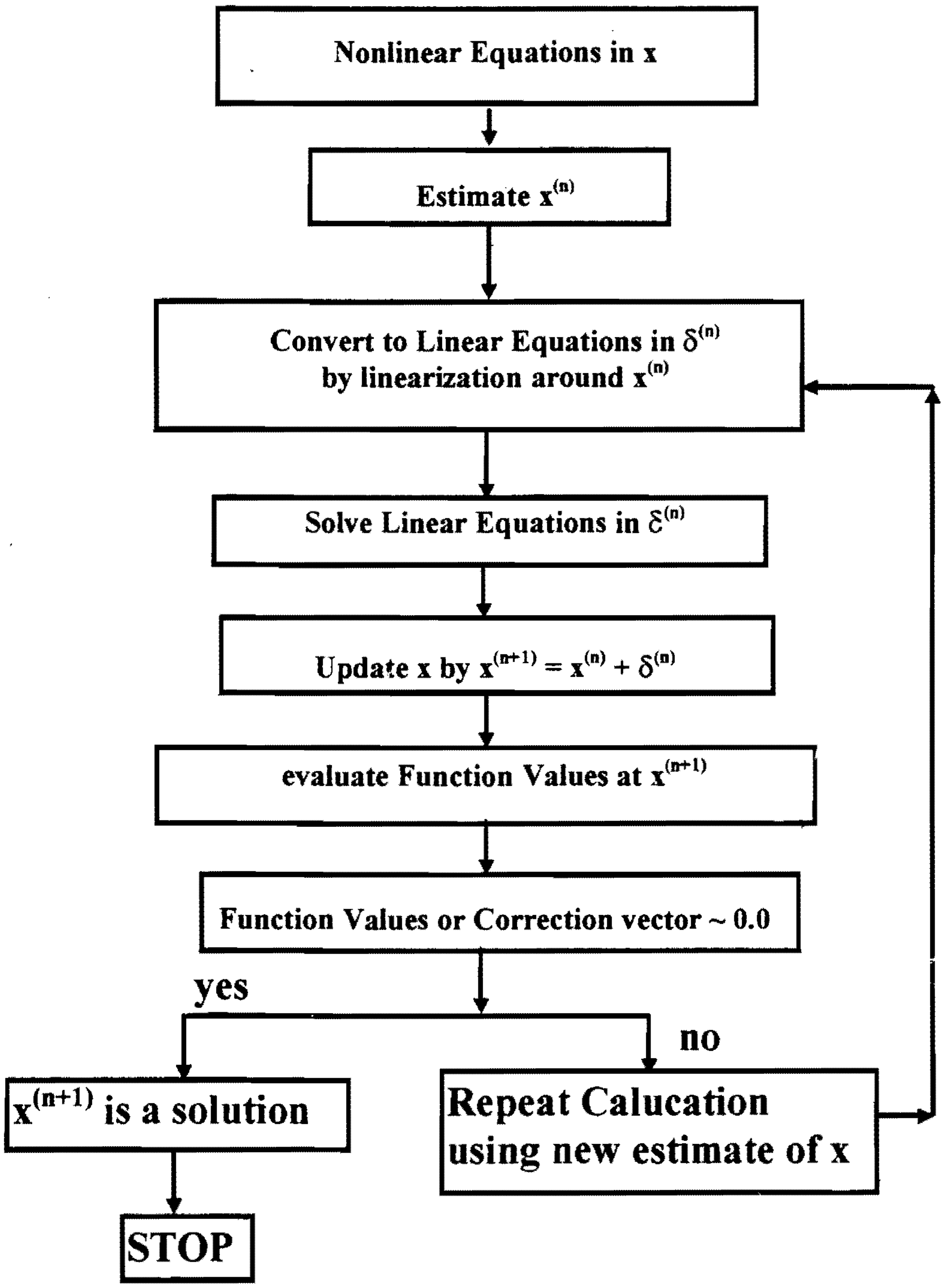
STEPS TO FOLLOW TO SOLVE NONLINEAR ALGEBRAIC EQUATIONS



Study on Your Own:
Other methods of solution of linear and non-linear algebraic equations.

