 1(1 - a) = 2a + 5(1 - a)$ which obtains $a^* = 2/3$.
- Notice that each firm is more likely (but not certain) to choose its own technology in this strategic randomization.

Tech Duopoly: Sequential Formulations



		Roogle	
Abble		A	G
	A	4 6	2 3
	G	1 2	5 4

- How much would each firm be willing to pay to invest in a superior technology R&D team which allows it to move first, assuming the other firm will move first without this investment?

Tech Duopoly: Sequential Formulations - Solution

		Roogle	
		A	G
Abble	A	4 6	2 3
	G	1 2	5 4

- How much would each firm be willing to pay to invest in a superior technology R&D team which allows it to move first, assuming the other firm will move first without this investment?
  - Each firm will choose its own technology if it gets to move first, and in either case the other firm will copy the first-mover.
  - Abble gains 2 by moving first and inducing Roogle to choose A compared to the outcome where both choose G. Roogle only gains 1 by moving first and inducing Abble to choose G compared to the first outcome where both choose A.

Tech Duopoly: Comparison of Outcomes

		Roogle	
		A	G
Abble	A	4 6	2 3
	G	1 2	5 4

- How do the outcomes for each firm compare across the different formulations of the game?
- If the firms could coordinate, like with the collusion examples earlier, is there anything that they could do to increase profits?

Tech Duopoly: Payoffs Comparison

Roogle

		A	G
Abble	A	4 6	2 3
	G	1 2	5 4

- The expected payoff from the MSE is 3.6 for Abble and 3.0 for Roogle
- They could alternate between the two pure NE outcomes every turn...

Discussion Points

- How does the ability of firms to coordinate and communicate affect consumers?
- How does corporate ownership and compensation structure affect executive decision-making relating to discounting?
- How does inflation and uncertainty affect patience in repeated games?
- How do changing expectations about world war and global stability affect the discount factor?
- How do you think the discount factor would compare across different forms of government around the world, holding all else constant?
- How does the discount factor relate to efforts to combat climate change?