

The Rigid Pham-Brieskorn Threefolds

Michael Chitayat and Adrien Dubouloz

Sami Shamoon College of Engineering and Université de Bourgogne

Overview
Let $S = (a_0, \dots, a_n) \in (\mathbb{N}^+)^{n+1}$. A Pham-Brieskorn ring is a ring of the following form: $B_S = B_{a_0, \dots, a_n} = \mathbb{C}[X_0, \dots, X_n] / \langle X_0^{a_0} + \dots + X_n^{a_n} \rangle$ where $a_0, \dots, a_n \in \mathbb{N}^+$. Observe that if we let $L = \text{lcm}(S)$ and define $w_i = \deg(X_i) = L/a_i$, then we can view $B_S = B_{a_0, \dots, a_n}$ as an $\mathbb{N}$ -graded ring and $\text{Proj } B_S \subset \mathbb{P}(w_0, \dots, w_n)$ as a weighted projective hypersurface.
Locally Nilpotent Derivations
- Let B be a commutative ring. A derivation $D : B \rightarrow B$ is <i>locally nilpotent</i> if for every $b \in B$ there exists $n \in \mathbb{N}^+$ such that $D^n(b) = 0$. Example: $B = \mathbb{C}[X_0, \dots, X_n]$. For any i, the derivation $D = \frac{\partial}{\partial X_i}$ is a locally nilpotent derivation. - A ring B is <i>rigid</i> if the only locally nilpotent derivation $D : B \rightarrow B$ is the zero derivation. (If B is a k-domain, $\text{Spec } B$ is <i>rigid</i> if it admits no non-trivial $\mathbb{G}_a$-actions.)
Rigidity Conjecture
Question: For which $(a_0, \dots, a_n) \in (\mathbb{N}^+)^{n+1}$ is $B_{a_0, \dots, a_n}$ rigid? The following Conjecture has been open for over 25 years. [6] Main Conjecture: Suppose that $n \geq 2$, $a_0 \leq \dots \leq a_n$. Then, $B_{a_0, \dots, a_n} \text{ is not rigid} \iff a_0 = 1 \text{ or } a_0 = a_1 = 2.$
Remarks: $(\Leftarrow)$ is easy to prove. - The $n = 2$ case of the conjecture is known to be true by Lemma 4 in [6]. Despite many partial results, there was no clear strategy for how to prove this conjecture in general.

A Reduction Theorem

The cotype of a tuple $(a_0, \dots, a_n)$

- Given $S = (a_0, \dots, a_n) \in (\mathbb{N}^+)^{n+1}$, let $S_i = (a_0, \dots, a_{i-1}, \hat{a}_i, a_{i+1}, \dots, a_n)$.
- Let $L = \text{lcm}(S)$ and let $L_i = \text{lcm}(S_i)$.
- Define $\text{cotype}(S) = \#\{i : L_i \neq L\}$.
- Given $S = (a_0, \dots, a_n) \in (\mathbb{N}^+)^{n+1}$ and $i \in \{0, \dots, n\}$, define $g_i(S) = \gcd(a_i, \text{lcm}(S_i))$.
- Let $S = (a_0, \dots, a_n)$ and $S' = (a'_0, \dots, a'_n)$ be elements of $(\mathbb{N}^+)^{n+1}$ and let $i \in \{0, \dots, n\}$. We write $S \leq^i S'$ if and only if

$$S_i = S'_i \quad \text{and} \quad g_i(S') \mid a_i \mid a'_i.$$

We write $S <^i S'$ if and only if $S \leq^i S'$ and $S \neq S'$.

Example: $(2, 3, 3, 2) <^3 (2, 3, 3, 4) <^0 (10, 3, 3, 4)$

Our first main result reduces the Main Conjecture to the cotype 0 cases.

Reduction Theorem: To prove the Main Conjecture, it suffices to prove it for the cotype 0 cases.

