A Plane-Strain Elasticity Analysis of Uniformly Loaded Solids with Internal Pressure

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I. Introduction

In applied mechanics and mechanical design, stress concentration in solids under externally applied loads have always been an important research topic to seek detailed insights such as onset of micro-cracking and initiation of fatigue failures, etc., so as to facilitate improved design guidance.

Classical theory of elasticity has long been considered an abundant source of such technical information. Although many elasticity solutions tend to be only applicable to simplified or limited cases, its capability of generating explicit analytical expressions and functional relationships among stress and strain components in term of material properties, geometry, and applied loadings, and to approximately predict certain structural behaviors can be quite practical and useful at times.

The problem of a thin plate with a circular hole under uniform tensile loads was solved by Kirsch (1898) [1,2] in which he resolved the boundary conditions into two parts and solving the problem by linearly combining the two simpler solutions to arrive at the desired solution. The first part is a solid "ring", formed by the outer circular boundary with its radius equal to half the plate width and the inner boundary of the hole. It is acted upon by a uniformly loaded constant radial normal stress equal to half the uniaxial normal load applied at the ends of the plate; the shear stress vanishes everywhere around the ring. The solution to the problem had been made available by Lamé (1852) [3]. The second part is the same solid ring subjected to radial normal and shear stresses sinusoidal applied all around the outer circular boundary. It is interesting to note that Timoshenko and Goodlier [4] stated that the solution "has been well confirmed many times by strain measurements and by the photo-elastic method" [2]. An important statement can now be made here: as long as the hole size is very small compared to the other dimensions, the stress concentration effect prevails only in the vicinity of the hole, but diminishes rapidly for regions away from the hole region. So, one may confidently conclude that an imaginary circular boundary outward from the center of the hole can be drawn to separate the loaded plate into a "high, complicated-stress" zone and a "low, simple-stress" zone.

This paper is now to apply the techniques of Kirsch in classical theory of elasticity to derive a set of governing equations and to arrive at a solution in a rigorous manner. First, the plane-strain problem is established as follows: a long pipe-like cylindrical structure of small radius a embedded at a depth **R** (R>> a) below the surface of a large piece of soil (or a matrix material) whose top surface is being loaded by a uniformly distributed load **W** while the pipe is carrying an internal pressure p_i (Fig.1). Due to the loading and the geometry conditions, and based upon results of Kirsch (1852) [3] and observations of finite element analysis by Hsu [10], It can now be premised , with confidence, that both displacement and normal strain in the longitudinal direction are negligibly small and essentially zero, and only in-plane displacements and strains are of significance. Hence, the complicated and mathematically intractable three dimensional problem may be simplified to a two-dimensional plan-strain problem by claiming that negligible axial normal strain and axial normal displacement is prevented by proper constraints applied at two longitudinal ends. The Material System Model of Fig.1 represent a typical vertical section of unit length in which the dotted circle of radius R is the afore-mentioned imaginary circular boundary.

