RIGIDITY OF AFFINE BRIESKORN-PHAM THREEFOLDS

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INTRODUCTION

An affine variety X defined over a field k of characteristic zero is called *rigid* if it does not admit a non-trivial action of the additive group $\mathbb{G}_{a,k}$. Over the last few decades, the study of the rigidity of the following class of varieties has been the source of many contributions involving a wide range of complementary techniques, from graded commutative algebra to birational geometry of mildy singular projective varieties.

Definition. An *affine Brieskorn-Pham hypersurface* is a variety $X_{a_0,\dots,a_n}$ in $\mathbb{A}_k^{n+1}$ defined by an equation of the form $X_0^{a_0}+\cdots+X_n^{a_n}=0$, where $n\geq 2$ and $a_0,\dots,a_n$ are positive integers.

If $a_i=1$ for some $i\in\{0,\dots,n\}$ then $X_{a_0,\dots,a_n}$ is isomorphic to $\mathbb{A}_k^n$, which is clearly not rigid. If k contains $\mathbb{Q}(i)$ and two of the a_i (say a_0 and a_1) equal 2, then $X_{a_0,\dots,a_n}$ is isomorphic to the hypersurface $uv+X_2^{a_2}+\cdots+X_n^{a_n}=0$ of $\mathbb{A}_k^{n+1}$, which is known to admit several non-trivial $\mathbb{G}_{a,k}$-actions. These observations inspired the following conjecture proposed by Flenner-Kaliman-Zaidenberg [10, 12]:

Rigidity Conjecture. *An affine Brieskorn-Pham hypersurface $X_{a_0,\dots,a_n}$ over an algebraically closed field of characteristic zero is rigid if and only if $\min\{a_0,\dots,a_n\}\geq 2$ and at most one element i of $\{0,\dots,n\}$ satisfies $a_i=2$.*

Due to the classical observation that for an affine variety X, the existence of a non-trivial action of $\mathbb{G}_{a,k}$ is equivalent to that of a principal open $\mathbb{A}^1$-*cylinder* in X, that is, a principal Zariski open subset U of X isomorphic to the product $Z\times_k\mathbb{A}_k^1$ for some affine variety Z, solving the Rigidity Conjecture amounts geometrically to showing that if $\min\{a_0,\dots,a_n\}\geq 2$ and $a_i=2$ for at most one $i\in\{0,\dots,n\}$, then $X_{a_0,\dots,a_n}$ does not contain any principal open $\mathbb{A}^1$-cylinder. The conjecture for Brieskorn-Pham surfaces $X_{a_0,a_1,a_2}\subset\mathbb{A}_k^3$ was settled affirmatively by Kaliman and Zaidenberg [12] by algebraic techniques but in fact, the result was already implicitly known much earlier as a consequence of the following geometric characterization due to Miyanishi [14]:

Theorem 1. *For a Brieskorn-Pham surface $X_{a_0,a_1,a_2}\subset\mathbb{A}_k^3$, the following hold:*

a) If $\frac{1}{a_0}+\frac{1}{a_1}+\frac{1}{a_1}\leq 1$ then the open subset $X_{a_0,a_1,a_2}\setminus\{(0,0,0)\}$ of X_{a_0,a_1,a_2} has non-negative logarithmic Kodaira dimension. In particular, X_{a_0,a_1,a_2} does contain an $\mathbb{A}^1$-cylinder.

b) If $\frac{1}{a_0}+\frac{1}{a_1}+\frac{1}{a_1}>1$ then (a_0,a_1,a_2) is one of the Platonic triplets $(2,3,3)$,$(2,3,4)$, $(2,3,5)$ and $(2,2,m)$, $m\geq 1$, the open subset $X_{a_0,a_1,a_2}\setminus\{(0,0,0)\}$ has negative logarithmic Kodaira dimension, but X_{a_0,a_1,a_2} contains an $\mathbb{A}^1$-cylinder if and only $(a_0,a_1,a_2)=(2,2,m)$ for some $m\geq 1$.