3.2 Solution Of Nonlinear Algebraic Equations

Example:

$$a_{11}x_1^2 + a_{12}x_1x_2 + c_1 = 0$$
$$a_{21}x_1^2 + a_{22}x_2 \quad + c_2 = 0$$

In the above set there are two variables (x_1, x_2) and two equations. The set can be written in function form as shown below.

$$f_1(x_1, x_2) = 0$$
$$f_2(x_1, x_2) = 0$$

Similarly a set of k nonlinear algebraic equations can be written as:

$$f_1(x_1, x_2, \ldots\ldots, x_k) = 0$$
$$f_2(x_1, x_2, \ldots\ldots, x_k) = 0$$
$$\vdots$$
$$\vdots$$
$$f_k(x_1, x_2, \ldots\ldots, x_k) = 0$$

We have k equations for k variables. Therefore, d.f. = 0

Note: You can no longer represent the set of equations in AX = C form (linear form).

How to solve?

There are many methods of solving nonlinear algebraic equations (**search in mathematics books!**). Here we will introduce the application of Taylor Series expansion on the equations.

Taylor Series Expansion

a) One equation - One variable

$f(x) = 0$

Series expansion around an initial guess ($x^{(1)}$) of the unknown variable x gives:

$$f(x) = 0 = f(x^{(1)}) + \frac{\partial f(x^{(1)})}{\partial x}.(x - x^{(1)}) + \ldots\ldots\ldots$$
$$= f(x^{(1)}) + f'(x^{(1)}).(x - x^{(1)}) + \ldots\ldots\ldots\ldots$$
$$\therefore f'(x^{(1)}).(x - x^{(1)}) = -f(x^{(1)})$$
$$\text{or, } f'(x^{(1)}).\partial^{(1)} = -f(x^{(1)})$$
$$\text{where, } f'(x^{(1)}) = \frac{\partial f(x^{(1)})}{\partial x} \quad ; \quad \partial^{(1)} = x - x^{(1)}$$

we neglect higher derivatives

Example:

$$f(x_1) = 5x_1^2 + 3x_1 - 2 = 0$$

Series expansion around $\left(x_1 = x_1^{(1)} = 3\right)$ will give:

$$f'(x_1^{(1)}).\partial_1^{(1)} = -f(x_1^{(1)})$$

Now $f(x_1^{(1)}) = 52$; $f'(x_1^{(1)}) = 10x_1 + 3 = 33$ (remember $x_1^{(1)} = 3$)

$\therefore$ $33\,\partial_1^{(1)} = -52$; $\partial_1^{(1)} = -52/33 = -1.576$

b) Two Equations-Two Variables System

$$\boxed{\begin{aligned} f_1(x_1, x_2) = 0 \\ f_2(x_1, x_2) = 0 \end{aligned}}$$

Taylor Series Expansion around $(x_1^{(1)}, x_2^{(1)})$ gives:

$$f_1(x_1, x_2) = 0 = f_1(x_1^{(1)}, x_2^{(1)}) + \frac{\partial f_1(x_1^{(1)})}{\partial x_1}.(x_1 - x_1^{(1)}) + \frac{\partial f_1(x_2^{(1)})}{\partial x_2}.(x_2 - x_2^{(1)})$$

$$= f_1(x_1^{(1)}, x_2^{(1)}) + f_1'(x_1^{(1)}).(x_1 - x_1^{(1)}) + f_1'(x_2^{(1)}).(x_2 - x_2^{(1)})$$

$$\therefore f_1'(x_1^{(1)}).(x_1 - x_1^{(1)}) + f_1'(x_2^{(1)}).(x_2 - x_2^{(1)}) = -f_1(x_1^{(1)}, x_2^{(1)})$$

or, $f_1'(x_1^{(1)}).\partial_1^{(1)} + f_1'(x_2^{(1)}).\partial_2^{(1)} = -f_1(x_1^{(1)}, x_2^{(1)})$

where, $f_1'(x_1^{(1)}) = \dfrac{\partial f_1(x_1^{(1)})}{\partial x_1}$; $f_1'(x_2^{(1)}) = \dfrac{\partial f_1(x_2^{(1)})}{\partial x_2}$;