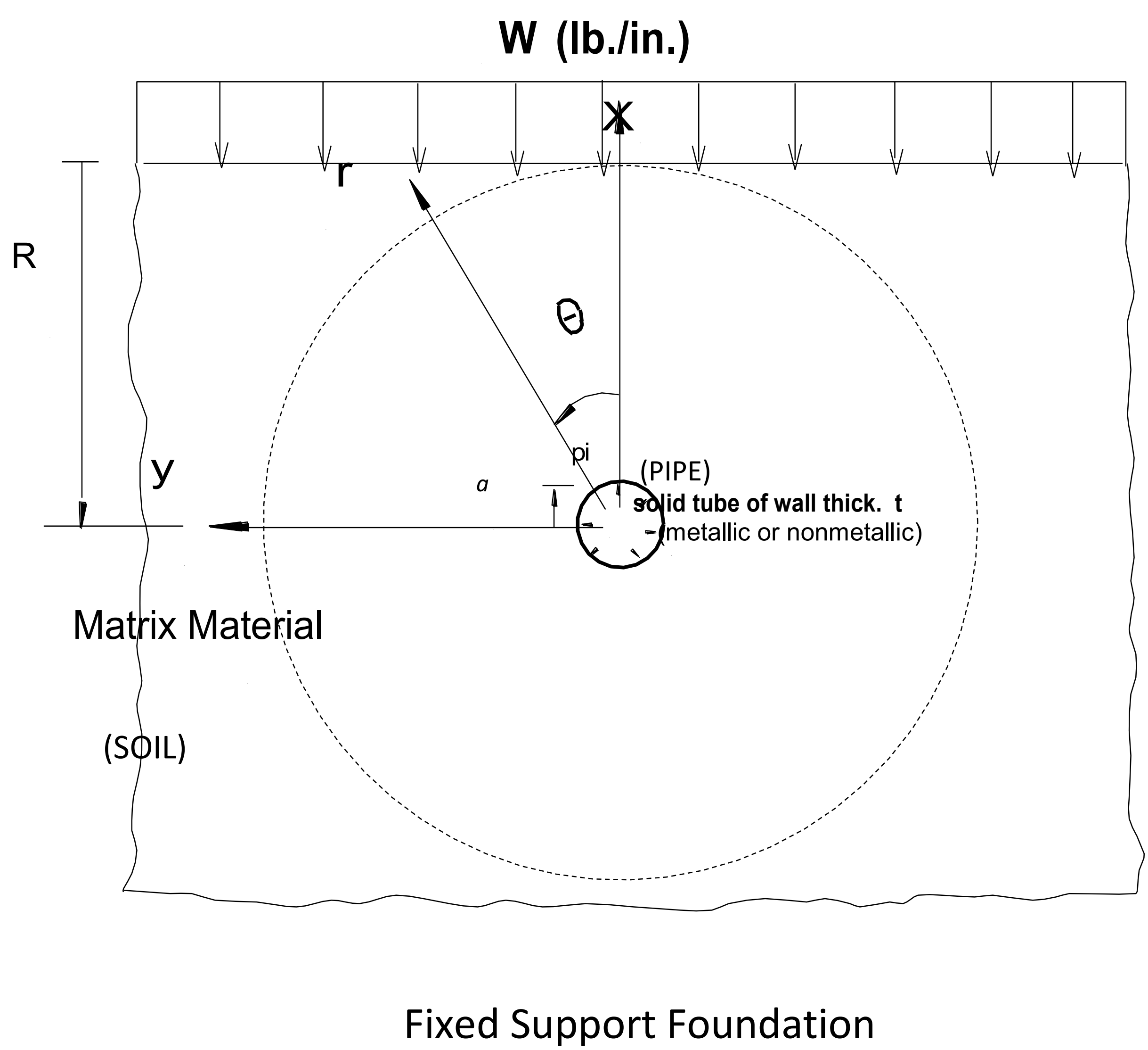

Figure1 The Material System Model – Vertical Cross Section

II. Plane-Strain Solution and Applications

A. Pipe/Soil Model Problem

In Figure 2, the model is applied to a real buried pipe and soil problem where the trench wall exerts a lateral normal reactive force in the hydrostatic manner. The vertical load and the gravity of soil also render the simple vertical normal stress to vary hydrostatically. So the stress boundary conditions at the imaginary boundary, r=R, are:

$$\sigma_x = -w - \gamma_s(R-x)$$

$$\sigma_y = -\gamma_s(R-x)$$

$$\tau_{xy} = 0$$

where γ_s stands for the specific weight of the soil.

This then leads to the stress components in their polar coordinate forms, followed by resolving the problem into three simpler problems: a Lamé (1852) [3] problem with stress function $\emptyset_0(r)$ independent of the angular coordinate θ, a second problem with stress function $\emptyset_1(r, \theta)$ and a third problem with stress function $\emptyset_2(r, \theta)$. These stress functions will lead to a set of three solutions whose linear combination will yield the following in-plane stress distributions:

$$\sigma_r = [a^2R^2(p_o-p_i)/r^2 + (p_i a^2 - p_o R^2)]/(R^2-a^2) + (B/r - 2C/r^3 + 2Dr)\cos\theta + (-4A'/r^2 - 2C' - 6D'/r^4)\cos 2\theta$$

$$\sigma_\theta = [-a^2R^2(p_o-p_i)/r^2 + (p_i a^2 - p_o R^2)]/(R^2-a^2) + (B/r + 2C/r^3 + 6Dr)\cos\theta + (12B'r^2 + 2C' + 6D'/r^4)\cos 2\theta$$

$$\tau_{r\theta} = -(-B/r + 2C/r^3 - 2Dr)\sin\theta - (2A'/r^2 - 6B'r^2 - 2C' + 6D'/r^4)\sin 2\theta$$

where the unknown coefficients will be determined by stress boundary conditions, radial and transverse normal stress and radial normal strain continuities at the material interface r= a +t , where t is the pipe wall thickness.

