DERIVATION OF THE LORENTZ TRANSFORMATION EQUATIONS

ABSTRACT

This paper derives the Lorentz transformation equations by using Einstein's two postulates of special relativity. The derivation demonstrates how the classical Galilean transformations are incompatible at near light speeds and must, therefore, be replaced. The resulting equations, the Lorentz transformation equations, preserve the constancy of the speed of light for all inertial reference frames. Furthermore, the Lorentz transformations can numerically express how measurements of time and length are shifted when travelling at near light speeds. These equations set the foundation for Special Relativity, and they have a crucial role in time dilation, length contraction, and the relativity of simultaneity.

INTRODUCTION

Before the twentieth century, Newton's laws of the physical world were considered the most accurate description for how physics worked. Space and time were seen as absolute quantities, meaning that all observers agreed on distance and time intervals. Additionally, the Galilean transformation equations explained how measurements of position changed between observers in different reference frames:

$$x' = x - vt$$

$$t' = t$$

where v is the velocity between the two frames. These equations successfully described the relative motion of objects at normal speeds and so they became widely accepted as the standard.

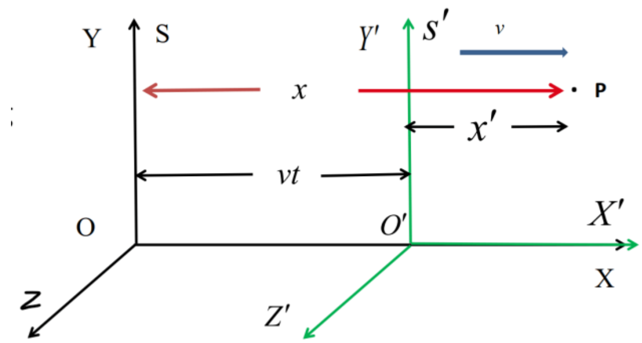
However, the development of Maxwell's theory of electromagnetism highlighted an important inconsistency with the standards in physics at the time. Maxwell's theory suggested that the speed of light c in a vacuum is constant for all observers. On the contrary, the Galilean transformations imply that the speed of light can change and vary based on the motion of an observer. Since the two findings contradicted each other, either classical physics had a major flaw in it, or Maxwell's theory was incorrect. The Michelson-Morley experiment was conducted to measure Earth's velocity through the hypothesized ether to test if it existed. However, the experiment failed to detect any evidence of ether, paving the way for Einstein's theory of Special Relativity.

In 1905, Einstein resolved this contradiction in his theory of Special Relativity, which was founded by his two postulates. His first postulate stated that the laws of physics are the same in all inertial reference frames, also called the principle of relativity. This idea was originally introduced by Galileo. His second postulate is that the speed of light in a vacuum maintains a constant speed for all observers equivalent to around 300,000 km/s. He arrived at this conclusion through Maxwell's theory of electromagnetism, and then the failed Michelson-Morley experiment.

If Einstein's postulates are correct and the speed of light is a fixed, finite speed, the Galilean transformation equations can no longer be used because they wouldn't be able to accurately relate coordinates of events between two reference frames at relativistic speeds. The Lorentz transformation equations satisfy the principle of relativity while also preserving the constancy of the speed of light. The purpose of this paper is to derive these equations directly from Einstein's postulates and the Galilean transformation equations.

METHODS

For transformations, there are two inertial reference frames, *S* and *S'*, where frame *S'* moves at a velocity *v* to the right relative to frame *S*. At $t = 0$, the origins meet, meaning the time is the same. After a certain amount of time *t*, the origin of the second reference frame, *O'*, has moved a certain distance *vt* (distance is equal to velocity multiplied by time), relative to frame *S*, as shown in the image.



In the image, *P* is an object whose coordinates are represented by *x* in frame *S* and *x'* in frame *S'*. Geometrically, this means that the distance *x* is equal to *x'* and *vt*. Therefore, Galileo proved that

$$x = x' + vt$$

which can then be arranged to get the classical transformation,

$$x' = x - vt$$

Since time was assumed to always be constant in classical relativity,

$$t' = t$$

These equations don't work anymore at near-light speeds. For example, imagine a beam of light is emitted at $t = 0$. The beam of light would have a velocity *c* because light travels at speed *c*, meaning

$$x = ct$$

Substitute this into the Galilean transformation equation:

$$x' = ct - vt$$

Factor out *t*:

$$x' = t(c - v)$$

The Galilean transformation equation for time *t* is

$$t' = t$$

This means the speed of light in frame *S'* (solved from x' divided by *t'*) is equal to

$$\frac{x'}{t'} = \frac{t(c - v)}{t} = c - v$$

Galilean relativity predicts that the speed of light according to a moving observer will be $c - v,$ but Einstein's second postulate states that the speed of light is constant for all observers. Therefore, the Galilean transformation equations do not work for relativistic speeds and must be replaced.