$\partial_1^{(1)} = x_1 - x_1^{(1)}$; $\partial_2^{(1)} = x_2 - x_2^{(1)}$

we neglect higher derivatives

similarly for the second function,

$$f_2(x_1, x_2) = 0 = f_2(x_1^{(1)}, x_2^{(1)}) + \frac{\partial f_2(x_1^{(1)})}{\partial x_1}.(x_1 - x_1^{(1)}) + \frac{\partial f_2(x_2^{(1)})}{\partial x_2}.(x_2 - x_2^{(1)})$$

$$= f_2(x_1^{(1)}, x_2^{(1)}) + f_2'(x_1^{(1)}).(x_1 - x_1^{(1)}) + f_2'(x_2^{(1)}).(x_2 - x_2^{(1)})$$

$$\therefore f_2'(x_1^{(1)}).(x_1 - x_1^{(1)}) + f_2'(x_2^{(1)}).(x_2 - x_2^{(1)}) = -f_2(x_1^{(1)}, x_2^{(1)})$$

or, $f_2'(x_1^{(1)}).\partial_1^{(1)} + f_2'(x_2^{(1)}).\partial_2^{(1)} = -f_2(x_1^{(1)}, x_2^{(1)})$

where, $f_2'(x_1^{(1)}) = \dfrac{\partial f_2(x_1^{(1)})}{\partial x_1}$; $f_2'(x_2^{(1)}) = \dfrac{\partial f_2(x_2^{(1)})}{\partial x_2}$;

Finally what we get after the series expansion? We get a set of linear algebraic equations in terms of variables $(\partial_1^{(1)}, \partial_2^{(1)})$:

$$\boxed{\begin{aligned} f_1'(x_1^{(1)}).\partial_1^{(1)} + f_1'(x_2^{(1)}).\partial_2^{(1)} = -f_1(x_1^{(1)}, x_2^{(1)}) \\ f_2'(x_1^{(1)}).\partial_1^{(1)} + f_2'(x_2^{(1)}).\partial_2^{(1)} = -f_2(x_1^{(1)}, x_2^{(1)}) \end{aligned}}$$

Example:

$$f_1(x_1,x_2) = 2x_1^2 - 3x_1x_2 + 5 = 0$$
$$f_2(x_1,x_2) = -x_1^2 + x_2 + 2 = 0$$

Taylor series expansion around the initial guesses $x_1^{(1)} = 1$; $x_2^{(1)} = 1$ gives:

$$f_1'(x_1^{(1)}).\partial_1^{(1)} + f_1'(x_2^{(1)}).\partial_2^{(1)} = -f_1(x_1^{(1)},x_2^{(1)})$$
$$f_2'(x_1^{(1)}).\partial_1^{(1)} + f_2'(x_2^{(1)}).\partial_2^{(1)} = -f_2(x_1^{(1)},x_2^{(1)})$$

where,

$$f_1'(x_1^{(1)}) = 4x_1 - 3x_2 = 1;\ f_1'(x_2^{(1)}) = -3x_1 = -3;\ -f_1(x_1^{(1)},x_2^{(1)}) = -4$$

$$f_2'(x_1^{(1)}) = -2x_1 = -2;\ f_2'(x_2^{(1)}) = 1;\ \ -f_2(x_1^{(1)},x_2^{(1)}) = -2$$

$$\partial_1^{(1)} - 3\partial_2^{(1)} = -4$$
$$-2\partial_1^{(1)} + \partial_2^{(1)} = -2$$

$$\longrightarrow \begin{bmatrix} 1 & -3 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} \partial_1^{(1)} \\ \partial_2^{(1)} \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \end{bmatrix}$$

Solution of which is:

$$\underline{A} = \begin{bmatrix} 1 & -3 & -4 \\ -2 & 1 & -2 \end{bmatrix} \overset{r_2' = 2r_1 + r_2}{=} \begin{bmatrix} 1 & -3 & -4 \\ 0 & -5 & -10 \end{bmatrix} \overset{r_2' = r_2/(-5)}{=} \begin{bmatrix} 1 & -3 & -4 \\ 0 & 1 & 2 \end{bmatrix} \overset{r_1' = r_1 + 3r_2}{=} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