Despite significant progress that has led to new techniques and partial results over the last decade, the conjecture remained open in dimension $n\geq 3$ (see [1, Conjecture 1.22]). Combining these techniques with a new observation that allows us to inductively reduce the verification of the Rigidity Conjecture in every dimension ≥ 3 to its verification for a natural sub-class of Brieskorn-Pham hypersurfaces called *well-formed affine Brieskorn-Pham hypersurfaces*, we were able in [8] to complete a proof of the Rigidity Conjecture for affine Brieskorn-Pham threefolds:

Main Theorem. *Let $X_{a_0,a_1,a_2,a_3}\subset\mathbb{A}_k^4$ be an affine Brieskorn-Pham threefold hypersurface. If $\min\{a_0,a_1,a_2,a_3\}\geq 2$ and at most one element i of $\{0,1,2,3\}$ satisfies $a_i=2$, then X_{a_0,a_1,a_2,a_3} is rigid.*

We now proceed to give an overview of the new input given by the notion of well-formed affine Brieskorn-Pham hypersurfaces and indicate the main geometric notions and remaining steps involved in the proof of the Main Theorem.

1. REDUCTION TO WELL-FORMED AFFINE BRIESKORN-PHAM HYPERSURFACES OF FANO TYPE

Every affine Brieskorn-Pham hypersurface $X=X_{a_0,\dots,a_n}\subset\mathbb{A}_k^{n+1}$ identifies with the affine cone over the quasi-smooth weighted hypersurface $\hat{X}$ defined by the weighted homogeneous equation $X_0^{a_0}+\cdots+X_n^{a_n}=0$ of degree $d=\mathrm{lcm}(a_0,\dots,a_n)$ in the weighted projective space $\mathbb{P}=\mathbb{P}(w_0,\dots,w_n)$, where $w_i=d/a_i$, with respect to the polarization by the ample $\mathbb{Q}$-Cartier Weil $\mathbb{Q}$-divisor $H|_{\hat{X}}$ where H is any Weil $\mathbb{Q}$-divisor on $\mathbb{P}$ such that $\mathscr{O}_{\mathbb{P}}(H)=\mathscr{O}_{\mathbb{P}}(1)$.

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Definition 2. An affine Brieskorn-Pham hypersurface $X = X_{a_0,\ldots,a_n} \subset \mathbb{A}_k^{n+1}$ is called *well-formed* if the corresponding quasi-smooth weighted hypersurface $\hat{X} \subset \mathbb{P}$ is well-formed that is, if $\gcd(w_0,\ldots,w_{i-1},\hat{w}_i,w_{i+1},\ldots,w_n) = 1$ for every $i = 0,\ldots,n$ and $\operatorname{codim}_{\hat{X}}(\hat{X} \cap \operatorname{Sing}(\mathbb{P})) \geq 2$.

The well-formed affine Brieskorn-Pham hypersurfaces $X_{a_0,\ldots,a_n}$ actually turn out to admit a particularly convenient alternative arithmetic characterization: they are those for which a_i divides $\operatorname{lcm}(a_0,\ldots,\hat{a}_i,\ldots a_n)$ for every $i = 0,\ldots,n$. With this notion at hand, we have the following reduction of the Rigidity Conjecture:

Theorem 3. *The Rigidity Conjecture holds in a given dimension $n \geq 3$ provided it holds both in dimension $n-1$ and for well-formed affine Brieskorn-Pham hypersurfaces of dimension n.*

Combined with the surface case of Theorem 1, we obtain that the Rigidity Conjecture holds in dimension 3 provided that it holds for well-formed affine Brieskorn-Pham threefolds and, arguing by induction, that it holds in all dimensions $n \geq 3$ provided that it holds for all well-formed affine Brieskorn-Pham hypersurfaces in all dimensions $n \geq 3$.