Locally Nilpotent Derivations and Polar cylinders

Polar cylinders

- Let X be a normal projective **k**-variety. A $\mathbb{Q}$ -divisor is a sum $D = \sum_{i=1}^n a_i C_i$ where each C_i is a prime divisor and $a_i \in \mathbb{Q}^+$. Two $\mathbb{Q}$ -divisors D and D' are $\mathbb{Q}$ -linearly equivalent (write $D \sim_{\mathbb{Q}} D'$) if there exists $n \in \mathbb{N}^+$ such that nD and nD' are linearly equivalent divisors.
- Let X be normal projective **k**-variety. An open subset $U \subset X$ is called a *cylinder* if $U \cong \mathbb{A}^1 \times Z$ for some affine variety Z .
- Let H be a $\mathbb{Q}$ -divisor of X . A cylinder $U \subset X$ is *H-polar* if $U = X \setminus \text{Supp}(D)$ for some effective $\mathbb{Q}$ -Cartier $\mathbb{Q}$ -divisor D such that $D \sim_{\mathbb{Q}} sH$ where $s \in \mathbb{Q}^+$.

There is an important connection between locally nilpotent derivations of graded rings and polar cylinders discovered by Kishimoto, Prokhorov and Zaidenberg in [7], and generalized in [5]. The theorem is as follows:

Cylinder Theorem: Assume $B = \bigoplus_{n \in \mathbb{N}} B_n$ satisfies $\text{trdeg}(B/B_0) \geq 2$, and is a normal finitely generated **k**-domain that is “saturated in codimension one”. Let $X = \text{Proj } B$ and let H be an ample $\mathbb{Q}$ -divisor such that $B \cong \bigoplus_{i \in \mathbb{N}} H^0(X, \mathcal{O}_X(nH))$.
$$B \text{ is not rigid} \iff X \text{ contains an } H\text{-polar cylinder.}$$

Proof of the 3 dimensional case

Using the previously discussed material, we offer a proof of the 3-dimensional case of the Main Conjecture. To derive this result, we first state the following result which is a corollary of the Cylinder Theorem.

Corollary 1: Let $B = B_{a_0, a_1, a_2, a_3}$ where $\text{cotype}(a_0, a_1, a_2, a_3) = 0$, let $\alpha = L - \sum_i \deg(x_i)$, and let $X = \text{Proj } B$. Then, $\mathcal{O}_X(K_X) \cong \mathcal{O}_X(\alpha)$ and

- (a) if $\alpha > 0$ then B is rigid $\iff X$ doesn't contain a K_X -polar cylinder;
(b) if $\alpha = 0$ then X is not birationally ruled, so B is rigid;
(c) if $\alpha < 0$ then B is rigid $\iff X$ doesn't contain a $-K_X$ -polar cylinder.

In view of Corollary 1, there are 2 cases to consider:

the $\alpha > 0$ case, and the $\alpha < 0$ case.

The $\alpha > 0$ case.

We want to prove that if $X = \text{Proj } B_{a_0, a_1, a_2, a_3}$, $\text{cotype}(a_0, a_1, a_2, a_3) = 0$ and $\alpha > 0$, then X doesn't contain a K_X -polar cylinder. But in fact, by a generalization of an argument in [2], X doesn't contain any cylinder, and so the $\alpha > 0$ case is complete by Corollary 1 (a).

The $\alpha < 0$ case.

If $X = \text{Proj } B_{a_0, a_1, a_2, a_3}$, $\text{cotype}(a_0, a_1, a_2, a_3) = 0$ and $\alpha < 0$, then X is a singular del Pezzo surface with quotient singularities. A number theoretic argument (Lemma 4.2.4 in [4]) shows that only 8 cases are possible. They are as follows:

- $(2, 3, 3, 6), \quad (2, 3, 6, 6), \quad (2, 4, 4, 4), \quad (3, 3, 3, 3), \quad (3, 3, 4, 4), \quad (3, 3, 5, 5)$
- $(2, 3, 4, 12)$
- $(2, 3, 5, 30)$

The first six cases are resolved by Cheltsov, Park and Won's classification of anti-canonical polar cylinders in del Pezzo surfaces with Du Val singularities [3] and by computations of log canonical threshholds [1]. It now remains to prove the $(2, 3, 4, 12)$ and $(2, 3, 5, 30)$ cases.

The (2,3,5,30) case.