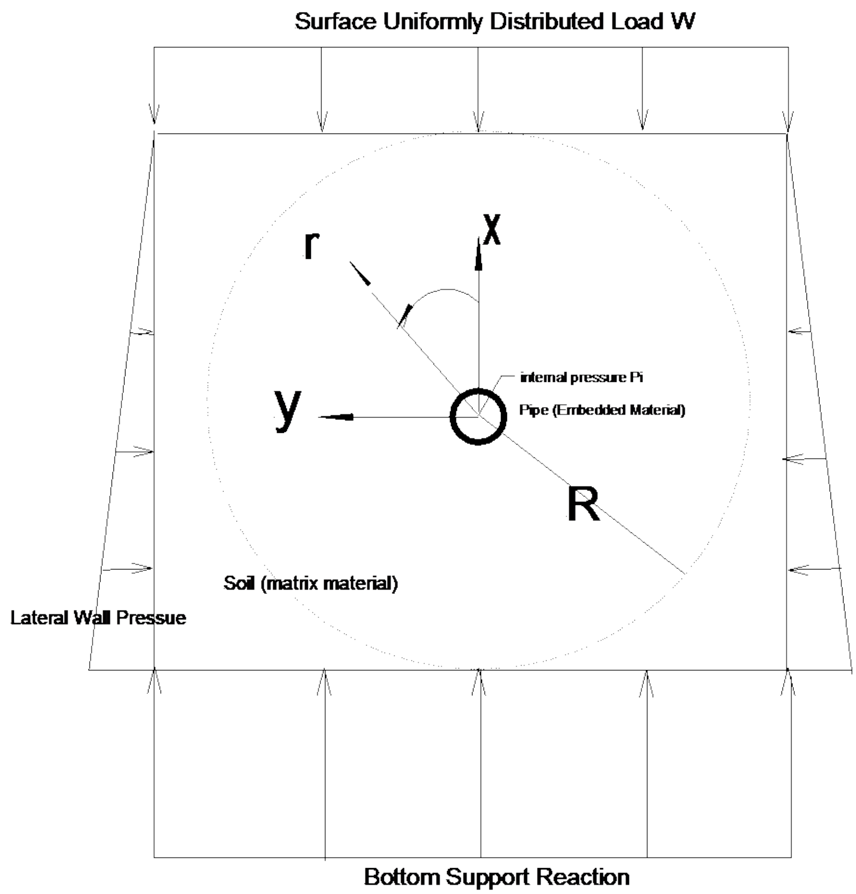

FIGURE 2 Pipe/Soil Structure with Internal Pressure, Surface Uniform Load, Lateral Reactions