The proof of Theorem 3 given in [8] uses the property that every affine Brieskorn-Pham hypersurface can be obtained from a well-formed one via a sequence of cyclic covers branched along coordinate axes. Ring theoretically, these covers correspond to sequences of Veronese subrings of the graded coordinate ring of the given hypersurface. Geometrically, two hypersurfaces in the same sequence arise as cones over the same sub-variety of the ambient weighted projective space with respect to different polarizations. The seemingly simple possibility of being able to relate the rigidity of an affine variety to that of some of its cyclic covers is in fact totally non-trivial, since additive group actions (equivalently, principal open $\mathbb{A}^1$-cylinders) do in general neither pullback nor descend through ramified cyclic covers. The argument of the proof depends on the application of deep techniques of equivariant graded commutative algebra progressively developed by Moser-Jauslin, Freudenburg, Daigle and Chitayat [6, 9, 11] over the last decade.

• A first key property of a well-formed affine Brieskorn-Pham hypersurface $X = X_{a_0,\ldots,a_n} \subset \mathbb{A}_k^{n+1}$ is that the existence of a non-trivial action of $\mathbb{G}_{a,k}$ on X is equivalent, through the application of a general correspondence first discovered by Kishimoto-Prokhorov-Zaidenberg [13] and later on clarified by Chitayat-Daigle [7], to the existence in its associated quasi-smooth weighted hypersurface $\hat{X} \subset \mathbb{P}$ of a so-called *$H|_{\hat{X}}$-polar $\mathbb{A}^1$-cylinder*, that is, an affine $\mathbb{A}^1$-cylinder $U \subset \hat{X}$ whose complement is equal to the support of an effective Weil $\mathbb{Q}$-divisor D on $\hat{X}$ which is $\mathbb{Q}$-linearly equivalent to $H|_{\hat{X}}$.

• A second key property is that the adjunction formula for $\hat{X} \subset \mathbb{P}$ holds in the form

$$K_{\hat{X}} \sim K_{\mathbb{P}}|_{\hat{X}} + dH|_{\hat{X}} \sim d(1 - \sum_{i=0}^{n} \frac{1}{a_i})H|_{\hat{X}},$$

leading to a dichotomy for $\hat{X}$ analogous to that of Theorem 1: the canonical divisor $K_{\hat{X}}$ of $\hat{X}$ is either pseudo-effective if $\sum_{i=0}^{n} \frac{1}{a_i} \leq 1$ or anti-ample if $\sum_{i=0}^{n} \frac{1}{a_i} > 1$. Using the fact that $\hat{X}$ has cyclic quotient singularities whence in particular klt singularities, it can be inferred by standard intersection theoretic arguments on an appropriate log-resolution of the singularities of $\hat{X}$ that the pseudo-effectivity of $K_{\hat{X}}$ rules out the possibility of $\hat{X}$ containing any open affine $\mathbb{A}^1$-cylinder.

In sum, the verification of the Rigidity Conjecture ultimately reduces to the study of the rigidity of the sub-family of well-formed affine Brieskorn-Pham hypersurfaces X of *Fano type*, that is, those for which $-K_{\hat{X}}$ is ample.

2. Exclusion of anti-canonical polar $\mathbb{A}^1$-cylinders in log del Pezzo surfaces

In light of the reductions made in the previous section, the remaining step to complete the proof of the Rigidity Conjecture for affine Brieskorn-Pham threefolds is to rule out for every well-formed affine Brieskorn-Pham threefold $X = X_{a_0,\ldots,a_3} \subset \mathbb{A}_k^4$ of Fano type such that $\min\{a_0,\ldots,a_3\} \geq 2$ and $a_i = 2$ for at most $i \in \{0,\ldots,3\}$ the possibility that its associated quasi-smooth del Pezzo surface $\hat{X} \subset \mathbb{P}(w_0,\ldots,w_3)$ with klt singularities contains a $-K_{\hat{X}}$-polar $\mathbb{A}^1$-cylinder. It turns out that up to a permutation of $a_0,\ldots,a_3$, there exists only finitely many such threefolds, for which the basic properties of the associated del Pezzo surfaces are summarized in Table 1.

