An Introduction to Optical Metamaterials

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We present an introduction to metamaterials, intended for undergraduate students who have taken courses in waves and in electricity and magnetism. The paper was written to accompany a presentation given as part of such a course. We have focused on theoretical and historical aspects of optical metamaterials which are relatively easy to understand, and provide references only to suitably non-technical articles and reviews.

I. INTRODUCTION

Metamaterials are media artificially engineered at a macrosopic scale to have specific, desired properties when interacting with certain types of waves. In particular, they should have properties not seen in natural materials. Our focus in this paper will be metamaterials which interact with the electromagnetic field.

Only very recently have nanofabrication, numerical modeling, and characterization tools developed enough to open the possibility of metamaterials. The technical details quickly become complicated. We have not shied away from including mathematics that should be accessible to an undergraduate who has completed a course in electricity and magnetism. However in the spirit of an introduction, we have focused on the principles of the field rather than the newest research. We feel that there are three generic questions one might ask in motivation: (i.) What kind of properties are allowed in theory, but are apparently not present in any natural materials? (ii.) Can one fashion a medium with those properties? (iii.) What can one do with it?

We suggest the kinds of answers so far available by tracing through the case of metamaterials with negative indices of refraction. Section II explains what this property is and why it may be of theoretical interest. Section III describes the first experimental demonstration of such a medium. Section IV discusses one of the suggested practical applications.

To conclude, we discuss the theory of cloaking in section V. Though it does not depend on a negative index of refraction, cloaking is one of the most exciting potential applications of metamaterials, and we could not resist presenting a short description. This is the outstanding example of what kinds of applications open up to us once we consider a medium's ϵ and μ as variable parameters that can be engineered as needed.

II. THEORY OF THE NEGATIVE INDEX

Since we tend to think of the refractive index n as an independent material parameter involved only in Snell's law, it is worth recalling briefly where it comes from. Starting from Maxwell's equations,

$$\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = \mathbf{0}, \tag{1a}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{1b}$$

$$\nabla \cdot \mathbf{D} = 4\pi\rho_f, \tag{1c}$$

$$\nabla \times \mathbf{H} - \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} = \frac{4\pi}{c}\mathbf{J}_f, \tag{1d}$$

we can derive a wave equation for the vectors $\mathbf{E}$ and $\mathbf{B}$. In linear, isotropic media, where $\mathbf{D} = \epsilon\mathbf{E}$ and $\mathbf{H} = \frac{1}{\mu}\mathbf{B}$, we can substitute for $\mathbf{D}$ and $\mathbf{H}$ in equations (1c) and (1d). Then, assuming no free charges or currents, we take the curl of equation (1d) and substitute the result into the curl of equation (1a). We then have the wave equation for $\mathbf{E}$:

$$\nabla^2\mathbf{E} = \frac{\epsilon\mu}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}. \tag{2}$$

A similar procedure results in the same wave equation for $\mathbf{B}$.

Identifying $v^2 = \frac{c^2}{\epsilon\mu}$ as the square of the phase velocity, we can define $n = \sqrt{\epsilon\mu}$ as the factor by which the phase velocity in the medium is reduced when compared with the speed of light in vacuum. Since n^2 rather than n enters the wave equations, one could have asked even in Maxwell's time, "Why not choose the negative square root?"

The negative square root must be chosen in certain situations[1]. In general, $\mathbf{E}$, $\mathbf{B}$, ϵ, and μ are all complex quantities. Then the index of refraction is the complex square root $n = \pm\sqrt{(\epsilon_r + i\epsilon_i)(\mu_r + i\mu_i)}$. Arguing that ϵ_i and μ_i are on the order of the small positive quantity δ, which corresponds to light damping[2], we can then expand the square root to first order in a Taylor series:

$$n = \pm\epsilon_r\mu_r\left(1 + i\frac{(\epsilon_r\mu_i + \epsilon_i\mu_r)}{(\epsilon_r\mu_r)}\right) + O(\delta^2) \tag{3}$$

[1] The argument we present here owes to [1], pp. 101-120. Veselago's argument in [9] is slightly more difficult.