$\therefore\ \partial_1^{(1)} = 2$ and $\partial_2^{(1)} = 2$

Therefore, $\partial_1^{(1)} = x_1 - x_1^{(1)};\ \therefore x_1 = x_1^{(1)} + \partial_1^{(1)}$

i.e. the new value of x_1, say $x_1^{(2)} = x_1^{(1)} + \partial_1^{(1)} = 1 + 2 = 3$

similarly, $\partial_2^{(1)} = x_2 - x_2^{(1)};\ \therefore x_2 = x_2^{(1)} + \partial_2^{(1)}$

i.e. the new value of x_2, say $x_2^{(2)} = x_2^{(1)} + \partial_2^{(1)} = 1 + 2 = 3$

We now have a new set of values for variables (x_1, x_2) as $(x_1^{(2)},x_2^{(2)})$. We again perform Taylor series expansion of the original functions around $(x_1^{(2)},x_2^{(2)})$ and repeat calculations as before.

Notice the superscripts of the variables $(x_1^{(2)},x_2^{(2)},\partial_1^{(2)},\partial_2^{(2)})$ denotes iteration no.

We continue until $(\partial_1^{(n)},\partial_2^{(n)})$ or function values become close to (0,0). The superscript n represents the number of iteration required to achieve a solution.

In fact, you can write a nice general computer program which will do all the boring calculations for you.

Check whether the new values of the variables are the solution of the equations.

$$f_1(x_1^{(2)},x_2^{(2)}) = -4; \quad f_2(x_1^{(2)},x_2^{(2)}) = -4$$

At the solution point the function values should be zero. Therefore, solution has not been found.

<u>2nd Iteration:</u> superscripts within brackets denote the iteration no.

$$\boxed{\begin{aligned} f_1(x_1,x_2) &= 2x_1^2 - 3x_1x_2 + 5 = 0 \\ f_2(x_1,x_2) &= -x_1^2 + x_2 + 2 = 0 \end{aligned}}$$

(original equations)

Taylor series expansion around $x_1^{(2)} = 3; x_2^{(2)} = 3$ gives:

$$\boxed{\begin{aligned} f_1'(x_1^{(2)}).\partial_1^{(2)} + f_1'(x_2^{(2)}).\partial_2^{(2)} &= -f_1(x_1^{(2)},x_2^{(2)}) \\ f_2'(x_1^{(2)}).\partial_1^{(2)} + f_2'(x_2^{(2)}).\partial_2^{(2)} &= -f_2(x_1^{(2)},x_2^{(2)}) \end{aligned}}$$

$f_1'(x_1^{(2)}) = 4x_1 - 3x_2 = 3;$ $f_1'(x_2^{(2)}) = -3x_1 = -9;$ $f_2'(x_1^{(2)}) = -2x_1 = -6;$ $f_2'(x_2^{(2)}) = 1$

$$\boxed{\begin{aligned} 3\partial_1^{(2)} - 9\partial_2^{(2)} &= 4 \\ -6\partial_1^{(2)} + \partial_2^{(2)} &= 4 \end{aligned}} \rightarrow \begin{bmatrix} 3 & -9 \\ -6 & 1 \end{bmatrix} \begin{bmatrix} \partial_1^{(2)} \\ \partial_2^{(2)} \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

Solution:

$$\underline{\mathbf{A}} = \begin{bmatrix} 3 & -9 & 4 \\ -6 & 1 & 4 \end{bmatrix} \overset{r_2' = 2r_1 + r_2}{=} \begin{bmatrix} 3 & -9 & 4 \\ 0 & -17 & 12 \end{bmatrix} \overset{\substack{r_2' = r_2/(-17) \\ r_1' = r_1/3}}{=} \begin{bmatrix} 1 & -3 & 1.33 \\ 0 & 1 & -0.70 \end{bmatrix} \overset{r_1' = r_1 + 3r_2}{=} \begin{bmatrix} 1 & 0 & -0.8 \\ 0 & 1 & -0.7 \end{bmatrix}$$

$\therefore \ \partial_1^{(2)} = -0.8$ and $\partial_2^{(2)} = -0.7$

Therefore, $\partial_1^{(2)} = x_1 - x_1^{(2)}; \quad \therefore x_1 = x_1^{(2)} + \partial_1^{(2)}$

i.e. the new value of x_1, $x_1^{(3)} = x_1^{(2)} + \partial_1^{(2)} = 2.2$ similarly, $x_2^{(3)} = x_2^{(2)} + \partial_2^{(2)} = 2.3$

Check:

$$f_1(x_1^{(3)},x_2^{(3)}) = -0.5; \quad f_2(x_1^{(3)},x_2^{(3)}) = -0.54$$

Therefore, solution has not been found.