The modern tools for the study of anti-canonical polar $\mathbb{A}^1$-cylinders in log del Pezzo surfaces S, mainly developped by Cheltsov, Park and Won [1, 3, 4, 5] during the last decade, are provided by techniques of log birational geometry.

A crucial fact in this context is that the existence of an anti-canonical polar $\mathbb{A}^1$-cylinder $U = S \setminus \operatorname{Supp}(D) \cong Z \times_k \mathbb{A}^1$, where D is an effective Weil $\mathbb{Q}$-divisor $\mathbb{Q}$-linearly equivalent to the anti-canonical Weil $\mathbb{Q}$-divisor $-K_S$ of S, imposes strong restrictions on the singularities of the log pair (S,D): given any log-resolution $\sigma : S' \to S$ of the rational map $\varphi : S \dashrightarrow \mathbb{P}^1$ induced by the projection $\mathrm{pr}_Z : U \to Z$, the log pair (S,D) is not log-canonical at every point of the image in S of the unique irreducible divisor $C_\infty \subset S' \setminus \sigma^{-1}(U)$ which is a rational section of the induced $\mathbb{P}^1$-fibration $\rho = \varphi \circ \sigma : S' \to \mathbb{P}^1$.

(a_0,a_1,a_2,a_3)	(w_0,w_1,w_2,w_3)	$\hat{X}$	Singularities of $\hat{X}$
$(2,3,6,6)$	$(3,2,1,1)$	del Pezzo surface of degree 1	Smooth
$(2,4,4,4)$	$(2,1,1,1)$	del Pezzo surface of degree 2	Smooth
$(3,3,3,3)$	$(1,1,1,1)$	del Pezzo surface of degree 3	Smooth
$(2,3,3,6)$	$(3,2,2,1)$	del Pezzo surface of degree 2	Three Du Val points A_1
$(3,3,4,4)$	$(4,4,3,3)$	del Pezzo surface of degree $\frac{1}{3}$	Non Du Val
$(3,3,5,5)$	$(5,5,3,3)$	del Pezzo surface of degree $\frac{1}{15}$	Non Du Val
$(2,3,4,12)$	$(6,4,3,1)$	del Pezzo surface of degree $\frac{2}{3}$	Non Du Val
$(2,3,5,30)$	$(15,10,6,1)$	del Pezzo surface of degree $\frac{2}{15}$	Non Du Val

TABLE 1. Del Pezzo surfaces associated to well-formed affine Brieskorn-Pham threefolds of Fano type.

• The above property immediately rules out for instance the possibility that a log del Pezzo surface whose global log-canonical threshold (also called α-invariant) is larger than or equal to 1 contains an anti-canonical polar $\mathbb{A}^1$-cylinder. The surfaces corresponding to the cases $(a_0,a_1,a_2,a_3)=(3,3,4,4)$ and $(3,3,5,5)$ appearing in Table 1 were verified earlier by Cheltsov-Park-Shramov [2] to have global log-canonical thresholds equal to 1 and 2 respectively, from which it follows that the corresponding two well-formed affine Brieskorn-Pham threefolds are rigid.

• On the other hand, the non-existence of anti-canonical polar $\mathbb{A}^1$-cylinders in log del Pezzo surfaces with at worst Du Val singularities appearing in Table 1 is covered by a series of more general results on the classification of anti-canonical polar cylinders in Du Val del Pezzo surfaces due to Cheltsov, Park and Won, rendering again the conclusion that the corresponding well-formed affine Brieskorn-Pham threefolds are rigid.

With these existing results at hand, finishing the proof of the Rigidity Conjecture for affine Brieskorn-Pham threefolds boils down to the study of the intriguing remaining two special cases $(a_0,a_2,a_2,a_3)=(2,3,4,12)$ and $(2,3,5,30)$, for which the corresponding log del Pezzo surfaces have non Du Val cyclic quotient singularities. The non-existence of anti-canonical polar $\mathbb{A}^1$-cylinders inside these surfaces is established in [8] by a combination of the aforementioned techniques and of the construction of specific birational transformations relating these surfaces to suitable auxiliary del Pezzo surfaces with Du Val singularities.

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