[2] Since they describe the dissipative effects of the medium, if a wave is to propagate at all, the imaginary parts of ϵ and μ must be small.

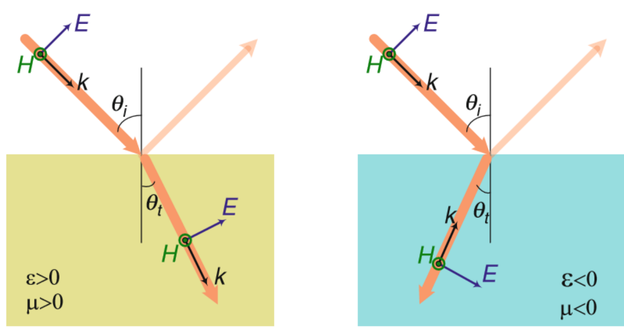

FIG. 1. Refraction into a medium **a.)** with positive index of refraction (left); **b.)** with negative index of refraction (right). From [1].

Now individual plane wave solutions to equation (2) are of the form:

$$\mathbf{E} = \mathbf{E}_0 e^{i(k_z z - \omega t)} = \mathbf{E}_0 e^{i(n_r k_z^0 z - \omega t)} e^{-n_i k_z^0 z}. \quad (4)$$

If the positive square root were chosen in equation (3) when ϵ_r and μ_r are negative, the imaginary part of n would be negative, leading in turn to an exponentially growing amplitude of the plane wave. This is unacceptable on physical grounds[3]. So negative values of ϵ_r and μ_r are together sufficient conditions for n to be negative[4].

There are a number of interesting consequences. The most important, on which most of the applications rely, is the effect on Snell's law. When considering the transmission of light across a boundary between two media with refractive indices n_1 and n_2 respectively, Snell's law is the familiar[5]

$$n_1 \sin\theta_i = n_2 \sin\theta_r, \quad (5)$$

where θ_i is the angle of incidence and θ_r is the angle of refraction. The angles are measured as shown in figure 1a.

The form of Snell's law is unchanged in the presence of negative index media. So if n_1 is positive, but $n_2 = -|n_2|$ is negative, θ_r must also be negative:

$$n_1 \sin\theta_i = -|n_2| \sin\theta_r = |n_2| \sin(-\theta_r). \quad (6)$$

The left hand side of this equation is the product of two positive quantities, and so is positive. The right hand side then must also be positive, and since θ_r is physically restricted to lie between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, this implies that θ_r is negative. This is pictured in figure 1b. Light refracts the "wrong" way in the negative index medium.

Another strange consequence is that the phase velocity of light in the medium will generally be opposed to the group velocity. As Veselago shows in [9], the vectors **E**, **H**, and **k** form a left-handed triple in a negative index medium, for which reason such media are sometimes called left-handed. However, **E**, **H**, and **S** still form a right-handed triple, since $\mathbf{S} = \frac{c}{4\pi}\mathbf{E} \times \mathbf{H}$. The flow of energy is opposed to the phase velocity of the individual plane waves[6].

Finally, we point out that the Doppler effect will be reversed in a negative index medium. The reasoning here is similar to that in the case of Snell's law. We consider a stationary source emitting radiation at a frequency ω_0. An observer moving toward the source with a speed v detects radiation with a frequency ω given by the familiar Doppler formula:

$$\omega = \omega_0 \left(1 + \frac{v}{c/n}\right) \quad (7)$$

Just as before, switching the sign of n reverses the usual effect.

III. EXPERIMENTAL DEMONSTRATION A NEGATIVE INDEX META-MATERIAL

We now suggest how metamaterials are made in practice by discussing the first experiment to demonstrate a medium with a negative index of refraction. The experiment was performed by Shelby, Shultz, and Smith in 2001 [7]. They imprinted a periodic lattice of copper split-ring resonators, or SRRs, on one side of a fiberglass plane, and a periodic lattice of single copper wires on the other. They then put many such fiberglass planes together in an interlocking array, as in figure 2. The SRRs on the lattice had a typical length scale of roughly 3-5 mm. The periodic elements here, SRRs and copper wires, could be considered the basic "atomic" units of the metamaterial[7].