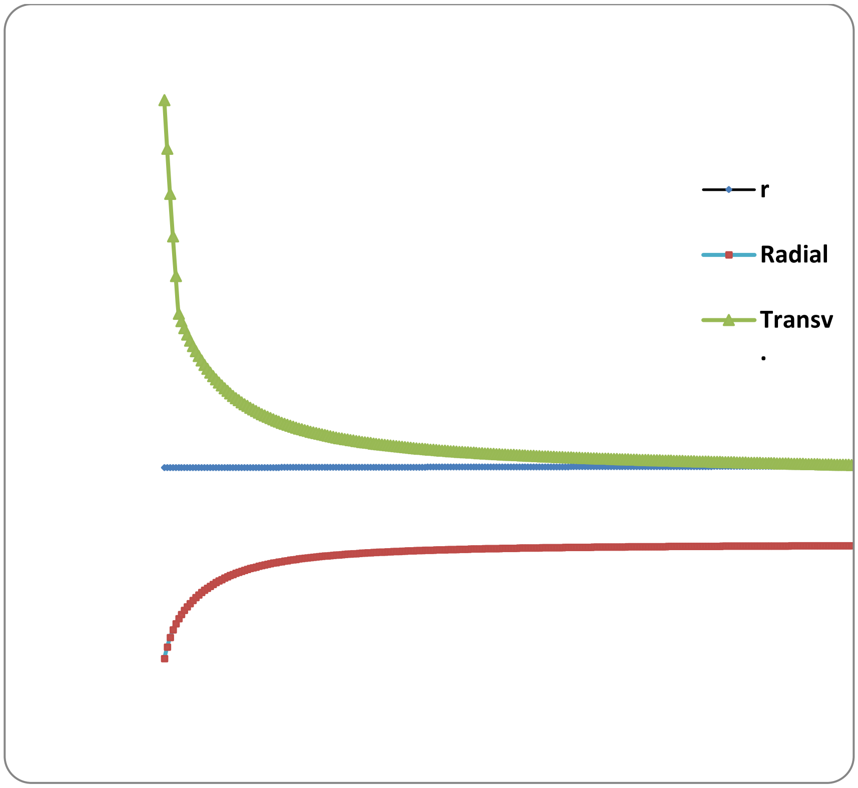

Figure 3 Pipe Internal Edge Top ($\theta=0^0$) Radial Normal Stress (σ_r), and Transverse Normal Stress (σ_θ), vs. radial distance from the pipe edge (r=4in.) to imaginary circular Boundary (r=28in.) (Pipe wall thickness=0.5 in.)

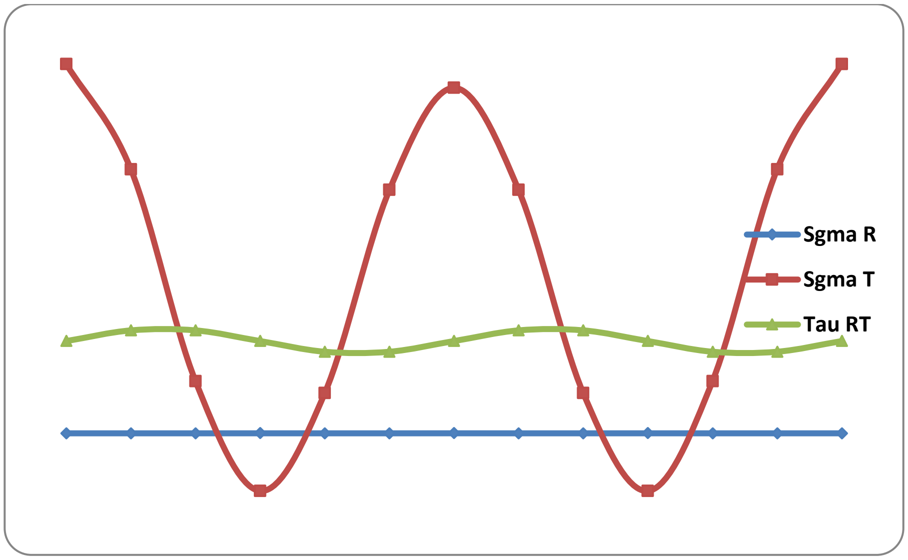

Figure 4 Normal Stresses: Transverse ($\sigma\theta$), Radial (σr), and Shear Stress ($\tau_{r\theta}$) vs. Angular Location around Pipe Inner Boundary r=4 in. (wall thickness 0.25in.)

Results and Discussion

Sample data currently applied are as follows:

Pipe Material: Steel E_p=30x10E6 psi, $\nu_p = 0.3$; soil E_s=2x10e5 psi (assumed), $\nu_s = 0.42$ (assumed), soil sp. weight γs=.0694 lb/cu in. ; Trench width Bw=30 in., Geometrical dimensions: Hole radius a=4 in., pipe wall thickness t=0.5 in., Imaginary circle radius R=28 in.. While the in-plane shear stress is everywhere very insignificant (near 0 psi), the radial normal stress and transverse normal stress are the two significant stress components whose magnitudes are determined and analyzed in detail in the current analysis. Fig. 3 shows the distributions of the two stress components where Point 1 location on the horizontal axis corresponds to the exact internal boundary edge r= a. It is clear that at the 12-oclock location (edge top), the radial normal stress (the bottom curve) equals the internal pressure while the transverse normal stress (the top curve) descends from a peak positive (tensile) value at the edge. As expected, away from the edge area, both stresses quickly ease their drastic change and stay at a steady constant level radically outward through and beyond the imaginary circle r=R. These behaviors further confirm what have been reported in previous publications of Hsu [6, 7, 8, 10] where the transverse normal stress σ_θ appeared to display significantly drastic variation as a function of the polar angular location θ as shown in Fig. 3. It is worth noting that while it reaches maximum tensile (positive) at edge top (θ=0) and edge bottom (θ=180), it is actually at a maximum compressive (negative) intensity at edge left (θ=90) and edge right (θ=270). It may be pointed out that such stress nature (sign) change involving peak magnitude response (variation) and is quite significant when the possible onset of micro-cracking forming and/or initiation of fatigue failure mode due to an repeatedly applied load.