3rd Iteration:

$$\boxed{\begin{aligned} f_1(x_1,x_2) &= 2x_1^2 - 3x_1x_2 + 5 = 0 \\ f_2(x_1,x_2) &= -x_1^2 + x_2 + 2 = 0 \end{aligned}}$$

(original equations)

Taylor series expansion around $x_1^{(3)} = 2.2; \ x_2^{(3)} = 2.3$:

$$\boxed{\begin{aligned} f_1'(x_1^{(3)}).\partial_1^{(3)} + f_1'(x_2^{(3)}).\partial_2^{(3)} &= -f_1(x_1^{(3)},x_2^{(3)}) \\ f_2'(x_1^{(3)}).\partial_1^{(3)} + f_2'(x_2^{(3)}).\partial_2^{(3)} &= -f_2(x_1^{(3)},x_2^{(3)}) \end{aligned}}$$

$f_1'(x_1^{(3)}) = 4x_1 - 3x_2 = 1.9;$ $f_1'(x_2^{(3)}) = -3x_1 = -6.6;$ $f_2'(x_1^{(3)}) = -2x_1 = -4.4;$ $f_2'(x_2^{(3)}) = 1$

$$1.9\partial_1^{(3)} - 6.6\partial_2^{(3)} = 0.5$$
$$-4.4\partial_1^{(3)} + \partial_2^{(3)} = 0.54$$

$$\longrightarrow \begin{bmatrix} 1.9 & -6.6 \\ -4.4 & 1 \end{bmatrix} \begin{bmatrix} \partial_1^{(3)} \\ \partial_2^{(3)} \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.54 \end{bmatrix}$$

Solution:

$$\underline{A} = \begin{bmatrix} 1.9 & -6.6 & 0.5 \\ -4.4 & 1 & 0.54 \end{bmatrix} \quad \overset{r_1' = r_1/1.9}{=} \begin{bmatrix} 1 & -3.45 & 0.26 \\ -4.4 & 1 & 0.54 \end{bmatrix} \quad \overset{r_2' = r_2 + 4.4r_1}{=} \begin{bmatrix} 1 & -3.45 & 0.26 \\ 0 & -14.18 & 1.684 \end{bmatrix}$$

$$\overset{r_2' = r_2/(-14.18)}{=} \begin{bmatrix} 1 & -3.45 & 0.26 \\ 0 & 1 & -0.12 \end{bmatrix} \quad \overset{r_1' = 3.45r_2 + r_1}{=} \begin{bmatrix} 1 & 0 & -0.154 \\ 0 & 1 & -0.12 \end{bmatrix}$$

$\therefore\ \partial_1^{(3)} = -0.154;\ \partial_2^{(3)} = -0.12$

Therefore, $\partial_1^{(3)} = x_1 - x_1^{(3)};\quad \therefore x_1 = x_1^{(3)} + \partial_1^{(3)}$

i.e. $x_1^{(4)} = x_1^{(3)} + \partial_1^{(3)} = 2.046;$ similarly, $x_2^{(4)} = x_2^{(3)} + \partial_2^{(3)} = 2.18$

Check:

$f_1(x_1^{(4)}, x_2^{(4)}) = -0.0086;\quad f_2(x_1^{(4)}, x_2^{(4)}) = -0.0061$

Function values are close to zero. Therefore, solution has been found.