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[3] Note that Cai and Shalaev claim in [1] that this choice would violate causality. We were unable to determine what was meant by this. Our argument would only really be true in the case of an infinite negative-index material; the possibility of an exponential growth does seem to enter into the applications. See the section on Superlenses below.

[4] Note that the more general condition for negative n is $\epsilon_r\mu_i + \epsilon_i\mu_r < 0$. See [1] and [2] for further details.

[5] We consider here only the case of real n, namely non-dissipative media.

[6] This is discussed further in [5]. We were unaware that this is possible even in principle, and find it fascinating. See William Schaich's web-page for some example video clips demonstrating one- and two-dimensional examples of wave packets propagating into a negative index medium: http://www.physics.indiana.edu/ schaich/ajp/ajp.html

[7] It is worth emphasizing here that metamaterial media are merely analogous to material media. Few authors resist the temptation to shorten "metamaterial" to "material," but this is highly misleading for the beginner. Metamaterial properties, as here,

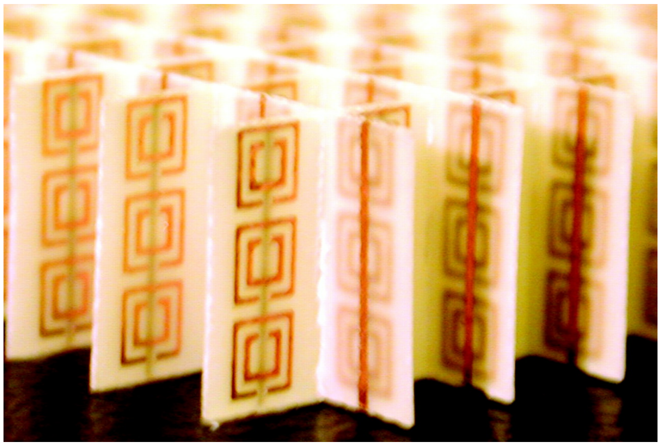

FIG. 2. The first negative index medium. Fiberglass was used as the dielectric material. A periodic array of copper split-ring resonators was placed on the front of each fiberglass plane, and a periodic array of thin copper wires was placed on the back. The height of the structure is about 1 cm. From [7].

Individual SRRs can be modeled as LC circuits in the presence of changing fields. In a periodic array however, they can interact coherently, since the field of one SRR can affect the field seen by its nearest neighbors. In [4], Pendry et al. argued that suitable periodic arrays of SRRs can display an effective μ that is negative[8]. Similarly, a periodic array of copper wires can display an effective ϵ that is negative. Combining the two in one lattice, should create a medium with simultaneously negative values of ϵ and μ, and so a negative n as argued in section II.

Shelby, Shultz, and Smith used a set-up as depicted in figure 3. A triangular prism was made from the negative index metamaterial and placed at the end of a metal waveguide with variable dimensions (to control the microwave frequencies used). A detector was mounted on a swiveling disk, and was rotated through the full range of angles to measure the intensity of the microwaves emerging from the prism. The angle theta was measured clockwise from the pictured surface normal. The diagram shows a refracted ray as we would expect from a positive index material.

The results of this experiment are shown in figure 4. The observed effect was a spectacular confirmation that the medium exhibits a negative index of refraction. But this was only so for a limited range of microwave frequencies. One of the remaining challenges is to design media that display such an effect over a larger range of frequencies, particularly at optical frequencies.

IV. SUPERLENSES

We now discus the exciting possibility of metamaterial superlenses[9]. The German physicist Ernst Abbe discovered in 1873 a limit to the size of structures one can resolve with traditional optical lenses, the so-called Abbe Diffraction Limit. One can only resolve structures greater than roughly one half the wavelength of the light used in imaging, which means about 200-300 nm for optical imaging[10]. Using metamaterials with negative indices of refraction, it is possible to overcome this limit, possibly even by several orders of magnitude. This comes from the familiar equation for a propagating wave:

$$\mathbf{E}(x,y,z,t) = \mathbf{E}_0 \; e^{i(k_x x + k_y y + k_z z - \omega t)}. \tag{8}$$