B. The Pressurized Cylindrical Inclusion Model

In Fig.5 , the free edges of the sides of the structural system under internal pressure and the hydro-statically varying vertical normal stress $\boldsymbol{\sigma}_x$ will actually give rise to a fourth stress function in addition to the three established in Sec. II.A. This results in additional cos3θ and sin3θ related terms, and the linear combination of all four partial problems yields the desired true solution for the three in-plane stress distribution as solution to the current problem.

Results and Discussion

Current on-going effort of data applied: materials applied: E fiberglass E_p=10x10E6 psi, $\nu_{\mathbf{p}}=0.3$ polyester E_s=2x10e3 psi, $\nu_{\mathbf{s}}=0.46$ **(assumed)** ; specific weight γ_s=.0361 lb/cu in. ; Geometrical dimensions: Hole radius $\boldsymbol{a}=2$ in., Imaginary circle radius R=20 in. Preliminary results have been obtained for this model but not exhibited in this article for simplicity. Follow-ups will appear in detail later.

III. Conclusion

The pipe/soil model problem was established and solved by classical theory of elasticity. Exact boundary conditions and material interface continuities are rigorously imposed. The problem of a previously presented model and solution [8] was briefly discussed. The capability of the analytical method to yield point-wise stress intensities at their specific precise locations suggests that the method can be a useful design tool for assessing the stress concentration effects in practical applications.

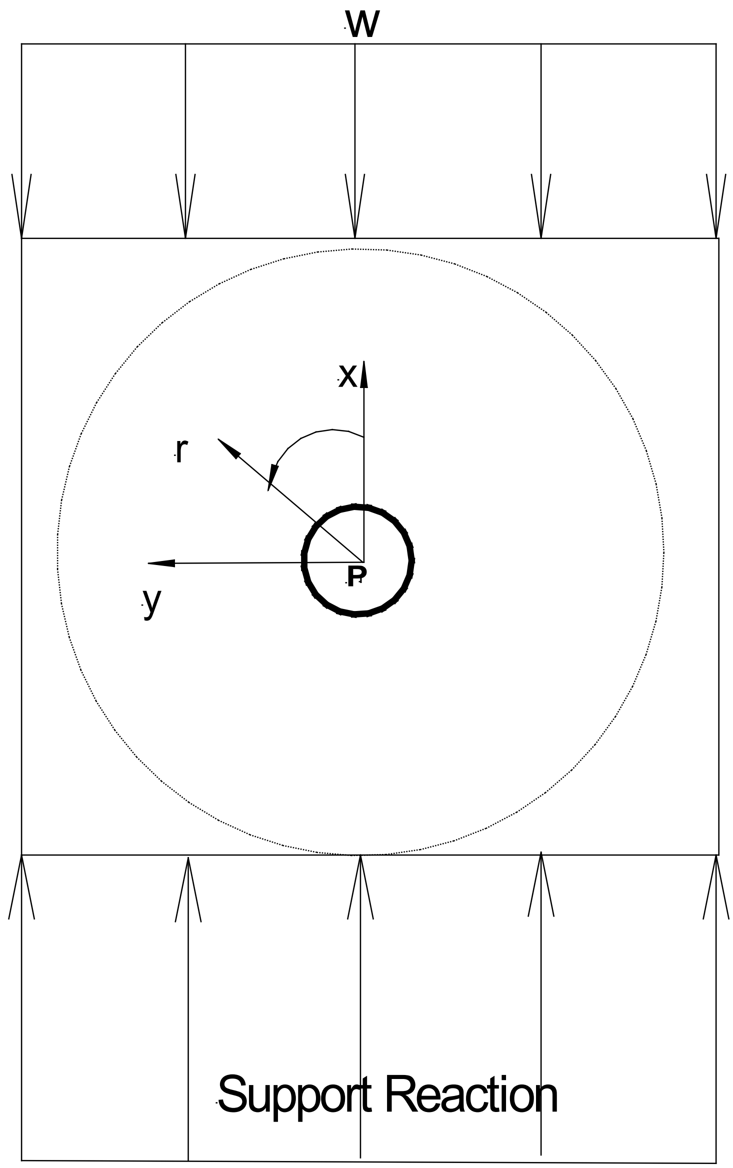

Figure 5 Matrix with Embedded Cylinder System Free of Lateral Forces

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