By Corollary 1 (c), it suffices to show that $X = \text{Proj } B_{2,3,5,30}$ does not contain a $-K_X$ -polar cylinder. Unfortunately, we cannot apply the classification in [3] because X does not have Du Val singularities.

Facts about $X = \text{Proj } B_{2,3,5,30}$

- X has 3 singular points $\{P_2, P_3, P_5\}$ of type $\frac{1}{2}(1, 1), \frac{1}{3}(1, 1), \frac{1}{5}(1, 1)$;
- the curve $\Delta = V_+(x_3) \subset X$ is isomorphic to $\mathbb{P}^1$ and satisfies $\text{mult}_{P_3}(\Delta) = \text{mult}_{P_3}(\Delta) = \text{mult}_{P_3}(\Delta) = 1$;
- if $\tilde{X} \rightarrow X$ is the minimal resolution of singularities of X , then the exceptional locus consists of three curves $\tilde{E}_2, \tilde{E}_3$ and $\tilde{E}_5$, each isomorphic to $\mathbb{P}^1$.

Lemma: If X contains a $-K_X$ -polar cylinder $U = X \setminus \text{Supp}(D)$ where $D = \sum_{i \in I} a_i C_i$, then $\Delta \subseteq \text{Supp}(D)$.

We now assume for the sake of a contradiction, that X contains a $-K_X$ -polar cylinder $U = X \setminus \text{Supp}(D)$ where D is an effective anti-canonical $\mathbb{Q}$ -divisor. By the Lemma, D contains Δ in its support, one can show that $\tilde{X}$ contains a $-K_{\tilde{X}}$ -polar cylinder whose complement contains $\tilde{\Delta}, \tilde{E}_2$ and $\tilde{E}_3$ in its support. Moreover, these three curves can be contracted via a morphism $\tau : \tilde{X} \rightarrow \hat{X}$. Since these three curves are contained in the complement of the $-K_{\tilde{X}}$ -polar cylinder, it turns out that $\hat{X}$ is a smooth degree 1 del Pezzo surface which too contains a $-K_{\hat{X}}$ -polar cylinder. But this contradicts Cheltsov, Park and Won's classification of del Pezzo surfaces that contain anti-canonical polar cylinders established in [3]. We thus conclude that X itself could not have contained an anti-canonical polar cylinder to begin with. We obtain:

Proposition: The surface $X = \text{Proj } B_{2,3,5,30}$ does not contain an anti-canonical polar cylinder. Consequently, $B_{2,3,5,30}$ is rigid.

Remark: The proof that $B_{2,3,4,12}$ is rigid is done in the same way. We conclude:

Main Theorem
Suppose that $a_0 \leq a_1 \leq a_2 \leq a_3$. Then, $B_{a_0, a_1, a_2, a_3} \text{ is not rigid} \iff a_0 = 1 \text{ or } a_0 = a_1 = 2.$

References

- [1] Ivan Cheltsov, Jihun Park, and Constantin Shramov. Exceptional del Pezzo hypersurfaces. *Journal of Geometric Analysis*, 20(4):787–816, 2010.
- [2] Ivan Cheltsov, Jihun Park, and Joonyeong Won. Cylinders in del Pezzo Surfaces. *International Mathematics Research Notices*, 2017(4):1179–1230, 05 2016.
- [3] Ivan Cheltsov, Jihun Park, and Joonyeong Won. Cylinders in singular del Pezzo surfaces. *Compositio Mathematica*, 152(6):1198–1224, 2016.
- [4] Michael Chitayat. Rigidity of Pham-Brieskorn Threefolds. <https://ruor.uottawa.ca/handle/10393/44886>, 2023. University of Ottawa, Theses and Dissertations.
- [5] Michael Chitayat and Daniel Daigle. Locally nilpotent derivations of graded integral domains and cylindricity. *Transformation Groups*, 2022.
- [6] Shulim Kaliman and Mikhail Zaidenberg. Miyanishi's characterization of the affine 3-space does not hold in higher dimensions. In *Annales de l'institut Fourier*, volume 50, pages 1649–1669, 2000.
- [7] Takashi Kishimoto, Yuri Prokhorov, and Mikhail Zaidenberg. G_a -actions on affine cones. *Transformation Groups*, 18:1137–1153, 2013.