Starting from the wave equation and using the above value of the E-field, one can derive the value of k_z to be

$$k_z = -\sqrt{\frac{\epsilon\mu\omega^2}{c^2} - (k_x^2 + k_y^2)}. \tag{9}$$

Normally, the square root takes the positive value, since with traditional materials the k_z vector is parallel to the direction of motion. For $(k_x^2 + k_y^2) > \frac{\omega^2}{c^2}$, we find an evanescent wave which decays exponentially. This poses a problem for imaging since it is the evanescent wave that carries information about small scale structures in the object. However, for a metamaterial with a negative index of refraction, k_z is antiparallel to the direction of propagation and so instead of a decaying wave, there would be

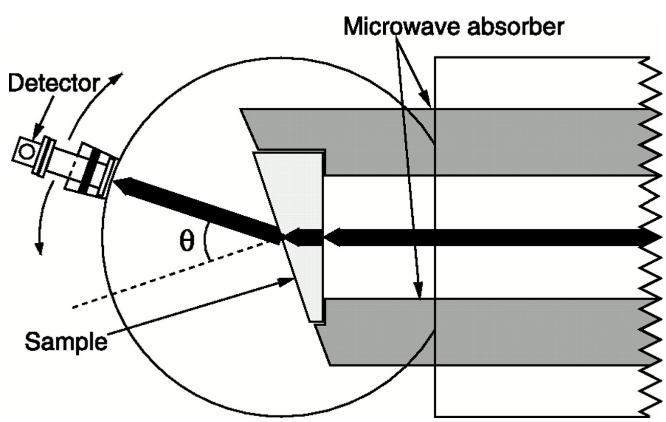

FIG. 3. Shelby, Smith and Shultz's experimental set-up.

result from coherent interactions of elements much larger than any individual atoms. Metamaterial properties are not chemical properties. Hence our generic use of the word "medium" throughout.

[8] There are natural materials with negative ϵ, such as gold and silver at optical frequencies. However, there are no known natural materials with negative μ, so this revelation is really the theoretical insight that opened current metamaterials research.

[9] Predicted in [3]. Most of the material in this section is drawn from [1], pp. 137-9.

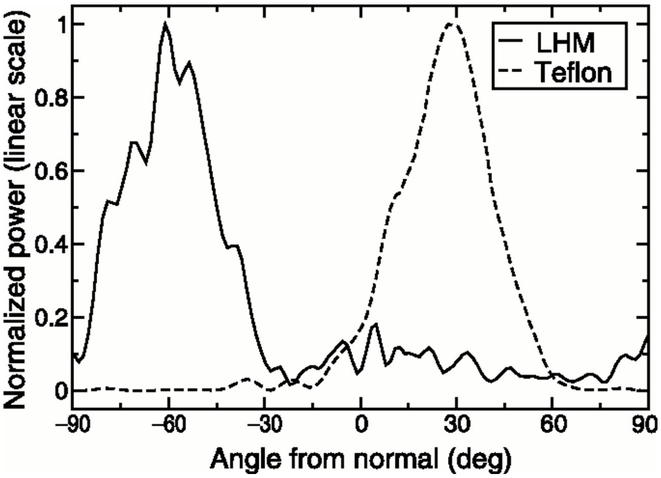

FIG. 4. Peak intensity measured at roughly $-60°$, as predicted in the case of a prism with negative n.

a growing wave within the medium. This means that the evanescent wave will grow significantly before reaching the object, allowing them to be used in imaging. This is why there is no Abbe diffraction limit for metamaterials with negative indices of refraction.