The solution is: $x_1 = 2.046 \quad x_2 = 2.18$

<u>Summary:</u>

A set of nonlinear algebraic equations

$$f_1(x_1, x_2) = 0$$
$$f_2(x_1, x_2) = 0$$

can be converted to a set of linear algebraic equations

$$f_1'(x_1^{(1)}).\partial_1^{(1)} + f_1'(x_2^{(1)}).\partial_2^{(1)} = -f_1(x_1^{(1)}, x_2^{(1)})$$
$$f_2'(x_1^{(1)}).\partial_1^{(1)} + f_2'(x_2^{(1)}).\partial_2^{(1)} = -f_2(x_1^{(1)}, x_2^{(1)})$$

by linearization using Taylor Series Expansion around $(x_1^{(1)}, x_2^{(1)})$.

Dropping the superscripts and writing in a simple form we get:

$$f_1'(x_1).\partial_1 + f_1'(x_2).\partial_2 = -f_1$$
$$f_2'(x_1).\partial_1 + f_2'(x_2).\partial_2 = -f_2$$

and in Matrix form we get:

$$\begin{bmatrix} f_1'(x_1) & f_1'(x_2) \\ f_2'(x_1) & f_2'(x_2) \end{bmatrix} \begin{bmatrix} \partial_1 \\ \partial_2 \end{bmatrix} = \begin{bmatrix} -f_1 \\ -f_2 \end{bmatrix}$$

<u>k x k system:</u>

$$\begin{aligned} f_1(x_1, x_2, \ldots\ldots, x_k) &= 0 \\ f_2(x_1, x_2, \ldots\ldots, x_k) &= 0 \\ &\vdots \\ &\vdots \\ f_k(x_1, x_2, \ldots\ldots, x_k) &= 0 \end{aligned}$$

Linearization by Taylor Series will result:

$$\begin{aligned} f_1'(x_1).\partial_1 + f_1'(x_2).\partial_2 + \ldots\ldots + f_1'(x_k).\partial_k &= -f_1 \\ f_2'(x_1).\partial_1 + f_2'(x_2).\partial_2 + \ldots\ldots + f_2'(x_k).\partial_k &= -f_2 \\ &\vdots \\ &\vdots \\ f_k'(x_1).\partial_1 + f_k'(x_2).\partial_2 + \ldots + f_k'(x_k).\partial_k &= -f_k \end{aligned}$$

In Matrix Form:

$$\begin{bmatrix} f_1'(x_1) & f_1'(x_2)\ldots\ldots f_1'(x_k) \\ f_2'(x_1) & f_2'(x_2)\ldots\ldots f_2'(x_k) \\ & \vdots \\ & \vdots \\ f_k'(x_1) & f_k'(x_2)\ldots\ldots f_k'(x_k) \end{bmatrix} \begin{bmatrix} \partial_1 \\ \partial_2 \\ \vdots \\ \vdots \\ \partial_k \end{bmatrix} = \begin{bmatrix} -f_1 \\ -f_2 \\ \vdots \\ \vdots \\ -f_k \end{bmatrix}$$

↓

$\underline{J}\,\underline{\delta} = -\underline{f}$ (very similar to **AX=C**)

where, $\underline{J}$ is the jacobian matrix as above
$\underline{\delta}$ is the correction vector
$\underline{f}$ is the vector of functions

Exercise/ Tutorial -1: Revisit the Flash Problem

A benzene-toluene mixture containing 50% benzene by molefraction is fed to a flash unit.

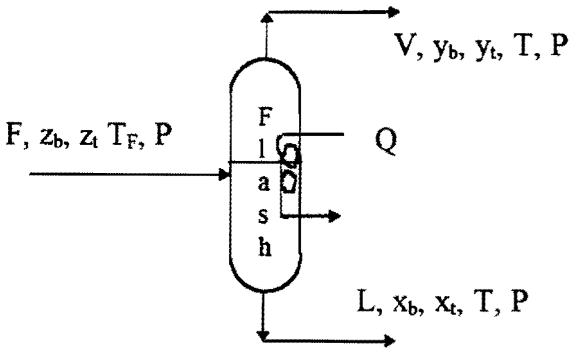


Figure 3.2: A Flash Vessel

Given: F = 100.0 kmol/hr

$z_b = 0.50$ molefraction benzene

$z_t = 0.50$ molefraction toluene

P = 1.013 bar $T_F = 350$ K $Q = 10^6$ kJ/hr

The model equations for the flash unit can be written as:

