Topic: Eventual Positivity
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A linear evolution equation on a Banach space E is most naturally viewed through its associated C_0-semigroup. The usual starting point is a (typically unbounded) linear operator $A : E \supseteq \mathrm{dom}(A) \to E$, consider the abstract Cauchy problem

$$\dot{u}(t) = Au(t), \qquad u(0) = u_0 \in E,$$

and studies the a family of bounded operators $(T(t))_{t\geq 0}$ – called the C_0-semigroup generated by A. Many classical differential equations become instances of this abstract setting once the state space E is chosen appropriately; for example:

- the heat semigroup generated by the Laplacian on L^p-spaces,
- transport and kinetic semigroups associated with first-order differential operators,
- the transition semigroup of a continuous-time Markov process.

Although these examples look different analytically, they share a structural feature at the semigroup level: if $u_0 \geq 0$, then $T(t)u_0 \geq 0$ for all $t \geq 0$. Semigroups with this property are called positive semigroups, and their theory is by now well developed — drawing on spectral theory, order theory, and fixed-point methods.

A rather subtler phenomenon emerges for certain generators:

- matrices whose off-diagonal signs violate classical monotonicity conditions,
- operators involving fourth-order derivatives,
- diffusion operators with non-local boundary interactions.

Here one encounters semigroup $(T(t))_{t\geq 0}$ that are not positive, yet satisfy a weaker but striking property: for every $u_0 \geq 0$, the orbit $t \mapsto T(t)u_0$ may initially leave the positive cone, but eventually returns to it and stays there. Such behaviour is captured by the notion of an eventually positive semigroup. The systematic study of these semigroups — especially in infinite dimensions — began only about a decade ago and has developed into a vibrant and still growing area.

What makes eventually positive semigroups intriguing is that they are notoriously difficult to characterize and that they show a much more subtle behaviour than their positive counterparts, for instance when it comes to perturbation theory. Large parts of the theory of eventually positive semigroups thus have a different focus and a different flavour than the theory of positive semigroups.

The central aim of the project is to understand how eventual positivity can be detected and characterized from the operator–theoretic data of the generator and the underlying cone. We will begin in finite dimensions, where even basic questions are still unresolved: for example, when does eventual positivity hold only pointwise (i.e., for each positive vector separately) and when does it hold uniformly across the entire cone? This turns out to depend sensitively on the geometry of the cone, and while the situation is reasonably well understood for polyhedral cones and in low dimensions, the general picture – especially for non-polyhedral cones or higher dimensions – remains wide open. Progress in this setting is not only interesting in its own right, but is expected to guide the development of a broader theory in infinite-dimensional ordered Banach spaces, where eventual positivity exhibits entirely new phenomena beyond the classical theory of positive semigroups.