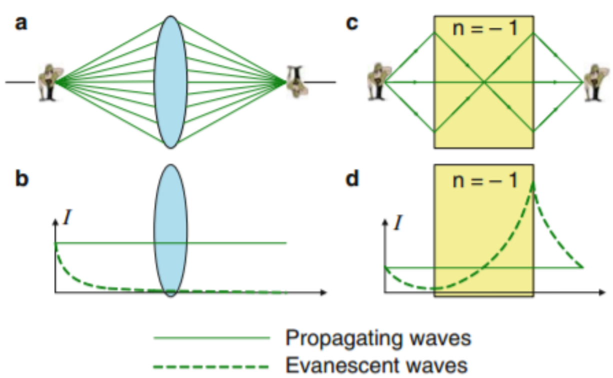

FIG. 5. Image **a** displays a standard lens while image **c** shows a lens made of metamaterials. The difference between image **b** and **d**, the growth of the evanescent wave in the metamaterial, is the reason a superlens is possible. From [1].

In addition, superlenses do not need to be shaped like ordinary lenses. They are generally completely flat but have been designed so that light bends the same way it would in a curved traditional lens. As a result, these lenses can be made very thin which is helpful for manufacturing. The hope is that these superlenses can be used for medical imaging and biological research due to their potential to resolve single molecules, DNA strands, viruses, or even spot cancerous cells with visible light.

V. CLOAKING

We now discuss the theory of cloaking[10]. To design a cloak, we will start by considering straight-line trajectories in an isotropic medium, then apply a pair of coordinate transformations which will exclude these trajectories from the target region, and then finally deduce what ϵ and μ would have to be to lead to such trajectories for light waves. We will discuss this first with a cylindrical region, and then will simply state the results in spherical coordinates.

We can apply our basic Maxwell equations in Cartesian coordinates to get our E and B fields, then transform our coordinates. In the transformed coordinates,

$$\epsilon' = \frac{G\epsilon G^T}{|G|} \quad , \quad \mu' = \frac{G\mu G^T}{|G|}, \tag{10}$$

$$\mathbf{E}' = (G^T)^{-1}\mathbf{E} \quad , \quad \mathbf{H}' = (G^T)^{-1}\mathbf{H}, \tag{11}$$

where G is the Jacobian matrix. After transforming to the primed cylindrical coordinates, we then map the radial component r to the new parameter r:

$$r = (1 - \frac{a}{b})r' + a \tag{12}$$

This hollows out the cylinder by creating a volume into which light cannot penetrate. In this new r transformation, a is the radius of the hollowed out inner cylinder and b is the radius of the entire cylinder. It essentially compresses the cylinder. This can be seen in figure 7. Now we calculate the components of G. These are:

$$g_{ij} = \sum_l (\frac{\partial x_l}{\partial q_i})(\frac{\partial x_l}{\partial q_j}) \tag{13}$$

The medium in old coordinates was isotropic and homogeneous, which implies that the Jacobian matrix is diagonal. We now calculate the elements of the transformation matrix:

$$h_r = \sqrt{g_{rr}} = \frac{b}{b-a} \tag{14a}$$

$$h_\theta = \sqrt{g_{\theta\theta}} = \frac{b}{b-a} \cdot \frac{r-a}{r} \tag{14b}$$

$$h_z = \sqrt{g_{zz}} = 1. \tag{14c}$$

And from here, we can get the needed values for ϵ:

$$\epsilon_r = \mu_r = \frac{r-a}{r} \tag{15a}$$

$$\epsilon_\theta = \mu_\theta = \frac{r}{r-a} \tag{15b}$$

$$\epsilon_z = \mu_z = (\frac{b}{b-a})^2 \cdot \frac{r-a}{r}. \tag{15c}$$

[10] The material in this section draws mostly from [1], pp. 159-67.

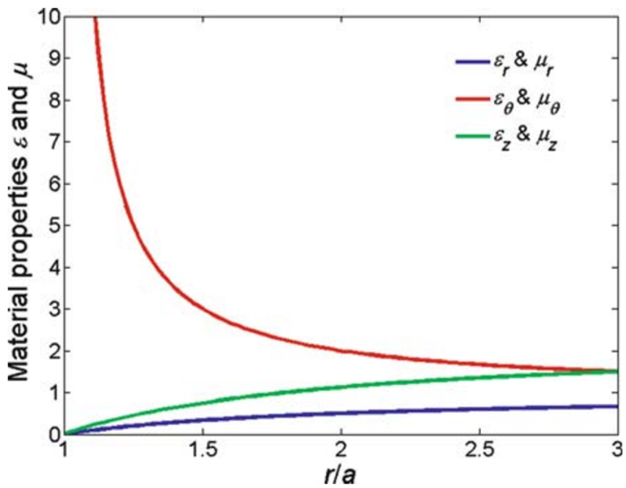

FIG. 6. ϵ_r decreases as r/a increases. From [1]