total mass balance:	$F = V + L$	(eq. 1)
benzene mass balance:	$Fz_b = Vy_b + Lx_b$	(eq. 2)
toluene mass balance:	$Fz_t = Vy_t + Lx_t$	(eq. 3)
phase equilibrium:	$y_b = g_1(T, P)\, x_b$	(eq. 4)
	$y_t = g_2(T, P)\, x_t$	(eq. 5)
restrictions:	$y_b + y_t = 1$	(eq. 6)
energy balance:		
	$F\, g_5(z_b, z_t, T_F) + Q = V.g_3(y_b, y_t, T) + L\, g_4(x_b, x_t, T)$	(eq. 7)

g_1, g_2, etc., are functions of the variables shown within brackets.

Function g_1: $g_1(T, P) = k_b = \dfrac{p_b^v}{P}$

where p_b^v is the pure component vapour pressure of benzene in mm Hg and P is the total pressure in mm Hg.

where, $\text{Log}_{10} p_b^v = A_b - \left(\dfrac{B_b}{C_b + TC}\right)$, TC is temperature in C.

Function g_2: $g_2(T, P) = k_t = \dfrac{p_t^v}{P}$

where p_t^v is the pure component vapour pressure of Toluene in mm Hg and P is the total pressure in mm Hg.

where, $\text{Log}_{10} p_t^v = A_t - \left(\dfrac{B_t}{C_t + TC}\right)$, TC is temperature in C.

The Antoine's constants are:

Component	A_i	B_i	C_i
Benzene	6.90565	1211.022	220.79
Toluene	6.95334	1343.943	219.377

Function g_3:

$g_3(y_b, y_t, T) = = 2.33\ \{y_b\ (12669.90 - 15.73T) + y_t\ (16285.25 + 12.5T)\ \}$; T is in K

Function g_4:

$g_4(x_b, x_t, T) = 2.33\ \{\ x_b\ (-\ 10486.0 + 21.55T) + x_t\ (-\ 5920.2 + 27.92T)\ \}$; T is in K

Function g_5:

$g_5(z_b, z_t, T_F) = 2.33\{z_b(-10486.0 + 21.55\ T_F) + z_t(-\ 5920.2 + 27.92\ T_F)\ \}$: T_F is in K

The unknown variables are $\{V, y_b, y_t, T, L, x_b, x_t\}$

Let us denote $\{V,\ y_b,\ y_t,\ T,\ L,\ x_b,\ x_t\}$ by

$\{X_1, X_2, X_3, X_4, X_5, X_6, X_7\}$ respectively

Often you need to do variable transformation to transform all the variables of the model into a single variable form (e.g. all in X) so that you can use a standard off-the-shelf mathematical software. This is one of the activities you often need to carry out to transform an Engineering Problem into a Mathematical Problem (see Figure 1 in page 2). Once the solution of the mathematical problem is found you have to perform reverse variable transformation to interpret mathematical solution as an engineering solution.

Rewrite the model equations in function form (i.e. f(x) = 0) in terms of the unknown variables.

$f_1(X_1, X_5)$	$= X_1 + X_5 - F$	$= 0$	(eq. 1)
$f_2(X_1, X_2, X_5, X_6)$	$= X_1X_2 + X_5X_6 - Fz_b$	$= 0$	(eq. 2)
$f_3(X_1, X_3, X_5, X_7)$	$= X_1\ X_3 + X_5\ X_7 - Fz_t$	$= 0$	(eq. 3)
$f_4(X_2, X_4, X_6)$	$= g1(X_4, P)\ X_6 - X_2$	$= 0$	(eq. 4)
$f_5(X_3, X_4, X_7)$	$= g2(X_4, P)\ X_7 - X_3$	$= 0$	(eq. 5)
$f_6(X_2, X_3)$	$= X_2 + X_3 - 1.0$	$= 0$	(eq. 6)
$f_7(X_1, X_2, X_3, X_4, X_5, X_6, X_7) = X_1.g3(X_2, X_3, X_4)$ $+ X_5.g4(X_6, X_7, X_4) - F.\ g5(z_b, z_t, T_F) - Q$		$= 0$	(eq. 7)

There are 7 variables and 7 equations, so d.f. = 0.