To find the new transformation equations, it is helpful to assume that the Galilean transformation equations were multiplied by some factor to get the Lorentz transformations, so that

$$x = A(x' + vt')$$

where A is an unknown factor. This means that the Lorentz transformation equation is a linear transformation of the Galilean transformation equation. Additionally, solve for *x'*:

$$x' = A(x - vt)$$

Remember from earlier:

$$x = ct$$

Substitute *x* into equations:

$$ct = A(ct' + vt')$$

$$ct' = A(ct - vt)$$

Simplify both equations:

$$ct = At'(c + v)$$

$$ct' = At(c - v)$$

Solve for *t'* in the second equation:

$$t' = \frac{At}{c}(c - v)$$

Substitute *t'* into the first equation:

$$ct = A\left(\frac{At}{c}(c - v)\right)(c + v)$$

Distribute and simplify:

$$ct = \frac{A^2t}{c}(c - v)(c + v)$$

$$c = \frac{A^2}{c}(c - v)(c + v)$$

$$c^2 = A^2(c - v)(c + v)$$

Solve for A^2:

$$A^2 = \frac{c^2}{(c-v)(c+v)}$$

Simplify:

$$A^2 = \frac{c^2}{(c^2 - v^2)}$$

Factor out c^2 from the denominator to cancel the c^2 that is in the numerator:

$$A^2 = \frac{c^2}{c^2\left(1 - \frac{v^2}{c^2}\right)}$$

Simplify:

$$A^2 = \frac{1}{1 - \frac{v^2}{c^2}}$$

Solve for *A:*

$$\sqrt{A^2} = \sqrt{\frac{1}{1 - \frac{v^2}{c^2}}}$$

$$A = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

This factor is called the Lorentz factor and is notated by the lowercase Greek letter gamma

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Now that the unknown factor was derived, the Lorentz transformation equation for x' and x are fully complete

$$x = \gamma(x' + vt')$$

$$x' = \gamma(x - vt)$$

From these equations, the equations for t and t' can be derived. Substitute x' into the equation for x:

$$x = \gamma(\gamma(x - vt) + vt')$$

Distribute and simplify the equation:

$$x = \gamma(\gamma x - \gamma vt) + \gamma vt'$$

$$x = \gamma^2 x - \gamma^2 vt + \gamma vt'$$

Solve for $\gamma vt'$ to try to isolate t' later in the future:

$$\gamma vt' = x + \gamma^2 vt - \gamma^2 x$$

Solve for t':

$$t' = \frac{x}{\gamma v} + \gamma t - \frac{\gamma x}{v}$$

Factor out γ:

$$t' = \gamma\left(\frac{x}{\gamma^2 v} + t - \frac{x}{v}\right)$$

Simplify the γ^2 to potentially cancel elements of the equation:

$$t' = \gamma\left(\frac{\frac{x}{v}}{\left(1-\frac{v^2}{c^2}\right)} + t - \frac{x}{v}\right)$$

$$t' = \gamma\left(\frac{x\left(1-\frac{v^2}{c^2}\right)}{v} + t - \frac{x}{v}\right)$$

$$t' = \gamma\left(\frac{x\left(1-\frac{v^2}{c^2}\right) - x}{v} + t\right)$$

$$t' = \gamma\left(\frac{x - \frac{xv^2}{c^2} - x}{v} + t\right)$$

$$t' = \gamma\left(t - \frac{xv}{c^2}\right)$$

Solve for *t*:

$$t = \gamma\left(t' + \frac{xv}{c^2}\right)$$

Both the *x'* and *t'* Lorentz transformation equations have been derived. To recap, here are the four equations (*x'*, *x*, *t'*, *t*)

$$x' = \gamma(x - vt)$$

$$x = \gamma(x' + vt')$$

$$t' = \gamma\left(t - \frac{xv}{c^2}\right)$$

$$t = \gamma\left(t' + \frac{xv}{c^2}\right)$$

RESULTS/CONCLUSION

Through the application of Einstein's two postulates of Special Relativity, the Lorentz transformation equations were successfully derived. Unlike the Galilean transformation equations, this set of equations can not only preserve the constancy of the speed of light, but it also still produces accurate results when calculating non-relativistic speeds. Another crucial finding is that the Lorentz transformation equations mathematically back the fact that space and time are not fixed quantities, but rather they are highly variant. Another problem that Lorentz transformations can solve is the relativity of simultaneity. These equations can mathematically explain how simultaneous events may appear different in alternate reference frames. Time dilation, length contraction, relativity of simultaneity, relativistic momentum, and relativistic energy ultimately all find their roots in the Lorentz transformation equations.

Now, although relativistic effects are practically nonexistent in everyday life, Lorentz transformations are becoming increasingly significant, especially in modern technologies. In conclusion, the Lorentz transformations not only replaced the former Galilean transformation equations, but simultaneously redefined the relationship between space and time, setting a foundation for modern physics.