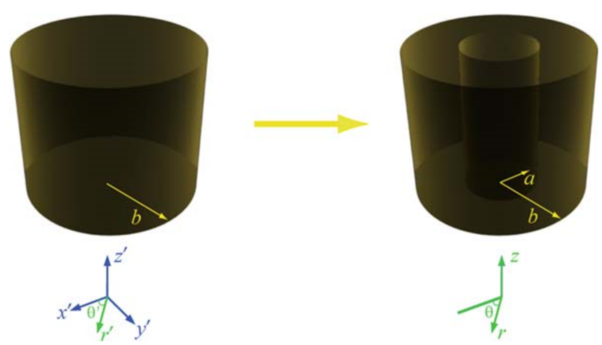

FIG. 7. Cylindrical coordinate transformation. From [1].

Now we can plot these values for the various ϵ versus r/a. This is given in figure 6.

It is now possible to design a metamaterial that can fit this requirement. The light would bend around the cylinder as desired. This is perhaps more apparent in spherical coordinates, as it is a more familiar coordinate system. Here, if we went through the same process, but with the conversion to spherical coordinates, we would find ϵ to be:

$$\epsilon_r = \mu_r = \frac{b}{b-a} \cdot (\frac{r-a}{r})^2 \tag{16a}$$

$$\epsilon_\theta = \mu_\theta = \frac{b}{b-a} \tag{16b}$$

$$\epsilon_\phi = \mu_\phi = \frac{b}{b-a}. \tag{16c}$$

Again, after going to spherical coordinates, the transformation

$$r = (1 - \frac{a}{b})r' + a \tag{17}$$

was used. Here, a is the radius of the hollowed out inner sphere and b is the radius of the overall sphere [1]. That is, light will not enter the sphere of radius a. Figure 7 is a plot of the trajectory of light through a medium satisfying these constraints on ϵ. The light encircles the ball, but never penetrates it, and comes out the other side without interacting with the cloaked object. This effect is independent of the direction of incident light since ϵ is independent of θ and ϕ.

We now have a prescription for engineering a metamaterial that will allow us to cloak a cylindrical or spherical region. The challenges of doing this in practice are not insignificant[11]. But this procedure can be generalized for various shapes. Cloaking has many possible applications, ranging from stealth technology for military purposes to some possibly surprising medical purposes: a doctor could see past certain organs by placing a cloak around them. The benefits of cloaking make it not only an entertaining, but active and beneficial area of modern research.

VI. CONCLUSION

This discussion of metamaterials was short, but we feel we have covered the principles of the field enough to give a good taste. Advanced nanofabrication and other structural methods allow us to place elaborate spacial dependence on material parameters, but many engineering challenges remain[12]. But beyond the engineering difficulties, the field is limited only by our creativity in rethinking our physical theories, and in finding potential new applications.

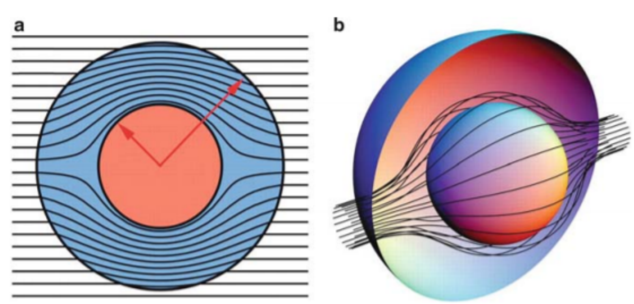

FIG. 8. In 2D and 3D. Light wraps around the ball and comes out on the other side without interacting with the inner ball.

[11] See [1], pp. 168 and following

[12] See [6] for a nice review of metamaterial plasmonics, which discusses many traditional and current engineering challenges.

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