Convert the set of linear and nonlinear equations to a set of linear equations by linearization around an initial guess of all the variables:

Applying Taylor Series expansion we get:

$$\begin{bmatrix} f_1'(x_1) & f_1'(x_2) & f_1'(x_3). & f_1'(x_4)\ldots\ldots & f_1'(x_7) \\ f_2'(x_1) & f_2'(x_2) & f_2'(x_3). & f_2'(x_4)\ldots\ldots & f_2'(x_7) \\ & & \vdots & & \\ & & \vdots & & \\ f_{71}'(x_1) & f_7'(x_2) & f_7'(x_3). & f_7'(x_4)\ldots\ldots & f_7'(x_7) \end{bmatrix} \begin{bmatrix} \partial_1 \\ \partial_2 \\ \vdots \\ \vdots \\ \partial_7 \end{bmatrix} = \begin{bmatrix} -f_1 \\ -f_2 \\ \vdots \\ \vdots \\ -f_7 \end{bmatrix}$$

The jacobian matrix can be written as:

$$
J = \begin{bmatrix}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
 & 1.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\
 & X_2 & X_1 & 0.0 & 0.0 & X_6 & X_5 & 0.0 \\
 & X_3 & 0.0 & X_1 & 0.0 & X_7 & 0.0 & X_5 \\
 & 0.0 & -1.0 & 0.0 & X_6\dfrac{\partial g_1}{\partial X_4} & 0.0 & g_1 & 0.0 \\
 & 0.0 & 0.0 & -1.0 & X_7\dfrac{\partial g_2}{\partial X_4} & 0.0 & 0.0 & g_2 \\
 & 0.0 & 1.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
 & g_3 & X_1\dfrac{\partial g_3}{\partial X_2} & X_1\dfrac{\partial g_3}{\partial X_3} & \left(X_1\dfrac{\partial g_3}{\partial X_4}+X_5\dfrac{\partial g_4}{\partial X_4}\right) & g_4 & X_5\dfrac{\partial g_4}{\partial X_6} & X_5\dfrac{\partial g_4}{\partial X_7}
\end{bmatrix}
$$

Let us look at element j_{44} of the **J** matrix.

$$\frac{\partial g_1}{\partial X_4} = \frac{\partial g_1}{\partial T} = \frac{\partial k_b}{\partial T} = \frac{\partial(\frac{p_b^v}{P})}{\partial T} = \frac{1}{P}\frac{\partial p_b^v}{\partial T}$$

Now, $$\text{Log}_{10} p_b^v = A_b - \left(\frac{B_b}{C_b + T}\right)$$

Let say: $y = p_b^v$ and $z = A_b - \left(\frac{B_b}{C_b + T}\right)$

so, $\text{Log}_{10} y = z$ or, $y = 10^z$ and $\frac{dy}{dz} = \ln(10).10^z$

$$\frac{dy}{dT} = \frac{dy}{dz}.\frac{dz}{dT} \qquad \frac{dz}{dT} = +\frac{B_b}{(C_b+T)^2}$$

Therefore, $$\frac{1}{P}\frac{\partial p_b^v}{\partial T} = \frac{1}{P}\frac{dy}{dT} = \frac{1}{P}\frac{dy}{dz}.\frac{dz}{dT} = \frac{1}{P}(\ln(10).10^z.\frac{B_b}{(C_b+T)^2}) = k_b.\ln(10).\frac{B_b}{(C_b+T)^2}$$

so, $$j_{44} = X_6.\frac{\partial g_1}{\partial X_4} = X_6.\frac{\partial g_1}{\partial T} = X_6.\frac{\partial k_b}{\partial T} = X_6.k_b.\ln(10).\frac{B_b}{(C_b+T)^2}$$

Similarly evaluate other elements of the above jacobian matrix.

With the following initial guesses of the unknown variables solve the system of equations (1-7).

$$x_1^{(1)} = 30.0 \quad x_2^{(1)} = 0.7 \quad x_3^{(1)} = 0.3 \quad x_4^{(1)} = 350.0$$
$$x_5^{(1)} = 70.0 \quad x_6^{(1)} = 0.5 \quad x_7^{(1)} = 0.5$$

Hint: You have to solve a 7x7 system. Try at least to achieve two iterations.